COMP 3008 Homework #6 Krista Miller

2/23/2021

Part 1:

1. Prove that graph isomorphism is an equivalence relation.

   Definition of graph isomorphism: two graphs $G_1= (V_1, E_1)$ and $G_2= (V_2, E_2)$ are isomorphic if there is a **bijection** f from V_1 to V_2 so that $(a,b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$

   Definition of bijection: the function is injective (distinct vertices of G_1 are mapped to distinct vertices of G_2) and the function is surjective (every vertex in G_2 should get matched by some vertex in G_1). Therefore, there is an exact one-to-one correspondence between vertices in G_1 and G_2.

   Definition of an equivalence relation: An equivalence relation is a relationship on a set that is reflexive, symmetric, and transitive.

   Proof:

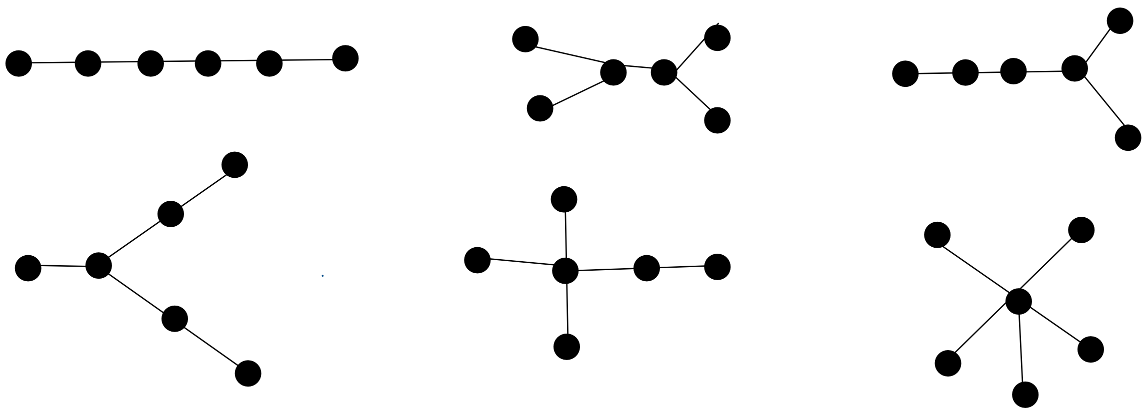
   Reflexivity: Claim: G_1 is isomorphic to G_1. This is true because it is the same graph. The function f preserves adjacency for the vertices and edges so $(a, b) \in E_1$ because $(f(a), f(b)) \in E_1$.

   Symmetric: Claim: G_1 is isomorphic to G_2 and G_2 is isomorphic to G_1.
   Suppose G_1 is isomorphic to G_2. By definition, there is a bijection f from V_1 to V_2 so that $(a,b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$. A bijection has an inverse, so there is also an isomorphism from G_2 to G_1. Therefore, G_2 is isomorphic to G_1.

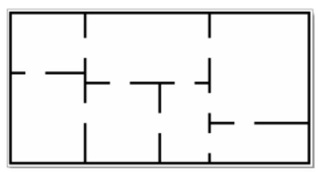
   Transitive: If G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3, then G_1 is isomorphic to G_3.
   Suppose G_1 is isomorphic to G_2. By definition, there is a bijection f from V_1 to V_2 so that $(a,b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$. A bijection has an inverse, so there is also an isomorphism from G_2 to G_1. Therefore, G_2 is isomorphic to G_1.
   Suppose G_2 is isomorphic to G_3. By definition, there is a bijection f from V_2 to V_3 so that $(a,b) \in E_2$ if and only if $(f(a), f(b)) \in E_3$. A bijection has an inverse, so there is also an isomorphism from G_3 to G_2. Therefore, G_3 is isomorphic to G_2.
   Because we can represent G_3 as G_2, then G_1 is isomorphic to G_3.

   So, we can conclude that two isomorphic graphs have an equivalence relation. ■

2. Find the 6 non-isomorphic trees with 6 vertices.

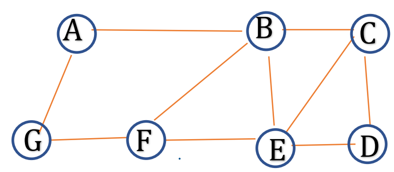


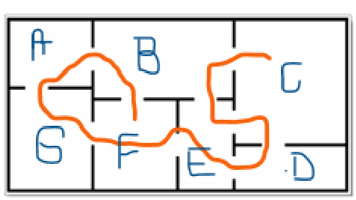
3. Edward A. Mouse has just finished his brand-new house. The floor plan is shown below:



a. Edward wants to give a tour of his house to a friend. Is it possible for them to walk through every doorway exactly once? If so, in which rooms must they begin and end the tour? Explain.

