COMP 3008 Homework #8 Krista Miller

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Part 1:

1. Suppose $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$=11. Determine the determinant of each of the following matrices.

   a. $\begin{bmatrix} a-d & b-e & c-f \\ d & e & f \\ g & h & i \end{bmatrix}$ row 1= row 1 – row 2

      Determinant remains unchanged= **11**

   b. $\begin{bmatrix} d & e & f \\ 2a & 2b & 2c \\ g & h & i \end{bmatrix}$ row 2 swapped with row 1*2

      Determinant has a sign switch and increases by a factor of 2= **-22**

   c. $\begin{bmatrix} b & e & h \\ a & d & g \\ c & f & i \end{bmatrix}$ transposed, row 1 swapped with row 2

      Determinant has a sign switch= **-11**

2. Use the determinant to determine whether the matrix is invertible

   a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ -2 & 3 & 4 & 6 \end{bmatrix}$

      This matrix is in upper triangular form.
      det = diagnonal product of the row echelon matrix=12, therefore matrix is invertible

   b. $det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 1 \cdot det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - 0 \cdot det \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} + 1 \cdot det \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

      =1(1)-0(4)+1(2)

      =3, therefore matrix is invertible

   c. $det \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = (\cos\theta \cdot \cos\theta) - (\cos\theta \cdot -\sin\theta)$

      =1, therefore matrix is invertible

3. Find conditions on λ so that the matrix $\begin{bmatrix} \lambda-1 & 0 & 0 \\ 1 & \lambda-2 & 1 \\ 3 & 1 & \lambda-1 \end{bmatrix}$ is invertible.

$$\det \begin{bmatrix} \lambda-1 & 0 & 0 \\ 1 & \lambda-2 & 1 \\ 3 & 1 & \lambda-1 \end{bmatrix} = (\lambda-1) \cdot det \begin{bmatrix} \lambda-2 & 1 \\ 1 & \lambda-1 \end{bmatrix} - 0 \cdot det \begin{bmatrix} 1 & 1 \\ 3 & \lambda-1 \end{bmatrix} +$$

$$0 \cdot \ det \begin{bmatrix} 1 & \lambda - 1 \\ 3 & 1 \end{bmatrix}$$

$=(\lambda - 1) \ \cdot det \begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 1 \end{bmatrix}$; for this matrix to be invertible, we must be able to calculate $det \begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 1 \end{bmatrix}$.

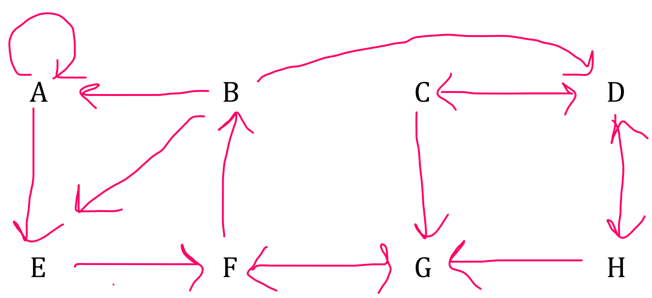
Therefore, ad-bc cannot equal 0

$\boldsymbol{[(\lambda - 2)(\lambda - 1) - 1] \neq 0; \lambda^2 \text{-} 3\lambda + 1 \neq 0}$

4. A directed graph has the following adjacency matrix.

$$\begin{array}{cc} & \text{TO} \\ \text{FROM} & \begin{bmatrix} X & a & b & c & d & e & f & g & h \\ a & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ d & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ e & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ f & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ g & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ h & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

a. Draw a picture of this directed graph. Note that there is at least one self-loop.



b. Determine the total number of paths of length **at most 2**.
Definition of path: a path from vertex u to vertex v in a graph is a sequence of edges $e_1 = (x_0, x_1)$, $e_2 = (x_1, x_2)$, …, $e_n(x_{n-1}, x_n)$, where $u = x_0$ and $v = x_n$.

<u>Two total paths of length at most 2:</u>

$Path_1 = D \to H \to G$

$Path_2 = D \to C \to G$

c. Determine the total number of paths of length **at most 3** starting at vertex v6.

Vertex v6 corresponds to "F" in my directed graph:

<u>Three total paths of length at most 3 starting at vertex v6:</u>

$Path_1 = F \to B \to E \to F$

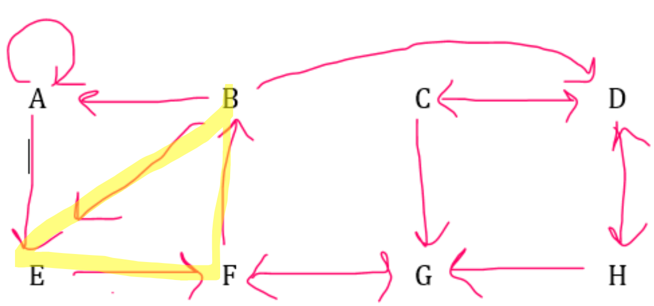
$Path_2 = F \to B \to A \to E$

Path$_3$= F→B→A→A

d. How many circuits of length 3 exist? For which vertices does such a circuit exist? Definition of a circuit: a path from vertex u to vertex v in a graph is a sequence of edges e_1= (x_0, x_1), e_2= (x_1, x_2), …, $e_n(x_{n-1}, x_n)$, where u= x_0 and v=x_n. A path is called a circuit if u=v.

