INF3410 — Fall 2015

Book Chapter 4: Frequency Response of Linear Systems

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Content

Convolution

$$
\begin{aligned}
(f * g)(t) &= \int_{-\infty}^{\infty} f(t-s)g(s)\mathrm{d}s \\
&= \int_{-\infty}^{\infty} f(s)g(t-s)\mathrm{d}s
\end{aligned}
$$

Impulse Response of a Linear Filter

A linear filter is an operator that maps one function of time $x(t)$ onto another function of time $y(t)$ and that can be defined as the convolution of an impulse response $h(t)$ with an input $x(t)$. The impulse response of a filter is the output $h(t)$ it would produce when the input is a Dirac delta-function $\delta(t)$. Incidentaly the impulse response actually defines the linear filter:

$$y(t) = (x * h)(t) \tag{1}$$

Example 1st order passive low-pass filter

The impulse response of a passive (no amplification) first order low pass filter (e.g. an RC low-pass filter) is an exponential decay:

$$h(t) = \omega_0 u(t) e^{-t\omega_0}$$

Where $u(t)$ is the Heaviside step function ($u(t) = 0$ for $t \leq 0$ and $u(t) = 1$ for $t > 0$). Dependent on context: $\omega_0 = \frac{1}{\tau} = \frac{1}{RC}$
Here follows a little matlab animated demonstration...

Content

Laplace Transform

$$x(t) \rightarrow X(s) = \int_0^{\infty} e^{-st} x(t) dt$$

Evaluated for $s = j\omega$ (i.e. along the imaginary axis only) and the integral from $-\infty$ (bilateral Laplace transform) this corresponds to the Fourrier-Transform.

Fourrier Transform

The Fourrier transform allows to decompose a waveform into an integral (a sum for the discrete Fourrier Transform) of pure weighted sine-waves where their frequency is the integration variable. You may have stumbled over 'power density spectra' which are the result of a Fourrier transformation of a signal. In other words: any waveform can be represented by a sum/integral of sine-waves. Thus if you make two plots with frequency ω on the x-axis any waveform in the time domain can be represented in frequency space: the first plot marks the power of a particular frequency and the second marks the phase (Bode plots).
Some illustrations on the white board...

Complex Numbers

With the example on the next slide there may be some repetition of rules concerning computation with complex numbers on the white board be in order ...

Example: Sinusoid

$$x(t) = A\cos\omega_0 t = \frac{A}{2}e^{j\omega_0 t} + e^{-j\omega_0 t}$$

Evaluating on the imaginary axis means just for $s = j\omega$ (equivalent to Fourier transform)

$$\begin{aligned}
X(s) &= A\int_0^\infty e^{-st}\cos(\omega_0 t)\,\mathrm{d}t \\
X(j\omega) &= A\int_0^\infty e^{-j\omega t}\cos(\omega_0 t)dt \\
&= A\int_0^\infty (\cos(-\omega t) + j\sin(-\omega t))\cos(\omega_0 t)\,\mathrm{d}t \\
&= A\int_0^\infty \cos(-\omega t)\cos(\omega_0 t)\,\mathrm{d}t \\
&\quad + j\int_0^\infty \sin(-\omega t)\cos(\omega_0 t)\,\mathrm{d}t \\
&= \begin{cases} \infty & \text{if } \omega = \pm\omega_0 \\ 0 & \text{if } \omega \neq \omega_0 \end{cases}
\end{aligned}$$

Transfer Function

The transfer function $H(s)$ of a linear filter is

- the Laplace transform of its impulse reponse $h(t)$.
- the Laplace transform of the differential equation describing the I/O realtionship that is then solved for $\frac{V_{out}(s)}{V_{in}(s)}$
- (easiest!!!) the circuit diagram solved quite normally for $\frac{V_{out}(s)}{V_{in}(s)}$ by putting in impedances $Z(s)$ for all linear elements according to some simple rules (next page).

