This deceptively simple result is quite profound. It tells us that the $|b'\rangle$'s are eigenkets of $UAU^{-1}$ with *exactly the same eigenvalues* as the $A$ eigenvalues. In other words, *unitary equivalent observables have identical spectra.*

The eigenket $|b^{(l)}\rangle$, by definition, satisfies the relationship

$$B|b^{(l)}\rangle = b^{(l)}|b^{(l)}\rangle. \tag{1.5.26}$$

Comparing (1.5.25) and (1.5.26), we infer that $B$ and $UAU^{-1}$ are simultaneously diagonalizable. A natural question is, Is $UAU^{-1}$ the same as $B$ itself? The answer quite often is yes in cases of physical interest. Take, for example, $S_x$ and $S_z$. They are related by a unitary operator, which, as we will discuss in Chapter 3, is actually the rotation operator around the $y$-axis by angle $\pi/2$. In this case $S_x$ itself is the unitary transform of $S_z$. Because we know that $S_x$ and $S_z$ exhibit the same set of eigenvalues—namely, $+\hbar/2$ and $-\hbar/2$—we see that our theorem holds in this particular example.

## 1.6 ■ POSITION, MOMENTUM, AND TRANSLATION

### Continuous Spectra

The observables considered so far have all been assumed to exhibit discrete eigenvalue spectra. In quantum mechanics, however, there are observables with continuous eigenvalues. Take, for instance, $p_z$, the $z$-component of momentum. In quantum mechanics this is again represented by a Hermitian operator. In contrast to $S_z$, however, the eigenvalues of $p_z$ (in appropriate units) can assume any real value between $-\infty$ and $\infty$.

The rigorous mathematics of a vector space spanned by eigenkets that exhibit a continuous spectrum is rather treacherous. The dimensionality of such a space is obviously infinite. Fortunately, many of the results we worked out for a finite-dimensional vector space with discrete eigenvalues can immediately be generalized. In places where straightforward generalizations do not hold, we indicate danger signals.

We start with the analogue of eigenvalue equation (1.2.5), which, in the continuous-spectrum case, is written as

$$\xi|\xi'\rangle = \xi'|\xi'\rangle, \tag{1.6.1}$$

where $\xi$ is an operator and $\xi'$ is simply a number. The ket $|\xi'\rangle$ is, in other words, an eigenket of operator $\xi$ with eigenvalue $\xi'$, just as $|a'\rangle$ is an eigenket of operator $A$ with eigenvalue $a'$.

In pursuing this analogy we replace the Kronecker symbol by Dirac's $\delta$-function—a discrete sum over the eigenvalues $\{a'\}$ by an integral over the *continuous variable* $\xi'$—so

$$\langle a'|a''\rangle = \delta_{a'a''} \to \langle\xi'|\xi''\rangle = \delta(\xi' - \xi''), \tag{1.6.2a}$$

$$\sum_{a'} |a'\rangle\langle a'| = 1 \to \int d\xi' |\xi'\rangle\langle\xi'| = 1, \tag{1.6.2b}$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \to |\alpha\rangle = \int d\xi' |\xi'\rangle\langle\xi'|\alpha\rangle, \tag{1.6.2c}$$

$$\sum_{a'} |\langle a'|\alpha\rangle|^2 = 1 \to \int d\xi' |\langle\xi'|\alpha\rangle|^2 = 1, \tag{1.6.2d}$$

$$\langle\beta|\alpha\rangle = \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle \to \langle\beta|\alpha\rangle = \int d\xi' \langle\beta|\xi'\rangle\langle\xi'|\alpha\rangle, \tag{1.6.2e}$$

$$\langle a''|A|a'\rangle = a'\delta_{a'a''} \to \langle\xi''|\xi|\xi'\rangle = \xi'\delta(\xi''-\xi'). \tag{1.6.2f}$$

Notice in particular how the completeness relation (1.6.2b) is used to obtain (1.6.2c) and (1.6.2e).

### Position Eigenkets and Position Measurements

In Section 1.4 we emphasized that a measurement in quantum mechanics is essentially a filtering process. To extend this idea to measurements of observables exhibiting continuous spectra, it is best to work with a specific example. To this end we consider the position (or coordinate) operator in one dimension.