<u>One total circuit of length 3 exists:</u>

Circuit$_1$= B→ E→ F →B



5. Recall that matrix multiplication can be thought of as the composition of two linear transformations. A linear transformation T: $\mathbb{R}^n \to \mathbb{R}^n$ is invertible if there exists a transformation S: $\mathbb{R}^n \to \mathbb{R}^n$ so that $S(T(\mathbf{x}))= \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

a. Prove that if A is the standard matrix for a linear transformation T, then T is invertible if and only if A is an invertible matrix. (problem done in office hours with Jurgen 3/9/2021).

<u>Invertible matrices: on A**x**=**b**</u>

$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$

$I\mathbf{x}= A^{-1}\mathbf{b}$

$\mathbf{x}= A^{-1}\mathbf{b}$

<u>Invertible transformation:</u>

Can "undo" a series of transformations

Suppose $T_A(\mathbf{x})= A(\mathbf{x})$ and A is an invertible matrix.

Define S: $S_{A^{-1}}(\mathbf{x})=A^{-1}(\mathbf{x})$

Then: $S_{A^{-1}}(T_A(\mathbf{x}))=A^{-1}A\mathbf{x}$

$= I\mathbf{x}= \mathbf{x}$

So, T_A is invertible

Suppose T_A has an inverse S_B.

T_A is onto for any **b**

$T_A(S_B(\mathbf{b}))=\mathbf{b}$

A is invertible iff there exists some matrix A^{-1} in R^n such that AA^{-1}= I given that A is the standard matrix of T, $A\mathbf{x}=T(\mathbf{x})$. Therefore, A^{-1} must be the standard matrix of S such that $A^{-1}\mathbf{x}=S(\mathbf{x})$. $S(T(\mathbf{x}))= A^{-1}\, T(\mathbf{x})= A^{-1}\, A(\mathbf{x})= I(\mathbf{x})=\mathbf{x}$. So, A is invertible.

If A is singular, there exists no A^{-1} in R^n such that A A^{-1} =I. So, A A^{-1} =C and S(T(**x**))= A^{-1} T(**x**)= A^{-1} A(**x**)=C(**x**)=**y**, which is a contradiction given the initial premise of the problem.

b. When *T* is invertible, then S(**x**)= A^{-1}**x** is the unique inverse of *T*. (Hint: To prove uniqueness, assume *S*(*T*(**x**))=**x**=*U*(*T*(**x**)) for all **x** ∈ $\mathbb{R}^n$. Show that for any **v** ∈ $\mathbb{R}^n$, *S*(**v**)=*U*(**v**).)

A linear transformation T is invertible if there is a companion linear transformation S which undoes the action of T.
S(T(**x**))= U(T(**x**))=**x** iff S=U
BA**x**= **x**= CA**x** then B must =C where B is the standard vector of S and C is the standard vector of U.
There must exist some matrix S(T(**x**))= A^{-1}(T(**x**))=**x**

6. Let *T*: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation that first performs a horizontal shear so that $e_2 \rightarrow e_2+2e_1$ (leaving e_1 unchanged) and then reflects points through the line y=x.
   a. Find the standard matrix *A* for *T*. (Hint: Think of the composition as matrix multiplication.

$e_1=\begin{bmatrix}1\\0\end{bmatrix}$ $e_2=\begin{bmatrix}0\\1\end{bmatrix}$

Horizontal shear: $T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}0\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\end{bmatrix}= \begin{bmatrix}2\\1\end{bmatrix}$

Unchanged e_1 under T: $T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix}$

Horizontal shear= $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$= B

Reflection about the line y=x =$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$=C

A=CB

standard matrix $A=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 2\end{bmatrix}$

   b. Find A^{-1}

$A^{-1}=\begin{bmatrix}0 & 1 & 1 & 0\\1 & 2 & 0 & 1\end{bmatrix} E_{21}\rightarrow \begin{bmatrix}1 & 2 & 0 & 1\\0 & 1 & 1 & 0\end{bmatrix} E_{12}(-2)\rightarrow\begin{bmatrix}1 & 0 & -2 & 1\\0 & 1 & 1 & 0\end{bmatrix}$