Impedances of Linear Circuit Elements

resistor:	R
capacitor:	$\frac{1}{sC}$
inductor:	sL

Ideal sources (e.g. the $i_d = g_m v_{gs}$ sources in small signal models of FETs) are left as they are.

Transfer Function

The laplace transform $Y(s)$ of the response $y(t)$ to input $x(t)$ is the *product* (!!!) of the Laplace transform of the input and the transfer function $H(s)X(s)$. Much simpler than performing a convolution $y(t) = (x * h)(t)$!!! You can do this for each individual frequency component of the input individually, i.e. set $s = j\omega$. Thus you can multiply each point in the power Bode plots and add the phase Bode plots of $X(s)$ and $H(s)$ to get the Bode plot of $Y(s)$. If you really want to, you can then apply the inverse Laplace/Fourrier transform to get back $y(t)$.

Transfer Function

Transfer functions $H(s)$ for linear electronic circuits can be written as dividing two polynomials of s.

$$H(s) = \frac{a_0 + a_1 s + \ldots + a_m s^m}{1 + b_1 s + \ldots + b_n s^n}$$

$H(s)$ is often written as products of first order terms in both nominator and denominator in the following *root form*, which is conveniently showing some properties of the Bode-plots. More of that later.

$$H(s) = a_0 \frac{(1 + \frac{s}{z_1})(1 + \frac{s}{z_2})\ldots(1 + \frac{s}{z_m})}{(1 + \frac{s}{\omega_1})(1 + \frac{s}{\omega_2})\ldots(1 + \frac{s}{\omega_n})}$$

1st order filters

That means s does only occur to the power of 1 , i.e s^1, in the denominator

Example 1st order passive low-pass filter

Here derived from the differential equation:

$$\dot{v}_{out} = (v_{in} - v_{out})\frac{1}{RC}$$

Laplace Transform (replace $\dot{x}$ with sx):

$$sV_{out} = (V_{in}(s) - V_{out}(s))\frac{1}{RC}$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{1}{1 + sRC}$$

Example 1st order passive high-pass filter

Here derived from the differential equation:

$$\dot{v}_{out} = \dot{v}_{in}\frac{C}{C_{tot}} - v_{out}\frac{1}{RC_{tot}} = \dot{v}_{in} - v_{out}\frac{1}{RC}$$

Laplace Transform:

$$sV_{out} = sV_{in}(s) - V_{out}(s)\frac{1}{RC}$$

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{s}{s + \frac{1}{RC}} = \frac{sRC}{1 + sRC}$$

Simple rule to obtain 1st order passive transfer functions

$$H(s) = \frac{G_1(s)}{G_1(s) + G_2(s)}$$

where $G_1(s)$ is the Admittance between input and output and $G_2(s)$ is the total admittance seen from the output node. Resistive Admittances are 1/R, capacitive are sC and inductive 1/Ls. Combining Admittances in parallel one has to simply do a sum. Combining in series one has to sum the inverse expressions (Impedances) and take the inverse.

Simple rule to obtain any transfer functions

Replace all capacitors, inductors and resistors with impedances or admitances and compute the I/O relationship as if they were resistances or conductances. Solve for $H(s) = \frac{V_{out}(s)}{V_{in}(s)}$

Book Example 4.4

Live on the white board...

Properties of 1st order low pass filters

Many single stage amplifiers can be considered to be 1st order low pass filters and have a transfer function:

$$H(s) = \frac{A_0}{1 + \frac{s}{\omega_0}}$$

Thus, they have a single 'pole' ω_0 (actually the pole is $-j\omega_0$) that is equal to the cutoff frequency $\omega_{-3dB} = \omega_0$. At cutoff the phase shift is -45^o and converges to -90^o at high frequencies. The unity gain frequency is $\omega_{ta} = A_0\omega_0$. Their 'roll-off' is 20dB per decade.