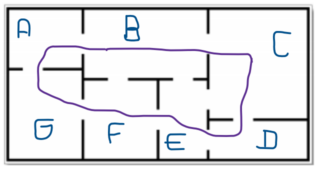
Room (VERTICES)	# of doors (EDGES)	Adjacent rooms
A	2	B, G
B	4	A, C, E, F
C	3	B, D, E
D	2	C, E
E	4	B, C, D, F
F	3	B, E, G
G	2	A, F

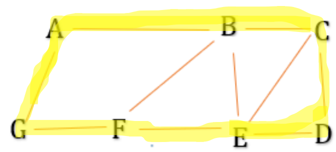




Yes, it is possible (see above). We must start and end in rooms with an odd number of doors. In this case, we must start and end at either room C or F.

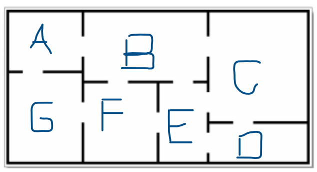
b. Is it possible to tour the house visiting each room exactly once (not necessarily using every doorway)? Explain.





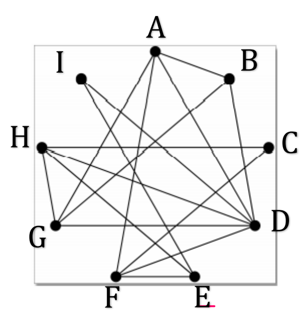
Yes, a circuit is possible along the following path= (A,B,C,D,E,F,G). The edges {{B,F}, {B,E}, {C,E}} would not be traversed.

c. After a few mouse-years, Edward decides to remodel. He would like to add some new doors between the rooms he has. Of course, he cannot add any doors to the exterior of the house. Is it possible for each room to have an odd number of doors? Explain.



No, we cannot add another door to room D, since the only adjacent rooms to D are C and E. Since D cannot have more than 2 doors, it does not satisfy the definition of odd and this remodel is not possible.

4. Below is a graph representing friendships between a group of students (each vertex is a student and each edge is a friendship). Is it possible for the students to sit around a round table in such a way that every student sits between two friends? What does this question have to do with paths?

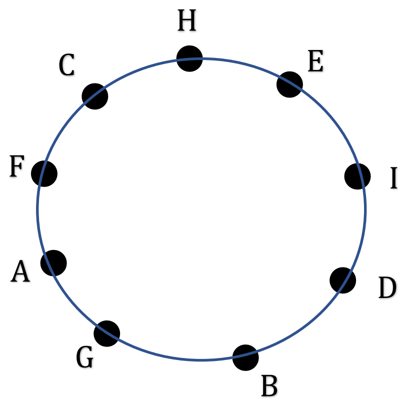


Friend Vertex	# of Edges	Friendship Adjacencies
A	3	B, F, G
B	3	A, G, D
C	**2**	**H, F**
D	6	B, A, I, H, G, F
E	3	I, H, F
F	4	E, D, C, A
G	4	D, B, A, H
H	3	C, D, G
I	**2**	**D, E**

Yes, it is possible for students to sit around a round table in such a way that every student sits between two friends, as shown below. In other words, we can create an unbroken circuit of edges between the friend vertices.

Definition of a path: a path from vertex u to vertex v in a graph is a sequence of edges $e_1 = (x_0, x_1)$, …, $e_n = (x_{n-1}, x_n)$, where $u = x_0$ and $v = x_n$. A path is called a circuit if $u = v$.

Path= (H,E,I,D,B,G,A,F,C). Consecutive vertices are adjacent in the graph is joined by an edge. No vertex is repeated. When the friends are arranged this way, this path is a circuit because you can traverse from any of the vertices back to itself with a sequence of 9 edges.