$A^{-1}= \begin{bmatrix}-2 & 1\\1 & 0\end{bmatrix}$

   c. Find the standard matrix for the inverse mapping directly by finding a transformation that first undoes the reflection through the line y=x and then undoes the horizontal shear. Show that this matrix is the same as the matrix you found for A^{-1}

$A^{-1}= B^{-1}\ C^{-1}$

Undoes horizontal shear= $B^{-1}=\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$

Undoes reflection= $C^{-1}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

$B^{-1}C^{-1}=\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}$, which is the same matrix that was found for A^{-1}

7.

a. Find the inverse (mod 26) of the matrix $K=\begin{bmatrix}3 & 7\\ 5 & 18\end{bmatrix}$

Hint: the familiar formula for finding the inverse of a 2x2 matrix still works, but instead of dividing by ad-bc, you multiply by $(ad-bc)^{-1}$ (mod 26)

$\det K=\begin{bmatrix}3 & 7\\ 5 & 18\end{bmatrix}=(3\cdot 18)-(7\cdot 5)=19$

modular multiplicative inverse 19(mod 26)= 11

$11\cdot\begin{bmatrix}18 & -7\\ -5 & 3\end{bmatrix}=\begin{bmatrix}198 & -77\\ -55 & 33\end{bmatrix}$, simplifies mod 26 to $\begin{bmatrix}\mathbf{16} & \mathbf{1}\\ \mathbf{23} & \mathbf{7}\end{bmatrix}$

b. A cipher system can be constructed as follows: represent the letters 'a' through 'z' with the numbers 0 through 25(mod 26). Break the original message into blocks of 2, writing the message as column vectors. For example, the text "hi" is represented by the vector $\mathbf{x}=\begin{bmatrix}h\\ i\end{bmatrix}=\begin{bmatrix}7\\ 8\end{bmatrix}$. Encrypt the plaintext **x** to the ciphertext **y** with the formula **y**=K**x**(mod 26). Traditionally we write the plaintext with lowercase letters and the ciphertext with uppercase letters. Use the key K from the previous part to encrypt the text "hi".

$y=\begin{bmatrix}16 & 1\\ 23 & 7\end{bmatrix}\begin{bmatrix}7\\ 8\end{bmatrix}=\begin{bmatrix}120\\ 217\end{bmatrix}(\text{mod } 26)=\begin{bmatrix}16\\ 9\end{bmatrix}$

c. Using the same encryption key as above, decrypt the ciphertext "IMIP".

$\begin{bmatrix}16 & 1\\ 23 & 7\end{bmatrix}\begin{bmatrix}8 & 8\\ 12 & 15\end{bmatrix}=\begin{bmatrix}140 & 143\\ 268 & 289\end{bmatrix}(\text{mod } 26)=\begin{bmatrix}10 & 13\\ 8 & 3\end{bmatrix}$ **KIND**

8. Factorizing a matrix A into a product of two or more matrices is a way to decompose a matrix into parts that are somehow more useful than the original matrix. For example, perhaps using the decomposed matrix can simplify more complex computations or can eliminate some redundant information. We will see some other factorizations later in the course. This exercise will walk you through LU factorization (or decomposition). That is, A will be decomposed into a product of a Lower triangular matrix times an Upper triangular matrix.

-The matrix U is simply an echelon form of A produced by only using the row replacement operations $E_{ij}(d)$. So, for some sequence of row replacement operations,

$U=E_kE_{k-1}\dots E_2E_1A$ or, equivalently, $(E_kE_{k-1}\dots E_2E_1)^{-1}U=A$

(Note that since only row replacement operations are used, entries along the main diagonal of U need not equal to 1.)

-The choice for L is then clear. In particular,

$L=(E_kE_{k-1}\dots E_2E_1)^{-1}$ or, equivalently, $L=E_1^{-1}E_2^{-1}\dots E_{k-1}^{-1}E_k^{-1}$

-Note that $(E_{ij}(d))^{-1}=E_{ij}(-d)$.

a. Using only a sequence of row replacement operation $E_{ij}(d)$, find U which is the row echelon form of

$$A=\begin{bmatrix}4 & 3 & -5\\ -4 & -5 & 7\\ 8 & 10 & -8\end{bmatrix} E_{21}(1)\rightarrow \begin{bmatrix}4 & 3 & -5\\ 0 & -2 & 2\\ 8 & 10 & -8\end{bmatrix} E_{31}(-2)\rightarrow \begin{bmatrix}4 & 3 & -5\\ 0 & -2 & 2\\ 0 & 4 & 2\end{bmatrix} E_{32}(2)\rightarrow$$

$$\begin{bmatrix}4 & 3 & -5\\ 0 & -2 & 2\\ 0 & 0 & 6\end{bmatrix} = U = \text{ upper triangular matrix}$$

Keep track of each step and write the sequence $E_{ij}(d)$ row operations used (in order).

