

Perform analytical calculation of autocorrelation function and response function in Fourier space for a 2D non-reciprocal generic model.

$$\begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \begin{bmatrix} \zeta_0 \\ \zeta_1 \end{bmatrix}$$

$$\gamma_0 \dot{x}_0 = a_{00} x_0 + a_{01} x_1 + \zeta_0 \quad (1)$$

$$\gamma_1 \dot{x}_1 = a_{10} x_0 + a_{11} x_1 + \zeta_1 \quad (2)$$

Transform eq(1) and eq(2) to Fourier space  $A(\omega) = \int_{-\infty}^{\infty} dt a(t) e^{i\omega t}$

$$-i\omega \gamma_0 \tilde{x}_0 = a_{00} \tilde{x}_0 + a_{01} \tilde{x}_1 + \tilde{\zeta}_0 \quad (3)$$

$$-i\omega \gamma_1 \tilde{x}_1 = a_{10} \tilde{x}_0 + a_{11} \tilde{x}_1 + \tilde{\zeta}_1 \quad (4)$$

Solve for the Fourier transformed position:

Solve for  $\tilde{x}_1$ :

$$\begin{aligned} -i\omega \gamma_0 \tilde{x}_0 &= a_{00} \tilde{x}_0 + a_{01} \tilde{x}_1 + \tilde{\zeta}_0 \\ + -i\omega \gamma_1 \tilde{x}_1 &= a_{10} \tilde{x}_0 + a_{11} \tilde{x}_1 + \tilde{\zeta}_1 \end{aligned}$$


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$$-i\omega \gamma_0 \tilde{x}_0 - i\omega \gamma_1 \tilde{x}_1 = (a_{00} + a_{10}) \tilde{x}_0 + (a_{01} + a_{11}) \tilde{x}_1 + \tilde{\zeta}_0 + \tilde{\zeta}_1$$

$$\begin{aligned} i\omega \gamma_1 \tilde{x}_1 + (a_{01} + a_{11}) \tilde{x}_1 &= -i\omega \gamma_0 \tilde{x}_0 - (a_{00} + a_{10}) \tilde{x}_0 - \tilde{\zeta}_0 - \tilde{\zeta}_1 \\ \tilde{x}_1 (i\omega \gamma_1 + a_{01} + a_{11}) &= \tilde{x}_0 (-i\omega \gamma_0 - a_{00} - a_{10}) - \tilde{\zeta}_0 - \tilde{\zeta}_1 \end{aligned}$$

$$(5) \quad \tilde{x}_1 = \frac{\tilde{x}_0 (-i\omega \gamma_0 - a_{00} - a_{10}) - \tilde{\zeta}_0 - \tilde{\zeta}_1}{i\omega \gamma_1 + a_{01} + a_{11}}$$

Substitute  $\tilde{x}_1$  in eq(3):

$$-i\omega \gamma_0 \tilde{x}_0 = a_{00} \tilde{x}_0 + a_{01} \tilde{x}_1 + \tilde{\zeta}_0 \quad (3)$$

$$-\omega \gamma_0 \tilde{x}_0 = a_{00} \tilde{x}_0 + a_{01} \left[ \frac{\tilde{x}_0 (-\omega \gamma_0 - a_{00} - a_{10}) - \tilde{\zeta}_0 - \tilde{\zeta}_1}{\omega \gamma_1 + a_{01} + a_{11}} \right] + \tilde{\zeta}_0$$

Multiply  $\omega \gamma_1 + a_{01} + a_{11}$  on both sides of the equation:

$$\begin{aligned} -\omega \gamma_0 \tilde{x}_0 (\omega \gamma_1 + a_{01} + a_{11}) &= a_{00} \tilde{x}_0 (\omega \gamma_1 + a_{01} + a_{11}) \\ &+ \tilde{x}_0 (-\omega \gamma_0 a_{01} - a_{00} a_{01} - a_{10} a_{01}) - \underline{\tilde{\zeta}_0 a_{01}} - \underline{\tilde{\zeta}_1 a_{01}} \\ &+ \underline{\tilde{\zeta}_0} (\omega \gamma_1 + a_{01} + a_{11}) \end{aligned}$$