Decibel

A logarithmic scale comparing two values: 10dB difference corresponds to a factor 10 in power of a sinusoidal signal, 20dB difference to a factor 10 in amplitude.

$$L_{dB} = 10 \log_{10} \frac{A_1^2}{A_0^2} = 20 \log_{10} \frac{A_1}{A_0}$$

2nd order filters

Combinations of 2 first order low pass circuits in series are 2nd order filters . Many have a simple 2nd order transfer function with no zeros:

$$H(s) = \frac{A_{cl}}{(1+\frac{s}{\omega_{p1}})(1+\frac{s}{\omega_{p2}})} = \frac{A_{cl}\omega_0^2}{\omega_0^2 + s\frac{\omega_0}{Q} + s^2}$$

Where $\omega_0 = \omega_{p1}\omega_{p2}$ and $Q = \frac{\omega_{p1}\omega_{p2}}{\omega_{p1}+\omega_{p2}}$.

$$\|H(\omega)\| = \frac{A_{cl}}{\sqrt{1+\left(\frac{\omega}{\omega_{p1}}\right)^2}\sqrt{1+\left(\frac{\omega}{\omega_{p2}}\right)^2}}$$

$$\angle H(\omega) = -tan^{-1}\left(\frac{\omega}{\omega_{p1}}\right) - tan^{-1}\left(\frac{\omega}{\omega_{p2}}\right)$$

Bode Plots

Plots of magnitude (e.g. $|H(s)|$) in dB and phase e.g. $\angle H(s)$ or ϕ) vs. $\log(\omega)$. Example 4.6:

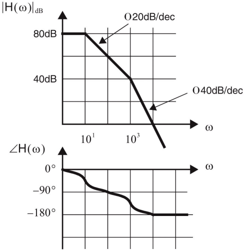


Figure 4.4

In general for transfer functions with only real poles and no zeros: a) $\omega \rightarrow 0_+$ $|H(s)|$ is constant at the low frequency gain and $\angle H(s) = 0^o$ b) for each pole as ω increases the slope of $|H(s)|$ increses by $-\frac{20dB}{\text{decade}}$ c) each pole contributes -90^o to the phase, but in a smooth transistion so that at a frequency exactly at the pole it is exactly -45^o

Example 4.7: Feedback ideal amplifier (1/3)

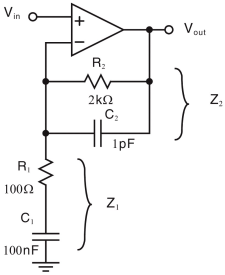


Figure 4.5

An ideal amplifier means infinite gain. Thus, the output can only be something else than $\pm\infty$ if the input (difference) is zero. The feedback loop is designed that the output moves wherever it needs to in order to keep the input zero.

$$H(s) = 1 + \frac{Z_2}{Z_1}$$

$$H(s) = \frac{1 + s[(R_1 + R_2)C_1 + R_2C_2] + s^2 R_1 R_2 C_1 C_2}{(1 + sR_1C_1)(1 + sR_2C_2)}$$

Example 4.7: Feedback ideal amplifier (2/3)

The nominator should match

$$N(s) = 1 + s\left(\frac{1}{z_1} + \frac{1}{z_2}\right) + s^2\frac{1}{z_1 z_2}$$

for *widely spaced* zeros $z_1 << z_2$ we can simplify

$$N(s) \approx 1 + s\left(\frac{1}{z_1}\right) + s^2\frac{1}{z_1 z_2} \quad (4.98)$$

Example 4.7: Feedback ideal amplifier (3/3)

$$z_1 \approx \frac{1}{(R_1+R_2)C_1+R_2C_2} \approx \frac{1}{(R_1+R_2)C_1} \quad (4.99)$$

$$z_2 \approx \frac{1}{z_1R_1R_2C_1C_2} \approx \frac{R_1+R_2}{R_1R_2C_2} = \frac{1}{(R_1 \parallel R_2)C_2} \quad (4.100)$$

$$\omega_{p1} = \frac{1}{R_1C_1} \quad \omega_{p2} = \frac{1}{R_2C_2} \quad (4.101-102)$$

General rules of thumb to use zeros and poles for Bode plots (1/3)