<u>Sequence</u>: $E_{21}(1)\rightarrow E_{31}(-2)\rightarrow E_{32}(2)$

$$\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\rightarrow\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}\rightarrow\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix}$$

b. Write the sequence of inverse elementary matrices in the reverse order.

$E_{32}(-2)\rightarrow E_{31}(2)\rightarrow E_{21}(-1)$

c. Find the product of these matrices to find L

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -2 & 1\end{bmatrix}\rightarrow\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & -2 & 1\end{bmatrix}\rightarrow\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & -2 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & -2 & 1\end{bmatrix} = L = \text{lower triangular matrix}$$

d. Verify that the product LU does equal the original matrix A.

$$\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & -2 & 1\end{bmatrix}\cdot\begin{bmatrix}4 & 3 & -5\\ 0 & -2 & 2\\ 0 & 0 & 6\end{bmatrix}=$$

$$\begin{bmatrix}1\cdot4+0\cdot0+0\cdot0 & 1\cdot3+0\cdot(-2)+0\cdot0 & 1\cdot(-5)+0\cdot2+0\cdot6\\ (-1)\cdot4+1\cdot0+0\cdot0 & (-1)\cdot3+1\cdot(-2)+0\cdot0 & (-1)(-5)+1\cdot2+0\cdot6\\ 2\cdot4+(-2)\cdot0+1\cdot0 & 2\cdot3+(-2)(-2)+1\cdot0 & 2(-5)+(-2)\cdot2+1\cdot6\end{bmatrix}$$

$$=\begin{bmatrix}4 & 3 & -5\\ -4 & -5 & 7\\ 8 & 10 & -8\end{bmatrix}$$

<u>Part 2:</u>

1. Let T be a linear transformation $T: V\rightarrow W$ and let U be a subspace of V. Prove that $T(U)$ is a subspace of range(T).

<u>Definition of range:</u> the range of a matrix is the set of all possible linear combinations of its column vectors. The range of a linear transformation T is the set of all vectors T(**v**) where **v** is any vector in its domain.

Definition of linear transformation: a transformation T is linear if, for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T and all scalars c and d
$T(c\mathbf{u}+d\mathbf{v})= cT(\mathbf{u})+ dT(\mathbf{v})$

Definition of subspace: a subset H of vector V is a subspace of V if and only if it has the following three properties:

1. The zero vector is in H.
2. H is closed under vector addition. That is, for each $\mathbf{u}$, $\mathbf{v}$ in H, $\mathbf{u}+\mathbf{v}$ is also an element of H.
3. H is closed under scalar multiplication. That is for each $\mathbf{u}$ in H and each scalar c, $c\mathbf{u}$ is an element of H.

Suppose $\mathbf{w}_1$ and $\mathbf{w}_2$ are elements of T(U). Then, there exists $\mathbf{u}_1$ and $\mathbf{u}_2$ in U such that
$\mathbf{w}_1= T(\mathbf{u}_1)$ and $\mathbf{w}_2= T(\mathbf{u}_2)$.
$c\mathbf{w}_1 + d\mathbf{w}_2 = cT(\mathbf{u}_1) +dT(\mathbf{u}_2)$; c and d are scalar constants
$= T(c\mathbf{u}_1) + T(d\mathbf{u}_2)$
$=T(c\mathbf{u}_1 + d\mathbf{u}_2)$
Since U is a subspace, $\mathbf{u}_1$, $\mathbf{u}_2$ are elements of U, so $c\mathbf{u}_1 +d\mathbf{u}_2$ are also elements of U.
Therefore, $c\mathbf{w}_1 +d\mathbf{w}_2$ is an element of T(U), since it is the image of the vector $c\mathbf{u}_1 +d\mathbf{u}_2$.

2. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors in a vector space *V* and let *H* be the subspace of *V* spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. Show that each vector $\mathbf{v}_j$ for $1 \le j \le p$ itself is in *H*.

In other words, prove that for any set of vectors $S= \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V, span(S) is a subspace of V.

Definition of subspace: a subset H of vector V is a subspace of V if and only if it has the following three properties:

4. The zero vector is in H.
5. H is closed under vector addition. That is, for each $\mathbf{u}$, $\mathbf{v}$ in H, $\mathbf{u}+\mathbf{v}$ is also an element of H.
6. H is closed under scalar multiplication. That is for each $\mathbf{u}$ in H and each scalar c, $c\mathbf{u}$ is an element of H.