Group terms w/  $\tilde{x}_0$  in one side of the equation:

$$\begin{aligned} \underline{\tilde{\zeta}_0} a_{01} + \underline{\tilde{\zeta}_1} a_{01} - \underline{\tilde{\zeta}_0} (\omega \gamma_1 + a_{01} + a_{11}) &= \\ -\omega^2 \gamma_0 \gamma_1 \tilde{x}_0 + \cancel{\omega \gamma_0 a_{01} \tilde{x}_0} + \cancel{\omega \gamma_0 a_{11} \tilde{x}_0} \\ + \cancel{\omega \gamma_1 a_{00} \tilde{x}_0} + \cancel{a_{01} a_{00} \tilde{x}_0} + \cancel{a_{11} a_{00} \tilde{x}_0} \\ - \cancel{\omega \gamma_0 a_{01} \tilde{x}_0} - \cancel{a_{00} a_{01} \tilde{x}_0} - a_{10} a_{01} \tilde{x}_0 \end{aligned}$$

Simplify terms:

$$\begin{aligned} \tilde{\zeta}_0 a_{01} + \tilde{\zeta}_1 a_{01} - \tilde{\zeta}_0 (\omega \gamma_1 + a_{01} + a_{11}) &= \\ \tilde{x}_0 (-\omega^2 \gamma_0 \gamma_1 + i\omega (\gamma_0 a_{11} + \gamma_1 a_{00}) + a_{11} a_{00} - a_{10} a_{01}) \end{aligned}$$

Finally:

$$\tilde{x}_0 = \frac{\tilde{\zeta}_0 a_{01} + \tilde{\zeta}_1 a_{01} - \tilde{\zeta}_0 (\omega \gamma_1 + a_{01} + a_{11})}{a_{11} a_{00} - a_{10} a_{01} + i\omega (\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2 \gamma_0 \gamma_1} \quad (6)$$

Sanity check if agree w/ eq(45) in Appendix C:

$$\begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ F \end{bmatrix} + \begin{bmatrix} n_x \\ \eta_F \end{bmatrix}$$

$$\tilde{x}_0 = \frac{\tilde{\eta}_x(1) + \tilde{\eta}_F(1) - \tilde{\eta}_x(i\omega\lambda + 1 - 1)}{k + \bar{k} + i\omega(\gamma(-1) + \gamma(-k)) - \omega^2\gamma\lambda}$$

$$= \frac{\tilde{\eta}_x + \tilde{\eta}_F - \tilde{\eta}_x(i\omega\lambda)}{k + \bar{k} - i\omega(\gamma + \gamma k) - \omega^2\gamma\lambda}$$

### C. Solutions of the linear model of hair bundle oscillations

We transform the model equations eq. (7) and eq. (8) to Fourier-space,

$$-i\omega\gamma\tilde{x} = -k\tilde{x} + \tilde{F} + \tilde{\eta}_x \quad (43)$$

and

$$-i\omega\lambda\tilde{F} = -\bar{k}\tilde{x} - \tilde{F} + \tilde{\eta}_F. \quad (44)$$

Now, we can solve for the Fourier-transformed position

$$\tilde{x} = \frac{\tilde{\eta}_F + \tilde{\eta}_x - i\lambda\tilde{\eta}_x\omega}{k + \bar{k} - i(\gamma + k\lambda)\omega - \gamma\lambda\omega^2}. \quad (45)$$

#### 1. Derivation of the autocorrelation function

The averaged autocorrelation function  $\tilde{C}_{xx}(\omega) = \langle \tilde{x}\tilde{x}^* \rangle$  is given by the averaged product of  $\tilde{x}$  from eq. (45) with its complex-conjugate  $\tilde{x}^*$ ,

$$\tilde{C}_{xx}(\omega) = \left\langle \frac{(\tilde{\eta}_F + \tilde{\eta}_x)^2 + \lambda^2\tilde{\eta}_x^2\omega^2}{2k\bar{k} + \gamma^2\omega^2 + (\bar{k} - \gamma\lambda\omega^2)^2 + k^2(1 + \lambda^2\omega^2)} \right\rangle. \quad (46)$$

Averaged autocorrelation function

$$C_{xx}(\omega) = \langle \tilde{x}\tilde{x}^* \rangle$$

$$\underline{C_{xx}(\omega)} = \langle \tilde{x}_0 \tilde{x}_0^* \rangle ?$$

how about  $x_1$ ?

