COMP 3008 Homework #9 Krista Miller

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Part 1:

1. Suppose $H \neq \{\mathbf{0}\}$ is a subspace of a finite dimensional vector space V and T is a one-to-one linear transformation of V into a finite dimensional vector space W. Prove that $\dim T(H) = \dim H$.

   Useful Definitions:
   - Given a subspace H with a basis $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ we say that the dimension of H is the cardinality of $\mathcal{B}$.
   - If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, then $T(x)=Ax$ for some A and satisfies the following properties: $T(x+y)=T(x)+T(y)$ and $T(ax)=aT(x)$.
   - T is one-to-one iff the rows of A are linearly independent.

   Proof:

   Assume $\dim H = k$ and $\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ be a basis for H.

   $T\{\mathbf{v}_1, \ldots \mathbf{v}_k\} = \{T(\mathbf{v}_1),, \ldots T(\mathbf{v}_k)\}$

   $c_1T(\mathbf{v}_1)+\ldots+c_kT(\mathbf{v}_k)=\mathbf{0}$

   $T(c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k)=\mathbf{0}$

   $c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k=\mathbf{0}$, so $c_1,\ldots c_k=\mathbf{0}$. Therefore $\{T(\mathbf{v}_1),, \ldots T(\mathbf{v}_k)\}$ is linearly independent set of T(H). Therefore, $\dim T(H) = \dim H$.

2. Given two vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$, the outer product of $\mathbf{u}$ and $\mathbf{v}$ is defined as the mxn matrix $\mathbf{u}\mathbf{v}^T$.

   a. Fined the outer product of $\mathbf{u}=\begin{bmatrix}2\\-1\\1\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}3\\4\\1\end{bmatrix}$.

   outer product= $\mathbf{u}\mathbf{v}^T = \begin{bmatrix}2\\-1\\1\end{bmatrix}\begin{bmatrix}3 & 4 & 1\end{bmatrix} = \begin{bmatrix}2\cdot 3 & 2\cdot 4 & 2\cdot 1\\(-1)\cdot 3 & (-1)\cdot 4 & (-1)\cdot 1\\1\cdot 3 & 1\cdot 4 & 1\cdot 1\end{bmatrix} =$
   $\begin{bmatrix}6 & 8 & 2\\-3 & -4 & -1\\3 & 4 & 1\end{bmatrix}$

   b. Determine the rank of $\mathbf{u}\mathbf{v}^T$

   $\begin{bmatrix}6 & 8 & 2\\-3 & -4 & -1\\3 & 4 & 1\end{bmatrix} E_{23}(1) \rightarrow \begin{bmatrix}6 & 8 & 2\\0 & 0 & 0\\3 & 4 & 1\end{bmatrix} E_{31}(-1/2) \rightarrow \begin{bmatrix}6 & 8 & 2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

   1 linearly independent row, therefore the rank of $\mathbf{u}\mathbf{v}^T=1$

   c. Prove that the rank of the outer product of any two vectors is either 1 or 0.

Useful Definitions:

- Transpose: a matrix operator that switches the row and column indices of matrix A by producing another matrix, denoted A^T
- Rank: the dimension of the vector space spanned by its columns, which is the number of linearly independent columns of A. This is identical to the dimension of the vector space spanned by its rows.

Proof:

Suppose **x** is a vector in $\mathbb{R}^m$ and **y** is a vector in $\mathbb{R}^n$. $\mathbf{xy}^T$ is the outer product of the vectors, given by the matrix with the entries:

$$\mathbf{xy}^T = \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 \cdot y_1 & \cdots. & x_1 \cdot y_n \\ \cdots. & \cdots & \cdots \\ x_m \cdot y_1 & \cdots & x_m \cdot y_n \end{bmatrix}$$

Each column of the outer product matrix will be a linear combination of the vector that created it. The columns of the outer product are all proportional to the first column so they are linearly independent.

If **x** and **y** are both nonzero, then the outer product matrix $\mathbf{xy}^T$ always has matrix rank 1.

