COMP 3008 Homework #10 Krista Miller

3/23/2021

<u>Part 1:</u>

1. Find the B-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ when is given by the indicated vectors $\mathbf{b}_i$.
   a. $A=\begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$ with $\mathbf{b}_1=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{b}_2=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   b. $A=\begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix}$ with $\mathbf{b}_1=\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2=\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, and $\mathbf{b}_3=\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$

2. Let $B=\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V. Let T: V→V be a linear transformation for which $[T]_B=\begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$. Find $T(3\mathbf{b}_1\text{-}4\mathbf{b}_2)$.

3. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^n$ of the same length, then $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{u}-\mathbf{v}$.

4. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, verify that

   $$||\,\mathbf{u}+\mathbf{v}\,||^2 + ||\,\mathbf{u}-\mathbf{v}\,||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$$

   Describe geometrically what this equation means.

5. Suppose vector $\mathbf{x}$ is orthogonal to vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Show that $\mathbf{x}$ is orthogonal to every $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

<u>Part 2:</u>

1. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is any angle θ satisfying $\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{||\boldsymbol{u}||\,||\boldsymbol{v}||} = \cos\theta$

   Let $\mathbf{u}=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
   a. Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

   b. Find unit vectors in the directions of $\mathbf{u}$ and $\mathbf{v}$.

   c. Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

2. Let V be a nonzero subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{v}_1, \ldots. \mathbf{v}_k\}$ and let T: $\mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(x) = \mathbf{proj}_V\mathbf{x}$. Show that T is a linear transformation.

3. Let U and V be orthogonal matrices.

a. Prove that the product UV is also orthogonal.

b. Find examples U and V for which U+V is not orthogonal.

4. Find the closest point to $\mathbf{x}=\begin{bmatrix}3\\1\\5\\1\end{bmatrix}$ in the subspace spanned by $\mathbf{w}_1=\begin{bmatrix}3\\1\\-1\\1\end{bmatrix}$ and $\mathbf{w}_2=\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$

5. Let A be an mxn matrix. Prove that $\mathbf{x} \in \mathbb{R}^n$ can be written in the form $\mathbf{p} + \mathbf{u}$, where $\mathbf{p} \in$ Row A and $\mathbf{u} \in$ Nul A. Further, show that if the equation A**x**=**b** is consistent, then there is a unique $\mathbf{p} \in$ Row A such that A**p**=**b**.

6. Apply the Gram-Schmidt process to produce an orthonormal basis for the subspace W spanned by the indicate vectors.
   a. $\mathbf{w}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{w}_2=\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{w}_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$,
   b. $\mathbf{w}_1=\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$, $\mathbf{w}_2=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$, $\mathbf{w}_3=\begin{bmatrix}2\\2\\1\\0\end{bmatrix}$, $\mathbf{w}_4=\begin{bmatrix}1\\2\\3\\1\end{bmatrix}$

7. For cases when the equation A**x**=**b** is inconsistent, there may still be a need to find an "approximate" solution. In that case, A**x**≠**b**, but the smaller the distance || A**x**-**b**|| is, the better the approximation is. The general least squares problem seeks to find a solution **x** that makes || A**x**-**b**|| as small as possible.
   One method to find a least-squares solution is to solve the equation
   $$A^T A\mathbf{x}=A^T\mathbf{b}$$
   Verify that A**x**=**b** is an inconsistent system; then using the method just described, find a least squares solution of A**x**=**b** when

   $A=\begin{bmatrix}1 & 3\\1 & -1\\1 & 1\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}5\\1\\0\end{bmatrix}$