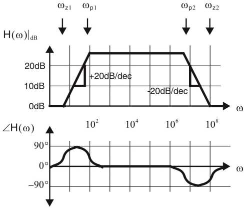


Figure 4.8

a) find a frequency ω_{mid} with equal number k of zeros and poles where $z_1, \ldots, z_k, \omega_1, \ldots, \omega_k < \omega_{mid} \Rightarrow$

$$|H(s)| \approx K \frac{|z_{(k+1)} \ldots |z_m|}{|\omega_{(k+1)}| \ldots |\omega_n|}$$

$$\angle H(s) \approx 0^o$$

and the gradient of both $|H(s)|$ and $\angle H(s)$ is zero

General rules of thumb to use zeros and poles for Bode plots (2/3)

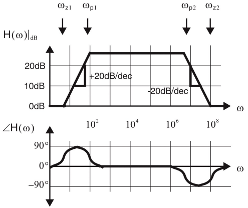
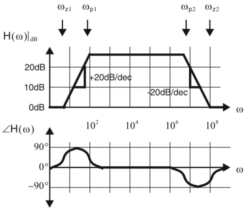


Figure 4.8

b) moving from ω_{mid} in the magnitude plot at each $|\omega_i|$ add -20dB/decade to the magnitude gradient and for each $|z_i|$ add +20dB/decade
c) moving from ω_{mid} in the phase plot to higher frequencies at each ω_i add -90^o to the phase in a smooth transition (respectively -45^o right *at* the poles) and vice versa towards lower frequencies.

General rules of thumb to use zeros and poles for Bode plots (3/3)



d) For the zeros towards higher frequencies if the nominater is of the form $(1+\frac{s}{z_i})$ add $+90^o$ and if its of the form $(1-\frac{s}{z_i})$ (refered to as right half plain zero as the solution for s of $0=(1-\frac{s}{z_i})$ is positive) add -90^o to the phase in a smooth transition (i.e. respectively $\pm 45^o$ right at the zeros) and vice versa towards lower frequencies.

Special Case: Complex Conjugated Poles (in 2nd order filters)

This happens when $Q > \frac{1}{2}$ in:

$$H(s) = \frac{K\omega_0^2}{\omega_0^2 + s\frac{\omega_0}{Q} + s^2} \quad (4.64)$$

Then the solution for the denominator is:

$$\omega_1, \omega_2 = \frac{\omega_0}{2Q}\left(1 \pm \sqrt{1 - 4Q^2}\right) = \omega_r \pm j\omega_q$$

Since $(Q > \frac{1}{2}) \Rightarrow (1 - 4Q^2) < 0$ the result is complex:

$$\omega_r = \frac{\omega_0}{2Q} \qquad \omega_q = \frac{\omega_0}{2Q}\sqrt{4Q^2 - 1} \quad (4.104, 4.105)$$

Step Response Dependent on Q

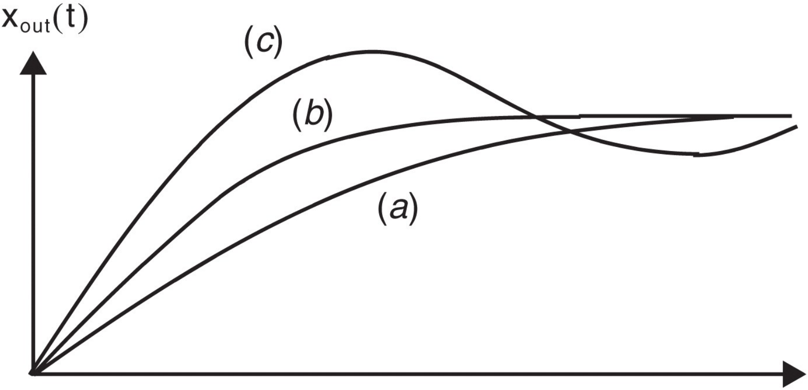


Figure 4.10

Step Response Dependent on Q

Correction of (4.107), problem pointed out by Kenneth

$$
\begin{aligned}
x_{out}(t) &= A_{in}K\left[1 - e^{-\omega_r t}\cos(\omega_q t) - \frac{\omega_r}{\omega_q} 1 - e^{-\omega_r t}\sin(\omega_q t)\right] \\
&= A_{in}K\left[1 - A_s e^{-\omega_r t}\cos(\omega_q t + \theta)\right] \quad (4.107)
\end{aligned}
$$

