FYS1120 - Elektromagnetisme

Oblig 1 Høsten 2015

Krister Borge

26. september 2015

0.1 oppgave 1

1. **a)**

(i)

$$f(x,y,z) = x^2y$$
$$\nabla f(x,y,z) = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$$
$$= 2xy\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

(ii)

$$g(x,y,z) = xyz$$
$$\nabla f(x,y,z) = (yz, xz, xy)$$

(iii)

$$h(r,\theta,\phi) = \frac{1}{r}e^{r^2}$$

Jeg bort i fra både θ og ϕ siden denne funksjonen kun beskriver avstanden av fra origo. Derfor kan jeg derivere med hensyn på r for funksjonen

$$h(z,\theta,\phi) = \frac{1}{r}e^{r^2}$$
$$= \left(\frac{e^{r^2}(2r^2-1)}{r^2} 0, 0\right)$$

Der $r = \sqrt{x^2+y^2+z^2}$

Bruker kartetiske koordinater og enhetsvektoren for å finne denne gradienten.

$$\nabla h(x,y,z) = \frac{e^{x^2+y^2+z^2}x(-1+2x^2 2y^2+2z^2)}{x^2+y^2+z^2}\mathbf{e}_x$$
$$+\frac{e^{x^2+y^2+z^2}y(-1+2x^2 2y^2+2z^2)}{x^2+y^2+z^2}\mathbf{e}_y$$
$$+\frac{e^{x^2+y^2+z^2}z(-1+2x^2 2y^2+2z^2)}{x^2+y^2+z^2}\mathbf{e}_z$$

b)

(i)

$$\mathbf{u}(x,y,z) = (2xy, x^2, 0)$$

$$\mathsf{div}\mathbf{u}(x,y,z) = \nabla \cdot \mathbf{u} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}x^2 + \frac{\partial}{\partial z}0 = 2y$$

det var divergensen, her kommer virvlinga:

$$\mathsf{curl}(\mathbf{u}(x,y,z)) = \nabla \times \mathbf{u}$$

$$\mathsf{curl}(\mathbf{u}(x,y,z)) = \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2zy & x^2 & 0 \end{pmatrix}$$

$$= 0 + 0 + 2x\hat{\mathbf{k}} - 2x\hat{\mathbf{k}} - 0 - 0 = 0$$

(ii)

$$\mathbf{v}(x,y,z) = (e^{yz}, ln(xy), z)$$

$$\mathsf{div}(\mathbf{v}(x,y,z)) = \nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(e^{yz}) + \frac{\partial}{\partial y}(ln(xy)) + \frac{\partial}{\partial z}(z) = \frac{1}{y} + 1$$

$$\mathsf{curl}(\mathbf{v}(x,y,z)) = \nabla \times \mathbf{v} = \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{yz} & ln(xy) & z \end{pmatrix}$$

$$= 0 + 0 + 0 - \frac{\partial}{\partial y}e^{yz}\hat{\mathbf{k}} - 0 - 0 = -ze^{zy}\hat{\mathbf{k}}$$

(iii)

$$\mathbf{w}(x,y,z) = (yz, xz, xy)$$

$$\mathsf{div}(\mathbf{w}(x,y,z)) = \nabla \cdot \mathbf{w} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$$

$$\mathsf{curl}(\mathbf{w}(x,y,z)) = \nabla \times \mathbf{w} = \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{pmatrix}$$

$$= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} - x\hat{\mathbf{i}} - y\hat{\mathbf{j}} - z\hat{\mathbf{k}} = 0$$