Definition of span: let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. The span of these vectors is the set of all possible linear combinations of these vectors. It is always the case that $\mathbf{0}$ is in element of the span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$

Let $\mathbf{u}$ and $\mathbf{w}$ be elements in the span of H. Then, there exist $c_1, \ldots, c_n$ and $d_1, \ldots d_n$ such that

$$\mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$$
$$\mathbf{w}= d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_n\mathbf{v}_n$$

$\mathbf{u}+\mathbf{w}= (c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n) + (d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_n\mathbf{v}_n)$
$\mathbf{u}+\mathbf{w}= (c_1+d_1)\mathbf{v}_1 + (c_2+d_2)\mathbf{v}_2 +\ldots+ (c_n+d_n)\mathbf{v}_n$ is an element in the span of H and so H is closed under vector addition.

$d\mathbf{u}= d(c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n)$
$d\mathbf{u}= d(c_1\mathbf{v}_1) + d(c_2\mathbf{v}_2)+\ldots+ d(c_n\mathbf{v}_n)$

d**u**= $(dc_1)\mathbf{v}_1+(dc_2)\mathbf{v}_2+\ldots+(dc_n)\mathbf{v}_n$ is an element in the span of H and so H is closed under scalar multiplication.

Therefore, H is a subspace of V.

3. For each of the following, determine whether the set W is a vector space. If it is not a vector space, then explain why it is not a vector space. If it is a vector space, justify why it is and then find a basis for W.

a. $W=\{\begin{bmatrix} a-2b \\ 5+b \\ a+3b \\ b \end{bmatrix}$: a, b $\in\mathbb{R}\}$

W is not a vector space because **0** does not exist.

Contradiction: take a=0, b=0; **u**=$\begin{bmatrix} 0-2(0) \\ 5+0 \\ 0+3(0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}$ =. . The last component of this matrix must equal 0, so we must set b=0. Now, notice the component (5+b). If b=0, then this element = 5. Therefore, we cannot generate the **0** vector.

b. $W=\{\begin{bmatrix} a-2b \\ -a+2b \\ 3a-6b \end{bmatrix}$: a, b $\in\mathbb{R}\}$. W is a vector space because it contains the **0** vector, is closed under addition and closed under scalar multiplication.

1. **0** vector is an element of W. Take a=0, b=0, **u**=$\begin{bmatrix} 0-2(0) \\ -0+2(0) \\ 3(0)-6(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2. **u**,**v**, $\in W$, $\mathbf{u}+\mathbf{v}\in W$

   u=$\begin{bmatrix} a_1-2b_1 \\ -a_1+2b_1 \\ 3a_1-6b_1 \end{bmatrix}$ **v**=$\begin{bmatrix} a_2-2b_2 \\ -a_2+2b_2 \\ 3a_2-6b_2 \end{bmatrix}$

   $\mathbf{u}+\mathbf{v}=\begin{bmatrix} a_1-2b_1+a_2-2b_2 \\ -a_1+2b_1+-a_2+2b_2 \\ 3a_1-6b_1+3a_2-6b_2 \end{bmatrix}$, $a_1, a_2, b_1, b_2 \in\mathbb{R}$

3. a $\in\mathbb{R}$, $\mathbf{u}\in W$, a**u** $\in W$

   u=$\begin{bmatrix} a_1-2b_1 \\ -a_1+2b_1 \\ 3a_1-6b_1 \end{bmatrix}$

   a**u**=a$\begin{bmatrix} a_1-2b_1 \\ -a_1+2b_1 \\ 3a_1-6b_1 \end{bmatrix} = \begin{bmatrix} aa_1-a2b_1 \\ a(-a_1)+a2b_1 \\ a3a_1-a6b_1 \end{bmatrix}$

Find a basis for W.

Basis for W, with a=1 and b=1: $\begin{bmatrix} 1-2(1) \\ -(1)+2(1) \\ 3(1)-6(1) \end{bmatrix}$

$$=\begin{bmatrix}-1\\1\\-3\end{bmatrix}$$

4. Find a basis and the dimension of Nul A, of Col A, and of Row A.

a. $A=\begin{bmatrix}1&2&-5&11&-3\\2&4&-5&15&2\\1&2&0&4&5\\3&6&-5&9&-2\\0&1&0&1&2\end{bmatrix}\rightarrow\text{rref}\begin{bmatrix}1&0&0&0&\frac{-4}{5}\\0&1&0&0&\frac{11}{10}\\0&0&1&0&\frac{143}{50}\\0&0&0&1&\frac{9}{10}\\0&0&0&0&0\end{bmatrix}$

$\begin{bmatrix}1&0&0&0&\frac{-4}{5}\\0&1&0&0&\frac{11}{10}\\0&0&1&0&\frac{143}{50}\\0&0&0&1&\frac{9}{10}\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$; x_5 is a free variable.