The eigenkets $|x'\rangle$ of the position operator $x$ satisfying

$$x|x'\rangle = x'|x'\rangle \tag{1.6.3}$$

are postulated to form a complete set. Here $x'$ is just a number with the dimension of length 0.23 cm, for example, whereas $x$ is an operator. The state ket for an arbitrary physical state can be expanded in terms of $\{|x'\rangle\}$:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx' |x'\rangle\langle x'|\alpha\rangle. \tag{1.6.4}$$

We now consider a highly idealized selective measurement of the position observable. Suppose we place a very tiny detector that clicks only when the particle is precisely at $x'$ and nowhere else. Immediately after the detector clicks, we can say that the state in question is represented by $|x'\rangle$. In other words, when the detector clicks, $|\alpha\rangle$ abruptly "jumps into" $|x'\rangle$ in much the same way as an arbitrary spin state jumps into the $S_z+$ (or $S_z-$) state when subjected to an SG apparatus of the $S_z$ type.

In practice, the best the detector can do is to locate the particle within a narrow interval around $x'$. A realistic detector clicks when a particle is observed to be located within some narrow range $(x'-\Delta/2, x'+\Delta/2)$. When a count is registered in such a detector, the state ket changes abruptly as follows:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx''|x''\rangle\langle x''|\alpha\rangle \xrightarrow{\text{measurement}} \int_{x'-\Delta/2}^{x'+\Delta/2} dx''|x''\rangle\langle x''|\alpha\rangle. \tag{1.6.5}$$

Assuming that $\langle x''|\alpha\rangle$ does not change appreciably within the narrow interval, the probability for the detector to click is given by

$$|\langle x'|\alpha\rangle|^2 dx', \tag{1.6.6}$$

where we have written $dx'$ for $\Delta$. This is analogous to $|\langle a'|\alpha\rangle|^2$ for the probability for $|\alpha\rangle$ to be thrown into $|a'\rangle$ when $A$ is measured. The probability of recording the particle *somewhere* between $-\infty$ and $\infty$ is given by

$$\int_{-\infty}^{\infty} dx' |\langle x'|\alpha\rangle|^2, \tag{1.6.7}$$

which is normalized to unity if $|\alpha\rangle$ is normalized:

$$\langle\alpha|\alpha\rangle = 1 \Rightarrow \int_{-\infty}^{\infty} dx' \langle\alpha|x'\rangle\langle x'|\alpha\rangle = 1. \tag{1.6.8}$$

The reader familiar with wave mechanics may have recognized by this time that $\langle x'|\alpha\rangle$ is the wave function for the physical state represented by $|\alpha\rangle$. We will say more about this identification of the expansion coefficient with the $x$-representation of the wave function in Section 1.7.

The notion of a position eigenket can be extended to three dimensions. It is assumed in nonrelativistic quantum mechanics that the position eigenkets $|\mathbf{x}'\rangle$ are complete. The state ket for a particle with internal degrees of freedom, such as spin, ignored can therefore be expanded in terms of $\{|\mathbf{x}'\rangle\}$ as follows:

$$|\alpha\rangle = \int d^3x' |\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle, \tag{1.6.9}$$

where $\mathbf{x}'$ stands for $x'$, $y'$, and $z'$; in other words, $|\mathbf{x}'\rangle$ is a *simultaneous* eigenket of the observables $x$, $y$, and $z$ in the sense of Section 1.4:

$$|\mathbf{x}'\rangle \equiv |x', y', z'\rangle, \tag{1.6.10a}$$

$$x|\mathbf{x}'\rangle = x'|\mathbf{x}'\rangle, \quad y|\mathbf{x}'\rangle = y'|\mathbf{x}'\rangle, \quad z|\mathbf{x}'\rangle = z'|\mathbf{x}'\rangle, \tag{1.6.10b}$$

To be able to consider such a simultaneous eigenket at all, we are implicitly assuming that the three components of the position vector can be measured simultaneously to arbitrary degrees of accuracy; hence, we must have

$$[x_i, x_j] = 0, \tag{1.6.11}$$

where $x_1$, $x_2$, and $x_3$ stand for $x$, $y$, and $z$, respectively.