Where $A_S = \frac{2Q}{\sqrt{4Q^2 - 1}}$ and $\theta = -\frac{\pi}{2(\sqrt{4Q^2-1}+1)}$

Overshoot for $Q > \frac{1}{2}$

Correction of (4.109), problem pointed out by Kenneth

$$x_{out}(t)|_{max} = A_{in}K\left(1 + e^{-\frac{\pi}{\sqrt{4Q^2-1}}}\right) \quad (4.109)$$

The expression in the book is **only** the overshoot and not the signal at its maximum!!!

Content

MOSFET High Frequency Small Signal Model

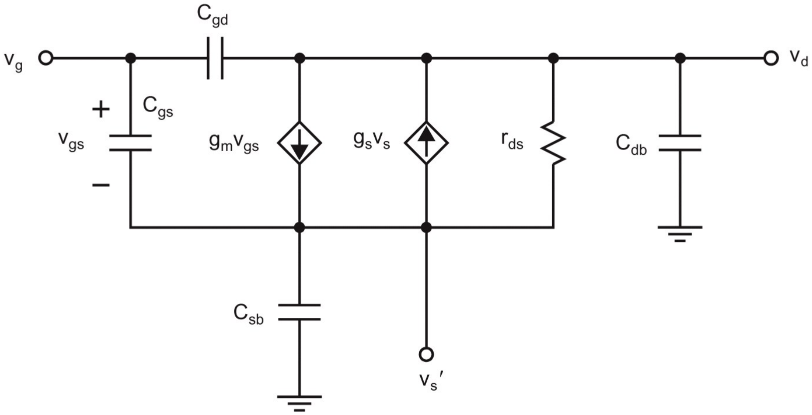


Figure 4.12

Common Source (1/2)

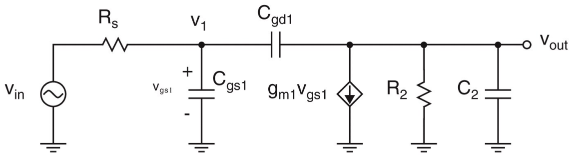


Figure 4.14

Follow the methodology for the Kirchoff node equations outlined on page 166/167! ($A(s) = H(s)$).

$$A(s) = \frac{v_{out}}{v_{in}} = -g_{m1}R_2 \frac{1 - s\frac{C_{gd1}}{g_{m1}}}{1 + sa + s^2 b}$$

where ...

Common Source (2/2)

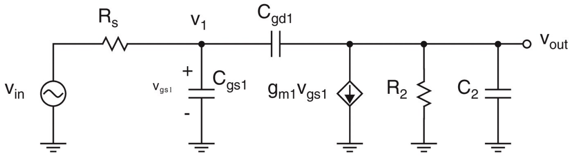


Figure 4.14

... where

$$
\begin{aligned}
a &= R_S[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)] + R_2(C_{gd1} + C_2) \\
b &= R_S R_2 (C_{gd1}C_{gs1} + C_{gs1}C_2 + C_{gd1}C_2)
\end{aligned}
$$

$\Rightarrow$ bloody complicated to get here! Can this be simplified?

