COMP 3008 Homework #7 Krista Miller

3/2/2021

<u>Part 1:</u>

1. Given that the vector **x**= $\begin{bmatrix} 3 \\ 0 \\ -1 \\ 6 \end{bmatrix}$ is a solution of the matrix equation A**x**=**b** and that the set of all solutions of the homogenous equation A**x**=**0** is equal to Span $\{\begin{bmatrix} 1 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}\}$, is it possible to determine whether **w**= $\begin{bmatrix} 5 \\ -4 \\ 7 \\ 5 \end{bmatrix}$ is also a solution of A**x**=**b**? Explain, and if it is possible to determine, then is it a solution of the matrix equation or not? *problem completed in office hours with Jurgen 3/2/2021.

$$\begin{bmatrix} 5 \\ -4 \\ 7 \\ 5 \end{bmatrix}=a\begin{bmatrix} 1 \\ 1 \\ 4 \\ 1 \end{bmatrix}+b\begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}+\begin{bmatrix} 3 \\ 0 \\ -1 \\ 6 \end{bmatrix}=\begin{bmatrix} 3+a \\ a+2b \\ 4a-1 \\ a-b+6 \end{bmatrix}$$

a=2

$$\begin{bmatrix} 3+2 \\ 2+2b \\ 4(2)-1 \\ 2-b+6 \end{bmatrix}=\begin{bmatrix} 5 \\ -4 \\ 7 \\ 5 \end{bmatrix}$$, b= -2 and 3 so **w** is not in the span

Yes, it is possible to determine if it is a solution. Vector **w** is not in the span because there is not one value for b that satisfies the function.

2. Prove that the columns of matrix A are linearly independent if and only if A**x**=**0** has only the trivial solution.

Null space of A: N(A)= $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$

A is mxn **x** is nx1 components

$$A=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$A=[\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \dots \quad \boldsymbol{v}_n]\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

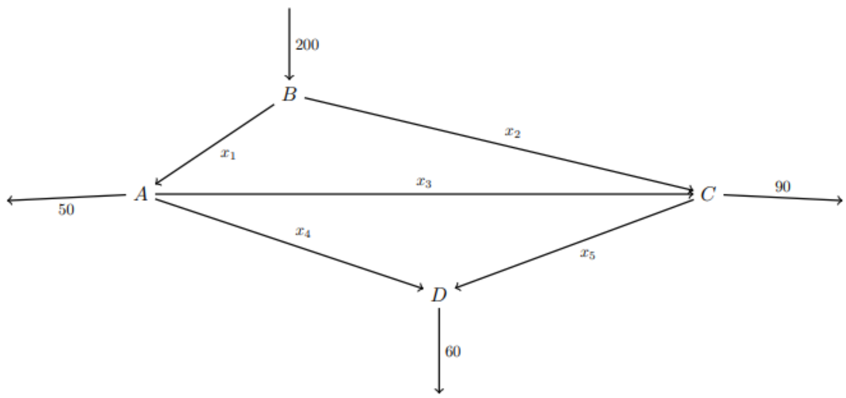
$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n=\mathbf{0}$

$\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n$ are linearly independent if and only if the weights to these vectors all equal zero. So, the only solution is $x_1, x_2, \dots x_n=0$. The only way to get the linear combinations of all the vectors to equal zero is if $x_1, x_2, \ldots x_n=0$. This is true if the null space of A contains only the zero vector.

3. Prove that any set of p vectors in $\mathbb{R}^n$ is linearly dependent if $p > n$.

A matrix with more columns than rows has linearly dependent columns. For example, 3 vectors in $\mathbb{R}^2$ are automatically linearly dependent because one of the three must be a linear combination of the others.

4. Consider the network shown here with the indicated flow rates.



a. Find the general flow pattern for the network.

Flow in = flow out

A: $x_1=50+x_3+x_4$

$x_1-x_3-x_4=50$

B: $200= x_1+x_2$

$x_1+x_2=200$

C: $x_2+x_3=90 +x_5$

$x_2+x_3-x_5=90$

D: $x_4 +x_5= 60$

$x_4 +x_5= 60$

Augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 50 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 90 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix}$$

Row-reduced augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 50 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 90 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix} \rightarrow E_{21}(-1) \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 50 \\ 0 & 1 & 1 & 1 & 0 & 150 \\ 0 & 1 & 1 & 0 & -1 & 90 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix}$$

