Afleveringsopgave 7

10. oktober 2018 08:12

KARSTEN BOSTEEN
19851608

1) Vi skal vise at $\phi_{nlm}(\vec{r})$ er en egenfunktion for $\hat{L}_+\hat{L}_-$. Dvs der gælder:
$\hat{L}_+\hat{L}_- \phi_{nlm} = \lambda \phi_{nlm}$, λ er egenværdien

$$\hat{L}_+\hat{L}_- \phi_{nlm}(r) = (\hat{L}^2 - \hat{L}_z^2 + \hbar \hat{L}_z)\phi_{nlm}$$
$$= (\hbar^2 l(l+1) - \hbar^2 m^2 + \hbar\hbar m)\phi_{nlm}$$, ϕ_{nlm} er en egenfunktion!
$$\lambda = \hbar^2(l(l+1) - m^2 + m)$$

2). $\psi(\vec{r}) = \frac{1}{\sqrt{2}}(\phi_{21-1}(\vec{r}) + i\phi_{211}(\vec{r})) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$ ✓

$\psi(\vec{r})$ er en egenfunktion af en operator $\hat{Q}$ hvis $\hat{Q}\psi(\vec{r}) = \lambda \cdot \psi(\vec{r})$

Energien er givet ved hammiltonoperatoren $\hat{H}$

$\hat{H}\psi(\vec{r}) = E\psi(\vec{r})$ så ja ψ er en egenfunktion af $\hat{H}$.

$\hat{L}^2\psi(\vec{r}) = \hat{L}^2 \frac{1}{\sqrt{2}}(\phi_{21-1}(\vec{r}) + i\phi_{211}(\vec{r}))$, $\hat{L}^2 f_l^m = \hbar^2 l(l+1)$
$= 2\hbar^2 \frac{1}{\sqrt{2}}(\phi_{21-1}(\vec{r}) + i\phi_{211}(\vec{r}))$ så ja egenværdi $2\hbar^2$

$\hat{L}_z\psi(\vec{r}) = \hat{L}_z \frac{1}{\sqrt{2}}(\phi_{21-1}(\vec{r}) + i\phi_{211}(\vec{r}))$, $L_z f_l^m = \hbar m f_l^m$
$= \frac{1}{\sqrt{2}}(\hbar(-1)\phi_{21-1} + i\hbar(1)\phi_{211}(\vec{r}))$
$\neq \lambda\psi(\vec{r})$, vektoren peger i en forkert retning.

3) $\mathcal{H}$ udspændt af $\{\phi_{21-1}, \phi_{210}, \phi_{211}\} = \{e_1, e_2, e_3\}$

Matrixrepræsentationen for en operator $\hat{Q}$ $= \left(\langle e_i|\hat{Q}|e_j\rangle \right)_{ij}$

$\underline{\hat{L}^2:}$

$\langle e_i|\hat{L}^2|e_j\rangle$ giver kun resultat for $i=j$ grundet orthogonalitet.
$\langle e_i|\hat{L}^2|e_i\rangle = \hbar^2 l(l+1) = 2\hbar^2$
$$\hat{L}^2 = \underline{\underline{\mathbb{1}\cdot 2\hbar^2}}$$

✓

$\hat{L}_z$:

$\langle e_i | \hat{L}_z | e_j \rangle = \langle e_i | \hbar m e_j \rangle$ giver også kun resultat for $i=j$.

$$L_z = \mathbb{1} \cdot \hbar m = \hbar \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\hat{L}_+$:

$\langle e_i | \hat{L}_+ | e_j \rangle = \langle e_i | \hbar \sqrt{l(l+1) + m(m+1)}\, e_{j+1} \rangle$
$\neq 0$ for $i = j+1$

$$\hat{L}_+ = \begin{array}{c|c} & i \\ \hline j & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \hbar \sqrt{2 + m(m+1)} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \hbar$$

$\hat{L}_-$: $\langle e_i | \hat{L}_+ | e_j \rangle = \langle e_i | \hbar \sqrt{l(l+1) + m(m-1)} \neq 0$ for $i = j-1$

$$\hat{L}_- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \hbar \sqrt{2 + m(m-1)} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \hbar$$

4) $\hat{L}_x$: $\hat{L}_x = \frac{L_+ + L_-}{2} = \frac{1}{2}\hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}$

5) Vi kan finde samtlige egenværdier λ for $\hat{L}_x$ ved at løse ligningen $|\hat{L}_x - \lambda \mathbb{1}| = 0$. $\lambda = \hbar m$

$$\left| \begin{pmatrix} -\hbar m & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\hbar m & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\hbar m \end{pmatrix} \right| = -\hbar m \left(\hbar^2 m^2 - \frac{1}{2} \right) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} (-\hbar m) \right)$$

$$= -\hbar^3 m^3 + \cancel{\frac{1}{2}\hbar m} - \cancel{\frac{1}{2}\hbar m} = 0$$

$m = 0$ eneste egenværdi

Vi finder egenvektorerne fra ligningen $(\hat{L}_x - \lambda \mathbb{1}) \vec{V} = \vec{0}$

$$\Rightarrow \hat{L}_x \vec{V} = \hbar \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} V_2 = 0 \\ V_1 = -V_3 \end{array}$$

Egenvekterne er alle vektorer $\begin{pmatrix} a \\ 0 \\ -a \end{pmatrix}$, $a \in \mathbb{C}$.
i basen $\{e_1, e_2, e_3\}$