(iv)

$$\mathbf{a}(x,y,z) = (y^2z, -z^2\mathsf{sin}(y) + 2xyz, 2z\mathsf{cos}(y) + y^2x)$$

$$\mathsf{div}(\mathbf{a}(x,y,z)) = \nabla \cdot \mathbf{a} = \frac{\partial}{\partial x}(y^2z) + \frac{\partial}{\partial y}(-z^2\mathsf{sin}(y) + 2xyz) + \frac{\partial}{\partial z}(2z\mathsf{cos}(y) + y^2x) = 0$$

$$= 0 + z\mathsf{cos}(y) + 2xz + 2\mathsf{cos}(y) + y^2$$

$$\mathsf{curl}(\mathbf{a}(x,y,z)) = \nabla\times\mathbf{a} = \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & -z^2\mathsf{sin}(y) + 2xyz & 2z\mathsf{cos}(y) + y^2 x \end{pmatrix}$$

$$= \frac{\partial}{\partial y}(2z\mathsf{cos}(y) + y^2 x)\hat{\mathbf{i}} + \frac{\partial}{\partial z}(y^2 z)\hat{\mathbf{j}} + \frac{\partial}{\partial x}(-z^2\mathsf{sin}(y) + 2xyz)\hat{\mathbf{k}}$$

$$-\frac{\partial}{\partial z}(-z^2\mathsf{sin}(y) + 2xyz)\hat{\mathbf{i}} - \frac{\partial}{\partial x}(2z\mathsf{cos}(y) + y^2 x)\hat{\mathbf{j}} - \frac{\partial}{\partial y}y^2 z\hat{\mathbf{k}}$$

$$= (2z\mathsf{cos}(y) + y)\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + 2yz\hat{\mathbf{k}} - (-2z\mathsf{sin}(y) + 2xy)\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} - 2yz\hat{\mathbf{k}}$$

$$= (-2z\mathsf{cos}(y)+y)\hat{\mathbf{i}} - (-2z\mathsf{sin}(y)+2xy)\hat{\mathbf{i}} = (-2z\mathsf{cos}(y)+2z\mathsf{sin}(y)+y-2xy)\hat{\mathbf{i}}$$

c)

Matematisk sett er et felt konservativt når det oppfyller baneuavhengig. Det kan også ses ved at virvlinga er lik 0-vektoren. Anta et vektorfelt **F** som er koblet til et skalarfelt ϕ ved $\mathbf{F} = \nabla\phi$, hvor $\nabla\phi$ eksisterer
i et område D. Da er **F** konservativt inne i D. På samme måte kan **F** skrives som gradienten til et skalarfelt $\mathbf{F} = \nabla\phi$
(Mathews, P. C. Vector Calculus, 7. utgave 2005)
Fra fysikken kan man tenke seg gravitasjonsfeltet som et konservativt felt. Ordet i seg selv betyr bevarende (konserverende). Siden man kan se på et gradienten til et potensial (skalarfelt) vil mekanisk energi være bevart.
Et objekt i et slikt felt er altså uavhengig av banen. Det er kun avhengig av start- og stoppunkt.

Skalarfeltene i 1a $f(x,y,z)$ og $h(x,y,z)$og har gradientene som tilsvarer vektorfeltene **u** og **w**. Nå har jeg et vektorfelt som er koblet med et skalarfelt:

$$\mathbf{u} = \nabla f(x,y,z)$$

og

$$\mathbf{w} = \nabla h(x,y,z)$$

Definisjonen til curl gjør at vi er avgrenset av et område C og siden vi da har curl =0 i begge vektorfelt kan jeg slutte at disse feltene er konservative.

d)

Vi ser her på laplaceoperatoren til

(i)

$$j(x,y,z) = x^2 + xy + yz^2$$

$$\nabla^2 j = \sum_{j=1}^{n} \frac{\partial^2 j}{\partial x\ _i} = \left(\frac{\partial^2 j}{\partial x}\right) + \left(\frac{\partial^2 j}{\partial y}\right) + \left(\frac{\partial^2 j}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}(2x+y)\frac{\partial}{\partial y}(x+z) + \frac{\partial}{\partial z}(2zy)\right) = 2$$