$\rightarrow E_{32}(-$

1) $\begin{bmatrix}1&0&-1&-1&0&50\\0&1&1&1&0&150\\0&0&0&-1&-1&-60\\0&0&0&1&1&60\end{bmatrix} \to E_{43}(1) \begin{bmatrix}1&0&-1&-1&0&50\\0&1&1&1&0&150\\0&0&0&1&1&60\\0&0&0&0&0&0\end{bmatrix} \to R_{21}(1)$

$\begin{bmatrix}1&0&-1&-1&0&50\\0&1&1&1&0&150\\0&0&0&1&1&60\\0&0&0&0&0&0\end{bmatrix} \to R_{13}(1) \begin{bmatrix}1&0&-1&0&1&110\\0&1&1&1&0&150\\0&0&0&1&1&60\\0&0&0&0&0&0\end{bmatrix} \to$

$R_{23}(-1) \begin{bmatrix}1&0&-1&0&1&110\\0&1&1&0&-1&90\\0&0&0&1&1&60\\0&0&0&0&0&0\end{bmatrix}$ x_3 and x_5 are free variables

$x_1 = 110 + x_3 - x_5$
$x_2 = 90 - x_3 + x_5$
$x_4 = 60 - x_5$

b. If a new road into *C* with a flow rate of 70 is added to the network, explain why there will be no viable flow pattern.
$C = 70 + x_2 + x_3 = 90 + x_5$

$x_2 + x_3 = 20 + x_5$

$\begin{bmatrix}1&0&-1&-1&0&50\\1&1&0&0&0&200\\0&1&1&0&-1&20\\0&0&0&1&1&60\end{bmatrix} \to R_{21}(-1) \begin{bmatrix}1&0&-1&-1&0&50\\0&1&1&1&0&150\\0&1&1&0&-1&20\\0&0&0&1&1&60\end{bmatrix} \to R_{32}(-1)$

$\begin{bmatrix}1&0&-1&-1&0&50\\0&1&1&1&0&150\\0&0&0&-1&-1&-130\\0&0&0&1&1&60\end{bmatrix}$ After row-reducing the augmented matrix, we see

that one of the equations in the system is: $-x_4 - x_5 = -130$. It doesn't make sense to simultaneously have negative flow into a node and negative flow out of a node.

c. How can the flow rate out of the network at *A* be changed (from 50) to accommodate this new road into *C*, and with that adjustment, what will the general flow pattern for the network be?

If we change the flow rate out of the network at *A* from 50 to 190, this would accommodate the new road into *C*. The general flow pattern for the network would now be:

$\begin{bmatrix}1&0&-1&-1&0&190\\0&1&1&1&0&10\\0&0&0&-1&-1&10\\0&0&0&1&1&60\end{bmatrix}$

$x_1 = 190 + x_3 - x_5$
$x_2 = 10 - x_3 + x_4$

x_4 =60-x_5

5. Suppose A= [$\mathbf{v}_1$ $\mathbf{v}_2$ $\mathbf{v}_3$ $\mathbf{v}_4$] and $\mathbf{v}_4$= $\mathbf{v}_1$-2$\mathbf{v}_2$. Find a nontrivial solution of the equation $A\mathbf{x}=\mathbf{0}$.

   A= [$\mathbf{v}_1$ $\mathbf{v}_2$ $\mathbf{v}_3$ ($\mathbf{v}_1$-2$\mathbf{v}_2$)]

   A is a 1x4 matrix, **x** must be 4 x 1 matrix

   $$[\ \mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ (\mathbf{v}_1\text{-}2\mathbf{v}_2)\ ]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\mathbf{0}$$

   $$\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 + \boldsymbol{v}_3 x_3 + (\boldsymbol{v}_1 2\boldsymbol{v}_2) x_4 = \mathbf{0}$$

   Notice that $\mathbf{v}_4$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so by the definition of linear independence, the vector equation has a nontrivial solution. Setting x_1=1, x_2=1, x_3=1, and x_4= 1 yields the nontrivial combination:

   $$\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 + (\boldsymbol{v}_1 2\boldsymbol{v}_2) = \mathbf{0}$$

6. The fact that the set of vectors $\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\}$ is linearly dependent can be justified with absolutely no computations whatsoever. Explain how. Then, determine a linear dependence relation among the vectors in the set. Make sure to show work to justify your reasoning and result.

   $$A=\begin{bmatrix} 2 & 3 & 2 & 1 \\ 1 & 4 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

   These four columns in $\mathbb{R}^3$ cannot be linearly independent. We do not have a pivot position in every column, which means that the system of equations has nontrivial solutions, which means that the given vectors are linearly dependent.