Part 2:

1. Apply operations to reduce the augmented matrix to **reduced echelon form**. Show your work, including stating which row operation is being applied at each step.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ -1 & -1 & 3 & 5 \\ 5 & 7 & -11 & -9 \end{bmatrix} E_{21}(1)\rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 8 \\ 5 & 7 & -11 & -9 \end{bmatrix} E_{31}(-5)\rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 8 \\ 0 & -3 & -6 & -24 \end{bmatrix}$$

$$E_{32}(3)\rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} E_{12}(-2) \begin{bmatrix} 1 & 0 & -5 & -13 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

According to this result, what is the solution for the system of linear equations below?

$$\begin{aligned} x_1 + 2x_2 \quad -x_3 &= 3 \\ -x_1 - x_2 \quad +3x_3 &= 5 \\ 5x_1 + 7x_2 - 11x_3 &= -9 \end{aligned}$$

There are infinitely many solutions to the system of equations:

$x_1 = 5x_3 - 13$
$x_2 = 2x_3 + 8$
x_3 is a free variable

2. Assuming that $a \neq 0$ and $ad-bc \neq 0$, perform the progression of row operations (in the order indicated) on the matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   a. $E_1(\frac{1}{a}) = \begin{bmatrix} \frac{a}{a} & \frac{b}{a} \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix}$

b. $E_{21}(-c)=\begin{bmatrix}1 & \frac{b}{a}\\ 0 & \frac{ad-bc}{a}\end{bmatrix}$

c. $E_2(\frac{a}{ad-bc})=\begin{bmatrix}1 & \frac{b}{a}\\ 0 & ((\frac{a}{ad-bc})\frac{ad-bc}{a}\end{bmatrix}=\begin{bmatrix}1 & \frac{b}{a}\\ 0 & 1\end{bmatrix}$

d. $E_{12}(-\frac{b}{a})=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

3. For a matrix A of size m × n, prove that the following statements are logically equivalent. (problem completed with Jurgen during office hours 2/23/21)

   a. For each $\mathbf{b} \in \mathbb{R}^m$, the matrix equation $A\mathbf{x}=\mathbf{b}$ has a solution.
   $\forall\, \mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x}=\mathbf{b}$ has a solution:

$$\mathbf{b}=\begin{bmatrix}y_1\\ y_2\\ \cdots\\ y_i\end{bmatrix}$$

$$A\mathbf{x}=\underset{A_{mxn}}{\begin{bmatrix}x_{11} & \cdots & x_{m1}\\ \vdots & \ddots & \vdots\\ x_{1n} & \cdots & x_{mn}\end{bmatrix}}\cdot\underset{\mathbf{x}_{1xm}}{\begin{bmatrix}y_1\\ y_2\\ \cdots\\ y_m\end{bmatrix}}=\begin{bmatrix}(x_{11}y_1 + & x_{21}y_2 \cdots & +x_{m1}y_m)\\ (x_{12}y_1 + & \cdots & +x_{m2}y_m)\\ \vdots & \ddots & \vdots\\ (x_{1n}y_1 + & \cdots & +x_{mn}y_m)\end{bmatrix}=\underset{\mathbf{b}_{1xn}}{\begin{bmatrix}z_1\\ z_2\\ \cdots\\ z_n\end{bmatrix}}$$

   b. Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.

$$\begin{bmatrix}x_{11} & \cdots & x_{m1}\\ \vdots & \ddots & \vdots\\ x_{1n} & \cdots & x_{mn}\end{bmatrix}\cdot\begin{bmatrix}x_{11}\\ x_{12}\\ \cdots\\ x_{1n}\end{bmatrix}=\begin{bmatrix}(x_{11}{}^2)+(x_{21}x_{12})+ & \cdots & +x_{m1}x_{1n})\\ (x_{12}x_n)+(x_{22}x_{12})+ & \cdots & +x_{m2}x_{1n})\\ \vdots & \ddots & \vdots\\ (x_{1n}x_n)+(x_{2n}x_{12})+ & \cdots & +x_{mn}x_{1n})\end{bmatrix}=\begin{bmatrix}z_1\\ z_2\\ \cdots\\ z_n\end{bmatrix}$$

   c. The columns of A span $\mathbb{R}^m$.

$$a\begin{bmatrix}x_1\\ \cdots\\ x_{1m}\end{bmatrix}+b\begin{bmatrix}x_{21}\\ \cdots\\ x_{2m}\end{bmatrix}+\cdots.+m.\begin{bmatrix}x_{n1}\\ \cdots\\ x_{nm}\end{bmatrix}=\begin{bmatrix}a'\\ \cdots\\ m'\end{bmatrix}$$

$$\begin{bmatrix}x_{11} & \cdots & x_{n1}\\ \vdots & \ddots & \vdots\\ x_{1m} & \cdots & x_{mn}\end{bmatrix}\cdot\begin{bmatrix}a\\ b\\ \cdots\\ m\end{bmatrix}=\begin{bmatrix}a'\\ b'\\ \cdots\\ m'\end{bmatrix}$$

$$\begin{bmatrix}x_{11}a+x_{21}b & \cdots & x_{n1}m\\ \vdots & \ddots & \vdots\\ x_{1m}a+ & \cdots & x_{mn}m\end{bmatrix}=\begin{bmatrix}a'\\ b'\\ \cdots\\ m'\end{bmatrix}$$

   d. The reduced echelon form of A has a pivot in every row.

