(b) Calculate the heat capacity for $T \ll T_F$, and compare to the experimental result $C_V = (2.8\ \text{K}^{-1})NkT$ (in the low-temperature limit). (Don't expect perfect agreement.)

(c) The entropy of *solid* ^{3}He below 1 K is almost entirely due to its multiplicity of nuclear spin alignments. Sketch a graph S vs. T for liquid and solid ^{3}He at low temperature, and estimate the temperature at which the liquid and solid have the same entropy. Discuss the shape of the solid-liquid phase boundary shown in Figure 5.13.

Problem 7.27. The argument given above for why $C_V \propto T$ does not depend on the details of the energy levels available to the fermions, so it should also apply to the model considered in Problem 7.16: a gas of fermions trapped in such a way that the energy levels are evenly spaced and nondegenerate.

(a) Show that, in this model, the number of possible system states for a given value of q is equal to the number of distinct ways of writing q as a sum of positive integers. (For example, there are three system states for $q = 3$, corresponding to the sums 3, $2 + 1$, and $1 + 1 + 1$. Note that $2 + 1$ and $1 + 2$ are not counted separately.) This combinatorial function is called the number of **unrestricted partitions** of q, denoted $p(q)$. For example, $p(3) = 3$.

(b) By enumerating the partitions explicitly, compute $p(7)$ and $p(8)$.

(c) Make a table of $p(q)$ for values of q up to 100, by either looking up the values in a mathematical reference book, or using a software package that can compute them, or writing your own program to compute them. From this table, compute the entropy, temperature, and heat capacity of this system, using the same methods as in Section 3.3. Plot the heat capacity as a function of temperature, and note that it is approximately linear.

(d) Ramanujan and Hardy (two famous mathematicians) have shown that when q is large, the number of unrestricted partitions of q is given approximately by

$$p(q) \approx \frac{e^{\pi\sqrt{2q/3}}}{4\sqrt{3}\,q}.$$

Check the accuracy of this formula for $q = 10$ and for $q = 100$. Working in this approximation, calculate the entropy, temperature, and heat capacity of this system. Express the heat capacity as a series in decreasing powers of kT/η, assuming that this ratio is large and keeping the two largest terms. Compare to the numerical results you obtained in part (c). Why is the heat capacity of this system independent of N, unlike that of the three-dimensional box of fermions discussed in the text?

The Density of States

To better visualize—and quantify—the behavior of a Fermi gas at small nonzero temperatures, I need to introduce a new concept. Let's go back to the energy integral (7.42), and change variables from n to the electron energy ϵ:

$$\epsilon = \frac{h^2}{8mL^2}n^2, \qquad n = \sqrt{\frac{8mL^2}{h^2}}\,\sqrt{\epsilon}, \qquad dn = \sqrt{\frac{8mL^2}{h^2}}\,\frac{1}{2\sqrt{\epsilon}}\,d\epsilon. \qquad (7.49)$$

With this substitution, you can show that the energy integral for a Fermi gas at zero temperature becomes

$$U = \int_0^{\epsilon_F} \epsilon \left[\frac{\pi}{2} \left(\frac{8mL^2}{h^2} \right)^{3/2} \sqrt{\epsilon} \right] d\epsilon \qquad (T = 0). \qquad (7.50)$$

The quantity in square brackets has a nice interpretation: It is the number of single-particle states per unit energy. To compute the total energy of the system we carry out a sum over all energies of the energy in question times the number of states with that energy.

The number of single-particle states per unit energy is called the **density of states**. The symbol for it is $g(\epsilon)$, and it can be written in various ways:

$$g(\epsilon) = \frac{\pi(8m)^{3/2}}{2h^3} V \sqrt{\epsilon} = \frac{3N}{2\epsilon_F^{3/2}} \sqrt{\epsilon}. \qquad (7.51)$$

The second expression is compact and handy, but perhaps rather confusing since it seems to imply that $g(\epsilon)$ depends on N, when in fact the N dependence is canceled by ϵ_F. I like the first expression better, since it shows explicitly that $g(\epsilon)$ is proportional to V and independent of N. But either way, the most important point is that $g(\epsilon)$, for a three-dimensional box of free particles, is proportional to $\sqrt{\epsilon}$. A graph of the function is a parabola opening to the right, as shown in Figure 7.13. If you want to know how many states there are between two energies ϵ_1 and ϵ_2, you just integrate this function over the desired range. The density of states is a function whose purpose in life is to be integrated.