   Augmented matrix:

   $$A=\begin{bmatrix} 2 & 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

   Row reduced matrix=

   $$\begin{bmatrix} 2 & 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow E_{21} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 2 & 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow E_{21}(-2) \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & -5 & 2 & -3 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow$$

   $$E_{31}(-1) \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & -5 & 2 & -3 & 0 \\ 0 & -3 & 1 & -1 & 0 \end{bmatrix} \rightarrow E_2(-1/5) \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 1 & -2/5 & 3/5 & 0 \\ 0 & -3 & 1 & -1 & 0 \end{bmatrix} \rightarrow$$

   $$E_{32}(3) \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 1 & -2/5 & 3/5 & 0 \\ 0 & 0 & -1/5 & 4/5 & 0 \end{bmatrix} \rightarrow E_3(-5) \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 1 & -2/5 & 3/5 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{bmatrix}$$

$x_1+6x_4=0$
$x_2-x_4= 0$
$x_3-4x_4=0$
x_4 is a free variable and can be expressed in terms of the other vectors, therefore vector $\begin{bmatrix}1\\2\\1\end{bmatrix}$ represents a redundancy.

Part 2:

1. Show that the mapping given by $T(x,y)=(2x-3y, x+4, 5y)$ is not a linear transformation.

   Definition of a linear transformation: a transformation T is linear if for all **u**, **v** in the domain of T and all scalars c:

   a. T(**u**+**v**)= T(**u**) +T(**v**)
   b. T(c**u**)= cT(**u**)

   Suppose $\mathbf{u}=\begin{bmatrix}2\\3\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}3\\4\end{bmatrix}$

   Therefore, $\mathbf{u}+\mathbf{v}=\begin{bmatrix}5\\7\end{bmatrix}$

   $$T(\mathbf{u}+\mathbf{v})=T(\begin{bmatrix}5\\7\end{bmatrix})=\begin{bmatrix}2(5)-3(7)\\5+4\\5(7)\end{bmatrix}=\begin{bmatrix}-11\\9\\35\end{bmatrix}$$

   $$T(\mathbf{u})= T(\begin{bmatrix}2\\3\end{bmatrix})=\begin{bmatrix}2(2)-3(3)\\2+4\\5(3)\end{bmatrix}=\begin{bmatrix}-5\\6\\15\end{bmatrix}$$

   $$T(\mathbf{v})= T(\begin{bmatrix}3\\4\end{bmatrix}) =\begin{bmatrix}2(3)-3(4)\\3+4\\5(4)\end{bmatrix}=\begin{bmatrix}-6\\7\\20\end{bmatrix}$$

   $$T(\mathbf{u}) +T(\mathbf{v})=\begin{bmatrix}-11\\13\\35\end{bmatrix}$$

   $$\begin{bmatrix}-11\\13\\35\end{bmatrix} \neq \begin{bmatrix}-11\\9\\35\end{bmatrix}$$

   Therefore, T is not linear because T(u) +T(v) $\neq$ T(u+v)

2. Let $T: \mathbb{R}^5 \rightarrow \mathbb{R}^5$ be the linear transformation with the standard matrix $A=$ $\begin{bmatrix}4 & -7 & 3 & 7 & 5\\6 & -8 & 5 & 12 & -8\\-7 & 10 & -8 & 9 & 14\\-5 & -6 & -8 & 9 & 8\\13 & 14 & 15 & 2 & 11\end{bmatrix}$. Is T **one-to-one**? Does T map **onto** $\mathbb{R}^5$? Explain.

   using a matrix row reduction calculator to check for pivots:

$$\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & 9 & 14 \\ -5 & -6 & -8 & 9 & 8 \\ 13 & 14 & 15 & 2 & 11 \end{bmatrix} \text{rref (T)}= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Because T contains a pivot in every row, T maps onto $\mathbb{R}^5$. Because T contains a pivot in very column, it is one-to-one.