### Translation

We now introduce the very important concept of translation, or spatial displacement. Suppose we start with a state that is well localized around $\mathbf{x}'$. Let us consider an operation that changes this state into another well-localized state, this time around $\mathbf{x}' + d\mathbf{x}'$, with everything else (for example, the spin direction) unchanged. Such an operation is defined to be an **infinitesimal translation** by $d\mathbf{x}'$, and the operator that does the job is denoted by $\mathscr{J}(d\mathbf{x}')$:

$$\mathscr{J}(d\mathbf{x}')|\mathbf{x}'\rangle = |\mathbf{x}' + d\mathbf{x}'\rangle, \tag{1.6.12}$$

where a possible arbitrary phase factor is set to unity by convention. Notice that the right-hand side of (1.6.12) is again a position eigenket, but this time with eigenvalue $\mathbf{x}' + d\mathbf{x}'$. Obviously $|\mathbf{x}'\rangle$ is *not* an eigenket of the infinitesimal translation operator.

By expanding an arbitrary state ket $|\alpha\rangle$ in terms of the position eigenkets, we can examine the effect of infinitesimal translation on $|\alpha\rangle$:

$$|\alpha\rangle \rightarrow \mathscr{J}(d\mathbf{x}')|\alpha\rangle = \mathscr{J}(d\mathbf{x}')\int d^3x'|\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle = \int d^3x'|\mathbf{x}' + d\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle. \tag{1.6.13}$$

We also write the right-hand side of (1.6.13) as

$$\int d^3x'|\mathbf{x}' + d\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle = \int d^3x'|\mathbf{x}'\rangle\langle\mathbf{x}' - d\mathbf{x}'|\alpha\rangle \tag{1.6.14}$$

because the integration is over all space and $\mathbf{x}'$ is just an integration variable. This shows that the wave function of the translated state $\mathscr{J}(d\mathbf{x}')|\alpha\rangle$ is obtained by substituting $\mathbf{x}' - d\mathbf{x}'$ for $\mathbf{x}'$ in $\langle\mathbf{x}'|\alpha\rangle$.

There is an equivalent approach to translation that is often treated in the literature. Instead of considering an infinitesimal translation of the physical system itself, we consider a change in the coordinate system being used such that the origin is shifted in the *opposite* direction, $-d\mathbf{x}'$. Physically, in this alternative approach we are asking how the *same* state ket would look to another observer whose coordinate system is shifted by $-d\mathbf{x}'$. In this book we try not to use this approach. Obviously it is important that we do not mix the two approaches!

We now list the properties of the infinitesimal translation operator $\mathscr{J}(-d\mathbf{x}')$. The first property we demand is the unitarity property imposed by probability conservation. It is reasonable to require that if the ket $|\alpha\rangle$ is normalized to unity, the translated ket $\mathscr{J}(d\mathbf{x}')|\alpha\rangle$ also be normalized to unity, so

$$\langle\alpha|\alpha\rangle = \langle\alpha|\mathscr{J}^\dagger(d\mathbf{x}')\mathscr{J}(d\mathbf{x}')|\alpha\rangle. \tag{1.6.15}$$

This condition is guaranteed by demanding that the infinitesimal translation be unitary:

$$\mathscr{J}^\dagger(d\mathbf{x}')\mathscr{J}(d\mathbf{x}') = 1. \tag{1.6.16}$$

Quite generally, the norm of a ket is preserved under unitary transformations. For the second property, suppose we consider two successive infinitesimal translations—first by $d\mathbf{x}'$ and subsequently by $d\mathbf{x}''$, where $d\mathbf{x}'$ and $d\mathbf{x}''$ need not be in the same direction. We expect the net result to be just a *single* translation operation by the vector sum $d\mathbf{x}' + d\mathbf{x}''$, so we demand that

$$\mathscr{J}(d\mathbf{x}'')\mathscr{J}(d\mathbf{x}') = \mathscr{J}(d\mathbf{x}' + d\mathbf{x}''). \tag{1.6.17}$$