The density-of-states idea can be applied to lots of other systems besides this one. Equation 7.51 and Figure 7.13 are for the specific case of a gas of "free" electrons, confined inside a fixed volume but not subject to any other forces. In more realistic models of metals we would want to take into account the attraction of the electrons toward the positive ions of the crystal lattice. Then the wavefunctions and their energies would be quite different, and therefore $g(\epsilon)$ would be a much more complicated function. The nice thing is that determining g is purely a problem of quantum mechanics, having nothing to do with thermal effects or temperature. And
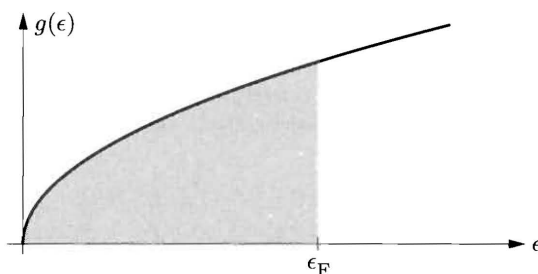

Figure 7.13. Density of states for a system of noninteracting, nonrelativistic particles in a three-dimensional box. The number of states within any energy interval is the area under the graph. For a Fermi gas at $T = 0$, all states with $\epsilon < \epsilon_F$ are occupied while all states with $\epsilon > \epsilon_F$ are unoccupied.

once you know g for some system, you can then forget about quantum mechanics and concentrate on the thermal physics.

For an electron gas at *zero* temperature, we can get the total number of electrons by just integrating the density of states up to the Fermi energy:

$$N = \int_0^{\epsilon_F} g(\epsilon)\, d\epsilon \qquad (T = 0). \tag{7.52}$$

(For a free electron gas this is the same as equation 7.50 for the energy, but without the extra factor of ϵ.) But what if T is nonzero? Then we need to multiply $g(\epsilon)$ by the *probability* of a state with that energy being occupied, that is, by the Fermi-Dirac distribution function. Also we need to integrate all the way up to infinity, since any state could conceivably be occupied:

$$N = \int_0^{\infty} g(\epsilon)\, \bar{n}_{FD}(\epsilon)\, d\epsilon = \int_0^{\infty} g(\epsilon)\, \frac{1}{e^{(\epsilon-\mu)/kT} + 1}\, d\epsilon \qquad (\text{any } T). \tag{7.53}$$

And to get the total energy of all the electrons, just slip in an ϵ:

$$U = \int_0^{\infty} \epsilon\, g(\epsilon)\, \bar{n}_{FD}(\epsilon)\, d\epsilon = \int_0^{\infty} \epsilon\, g(\epsilon)\, \frac{1}{e^{(\epsilon-\mu)/kT} + 1}\, d\epsilon \qquad (\text{any } T). \tag{7.54}$$

Figure 7.14 shows a graph of the integrand of the N-integral (7.53), for a free electron gas at nonzero T. Instead of falling immediately to zero at $\epsilon = \epsilon_F$, the number of electrons per unit energy now drops more gradually, over a width of a few times kT. The chemical potential, μ, is the point where the probability of a state being occupied is exactly $1/2$, and it's important to note that this point is no longer the same as it was at zero temperature:

$$\mu(T) \neq \epsilon_F \qquad \text{except when } T = 0. \tag{7.55}$$
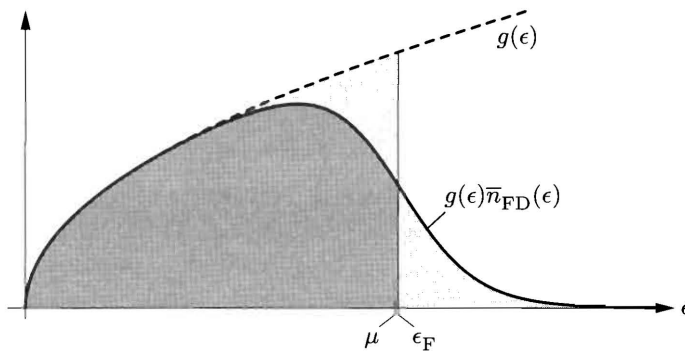