3. Find the inverse of each matrix using the row operations algorithm. Make sure to identify the row operation(s) applied in each step.

a. $A=\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to E_{12}(-2) \begin{bmatrix} 1 & 0 & -6 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\to E_{23}(-3) \begin{bmatrix} 1 & 0 & -6 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to E_{13}(6) \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 6 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}= \begin{bmatrix} 1 & -2 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

b. $A=\begin{bmatrix} -3 & 1 & 4 \\ 1 & 0 & -2 \\ 2 & -3 & 4 \end{bmatrix}$

$$\begin{bmatrix} -3 & 1 & 4 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \to E_{21} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 \\ -3 & 1 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$$\to E_{21}(3) \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \to E_{31}(-2) \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & -3 & 8 & 0 & -2 & 1 \end{bmatrix}$$

$$\to E_{32}(3) \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 3 & 7 & 1 \end{bmatrix} \to E_{3}(1/2) \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3/2 & 7/2 & 1/2 \end{bmatrix}$$

$$\to E_{23}(2) \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 & 10 & 1 \\ 0 & 0 & 1 & 3/2 & 7/2 & 1/2 \end{bmatrix} \to E_{13}(2) \begin{bmatrix} 1 & 0 & 0 & 3 & 8 & 1 \\ 0 & 1 & 0 & 4 & 10 & 1 \\ 0 & 0 & 1 & 3/2 & 7/2 & 1/2 \end{bmatrix}$$

$$A^{-1}= \begin{bmatrix} 3 & 8 & 1 \\ 4 & 10 & 1 \\ 3/2 & 7/2 & 1/2 \end{bmatrix}$$

4. Prove that if the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent and T is a linear transformation, then the set of vectors $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is also linearly dependent.

   If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then there exists $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1\mathbf{v}_1+a_2\mathbf{v}_2+a_3\mathbf{v}_3=\mathbf{0}$ where at least one of the values of "a" does not equal zero.

   If we apply T, then:
   $a_1\mathbf{v}_1+a_2\mathbf{v}_2+a_3\mathbf{v}_3=\mathbf{0}$
   $\mathrm{T}(a_1\mathbf{v}_1+a_2\mathbf{v}_2+a_3\mathbf{v}_3)= \mathrm{T}(\mathbf{0})$

   T is linear so $\mathrm{T}(\mathbf{0})=\mathbf{0}$. Therefore,
   $a_1\mathrm{T}(\mathbf{v}_1)+ a_2\mathrm{T}(\mathbf{v}_2)+ a_2\mathrm{T}(\mathbf{v}_2)=\mathbf{0}$ that is a linear combination of vectors $\mathrm{T}(v_1)$, $\mathrm{T}(v_2)$ and $\mathrm{T}(v_3)$ such that at least one coefficient does not equal zero. Therefore, $\{\mathrm{T}(\mathbf{v}_1), \mathrm{T}(\mathbf{v}_2), \mathrm{T}(\mathbf{v}_3)\}$ is linearly dependent.

5. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\3\end{bmatrix} \quad T\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\-1\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}0\\3\\1\end{bmatrix}\right)=\begin{bmatrix}5\\0\end{bmatrix}$$

   Find the standard matrix for T.

   T M = N

$$\mathrm{T}\begin{bmatrix}1 & 2 & 0\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 1 & 5\\3 & -1 & 0\end{bmatrix}$$

   $\mathrm{T}= \mathrm{M}^{-1}\mathrm{N}$

$$\text{To find } \mathrm{M}^{-1}=\begin{bmatrix}1 & 2 & 0 & 1 & 0 & 0\\0 & 1 & 3 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}\to R_{12}(-2)\begin{bmatrix}1 & 0 & -6 & 1 & -2 & 0\\0 & 1 & 3 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}\to R_{13}(6)$$

$$\begin{bmatrix}1 & 0 & 0 & 1 & -2 & 6\\0 & 1 & 3 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}\to R_{23}(-3)\begin{bmatrix}1 & 0 & 0 & 1 & -2 & 6\\0 & 1 & 0 & 0 & 1 & -3\\0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}$$

$$\mathrm{M}^{-1}=\begin{bmatrix}1 & -2 & 6\\0 & 1 & -3\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}2 & 1 & 5\\3 & -1 & 0\end{bmatrix}\begin{bmatrix}1 & -2 & 6\\0 & 1 & -3\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & -3 & 14\\3 & -7 & 21\end{bmatrix}$$

   Standard matrix for $\mathrm{T}=\begin{bmatrix}2 & -3 & 14\\3 & -7 & 21\end{bmatrix}$

6. Let $\mathbf{u}=\begin{bmatrix}3\\-2\\4\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$. Compute these four products: $\mathbf{u}^T\mathbf{v}$, $\mathbf{u}\mathbf{v}^T$, $\mathbf{v}^T\mathbf{u}$, and $\mathbf{v}\mathbf{u}^T$.