Miller Theorem

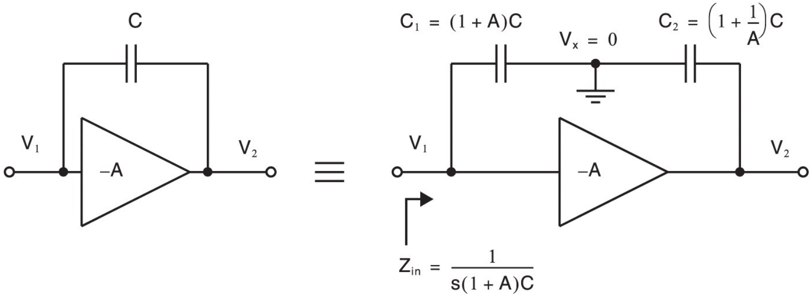


Figure 4.17

Miller Theorem and Common Source Amp (1/2)

Applied to the common source amplifier:

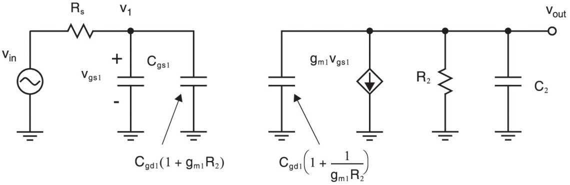


Figure 4.18

These are two 1st order low pass filters in series. The pole of the first is usually dominant, because of the large equivalent capacitance.

Miller Theorem and Common Source Amp (2/2)

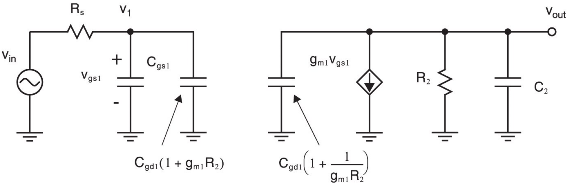


Figure 4.18

$$\omega_{-3\text{dB}} \approx \omega_1 = \frac{1}{R_S[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)]}$$

Zero-Value Time-Constant Analysis

a) set all independent sources to zero.
b) For each capacitor in turn, assume all other capacitances to be zero and find its corresponding time constant.
c) The -3DB frequency for the complete circuit is 1 devided by the sum of the all zero-value time-constants

Example Common Source Amp

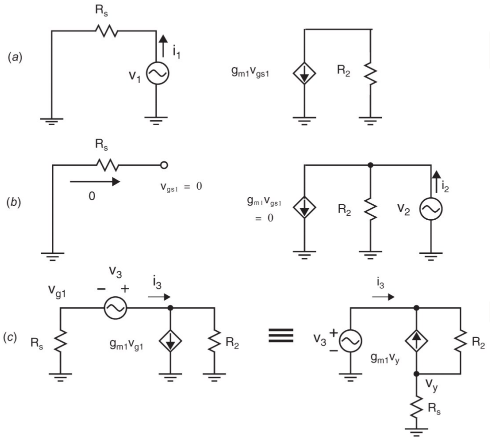


Figure 4.19

Example Common Source Amp

Zero-value time-constant analysis:

$$\omega_{-3dB} \approx \frac{1}{R_S[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)] + R_2(C_{gd1} + C_2)} \quad (4.152)$$

Complete analysis with dominant pole approximation:

$$\omega_{-3dB} \approx \frac{1}{R_S[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)] + R_2(C_{gd1} + C_2)} \quad (4.131)$$

Miller theorem (exact 1st pole):

$$\omega_{-3\mathrm{dB}} \approx \omega_1 = \frac{1}{R_S[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)]} \quad (4.144)$$

Common Gate Amp

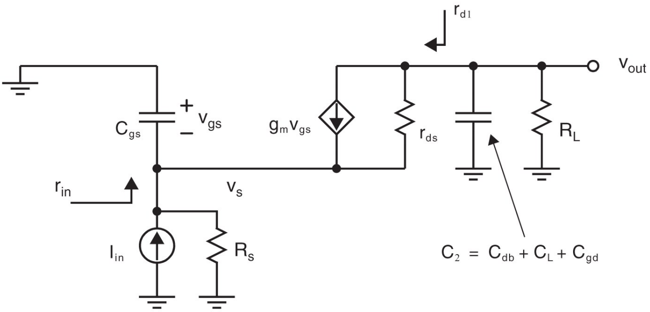


Figure 4.20

Zero-Value Time-Constant Analysis

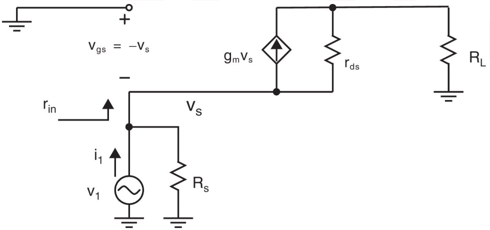


Figure 4.21

$$\tau_1 = (r_{in} \parallel R_S)C_{gs} \approx \frac{C_{gs}}{g_m} \quad (4.162)$$