Every column of A is a multiple of **x** and every row is a multiple of **y**. If **x** or **y** is zero vector, then the outer product matrix $\mathbf{xy}^T$ has matrix rank 0:

$$\mathbf{xy}^T = \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix} \begin{bmatrix} 0y_1 & \cdots & 0y_n \end{bmatrix} = \begin{bmatrix} x_1 \cdot 0y_1 & \cdots. & x_1 \cdot 0y_n \\ \cdots. & \cdots & \cdots \\ x_m \cdot 0y_1 & \cdots & x_m \cdot 0y_n \end{bmatrix} = \begin{bmatrix} 0 & \cdots. & 0 \\ \cdots. & \cdots & \cdots \\ 0 & \cdots & 0 \end{bmatrix}$$

3. Given two vectors **u** and **v** both in $\mathbb{R}^n$, the inner product of **u** and **v**, often written $\mathbf{u} \cdot \mathbf{v}$ is defined as the scalar $\mathbf{u}^T\mathbf{v}$. Prove that $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Useful Definitions:

- Transpose: a matrix operator that switches the row and column indices of matrix A by producing another matrix, denoted A^T

Proof:

Suppose $\mathbf{u} = [u_1, u_2, \ldots., u_n] = \mathbf{0}$. Therefore, $\mathbf{u} = [0, 0, \ldots., 0_n]$ and $\mathbf{u}^T = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$

$\mathbf{u} \cdot \mathbf{u} = \mathbf{u}^T\mathbf{u} = \sum_{i=1}^{n} \boldsymbol{u}_i^T \boldsymbol{u}_i$

$\quad = \mathbf{u}^T_1\mathbf{u}_1 + \mathbf{u}^T_2\mathbf{u}_2 + \ldots + \mathbf{u}^T_n\mathbf{u}_n = 0(0) + 0(0) + \ldots 0(0) = 0$

Conversely, suppose $\mathbf{u} = [u_1, u_2, \ldots., u_n] \neq \mathbf{0}$. Therefore $\mathbf{u} = [x, y, \ldots., z_n]$ and $\mathbf{u}^T = \begin{bmatrix} x \\ y \\ \cdots \\ z \end{bmatrix}$

$\mathbf{u} \cdot \mathbf{u} = \mathbf{u}^T\mathbf{u} = \sum_{i=1}^{n} \boldsymbol{u}_i^T \boldsymbol{u}_i$

$\quad = \mathbf{u}^T_1\mathbf{u}_1 + \mathbf{u}^T_2\mathbf{u}_2 + \ldots + \mathbf{u}^T_n\mathbf{u}_n = x(x) + y(y) + \ldots z(z) \neq 0$

Therefore, $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

4. Show that if the nxp matrix A with n>p has full rank p then the matrix $A^T A$ is invertible. (Hint: consider the matrix equation $A^T A\mathbf{x} = \mathbf{0}$ and introduce $\mathbf{x}^T$ to transform the equation to the inner product of $A\mathbf{x}$ and itself).

   Useful Definitions:

   - Transpose: a matrix operator that switches the row and column indices of matrix A by producing another matrix, denoted A^T
   - Rank: the dimension of the vector space spanned by its columns, which is the number of linearly independent columns of A. This is identical to the dimension of the vector space spanned by its rows.
   - Invertible matrix: an nxn square matrix A is invertible if there exists an nxn square matrix B such that $AB = BA = I_n$

   Proof:

   Suppose nxp matrix A with n>p has full rank p.

   $A^T A\mathbf{x} = \mathbf{0}$

   $(A^T)^{-1} = (A^{-1})^T$

   $(A^T)^{-1} A^T A\mathbf{x} = (A^T)^{-1}\, \mathbf{0}$

   $A\mathbf{x} = \mathbf{0}$

5. Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for $\mathbb{R}^2$.

   Suppose $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   a. Find the transition matrix from C to B.

   $$\left[\begin{array}{cc|cc} 7 & 2 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{array}\right]$$

   rref C:

   $$\begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/3 & -1/3 \\ -2/3 & 5/3 \end{bmatrix}$$

   b. Suppose $\mathbf{x} = 2\mathbf{c}_1 - 5\mathbf{c}_2$. Find $[\mathbf{x}]_B$

   $\mathbf{x} = 2\begin{bmatrix} 7 \\ -2 \end{bmatrix} - 5\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -4 \end{bmatrix} - \begin{bmatrix} 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

   $[\mathbf{x}]_B = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 5b_1 - b_2$

   c. Find the transition matrix from B to C.

   $$\left[\begin{array}{cc|cc} 1 & 1 & 7 & 2 \\ 0 & -1 & -2 & -1 \end{array}\right]$$

   rref B:

   $$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$$

d. Suppose $\mathbf{x}=2\,\mathbf{b}_1 - 5\,\mathbf{b}_2$. Find $[\mathbf{x}]_C$

$\mathbf{x}=2\begin{bmatrix}1\\0\end{bmatrix}-5\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}-\begin{bmatrix}5\\-5\end{bmatrix}=\begin{bmatrix}-3\\5\end{bmatrix}$