$\mathbf{u}^T = [3 \quad -2 \quad 4]$ $\mathbf{v}^T = [x_1 \quad x_2 \quad x_3]$

$\mathbf{u}^T\mathbf{v} = [3 \quad -2 \quad 4]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [3x_1 - 2x_2 + 4x_3]$

$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}[x_1 \quad x_2 \quad x_3] = \begin{bmatrix} 3x_1 & 3x_2 & 3x_3 \\ -2x_1 & -2x_2 & -2x_3 \\ 4x_1 & 4x_2 & 4x_3 \end{bmatrix}$

$\mathbf{v}^T\mathbf{u} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = [3x_1 - 2x_2 + 4x_3]$

$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}[3 \quad -2 \quad 4] = \begin{bmatrix} 3x_1 & -2x_1 & 4x_1 \\ 3x_2 & -2x_2 & 4x_2 \\ 3x_3 & -2x_3 & 4x_3 \end{bmatrix}$

7. For each pair of matrices, **compute the determinant** of the two matrices and compare the results. Then, make a guess about a general property of determinants based on your comparison.

   a. $\begin{bmatrix} 1 & 5 & -6 \\ 3 & 1 & 0 \\ 2 & -4 & 7 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 2 \\ 5 & 1 & -4 \\ -6 & 0 & 7 \end{bmatrix}$

   $$\det\begin{bmatrix} 1 & 5 & -6 \\ 3 & 1 & 0 \\ 2 & -4 & 7 \end{bmatrix} = 1det\begin{bmatrix} 1 & 0 \\ -4 & 7 \end{bmatrix} - 5det\begin{bmatrix} 3 & 0 \\ 2 & 7 \end{bmatrix} + (-6)det\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$$

   $$= 1(1 \cdot 7 - 0 \cdot 4) - 5(3 \cdot 7 - 0 \cdot 2) + (-6)((3 \cdot (-4)) - 1 \cdot 2)$$

   =7-105+84= **-14**

   $$\det\begin{bmatrix} 1 & 3 & 2 \\ 5 & 1 & -4 \\ -6 & 0 & 7 \end{bmatrix} = 1det\begin{bmatrix} 1 & -4 \\ 0 & 7 \end{bmatrix} - 3det\begin{bmatrix} 5 & -4 \\ -6 & 7 \end{bmatrix} + 2det\begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}$$

   $$=1(1 \cdot 7 - (-4) \cdot 0) - 3(5 \cdot 7 - ((-4) \cdot -6)) + 2(5 \cdot 0 - 1 \cdot -6)$$

   =1(7)-3(11)+2(6)=**-14**

   Property: transposing a matrix that has a determinant does not change the determinant.

   b. $\begin{bmatrix} 1 & 5 & -6 \\ 3 & 1 & 0 \\ 2 & -4 & 7 \end{bmatrix}$ and $\begin{bmatrix} 1 & 5 & -6 \\ 2 & -4 & 7 \\ 3 & 1 & 0 \end{bmatrix}$

   $$\det\begin{bmatrix} 1 & 5 & -6 \\ 3 & 1 & 0 \\ 2 & -4 & 7 \end{bmatrix} = \mathbf{-14}$$

$$\det\begin{bmatrix}1 & 5 & -6\\ 2 & -4 & 7\\ 3 & 1 & 0\end{bmatrix} = 1det\begin{bmatrix}-4 & 7\\ 1 & 0\end{bmatrix} - 5det\begin{bmatrix}2 & 7\\ 3 & 0\end{bmatrix} + (-6)det\begin{bmatrix}2 & -4\\ 3 & 1\end{bmatrix}$$

$$=1(-4\cdot 0-(7\cdot 1))-5(2\cdot 0-(7\cdot 3))+\ -6(2\cdot 1-(-4\cdot 3))$$

$=1\cdot(-7)$-5(-21)-6(14)=**14**

Property: swapping the rows of a matrix that has a determinant changes the sign of the determinant. We see in this case that swapping rows 2 and 3 results changes the determinant from negative to positive.