(ii)

$$k(r,\theta,\phi) = \frac{1}{r}e^{r^2}$$

i kulekoordinater blir dette:

$$k(r,\theta,\phi) = \frac{\partial^2 f}{\partial r}\left(\frac{e^{r^2}(2r^2+1)}{r^2}\right)$$

$$= \frac{2e^{r^2}(2r^4 - r^2 + 1)}{r^3}$$

0.2 oppgave 2

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

Der vektorene er $\mathbf{a} = (a_1, a_2, a_3)$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (a_1, a_2, a_3) \times ((b_1, b_2, b_3) \times (c_1, c_2, c_3))$$

$$\begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$= \hat{\mathbf{i}}b_2c_3 + \hat{\mathbf{j}}b_3c_1 + \hat{\mathbf{k}}b_1c_2 - \hat{\mathbf{i}}b_3c_2 - \hat{\mathbf{j}}b_1c_3 - \hat{\mathbf{k}}b_1c_2$$

$$= \hat{\mathbf{i}}(b_2c_3 - b_1c_3) + \hat{\mathbf{j}}(b_3c_1 - b_1c_3) + \hat{\mathbf{k}}(b_1c_2 - b_2c_1)$$

$$\begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_1c_3 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{pmatrix}$$

$$= \hat{\mathbf{i}}a_2(b_1c_2 - b_1c_3) + \hat{\mathbf{j}}a_3(b_2c_3 - b_1c_3) + \hat{\mathbf{k}}a_1(b_3c_1 - b_1c_3)$$

$$-\hat{\mathbf{i}}a_3(b_3c_1 - b_1c_3) - \hat{\mathbf{j}}a_1(b_1c_2 - b_1c_2) - \hat{\mathbf{k}}a_2(b_2c_3 - b_1c_3)$$

$$= \hat{\mathbf{i}}(a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3))$$

$$+\hat{\mathbf{j}}(a_3(b_2c_3 - b_1c_3) - a_1(b_1c_2 - b_1c_2))$$

$$+\hat{\mathbf{k}}(a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_1c_3)) = \hat{\mathbf{i}}(a_2b_2c_1 - a_2b_1c_2 - a_3b_3c_1 + a_3b_1c_3)$$

$$+\hat{\mathbf{j}}(a_3b_2c_3 - a_3b_1c_3 - a_1b_1c_2 + a_1b_1c_2) + \hat{\mathbf{k}}(a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_1c_3)$$

Tar for meg høyre siden nå:

$$= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) =$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right) - \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right)$$

$$= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} (a_1c_1 + a_2c_2 + a_3c_3) - \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} (a_1b_1 + a_2b_2 + a_3b_3)$$

$$= \begin{pmatrix} b_1 \cdot (a_1c_1 + a_2c_2 + a_3c_3) \\ b_2 \cdot (a_1c_1 + a_2c_2 + a_3c_3) \\ b_3 \cdot (a_1c_1 + a_2c_2 + a_3c_3) \end{pmatrix} - \begin{pmatrix} c_1 \cdot (a_1b_1 + a_2b_2 + a_3b_3) \\ c_2 \cdot (a_1b_1 + a_2b_2 + a_3b_3) \\ c_3 \cdot (a_1b_1 + a_2b_2 + a_3b_3) \end{pmatrix}$$

$$= \begin{pmatrix} b_1 \cdot (a_1c_1 + a_2c_2 + a_3c_3) - c_1 \cdot (a_1b_1 + a_2b_2 + a_3b_3) \\ b_2 \cdot (a_1c_1 + a_2c_2 + a_3c_3) - c_2 \cdot (a_1b_1 + a_2b_2 + a_3b_3) \\ b_3 \cdot (a_1c_1 + a_2c_2 + a_3c_3) - c_3 \cdot (a_1b_1 + a_2b_2 + a_3b_3) \end{pmatrix}$$

$$= \begin{pmatrix} b_1a_2c_2 + b_1a_3c_3 - c_1a_2b_2 - c_1a_3b_3 \\ b_2a_1c_1 + b_2a_3c_3 - c_2a_1b_1 - c_2a_3b_3 \\ b_3a_1c_1 + b_3a_2c_2 - c_3a_1b_1 - c_3a_2b_2 \end{pmatrix} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

0.3 oppgave 3: Fluksintegral og Gauss' teorem

a)

$$\mathbf{f}(\mathbf{x}) = (y, x, z - x)$$

Enhetskuben er gitt ved $(x, y, z) \in [0, 1]$
Finner divergensen til $f(\mathbf{x})$

$$\mathsf{div} f(\mathbf{x}) = (0 + 0 + 1) = 1$$

Regner så ut fluksen for hver av de seks sidene til kuben:

$$\iint_A = \mathbf{v} \cdot \mathbf{n} dA =$$

$$\iiint_0^1 \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial y} dxdydz$$