$[\mathbf{x}]_C= -3\begin{bmatrix}7\\-2\end{bmatrix}+5\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}-11\\1\end{bmatrix}= -3c_1+5c_2$

6. Let $M=\begin{bmatrix}1&2&-1\\-3&-5&0\\4&6&1\end{bmatrix}$ and let $\mathbf{v}_1=\begin{bmatrix}5\\-3\\4\end{bmatrix}$, $\mathbf{v}_2=\begin{bmatrix}-8\\2\\7\end{bmatrix}$, and $\mathbf{v}_3=\begin{bmatrix}2\\6\\-1\end{bmatrix}$

a. Find a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ so that M is the transition matrix from $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$U=M^{-1}VM$

$M^{-1}=\begin{bmatrix}5&8&5\\-3&-5&-3\\-2&-2&-1\end{bmatrix}$

$U=\begin{bmatrix}5&8&5\\-3&-5&-3\\-2&-2&-1\end{bmatrix}\begin{bmatrix}5&-8&2\\-3&2&6\\4&7&-1\end{bmatrix}\begin{bmatrix}1&2&-1\\-3&-5&0\\4&6&1\end{bmatrix}$

$U=\begin{bmatrix}200&305&32\\-123&-187&-21\\-83&-131&-7\end{bmatrix}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}=\{\begin{bmatrix}200\\-123\\-83\end{bmatrix},\begin{bmatrix}305\\-187\\-131\end{bmatrix},\begin{bmatrix}32\\-21\\-7\end{bmatrix}\}$

b. Find a basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ so that M is the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

$W=MVM^{-1}$

$W=\begin{bmatrix}1&2&-1\\-3&-5&0\\4&6&1\end{bmatrix}\begin{bmatrix}5&-8&2\\-3&2&6\\4&7&-1\end{bmatrix}\begin{bmatrix}5&8&5\\-3&-5&-3\\-2&-2&-1\end{bmatrix}$

$W=\begin{bmatrix}-22&-15&-7\\30&2&-6\\-17&27&26\end{bmatrix}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}=\{\begin{bmatrix}-22\\30\\-17\end{bmatrix},\begin{bmatrix}-15\\2\\27\end{bmatrix},\begin{bmatrix}-7\\-6\\26\end{bmatrix}\}$

Part 2:

1. Prove that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

Useful Definitions:

- Transpose: a matrix operator that switches the row and column indices of matrix A by producing another matrix, denoted A^T
- Eigenvalue: a scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of the equation $A\mathbf{x}= \lambda\mathbf{x}$, and such an $\mathbf{x}$ is called an eigenvector corresponding to λ ($\mathbf{x}\neq\mathbf{0}$).

Proof:

Suppose λ is an eigenvalue of A.

Therefore, the $\det(A-\lambda I)=0$. This means the matrix $(A- \lambda I)$ is not invertible.

so, $(A- \lambda I)^T$ is equivalent to $(A^T - \lambda I)$, and is not invertible

Therefore, λ is an eigenvalue of A^T.

2. Prove that if λ is an eigenvalue of an invertible matrix A, then $1/\lambda$ is an eigenvalue for A^{-1}.

<u>Useful Definitions:</u>

- Invertible matrix: an nxn square matrix A is invertible if there exists an nxn square matrix B such that $AB=BA=I_n$
- Eigenvalue: a scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of the equation $A\mathbf{x}=\lambda\mathbf{x}$, and such an $\mathbf{x}$ is called an eigenvector corresponding to λ ($\mathbf{x}\neq\mathbf{0}$).

Suppose A is an invertible matrix and λ is an eigenvalue:

$Ax=\lambda x$
$A^{-1}Ax=A^{-1}\lambda x$
$Ix=A^{-1}\lambda x$
$x=\lambda A^{-1}x$ (Since λ is a scalar, we can divide both sides by λ)
$\frac{1}{\lambda}x=A^{-1}x$

So, $\frac{1}{\lambda}$ is an eigenvalue for A^{-1}.