Using the following matrix operations, we can convert the matrix to the identity matrix, which has a pivot in every row:

$$E_1(\frac{1}{a}) = \begin{bmatrix} \frac{a}{a} & \frac{b}{a} \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix}$$

$$E_{21}(-c) = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$$

$$E_2(\frac{a}{ad-bc}) = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & ((\frac{a}{ad-bc})\frac{ad-bc}{a} \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$$

$$E_{12}(-\frac{b}{a}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This row-reduction algorithm can be applied to any matrix:

$$\begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1m} & \cdots & x_{mn} \end{bmatrix}$$

Which results in

$$\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix}$$

4. Find two vectors in $\mathbb{R}^3$ that are not in Span$\{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ -4 \end{bmatrix}\}$

Make sure to justify your reasoning and result. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ -1 \\ -4 \end{bmatrix}$. First, we want to find a vector $\mathbf{v}_3$ that is linearly independent from $\mathbf{v}_1$ and $\mathbf{v}_2$. All linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ can be expressed in the form $\begin{bmatrix} c_1 + 5c_2 \\ 2c_1 - c_2 \\ -3c_1 + 4c_2 \end{bmatrix}$. Therefore, if we want to find a $\mathbf{v}_3$ that is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$, it cannot be expressed as a linear combination of v_1 and v_2. For simplicity, we can choose $c_1=1$ and $c_2=1$, and select something for the third row that cannot not be expressed by those values. In this case, I selected 2. $2 \neq -3(1)+4(1)$.

$$\mathbf{v}_3 = \begin{bmatrix} 1+5(1) \\ 2(1)-1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$$

To find another vector that is linearly independent from $\mathbf{v}_1$ and $\mathbf{v}_2$, we can just scale the linearly independent vector $\mathbf{v}_3$ that we found. For example, we can multiply $\mathbf{v}_3$ by 2, which will also be linearly independent from $\mathbf{v}_1$ and $\mathbf{v}_2$,

So, $\mathbf{v}_4 = \begin{bmatrix} 12 \\ 2 \\ 4 \end{bmatrix}$

5. Prove that for A an m × n matrix, u and v vectors in $\mathbb{R}^n$, and c a scalar:
   a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

$$A(\mathbf{u}+\mathbf{v}) = A(\begin{bmatrix} u_1 \\ \cdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \cdots \\ v_n \end{bmatrix})$$

$$= A(\begin{bmatrix} u_1 \\ \cdots \\ u_n \end{bmatrix}) + A(\begin{bmatrix} v_1 \\ \cdots \\ v_n \end{bmatrix})$$

$$= A\mathbf{u} + A\mathbf{v}$$

   b. $A(c\mathbf{u}) = c(A\mathbf{u})$

$$\begin{array}{ccc} A & \mathbf{u} & \mathbf{v} \\ \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1m} & \cdots & x_{nm} \end{bmatrix} & \begin{bmatrix} y_1 \\ \cdots \\ y_u \end{bmatrix} & \begin{bmatrix} z_1 \\ \cdots \\ z_v \end{bmatrix} \end{array}$$

$A(c\mathbf{u}) = c(A\mathbf{u})$

$$\begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1m} & \cdots & x_{nm} \end{bmatrix} \left(c \begin{bmatrix} y_1 \\ \cdots \\ y_u \end{bmatrix} \right) = c \left(\begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1m} & \cdots & x_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ \cdots \\ y_u \end{bmatrix} \right)$$

$$\begin{bmatrix} cx_{11}y_1 & \cdots & cx_{n1}y_u \\ \vdots & \ddots & \vdots \\ cx_{1m}y_1 & \cdots & cx_{nm}y_u \end{bmatrix} = \begin{bmatrix} cx_{11}y_1 & \cdots & cx_{n1}y_u \\ \vdots & \ddots & \vdots \\ cx_{1m}y_1 & \cdots & cx_{nm}y_u \end{bmatrix}$$