c. $\begin{bmatrix}1 & 5 & -6\\ 3 & 1 & 0\\ 2 & -4 & 7\end{bmatrix}$ and $\begin{bmatrix}1 & 5 & -6\\ 15 & 5 & 0\\ 2 & -4 & 7\end{bmatrix}$

$$\det\begin{bmatrix}1 & 5 & -6\\ 3 & 1 & 0\\ 2 & -4 & 7\end{bmatrix} = -\mathbf{14}$$

$$\det\begin{bmatrix}1 & 5 & -6\\ 15 & 5 & 0\\ 2 & -4 & 7\end{bmatrix} = 1det\begin{bmatrix}5 & 0\\ -4 & 7\end{bmatrix} - 5det\begin{bmatrix}15 & 0\\ 2 & 7\end{bmatrix} + (-6)det\begin{bmatrix}15 & 5\\ 2 & -4\end{bmatrix}$$

$$=1(5\cdot 7-(0\cdot -4))-5(15\cdot 7-(0\cdot 2)+\ -6(15\cdot -4-(5\cdot 2))$$

=35-525+420=**-70**

Property: if a row of a matrix with a determinant is scalar multiplied by a number, then the determinate is also multiplied by the same scalar number. We see that row 2 is multiplied by 5 for the second matrix, and the determinant also gets multiplied by a factor of 5.

8. In a fixed housing market, where there is a mix of single-family homes and multi-unit apartment buildings, a recent demographic study shows that each year about 22% of apartment dwellers move to single-family homes with the remaining 78% remaining in apartments, whereas 10% of single-family home dwellers move to apartments with the remaining 90% staying in a single-family home. If s_k is the proportion of the population living in single-family homes in year k and a_k is the proportion living in multi-unit apartment buildings in year k, the proportion of the population in each dwelling type in year k+1 can be represented by the system of equations:

Single-family homes: $s_{k+1}= 0.90s_k+0.22a_k$

Apartment homes: $a_{k+1} =0.10s_k + 0.78a_k$

a. If we let $x_k=\begin{bmatrix}s_k\\ a_k\end{bmatrix}$ be that "state" of the system in the kth year, then write a matrix equation of the form $\mathbf{x}_{k+1}= M\mathbf{x}_k$.

$\mathbf{x}_{k+1}= M\begin{bmatrix}s_k\\ a_k\end{bmatrix}$

$\mathbf{x}_{k+1}= \begin{bmatrix}0.90s_k + 0.22a_k\\ 0.10s_k + 0.78a_k\end{bmatrix} = \begin{bmatrix}0.90 & 0.22\\ 0.10 & 0.78\end{bmatrix}\begin{bmatrix}s_k\\ a_k\end{bmatrix}$

b. Suppose the initial state of the system is s_0= 40% and a_0=60%. What is the state of the system (that is, what proportion of the population will be living in single-family homes versus apartments) one year later? Two years later?

$\begin{bmatrix} s_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$

One year later:

Single-family homes: s_1= 0.90(0.40)+0.22(0.6)=0.492= 49.2%

Apartment homes: a_1 =0.10(0.40) + 0.78(0.6)=0.508= 50.8%

Two years later:

Single-family homes: s_2=0.90(0.492)+0.22(0.508)=0.5546= 55.46%

Apartment homes: a_2 =0.10(0.492) + 0.78(0.508)=0.4454= 44.54%

c. Assuming that this pattern does not change from year to year, what will happen as time passes? Is there a point at which the system will reach a steady state so that the market planners can predict how many of each housing type they will ultimately need? To answer this equation, solve the equation $M\mathbf{x}=\mathbf{x}$ or equivalently $(M\text{-}I)\mathbf{x}=\mathbf{0}$ (this element of the problem completed with Jurgen during office hours 3/2/2021).

s_k= 0.90s_k+0.22a_k

a_k=0.10s_k +0.78a_k

a_k= -0.78a_k =0.15 s_k

s_k= (a_k -0.78a_k)10

10(a_k-0.78a_k)= 0.90(a_k-0.78a_k)10)+0.22a_k

10a_k-7.8a_k=9a_k-7.02a_k+0.22a_k

2.2a_k=1.98a_k+0.22a_k

2.2a_k=2.2a_k, so a steady state does exist.

a_k=a_k

s_0=0.4, a_0=0.6

s_1=0.492, a_1= 0.508

s_2=0.5546, a_2=0.4454

X=MX

(M-I)X=0

$\begin{bmatrix} 0.90-1 & 0.22-0 \\ 0.10-0 & 0.78-1 \end{bmatrix} \begin{bmatrix} s_k \\ a_k \end{bmatrix} = 0$

-0.10s+ 0.22a =0 s= 2.2a

0.10s - 0.22a =0 a=$\frac{0.1}{0.2}$s=0.45s

Since s+a=1:

2.2a+a=1

3.2a=1

a=0.3125

s+0.3125=1, **s=0.6875**

steady state for s= 0.90(0.6875)+0.22(0.3125)

=0.61875+0.06875=0.6875

steady state for a= 0.1(0.6875)+0.78(0.3125)

=0.3125