For the third property, suppose we consider a translation in the opposite direction; we expect the opposite-direction translation to be the same as the inverse of the original translation:

$$\mathscr{J}(-d\mathbf{x}') = \mathscr{J}^{-1}(d\mathbf{x}'). \tag{1.6.18}$$

For the fourth property, we demand that as $d\mathbf{x}' \to 0$, the translation operation reduce to the identity operation

$$\lim_{d\mathbf{x}'\to 0} \mathcal{J}(d\mathbf{x}') = 1 \tag{1.6.19}$$

and that the difference between $\mathcal{J}(d\mathbf{x}')$ and the identity operator be of first order in $d\mathbf{x}'$.

We now demonstrate that if we take the infinitesimal translation operator to be

$$\mathcal{J}(d\mathbf{x}') = 1 - i\mathbf{K}\cdot d\mathbf{x}', \tag{1.6.20}$$

where the components of $\mathbf{K}$, $K_x$, $K_y$, and $K_z$, are **Hermitian operators**, then all the properties listed are satisfied. The first property, the unitarity of $\mathcal{J}(d\mathbf{x}')$, is checked as follows:

$$\begin{aligned}\mathcal{J}^\dagger(d\mathbf{x}')\mathcal{J}(d\mathbf{x}') &= (1 + i\mathbf{K}^\dagger\cdot d\mathbf{x}')(1 - i\mathbf{K}\cdot d\mathbf{x}')\\ &= 1 - i(\mathbf{K}-\mathbf{K}^\dagger)\cdot d\mathbf{x}' + 0[(d\mathbf{x}')^2]\\ &\simeq 1,\end{aligned} \tag{1.6.21}$$

where terms of second order in $d\mathbf{x}'$ have been ignored for an infinitesimal translation. The second property (1.6.17) can also be proved as follows:

$$\begin{aligned}\mathcal{J}(d\mathbf{x}'')\mathcal{J}(d\mathbf{x}') &= (1 - i\mathbf{K}\cdot d\mathbf{x}'')(1 - i\mathbf{K}\cdot d\mathbf{x}')\\ &\simeq 1 - i\mathbf{K}\cdot(d\mathbf{x}' + d\mathbf{x}'')\\ &= \mathcal{J}(d\mathbf{x}' + d\mathbf{x}'').\end{aligned} \tag{1.6.22}$$

The third and fourth properties are obviously satisfied by (1.6.20).

Accepting (1.6.20) to be the correct form for $\mathcal{J}(d\mathbf{x}')$, we are in a position to derive an extremely fundamental relation between the $\mathbf{K}$ operator and the $\mathbf{x}$ operator. First, note that

$$\mathbf{x}\mathcal{J}(d\mathbf{x}')|\mathbf{x}'\rangle = \mathbf{x}|\mathbf{x}' + d\mathbf{x}'\rangle = (\mathbf{x}' + d\mathbf{x}')|\mathbf{x}' + d\mathbf{x}'\rangle \tag{1.6.23a}$$

and

$$\mathcal{J}(d\mathbf{x}')\mathbf{x}|\mathbf{x}'\rangle = \mathbf{x}'\mathcal{J}(d\mathbf{x}')|\mathbf{x}'\rangle = \mathbf{x}'|\mathbf{x}' + d\mathbf{x}'\rangle; \tag{1.6.23b}$$

hence,

$$\left[\mathbf{x}, \mathcal{J}(d\mathbf{x}')\right]|\mathbf{x}'\rangle = d\mathbf{x}'|\mathbf{x}' + d\mathbf{x}'\rangle \simeq d\mathbf{x}'|\mathbf{x}'\rangle, \tag{1.6.24}$$

where the error made in approximating the last step of (1.6.24) is of second order in $d\mathbf{x}'$. Now $|\mathbf{x}'\rangle$ can be *any* position eigenket, and the position eigenkets are known to form a complete set. We must therefore have an **operator identity**

$$\left[\mathbf{x}, \mathcal{J}(d\mathbf{x}')\right] = d\mathbf{x}', \tag{1.6.25}$$

or

$$-i\mathbf{x}\mathbf{K}\cdot d\mathbf{x}' + i\mathbf{K}\cdot d\mathbf{x}'\mathbf{x} = d\mathbf{x}', \tag{1.6.26}$$