If x_5=r, $x_1=\frac{4}{5}$r, $x_2=-\frac{11}{10}$r, $x_3=-\frac{143}{50}$r, $x_4=-\frac{9}{10}$r, x_5=r

$$\begin{bmatrix}\frac{4r}{5}\\-\frac{11r}{10}\\-\frac{143r}{50}\\-\frac{9r}{10}\\r\end{bmatrix}=\begin{bmatrix}\frac{4}{5}\\-\frac{11}{10}\\-\frac{143}{50}\\-\frac{9}{10}\\1\end{bmatrix}r;$$

Basis for the null (A) $=\begin{bmatrix}\frac{4}{5}\\-\frac{11}{10}\\-\frac{143}{50}\\-\frac{9}{10}\\1\end{bmatrix}$

Dim Null(A)= number of free variables in row reduced form of A.

Dim Null(A)=1

A basis for Col(A) is given by the columns corresponding to the leading 1's in the row reduced form of A.

Basis for Col(A)={$\begin{bmatrix}1\\2\\1\\3\\0\end{bmatrix}$, $\begin{bmatrix}2\\4\\2\\6\\1\end{bmatrix}$, $\begin{bmatrix}-5\\-5\\0\\-5\\0\end{bmatrix}$, $\begin{bmatrix}11\\15\\4\\9\\1\end{bmatrix}$}

Dim Col(A)= 4

Basis for Row(A) is given by the nonzero rows in the reduced row-echelon form:

{[1,0,0,0,0,-4/5], [0,1,0,0,11/10], [0,0,1,0,143/50],[0,0,0,1,9/10]}

Dim Row(A)= 4

b. $A=\begin{bmatrix}2 & 1 & -2 & 0 & 1 & 2 & 1\\1 & 2 & 0 & -1 & 1 & 2 & 2\\3 & 2 & -1 & 1 & 1 & 8 & 9\\0 & 2 & 2 & -1 & 1 & 6 & 8\\0 & 3 & 3 & 3 & -3 & 0 & 3\end{bmatrix}$

$\begin{bmatrix}2 & 1 & -2 & 0 & 1 & 2 & 1\\1 & 2 & 0 & -1 & 1 & 2 & 2\\3 & 2 & -1 & 1 & 1 & 8 & 9\\0 & 2 & 2 & -1 & 1 & 6 & 8\\0 & 3 & 3 & 3 & -3 & 0 & 3\end{bmatrix}$→rref

$\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 2 & 3\\0 & 0 & 0 & 1 & 0 & 4 & 5\\0 & 0 & 0 & 0 & 1 & 6 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}$; x_6 and x_7 are free variables

If $x_6=r, x_7=s, x_1$=0, x_2=0, x_3=$-3s-2r$, x_4=$-5s-4r$, $x_5=-7s-6r, x_6=r, x_7=s$

$\begin{bmatrix}0\\0\\-3s-2r\\-5s-4r\\-7s-6r\\r\\s\end{bmatrix}=\begin{bmatrix}0\\0\\-2\\-4\\-6\\1\\0\end{bmatrix}$r + $\begin{bmatrix}0\\0\\-3\\-5\\-7\\0\\1\end{bmatrix}$s;

Basis for the null (A)={$\begin{bmatrix}0\\0\\-2\\-4\\-6\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\-3\\-5\\-7\\0\\1\end{bmatrix}$}

Dim Null(A)= number of free variables in row reduced form of A.

Dim Null(A)=2

Basis vectors correspond to the columns that have a pivot value:

Basis for Col(A)= $\{\begin{bmatrix}2\\1\\3\\0\\0\end{bmatrix},\begin{bmatrix}1\\2\\2\\2\\3\end{bmatrix},\begin{bmatrix}-2\\0\\-1\\2\\3\end{bmatrix},\begin{bmatrix}0\\-1\\1\\-1\\3\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\\-3\end{bmatrix}\}$

Dim Col(A)=5

Basis for Row(A) is given by the nonzero rows in the reduced row-echelon form:

Basis for Row (A)= {[1,0,0,0,0,0,0],[0,1,0,0,0,0,0],[0,0,1,0,0,2,3],[0,0,0,1,0,4,5],[0,0,0,0,1,6,7]}

Dim Row(A)= 5

5. Let

$$H=\text{Span}\{\mathbf{v}_1=\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \mathbf{v}_2=\begin{bmatrix}0\\1\\2\\-3\end{bmatrix}, \mathbf{v}_3=\begin{bmatrix}-1\\-2\\-1\\4\end{bmatrix}\}$$

a. Find a basis $\mathcal{B}$ of H

$$a\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}+b\begin{bmatrix}0\\1\\2\\-3\end{bmatrix}+c\begin{bmatrix}-1\\-2\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