Jeg tar for meg hver side av enhetskuben: For siden der$x = 0$ blir $y \in [0, 1]$ og $z \in [0, 1]$. og $\mathbf{n} = -\hat{\mathbf{i}}$

$$\mathbf{v} \cdot \mathbf{n} = -y$$

Fluxen blir

$$\iint\limits_{A_{\hat{\mathbf{i}}}} -y = \int\limits_0^1 \int\limits_0^1 -y dy dz = -\frac{1}{2}$$

For siden der$x = 1$ blir $y \in [0,1]$ og $z \in [0,1]$. og $\mathbf{n} = \hat{\mathbf{i}}$

$$\mathbf{v} \cdot \mathbf{n} = y$$

Fluxen blir

$$\iint\limits_{A_{\hat{\mathbf{i}}}} -y = \int\limits_0^1 \int\limits_0^1 y dy dz = \frac{1}{2}$$

For siden der$x \in [0,1]$, $y = 0$ og $z \in [0,1]$. og $\mathbf{n} = -\hat{\mathbf{j}}$

$$\mathbf{v} \cdot \mathbf{n} = -x$$

Fluxen blir

$$\iint\limits_{A_{\hat{\mathbf{j}}}} -y = \int\limits_0^1 \int\limits_0^1 y dx dz = -\frac{1}{2}$$

For siden der$x \in [0,1]$, $y = 1$ og $z \in [0,1]$. og $\mathbf{n} = \hat{\mathbf{j}}$

$$\mathbf{v} \cdot \mathbf{n} = x$$

Fluxen blir

$$\iint\limits_{A_{\hat{\mathbf{j}}}} -y = \int\limits_0^1 \int\limits_0^1 y dx dz = \frac{1}{2}$$

For siden der$x \in [0,1]$, $y \in [0,1]$og $z = 0$. og $\mathbf{n} = -\hat{\mathbf{k}}$

$$\mathbf{v} \cdot \mathbf{n} = -(z - x) = z + x$$

Fluxen blir

$$\iint\limits_{A_{\hat{\mathbf{k}}}} -y = \int\limits_0^1 \int\limits_0^1 z + x dx dz = 1$$

For siden der$x \in [0,1]$, $y \in [0,1]$og $z = 1$. og $\mathbf{n} = \hat{\mathbf{k}}$

$$\mathbf{v} \cdot \mathbf{n} = -(z - x) = z - x$$

Fluxen blir

$$\iint\limits_{A_{\hat{\mathbf{k}}}} -y = \int\limits_0^1 \int\limits_0^1 z - x dx dz = 0$$

Fluks ut av enhetskuben blir da

$$\iint\limits_A \mathbf{v}\cdot\mathbf{n} = -\frac{1}{2} + \frac{1}{2} + \left(-\frac{1}{2}\right) + \frac{1}{2} + 0 + 1 = 1$$

Gauss' teorem:

$$\iiint\limits_A \nabla\cdot\mathbf{v} dA = \iiint\limits_0^1 \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial y} dxdydz = \iiint\limits_0^1 0+0+1 dxdydz = [1]_0^1 = 1$$

0.4 Oppgave 4: Linjeintegral og Stokes' teorem

$$\mathbf{w}(x, y, z) = 2x - y, -y^2, -y^2z$$

a)
Regner ut divergensen til $\mathbf{w}$:

$$\mathsf{div}(\mathbf{w}(x, y, z) = \nabla \cdot \mathbf{w} = \frac{\partial}{\partial x}(2x - y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(-y^2z)$$

$$= 2 - 2y - y^2$$

b)
Regner ut curl til $\mathbf{w}$:

$$\mathsf{curl}(\mathbf{w}(x, y, z) = \nabla \times \mathbf{w}$$

$$= \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -y^2 & -y^2z \end{pmatrix}$$