Further simplify the Fourier-transformed position eq. (6)

$$\tilde{x}_0 = \frac{\tilde{\zeta}_0 a_{01} + \tilde{\zeta}_1 a_{01} - \tilde{\zeta}_0 (i\omega\gamma_1 + a_{01} + a_{11})}{a_{11}a_{00} - a_{10}a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1} \quad (6)$$

$$\tilde{x}_0 = \frac{\tilde{\zeta}_0 (a_{01} - a_{01} - a_{11}) + \tilde{\zeta}_1 a_{01} - i\omega\gamma_1 \tilde{\zeta}_0}{a_{11}a_{00} - a_{10}a_{01} + i\omega\gamma_0 a_{11} + i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1}$$

Solve for complex conjugate:

$$\tilde{x}_0^* = \frac{-a_{11} \tilde{\zeta}_0 + \tilde{\zeta}_1 a_{01} + i\omega\gamma_1 \tilde{\zeta}_0}{a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0 a_{11} - i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1} \quad (7)$$

then:

$$\langle \tilde{x}_0 \tilde{x}_0^* \rangle :$$

Numerator:

$$\begin{aligned}
 & (-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 - i\omega\gamma_1\tilde{\xi}_0) (-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 + i\omega\gamma_1\tilde{\xi}_0) \\
 = & -a_{11}\tilde{\xi}_0 (-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 + i\omega\gamma_1\tilde{\xi}_0) \\
 & + a_{01}\tilde{\xi}_1 (-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 + i\omega\gamma_1\tilde{\xi}_0) \\
 & - i\omega\gamma_1\tilde{\xi}_0 (-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 + i\omega\gamma_1\tilde{\xi}_0) \\
 = & a_{11}^2 \tilde{\xi}_0^2 - a_{01}a_{11}\tilde{\xi}_0\tilde{\xi}_1 - i a_{11}\omega\gamma_1\tilde{\xi}_0^2 \\
 & - a_{01}a_{11}\tilde{\xi}_1\tilde{\xi}_0 + a_{01}^2 \tilde{\xi}_1^2 + i a_{01}\omega\gamma_1\tilde{\xi}_0\tilde{\xi}_1 \\
 & + i a_{11}\omega\gamma_1\tilde{\xi}_0^2 - a_{01}i\omega\gamma_1\tilde{\xi}_0\tilde{\xi}_1 + \omega^2\gamma_1^2\tilde{\xi}_0^2 \\
 = & a_{11}^2 \tilde{\xi}_0^2 + a_{01}^2 \tilde{\xi}_1^2 - 2a_{01}a_{11}\tilde{\xi}_1\tilde{\xi}_0 + \omega^2\gamma_1^2\tilde{\xi}_0^2 \\
 = & (a_{11}\tilde{\xi}_0 - a_{01}\tilde{\xi}_1)^2 + \omega^2\gamma_1^2\tilde{\xi}_0^2
 \end{aligned}$$

Sanity check numerator:

$$\checkmark \quad = ((-1)\tilde{n}_x - \tilde{n}_f)^2 + \omega^2\gamma^2\tilde{n}_x^2 = (-n_x - n_f)^2 + \omega^2\gamma^2\tilde{n}_x^2$$

Denominator:

$$\tilde{x}_0 = \frac{\tilde{\xi}_0 (a_{11} - a_{01} - a_{11}) + \tilde{\xi}_1 a_{01} - i\omega\gamma_1\tilde{\xi}_0}{a_{11}a_{00} - a_{10}a_{01} + i\omega\gamma_0a_{11} + i\omega\gamma_1a_{00} - \omega^2\gamma_0\gamma_1}$$