1a+0b-1c=0

0a+1b-2c=0

-3a+2b-1c=0

2a-3b+4c=0

$$\begin{bmatrix}1&0&-1\\0&1&2\\-3&2&-1\\2&-3&4\end{bmatrix} E_{31}(3)\rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&2&-4\\2&-3&4\end{bmatrix} E_{21}(-2)\rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&2&-4\\0&-3&6\end{bmatrix}$$

$$E_{32}(-2)\rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\\0&-3&6\end{bmatrix} E_{42}(3)\rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\\0&0&0\end{bmatrix}$$

1a-1c=0

1b-2c=0

c= free variable, therefore the system has infinitely many solutions

Therefore the set $\{\begin{bmatrix}1\\0\\-3\\2\end{bmatrix},\begin{bmatrix}0\\1\\2\\-3\end{bmatrix}\}$ forms a basis $\mathcal{B}$ of H

b. Show that $\mathbf{x}=\begin{bmatrix}-4\\-3\\6\\1\end{bmatrix}$, is in H and find $[\mathbf{x}]_{\mathcal{B}}$

a$\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}$+b$\begin{bmatrix}0\\1\\2\\-3\end{bmatrix}$=$\begin{bmatrix}-4\\-3\\6\\1\end{bmatrix}$

a+0b=-4 a=-4

0a+1b=-3 b=-3

-4+0=-4

0+-3=-3

12+(-6)=6

-8+9=1; therefore $\mathbf{x}=\begin{bmatrix}-4\\-3\\6\\1\end{bmatrix}$ is in H

$[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}-4\\-3\end{bmatrix}$

6. Let $\mathcal{B}=\{\mathbf{b}_1=\begin{bmatrix}4\\1\end{bmatrix}, \mathbf{b}_2=\begin{bmatrix}-6\\1\end{bmatrix}\}$ and $\mathcal{C}=\{\mathbf{c}_1=\begin{bmatrix}1\\-1\end{bmatrix}, \mathbf{c}_2=\begin{bmatrix}3\\2\end{bmatrix}\}$ be two difference bases for $\mathbb{R}^2$. Suppose $\mathbf{x}=\begin{bmatrix}9\\1\end{bmatrix}$

   a. Find $[\mathbf{x}]_{\mathcal{B}}$

   $\begin{bmatrix}4 & -6\\1 & 1\end{bmatrix}\begin{bmatrix}j\\k\end{bmatrix}=\begin{bmatrix}9\\1\end{bmatrix}$

   Find inverse of $\mathcal{B}$:

   $\begin{bmatrix}4 & -6 & 1 & 0\\1 & 1 & 0 & 1\end{bmatrix}$ $E_1(1/4)\rightarrow$ $\begin{bmatrix}1 & -3/2 & 1/4 & 0\\1 & 1 & 0 & 1\end{bmatrix}$ $E_{21}(-1)\rightarrow$ $\begin{bmatrix}1 & -3/2 & 1/4 & 0\\0 & 5/2 & -1/4 & 1\end{bmatrix}$

   $E_2(2/5)\rightarrow$ $\begin{bmatrix}1 & -3/2 & 1/4 & 0\\0 & 1 & -1/10 & 2/5\end{bmatrix}$ $E_{12}(3/2)\rightarrow$ $\begin{bmatrix}1 & 0 & 1/10 & 3/5\\0 & 1 & -1/10 & 2/5\end{bmatrix}$

   Use inverse to solve above equation:

   $\begin{bmatrix}1/10 & 3/5\\-1/10 & 2/5\end{bmatrix}\begin{bmatrix}4 & -6\\1 & 1\end{bmatrix}\begin{bmatrix}j\\k\end{bmatrix}=\begin{bmatrix}1/10 & 3/5\\-1/10 & 2/5\end{bmatrix}\begin{bmatrix}9\\1\end{bmatrix}$

   $$\begin{bmatrix}j\\k\end{bmatrix}=[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}\left(\frac{1}{10}\right)9+\left(\frac{3}{5}\right)1\\\left(-\frac{1}{10}\right)9+\left(\frac{2}{5}\right)1\end{bmatrix}=\begin{bmatrix}3/2\\-1/2\end{bmatrix}$$

   b. Find $[\mathbf{x}]_{\mathcal{C}}$

   $\begin{bmatrix}1 & 3\\-1 & 2\end{bmatrix}\begin{bmatrix}j\\k\end{bmatrix}=\begin{bmatrix}9\\1\end{bmatrix}$

   Find inverse of $\mathcal{C}$:

   $\begin{bmatrix}1 & 3 & 1 & 0\\-1 & 2 & 0 & 1\end{bmatrix}$ $E_{21}(1)\rightarrow$ $\begin{bmatrix}1 & 3 & 1 & 0\\0 & 5 & 1 & 1\end{bmatrix}$ $E_2(1/5)\rightarrow$ $\begin{bmatrix}1 & 3 & 1 & 0\\0 & 1 & 1/5 & 1/5\end{bmatrix}$ $E_{21}(-3)\rightarrow$