where, on the right-hand sides of (1.6.25) and (1.6.26), $d\mathbf{x}'$ is understood to be the number $d\mathbf{x}'$ multiplied by the identity operator in the ket space spanned by $|\mathbf{x}'\rangle$. By choosing $d\mathbf{x}'$ in the direction of $\hat{\mathbf{x}}_j$ and forming the scalar product with $\hat{\mathbf{x}}_i$, we obtain

$$[x_i, K_j] = i\delta_{ij}, \tag{1.6.27}$$

where again $\delta_{ij}$ is understood to be multiplied by the identity operator.

### Momentum as a Generator of Translation

Equation (1.6.27) is the fundamental commutation relation between the position operators $x$, $y$, $z$ and the $K$ operators $K_x, K_y, K_z$. Remember that so far, the $K$ operator is *defined* in terms of the infinitesimal translation operator by (1.6.20). What is the physical significance we can attach to $\mathbf{K}$?

J. Schwinger, lecturing on quantum mechanics, once remarked, "... for fundamental properties we will borrow only names from classical physics." In the present case we would like to borrow from classical mechanics the notion that momentum is the generator of an infinitesimal translation. An infinitesimal translation in classical mechanics can be regarded as a canonical transformation,

$$\mathbf{x}_{\text{new}} \equiv \mathbf{X} = \mathbf{x} + d\mathbf{x}, \quad \mathbf{p}_{\text{new}} \equiv \mathbf{P} = \mathbf{p}, \tag{1.6.28}$$

obtainable from the generating function (Goldstein 2002, pp. 386 and 403)

$$F(\mathbf{x},\mathbf{P}) = \mathbf{x} \cdot \mathbf{P} + \mathbf{p} \cdot d\mathbf{x}, \tag{1.6.29}$$

where $\mathbf{p}$ and $\mathbf{P}$ refer to the corresponding momenta.

This equation has a striking similarity to the infinitesimal translation operator (1.6.20) in quantum mechanics, particularly if we recall that $\mathbf{x} \cdot \mathbf{P}$ in (1.6.29) is the generating function for the identity transformation ($\mathbf{X} = \mathbf{x}, \mathbf{P} = \mathbf{p}$). We are therefore led to speculate that the operator $\mathbf{K}$ is in some sense related to the momentum operator in quantum mechanics.

Can the $K$ operator be identified with the momentum operator itself? Unfortunately, the dimension is all wrong; the $K$ operator has the dimension of 1/length because $\mathbf{K} \cdot d\mathbf{x}'$ must be dimensionless. But it appears legitimate to set

$$\mathbf{K} = \frac{\mathbf{p}}{\text{universal constant with the dimension of action}}. \tag{1.6.30}$$

From the fundamental postulates of quantum mechanics there is no way to determine the actual numerical value of the universal constant. Rather, this constant is needed here because, historically, classical physics was developed before quantum mechanics using units convenient for describing macroscopic quantities—the circumference of the earth, the mass of 1 cc of water, the duration of a mean solar day, and so forth. Had microscopic physics been formulated before macroscopic physics, the physicists would have almost certainly chosen the basic units in such a way that the universal constant appearing in (1.6.30) would be unity.

An analogy from electrostatics may be helpful here. The interaction energy between two particles of charge $e$ separated at a distance $r$ is proportional to $e^2/r$; in unrationalized Gaussian units, the proportionality factor is just 1, but in rationalized mks units, which may be more convenient for electrical engineers, the proportionality factor is $1/4\pi\varepsilon_0$. (See Appendix A.)