3. Compute the eigenvalues of the matrix.

a. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$$0=\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 0-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}=(2-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix}-0\begin{vmatrix} 0 & 1 \\ 0 & 2-\lambda \end{vmatrix}+1\begin{vmatrix} 0 & 1 \\ -\lambda & 0 \end{vmatrix}$$

$=(2-\lambda)((-\lambda)(2-\lambda))+1(-(-\lambda))$
$=(2-\lambda)((-\lambda)(2-\lambda))+(\lambda)$
$=-(2-\lambda)\lambda(2-\lambda)+\lambda$
$=-\lambda(2-\lambda)^2+\lambda$
$=-4\lambda+4\lambda^2-\lambda^3+\lambda$
$0=-\lambda^3+4\lambda^2-3\lambda$
$0=-\lambda(\lambda-1)(\lambda-3)$

$\boldsymbol{\lambda_1=0, \lambda_2=1, \lambda_3=3}$

b. $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (upper triangular case, diagonal entries are eigenvalues)

$\boldsymbol{\lambda_1=2}$ (with multiplicity of 2), $\boldsymbol{\lambda_2=3}$

c. $\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ (lower triangular case, diagonal entries are eigenvalues)

$\boldsymbol{\lambda_1=-1}$(with multiplicity of 3)

4. For each matrix in the problem above, find the basis for the eigenspace corresponding to each eigenvalue.

a. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$

$\lambda_1 = 0$:

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 0-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} E_{31}(-1/2) \rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} E_{23} \rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} E_1(1/2), E_2(2/3), \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{12}(-1/2) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; if $v_2 = t$, then $v = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigenspace corresponding to eigenvalue of 0= Span$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\lambda_2 = 1$:

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 0-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} E_{31}(-1) \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} E_2(-1) \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; if $v_3 = t$, then $v = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Eigenspace corresponding to eigenvalue of 1= Span$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda_3 = 3$:

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 0-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -1 \end{bmatrix} E_{31}(1) \rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} E_1(-1), E_2(-3), \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; if $v_3 = t$, then $v = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Eigenspace corresponding to eigenvalue of 3= Span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

b. $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

λ₁ =2 (with multiplicity of 2)**:**

$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ E_{21}(-1)→ $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; if v_2=t, v_3=r, then v_1=t, v_2=-r, v_3=r, then **v**=$\begin{bmatrix} t \\ -r \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} r$

Eigenspace corresponding to eigenvalue of 0= Span{$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$}

λ₂=3:

$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ E_{32}(1) → $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; if v_2=t, then **v**=$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t$

Eigenspace corresponding to eigenvalue of 3= Span{$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$}

c. $\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

λ₁ =-1(with multiplicity of 3):

$\begin{bmatrix} -\lambda-1 & 0 & 0 \\ 1 & -\lambda-1 & 0 \\ 0 & 1 & -\lambda-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ E_{21}, E_{32}→ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; if v_3=t, then **v**=$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t$

Eigenspace corresponding to eigenvalue of -1= Span{$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$}

d. Diagonalize each matrix (if possible) in the problem above. If the matrix is not diagonalizable, then explain why

a. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$: has eigenvalues λ_1 =0 with eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, λ_2=1 with eigenvector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, λ_3=3 with eigenvector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Forming matrix P with the eigenvectors and a corresponding diagonal matrix D, whose element at row i, column i, is the i-th eigenvalue:

$A=PDP^{-1}$

$P=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$D=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$: has eigenvalues λ_1 =2 with eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and λ_2=3 with eigenvector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Forming matrix P with the eigenvectors and a corresponding diagonal matrix D, whose element at row i, column i, is the i-th eigenvalue:

$A=PDP^{-1}$

$P=\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

c. $\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$: has eigenvalue λ_1 =-1 and one eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The number of eigenvectors is less than the dimension of the matrix, so this matrix is not diagonalizable.

e. Show that matrix similarity is an equivalence relation on the set of nxn matrices.

Useful Definitions:

- Matrix similarity: given matrices A, B, we say that they are similar if there exists an invertible matrix P such that $A=P^{-1}BP$
- Equivalence relation: a relation that is reflexive, symmetric, and transitive

Proof:

Reflexivity:

$A=I^{-1}AI$

Symmetry:

$B=P^{-1}AP$

$PBP^{-1} = PP^{-1}APP^{-1}$

$=IAI$

$=A$

Transitive:

Suppose $B= P^{-1}AP$ and $C= P^{-1}BP$. Substituting B:

Then $C=P^{-1}\ P^{-1}APP$, so C is also similar to A