Figure 7.14. At nonzero T, the number of fermions per unit energy is given by the density of states times the Fermi-Dirac distribution. Because increasing the temperature does not change the total number of fermions, the two lightly shaded areas must be equal. Since $g(\epsilon)$ is greater above ϵ_F than below, this means that the chemical potential decreases as T increases. This graph is drawn for $T/T_F = 0.1$; at this temperature μ is about 1% less than ϵ_F.

Why not? Recall from Problem 7.12 that the Fermi-Dirac distribution function is symmetrical about $\epsilon = \mu$: The probability of a state above μ being occupied is the same as the probability of a state the same amount below μ being *unoccupied*. Now suppose that μ were to remain constant as T increases from zero. Then since the density of states is greater to the right of μ than to the left, the number of electrons we would be adding at $\epsilon > \mu$ would be greater than the number we are losing from $\epsilon < \mu$. In other words, we could increase the total number of electrons just by raising the temperature! To prevent such nonsense, the chemical potential has to decrease slightly, thus lowering all of the probabilities by a small amount.

The precise formula for $\mu(T)$ is determined implicitly by the integral for N, equation 7.53. *If* we could carry out this integral, we could take the resulting formula and solve it for $\mu(T)$ (since N is a fixed constant). Then we could plug our value of $\mu(T)$ into the energy integral (7.54), and try to carry out *that* integral to find $U(T)$ (and hence the heat capacity). The bad news is that these integrals cannot be evaluated exactly, even for the simple case of a free electron gas. The good news is that they *can* be evaluated approximately, in the limit $kT \ll \epsilon_F$. In this limit the answer for the energy integral is what I wrote in equation 7.47.

> **Problem 7.28.** Consider a free Fermi gas in two dimensions, confined to a square area $A = L^2$.
>
> **(a)** Find the Fermi energy (in terms of N and A), and show that the average energy of the particles is $\epsilon_F/2$.
>
> **(b)** Derive a formula for the density of states. You should find that it is a constant, independent of ϵ.
>
> **(c)** Explain how the chemical potential of this system should behave as a function of temperature, both when $kT \ll \epsilon_F$ and when T is much higher.
>
> **(d)** Because $g(\epsilon)$ is a constant for this system, it is possible to carry out the integral 7.53 for the number of particles analytically. Do so, and solve for μ as a function of N. Show that the resulting formula has the expected qualitative behavior.
>
> **(e)** Show that in the high-temperature limit, $kT \gg \epsilon_F$, the chemical potential of this system is the same as that of an ordinary ideal gas.

The Sommerfeld Expansion

After talking about the integrals 7.53 and 7.54 for so long, it's about time I explained how to evaluate them, to find the chemical potential and total energy of a free electron gas. The method for doing this in the limit $kT \ll \epsilon_F$ is due to Arnold Sommerfeld, and is therefore called the **Sommerfeld expansion**. None of the steps are particularly difficult, but taken as a whole the calculation is rather tricky and intricate. Hang on.

I'll start with the integral for N:

$$N = \int_0^\infty g(\epsilon)\, \overline{n}_{FD}(\epsilon)\, d\epsilon = g_0 \int_0^\infty \epsilon^{1/2}\, \overline{n}_{FD}(\epsilon)\, d\epsilon. \tag{7.56}$$

(In the second expression I've introduced the abbreviation g_0 for the constant that multiplies $\sqrt{\epsilon}$ in equation 7.51 for the density of states.) Although this integral