The universal constant that appears in (1.6.30) turns out to be the same as the constant $\hbar$ that appears in L. de Broglie's relation, written in 1924,

$$\frac{2\pi}{\lambda} = \frac{p}{\hbar}, \tag{1.6.31}$$

where $\lambda$ is the wavelength of a "particle wave." In other words, the $K$ operator is the quantum-mechanical operator that corresponds to the wave number—that is, $2\pi$ times the reciprocal wavelength, usually denoted by $k$. With this identification, the infinitesimal translation operator $\mathcal{J}(d\mathbf{x}')$ reads

$$\mathcal{J}(d\mathbf{x}') = 1 - i\mathbf{p}\cdot d\mathbf{x}'/\hbar, \tag{1.6.32}$$

where $\mathbf{p}$ is the momentum operator. The commutation relation (1.6.27) now becomes

$$[x_i, p_j] = i\hbar\delta_{ij}. \tag{1.6.33}$$

The commutation relations (1.6.33) imply, for example, that $x$ and $p_x$ (but not $x$ and $p_y$) are incompatible observables. It is therefore impossible to find simultaneous eigenkets of $x$ and $p_x$. The general formalism of Section 1.4 can be applied here to obtain the **position-momentum uncertainty relation** of W. Heisenberg:

$$\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \geq \hbar^2/4. \tag{1.6.34}$$

Some applications of (1.6.34) will appear in Section 1.7.

So far we have concerned ourselves with infinitesimal translations. A finite translation—that is, a spatial displacement by a finite amount—can be obtained by successively compounding infinitesimal translations. Let us consider a finite translation in the $x$-direction by an amount $\Delta x'$:

$$\mathcal{J}(\Delta x'\hat{\mathbf{x}})|\mathbf{x}'\rangle = |\mathbf{x}' + \Delta x'\hat{\mathbf{x}}\rangle. \tag{1.6.35}$$

By compounding $N$ infinitesimal translations, each of which is characterized by a spatial displacement $\Delta x'/N$ in the $x$-direction, and letting $N \to \infty$, we obtain

$$\begin{aligned}\mathcal{J}(\Delta x'\hat{\mathbf{x}}) &= \lim_{N\to\infty}\left(1 - \frac{ip_x\Delta x'}{N\hbar}\right)^N \\ &= \exp\left(-\frac{ip_x\Delta x'}{\hbar}\right).\end{aligned} \tag{1.6.36}$$

Here $\exp(-ip_x\Delta x'/\hbar)$ is understood to be a function of the *operator* $p_x$; generally, for any operator $X$ we have

$$\exp(X) \equiv 1 + X + \frac{X^2}{2!} + \cdots. \tag{1.6.37}$$

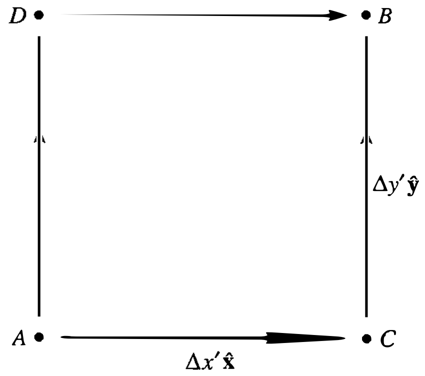


**FIGURE 1.9** Successive translations in different directions.

A fundamental property of translations is that successive translations in different directions, say in the $x$- and $y$-directions, commute. We see this clearly in Figure 1.9; in shifting from $A$ and $B$, it does not matter whether we go via $C$ or via $D$. Mathematically,

$$\begin{aligned}\mathscr{J}(\Delta y'\hat{\mathbf{y}})\mathscr{J}(\Delta x'\hat{\mathbf{x}}) &= \mathscr{J}(\Delta x'\hat{\mathbf{x}}+\Delta y'\hat{\mathbf{y}}),\\ \mathscr{J}(\Delta x'\hat{\mathbf{x}})\mathscr{J}(\Delta y'\hat{\mathbf{y}}) &= \mathscr{J}(\Delta x'\hat{\mathbf{x}}+\Delta y'\hat{\mathbf{y}}).\end{aligned} \tag{1.6.38}$$

This point is not so trivial as it may appear; we will show in Chapter 3 that rotations about different axes do *not* commute. Treating $\Delta x'$ and $\Delta y'$ up to second order, we obtain

$$\begin{aligned}\left[\mathscr{J}(\Delta y'\hat{\mathbf{y}}),\mathscr{J}(\Delta x'\hat{\mathbf{x}})\right] &= \left[\left(1-\frac{ip_y\Delta y'}{\hbar}-\frac{p_y^2(\Delta y')^2}{2\hbar^2}+\cdots\right),\right.\\ &\qquad \left.\left(1-\frac{ip_x\Delta x'}{\hbar}-\frac{p_x^2(\Delta x')^2}{2\hbar^2}+\cdots\right)\right]\\ &\simeq -\frac{(\Delta x')(\Delta y')[p_y,p_x]}{\hbar^2}.\end{aligned} \tag{1.6.39}$$

