Quantum Mechanics 6

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A particle moves along the x-axis in the potential $V(x) = kx^n$, where k is a positive constant and n is an even natural number.

1. It is assumed the particle is in a stationary state with energy E. We find expressions of $\langle T\rangle$ and $\langle V\rangle$ in terms of n and E.

   The virial theorem states

   $$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle = \left\langle nkx^n\right\rangle = n\langle V\rangle.$$

   From this we have

   $$\langle T\rangle = \frac{n}{2}\langle V\rangle, \qquad \langle V\rangle = \frac{2}{n}\langle T\rangle.$$

   The expected value of the Hamiltonian can be written as

   $$\langle H\rangle = \langle T\rangle + \langle V\rangle.$$

   Since the particle is in a stationary state with the energy E, we may also write

   $$\langle H\rangle = E.$$

   Putting it all together we gete

   $$\begin{aligned} & E = \langle T\rangle + \langle V\rangle = \langle T\rangle + \frac{2}{n}\langle T\rangle = \frac{n+2}{n}\langle T\rangle \\ \implies \qquad & \langle T\rangle = \frac{n}{n+2}E \end{aligned}$$

   and

   $$\begin{aligned} & E = \langle T\rangle + \langle V\rangle = \frac{n}{2}\langle V\rangle + \langle V\rangle = \frac{n+2}{2}\langle T\rangle \\ \implies \qquad & \langle V\rangle = \frac{2}{n+2}E \end{aligned}$$

   which are the wanted results.

2. We now consider an arbitrary time dependant wave equation $\Psi(x,t)$. We wish to compute

$$\frac{d}{dt}\langle x^l \rangle, \qquad \frac{d}{dt}\langle p \rangle$$

For this we use the generalized Ehrenfest theorem

$$\frac{d\langle Q \rangle}{dt} = \frac{i}{\hbar}\left\langle \left[\hat{H}, \hat{Q}\right] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

where $\hat{H}$ is the Hamiltonian

$$\hat{H} = \frac{p^2}{2m} + kx^n$$

Before continuing, we remind ourselves of the results in Griffiths exercise 3.14

$$\left[\hat{A} + \hat{B}, \hat{C}\right] = \left[\hat{A}, \hat{C}\right] + \left[\hat{B}, \hat{C}\right] \tag{1}$$

$$\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B} \tag{2}$$

$$\left[x^n, p\right] = i\hbar n x^{n-1} \tag{3}$$

We are now ready to compute $\frac{d}{dt}\langle x^l \rangle$.

$$\begin{aligned}
\frac{d}{dt}\langle x^l \rangle &= \frac{i}{\hbar}\left\langle \left[\frac{p^2}{2m} + kx^n, x^l\right] \right\rangle + \left\langle \frac{\partial}{\partial t} x^l \right\rangle \\
&\overset{(1)}{=} \frac{i}{\hbar}\left\langle \left[\frac{p^2}{2m}, x^l\right] + \left[kx^n, x^l\right] \right\rangle \\
&= \frac{i}{\hbar}\frac{1}{2m}\left\langle \left[p^2, x^l\right] \right\rangle \\
&\overset{(2)}{=} \frac{i}{\hbar}\frac{1}{2m}\left\langle p\left[p, x^l\right] + \left[p, x^l\right]p \right\rangle \\
&\overset{(3)}{=} \frac{i}{\hbar}\frac{1}{2m}\left\langle -pi\hbar l x^{l-1} - i\hbar l x^{l-1} p \right\rangle \\
&= \frac{l}{2m}\left\langle p x^{l-1} + x^{l-1} p \right\rangle
\end{aligned}$$

And $\frac{d}{dt}\langle p \rangle$

$$
\begin{aligned}
\frac{d}{dt}\langle p \rangle &= \frac{i}{\hbar}\left\langle\left[\frac{p^2}{2m} + kx^n, p\right]\right\rangle + \left\langle \frac{\partial}{\partial t} p \right\rangle \\
&\overset{(1)}{=} \frac{i}{\hbar}\left\langle\left[\frac{p^2}{2m}, p\right] + \left[kx^n, p\right]\right\rangle \\
&= \frac{ik}{\hbar}\left\langle\left[x^n, p\right]\right\rangle \\
&\overset{(3)}{=} \frac{ik}{\hbar}\left\langle i\hbar n x^{n-1}\right\rangle \\
&= -kn\left\langle x^{n-1}\right\rangle
\end{aligned}
$$

3. We use the results to find $\frac{d^2}{dt^2}\langle p \rangle$.

$$
\begin{aligned}
\frac{d^2}{dt^2}\langle p \rangle &= \frac{d}{dt}\left(-kn\left\langle x^{n-1}\right\rangle\right) \\
&= -kn\frac{d}{dt}\left\langle x^{n-1}\right\rangle \\
&= \frac{-kn(n-1)}{2m}\left\langle p x^{n-2} + x^{n-2} p\right\rangle
\end{aligned}
$$

If we let $n = 2$ and $k = \frac{1}{2}m\omega^2$ corresponding to the harmonic oscillator, we get

$$
\frac{d^2}{dt^2}\langle p \rangle = -\omega^2 \langle p \rangle \tag{4}
$$

To see, why this result makes sense, consider the classical harmonic oscillator.

$$
\begin{aligned}
& ma = -kx \\
\implies \quad & a = -\frac{k}{m}x = -\omega^2 x
\end{aligned}
$$

where $\omega = \sqrt{k/m}$. Both sides are multiplied by m.

$$
ma = -\omega^2 m x
$$

Newton's second law is used to rewrite the left side.

$$
\frac{dp}{dt} = -\omega^2 m x
$$

Both sides are differentiated with respect to time.

$$\frac{d^2p}{dt^2} = -\omega^2 p$$

This is the classical equivalent of equation (4), so equation (4) is another example of quantum mechanics behaving like classical mechanics, when it comes to expectation values (Ehrenfest's theorem.)