$$\tau_2 = (R_L \parallel r_{d1})C_2 \approx R_L C_2 \quad (4.165)$$

Cascode Gain Stage

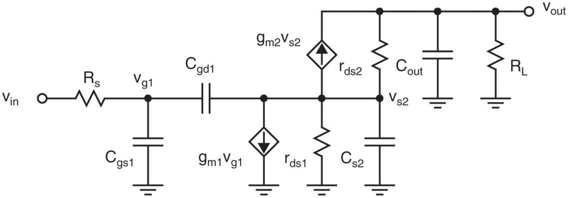


Figure 4.23

Zero-Value Time-Constant analysis result:

$$\tau_{\text{total}} \approx (g_{m2}r_{ds1}r_{ds2} \parallel R_L)C_{out} + C_{gs1}R_S$$
$$+(r_{in2} \parallel r_{ds1})C_{s2} + R_S[1 + g_m(r_{in2} \parallel r_{ds1})]C_{gd1}$$

Source Follower

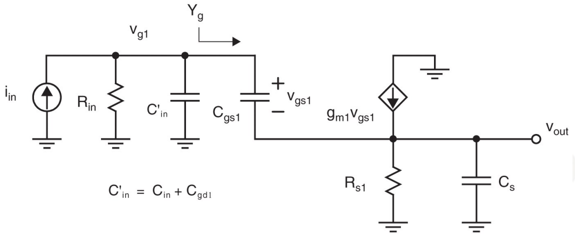


Figure 4.28

Solution in the book: (4.199). It may have $Q > \frac{1}{2} \Rightarrow$ ringing! Then one might explicitly add input or load capacitance (sacrificing speed) or a compensation network as shown in fig. 4.30 with values given in (4.216).

Symmetric Differantial Pair

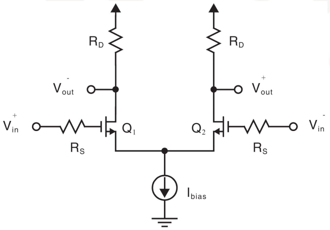


Figure 4.32

Assuming $v_{in}^- = -v_{in}^+$ one can analyse just half the circuit which is equivalent to a common source amplifier. Thus:

$$\omega_{-3dB} \approx \frac{1}{R_S[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)]} \quad (4.225)$$

Single Ended Differantial Amplifier

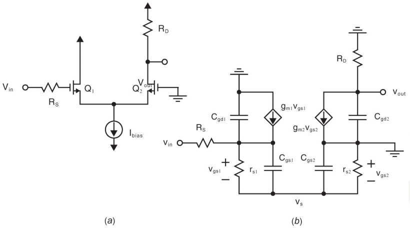


Figure 4.34

Avoiding Miller capacitrance between output and input increases bandwidth:

$$\omega_{-3dB} \approx \omega_1 = \frac{1}{R_S[\frac{C_{gs1}}{2} + C_{gd1}]} \quad (4.227)$$

Differantial Pair with active load

Often used as the input stage of a two stage amplifier.
⇒ A very large load capacitor that dominates entirely.
Thus a very simplified model is often sufficient:

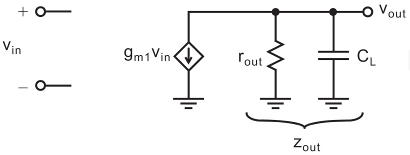


Figure 4.37

$$\omega_{-3dB} \approx \frac{1}{r_{out}C_L} = \frac{1}{(r_{ds2} \parallel r_{ds4})C_L} \quad (4.229)$$