$$= \hat{\mathbf{i}}(\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-y^2)) + \hat{\mathbf{j}}(\frac{\partial}{\partial z}(2x - y) - \frac{\partial}{\partial x}(-y^2z)) + \hat{\mathbf{k}}(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(2x - y))$$

$$= \hat{\mathbf{i}}(-2yz) + 0\hat{\mathbf{j}} + \hat{\mathbf{k}}(1) = (2xy, 0, 1)$$

c)
γ er gitt ved $x^2 - y^2 = 1$,$z = 1$ Dette er en sirkel i xy-planet En parametrisering av γ er gitt ved

$$\gamma(t) = (cos(t), sin(t), 1)t \in [0, 2\pi]$$

d)
Finner sirkulasjonen C, til $\mathbf{w}$ rundt γ. $\mathbf{w}(x, y, z)$

$$\int\limits_C \mathbf{w}d\mathbf{s} = \int\limits_0^{2\pi} \mathbf{w}(\gamma(t)) \cdot \gamma(t)' dt$$

$$\int\limits_0^{2\pi} \mathbf{w}(2\sin t - \cos t, -\cos^2 t, -\cos^2 t) \cdot (\cos t, -\sin t, 0) dt$$

$$= \int\limits_0^{2\pi} (\cos t(2\sin t - \cos t) + (-\cos^2 t)(-\sin t) dt$$

$$= 2\int\limits_0^{2\pi} \cos t \sin t dt - \int\limits_0^{2\pi} \cos^2 t dt + \int\limits_0^{2\pi} \cos^2 t \sin t dt$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin u(-\cos u)du - \int_0^{2\pi} \cos^2 t dt + \int_0^{2\pi} \cos^2 t \sin t dt$$

Siden $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin u(-\cos u)du = 0$

$$= \int_0^{2\pi} \cos^2 t dt - \int_0^{2\pi} \cos^2 t \sin t dt$$

$$= \int_0^{2\pi} \frac{1}{2}\cos 2t - \frac{1}{2}dt + \int_0^{2\pi} \cos^2 t \sin t dt$$

skriver om

$$= \frac{1}{2}\int_0^{2\pi} \cos 2t dt + \frac{1}{2}\int_0^{2\pi} 1 dt - \int_0^{2\pi} \cos^2 t \sin t dt$$

$$= \frac{1}{4}\int_0^{4\pi} \cos s ds + \frac{1}{2}\int_0^{2\pi} 1 dt - \int_0^{2\pi} \cos^2 t \sin t dt$$

$$= \left.\frac{\sin s}{4}\right|_0^{4\pi} + \frac{1}{2}\int_0^{2\pi} 1 dt - \int_0^{2\pi} \cos^2 t \sin t dt$$

$$= \frac{1}{2}\int_0^{2\pi} 1 dt - \int_0^{2\pi} \cos^2 t \sin t dt$$

$$= \left.\frac{t}{2}\right|_0^{2\pi} - \int_0^{2\pi} \cos^2 t \sin t dt$$

$$= \pi - \int_0^{2\pi} \cos^2 t \sin t dt$$

som gir

$$\int_C \mathbf{w} d\mathbf{s} = \int_0^{2\pi} \mathbf{w}(\gamma(t)) \cdot \gamma(t)' dt = \pi$$

e) Regner ut curl til **w** er $(2xy, 0, 1)$
Stokes teorem er

$$\int\limits_C \mathbf{w} d\mathbf{r} = \iint\limits_S \mathsf{curl}\mathbf{w} \cdot d\mathbf{S}$$

$$\iint\limits_S \mathsf{curl}\mathbf{w} \cdot d\mathbf{S} = \iint\limits_S (2xy, 0, 1) \cdot (z_x, z_y, 1) d\mathbf{S} = \iint\limits_S (2xy, 0, 1) \cdot (0, 0, 1) d\mathbf{S}$$

Siden **w** ligger på $z = 1$.

$$\iint\limits_S (2xy, 0, 1) \cdot (0, 0, 1) d\mathbf{S} = 1 * \pi = \pi$$

og dette stemmer med oppgave **d)**