Because $\Delta x'$ and $\Delta y'$ are arbitrary, requirement (1.6.38), or

$$\left[\mathscr{J}(\Delta y'\hat{\mathbf{y}}),\mathscr{J}(\Delta x'\hat{\mathbf{x}})\right]=0, \tag{1.6.40}$$

immediately leads to

$$[p_x,p_y]=0, \tag{1.6.41}$$

or, more generally,

$$[p_i,p_j]=0. \tag{1.6.42}$$

This commutation relation is a direct consequence of the fact that translations in different directions commute. Whenever the generators of transformations commute, the corresponding group is said to be **Abelian**. The translation group in three dimensions is Abelian.

Equation (1.6.42) implies that $p_x$, $p_y$, and $p_z$ are mutually compatible observables. We can therefore conceive of a simultaneous eigenket of $p_x$, $p_y$, $p_z$, namely,

$$|\mathbf{p}'\rangle \equiv |p'_x, p'_y, p'_z\rangle, \tag{1.6.43a}$$

$$p_x|\mathbf{p}'\rangle = p'_x|\mathbf{p}'\rangle, \quad p_y|\mathbf{p}'\rangle = p'_y|\mathbf{p}'\rangle, \quad p_z|\mathbf{p}'\rangle = p'_z|\mathbf{p}'\rangle. \tag{1.6.43b}$$

It is instructive to work out the effect of $\mathcal{J}(d\mathbf{x}')$ on such a momentum eigenket:

$$\mathcal{J}(d\mathbf{x}')|\mathbf{p}'\rangle = \left(1 - \frac{i\mathbf{p}\cdot d\mathbf{x}'}{\hbar}\right)|\mathbf{p}'\rangle = \left(1 - \frac{i\mathbf{p}'\cdot d\mathbf{x}'}{\hbar}\right)|\mathbf{p}'\rangle. \tag{1.6.44}$$

We see that the momentum eigenket remains the same even though it suffers a slight phase change, so, unlike $|\mathbf{x}'\rangle$, $|\mathbf{p}'\rangle$ *is* an eigenket of $\mathcal{J}(d\mathbf{x}')$, which we anticipated because

$$\left[\mathbf{p}, \mathcal{J}(d\mathbf{x}')\right] = 0. \tag{1.6.45}$$

Notice, however, that the eigenvalue of $\mathcal{J}(d\mathbf{x}')$ is complex; we do not expect a real eigenvalue here because $\mathcal{J}(d\mathbf{x}')$, though unitary, is not Hermitian.

### The Canonical Commutation Relations

We summarize the commutator relations we inferred by studying the properties of translation:

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}. \tag{1.6.46}$$

These relations form the cornerstone of quantum mechanics; in his book, P. A. M. Dirac calls them the "fundamental quantum conditions." More often they are known as the **canonical commutation relations** or the **fundamental commutation relations**.

Historically it was W. Heisenberg who, in 1925, showed that the combination rule for atomic transition lines known at that time could best be understood if one associated arrays of numbers obeying certain multiplication rules with these frequencies. Immediately afterward, M. Born and P. Jordan pointed out that Heisenberg's multiplication rules are essentially those of matrix algebra, and a theory based on the matrix analogues of (1.6.46) was developed; it is now known as **matrix mechanics**.*

Also in 1925, P. A. M. Dirac observed that the various quantum-mechanical relations can be obtained from the corresponding classical relations just by replacing classical Poisson brackets by commutators, as follows:

$$[\ ,\ ]_{\text{classical}} \to \frac{[\ ,\ ]}{i\hbar}, \tag{1.6.47}$$

*Appropriately, $pq - qp = h/2\pi i$ is inscribed on the gravestone of M. Born in Göttingen.