   $\begin{bmatrix}1 & 0 & 2/5 & -3/5\\0 & 1 & 1/5 & 1/5\end{bmatrix}$

Use inverse to solve above equation:

$$\begin{bmatrix}2/5 & -3/5\\1/5 & 1/5\end{bmatrix}\begin{bmatrix}1 & 3\\-1 & 2\end{bmatrix}\begin{bmatrix}j\\k\end{bmatrix}=\begin{bmatrix}2/5 & -3/5\\1/5 & 1/5\end{bmatrix}\begin{bmatrix}9\\1\end{bmatrix}$$

$$\begin{bmatrix}j\\k\end{bmatrix}=[\mathbf{x}]_{\mathcal{C}}=\begin{bmatrix}\left(\frac{2}{5}\right)9+\left(-\frac{3}{5}\right)1\\\left(\frac{1}{5}\right)9+\left(\frac{1}{5}\right)1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$$

c. Find $[\mathbf{b}_1]_{\mathcal{C}}$ and $[\mathbf{b}_2]_{\mathcal{C}}$ and let M= $[[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}}]$

$$[\mathbf{b}_1]_{\mathcal{C}}=\begin{bmatrix}2/5 & -3/5\\1/5 & 1/5\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}\left(\frac{2}{5}\right)4+\left(-\frac{3}{5}\right)1\\\left(\frac{1}{5}\right)4+\left(\frac{1}{5}\right)1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$[\mathbf{b}_2]_{\mathcal{C}}=\begin{bmatrix}2/5 & -3/5\\1/5 & 1/5\end{bmatrix}\begin{bmatrix}-6\\1\end{bmatrix}=\begin{bmatrix}\left(\frac{2}{5}\right)(-6)+\left(-\frac{3}{5}\right)1\\\left(\frac{1}{5}\right)(-6)+\left(\frac{1}{5}\right)1\end{bmatrix}=\begin{bmatrix}-3\\-1\end{bmatrix}$$

$$M=[[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}}]=\begin{bmatrix}1 & -3\\1 & -1\end{bmatrix}$$

d. Show that $M[\mathbf{x}]_{\mathcal{B}}=[\mathbf{x}]_{\mathcal{C}}$

$$\begin{bmatrix}1 & -3\\1 & -1\end{bmatrix}\begin{bmatrix}3/2\\-1/2\end{bmatrix}=\begin{bmatrix}1\left(\frac{3}{2}\right)+(-3)(-\frac{1}{2})\\1\left(\frac{3}{2}\right)+(-1)(-\frac{1}{2})\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$$, which equals $[\mathbf{x}]_{\mathcal{C}}=\begin{bmatrix}3\\2\end{bmatrix}$.

e. Note that this matrix M is called a **change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$**, and M multiplied by $[\mathbf{x}]_{\mathcal{B}}$ converts $\mathcal{B}$-coordinates of a vector $\mathbf{x}$ to $\mathcal{C}$-coordinates of $\mathbf{x}$. Find a matrix N that will convert $\mathcal{C}$-coordinates of a vector $\mathbf{x}$ to $\mathcal{B}$-coordinates of $\mathbf{x}$ and verify that $N[\mathbf{x}]_{\mathcal{C}}=[\mathbf{x}]_{\mathcal{B}}$

Find $[\mathbf{c}_1]_{\mathcal{B}}$ and $[\mathbf{c}_2]_{\mathcal{B}}$ and let N= $[[\mathbf{c}_1]_{\mathcal{B}} \quad [\mathbf{c}_2]_{\mathcal{B}}]$

$$[\mathbf{c}_1]_{\mathcal{B}}=\begin{bmatrix}1/10 & 3/5\\-1/10 & 2/5\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}\left(\frac{1}{10}\right)1+(\frac{3}{5})(-1)\\\left(-\frac{1}{10}\right)(1)+(\frac{2}{5})(-1)\end{bmatrix}=\begin{bmatrix}-1/2\\-1/2\end{bmatrix}$$

$$[\mathbf{c}_2]_{\mathcal{B}}=\begin{bmatrix}1/10 & 3/5\\-1/10 & 2/5\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}\left(\frac{1}{10}\right)3+(\frac{3}{5})(2)\\\left(-\frac{1}{10}\right)(3)+(\frac{2}{5})(2)\end{bmatrix}=\begin{bmatrix}3/2\\1/2\end{bmatrix}$$

$$N=[[\mathbf{c}_1]_{\mathcal{B}}\,[\mathbf{c}_2]_{\mathcal{B}}]=\begin{bmatrix}-1/2 & 3/2\\-1/2 & 1/2\end{bmatrix}$$