Solve for complex conjugate:

$$\tilde{x}_0^* = \frac{-a_{11}\tilde{\xi}_0 + \tilde{\xi}_1 a_{01} + i\omega\gamma_1\tilde{\xi}_0}{a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0a_{11} - i\omega\gamma_1a_{00} - \omega^2\gamma_0\gamma_1} \quad (?)$$

$$(a_{11}a_{00} - a_{10}a_{01} + iw\gamma_0 a_{11} + iw\gamma_1 a_{00} - w^2 \gamma_0 \gamma_1) \\ (a_{11}a_{00} - a_{10}a_{01} - iw\gamma_0 a_{11} - iw\gamma_1 a_{00} - w^2 \gamma_0 \gamma_1)$$

$$= a_{11}a_{00} (a_{11}a_{00} - a_{10}a_{01} - iw\gamma_0 a_{11} - iw\gamma_1 a_{00} - w^2 \gamma_0 \gamma_1) \\ - a_{10}a_{01} (a_{11}a_{00} - a_{10}a_{01} - iw\gamma_0 a_{11} - iw\gamma_1 a_{00} - w^2 \gamma_0 \gamma_1) \\ + iw\gamma_0 a_{11} (a_{11}a_{00} - a_{10}a_{01} - iw\gamma_0 a_{11} - iw\gamma_1 a_{00} - w^2 \gamma_0 \gamma_1) \\ + iw\gamma_1 a_{00} (a_{11}a_{00} - a_{10}a_{01} - iw\gamma_0 a_{11} - iw\gamma_1 a_{00} - w^2 \gamma_0 \gamma_1) \\ - w^2 \gamma_0 \gamma_1 (a_{11}a_{00} - a_{10}a_{01} - iw\gamma_0 a_{11} - iw\gamma_1 a_{00} - w^2 \gamma_0 \gamma_1)$$

$$= \cancel{a_{11}^2 a_{00}^2} - \cancel{a_{11} a_{00} a_{10} a_{01}} - i \cancel{a_{11}^2 a_{00}^2} w \gamma_0 - i \cancel{a_{00}^2 a_{11}^2} w \gamma_1 - \cancel{a_{11} a_{00} w^2 \gamma_0 \gamma_1} \\ - \cancel{a_{11} a_{00} a_{10} a_{01}} + \cancel{a_{10}^2 a_{01}^2} + i \cancel{a_{10} a_{01} a_{11} w \gamma_0} + i \cancel{a_{10} a_{01} a_{00} w \gamma_1} + \cancel{a_{10} a_{01} w^2 \gamma_0 \gamma_1} \\ + i \cancel{a_{11}^2 a_{00} w \gamma_0} - i \cancel{a_{10} a_{01} a_{11} w \gamma_0} + \cancel{w^2 \gamma_0^2 a_{11}^2} + \cancel{w^2 \gamma_1 \gamma_0 a_{00} a_{11}} - \cancel{w^3 \gamma_0^2 \gamma_1 a_{11}} \\ + i \cancel{a_{00}^2 a_{11}^2 w \gamma_1} - i \cancel{a_{00} a_{10} a_{01} w \gamma_1} + \cancel{w^2 \gamma_0 \gamma_1 a_{00} a_{11}} + \cancel{w^2 \gamma_1^2 a_{00}^2} - \cancel{i w^3 \gamma_0^2 \gamma_1^2 a_{00}} \\ - \cancel{a_{11} a_{00} w^2 \gamma_0 \gamma_1} + \cancel{a_{10} a_{01} w^2 \gamma_0 \gamma_1} + i \cancel{w^3 \gamma_0^2 \gamma_1 a_{11}} + i \cancel{w^3 \gamma_0 \gamma_1^2 a_{00}} + \cancel{w^4 \gamma_0^2 \gamma_1^2}$$

$$= \cancel{a_{11}^2 a_{00}^2} - \cancel{2 a_{11} a_{00} a_{10} a_{01}} + \cancel{a_{10}^2 a_{01}^2} + \cancel{2 a_{10} a_{01} w^2 \gamma_0 \gamma_1} \\ - \cancel{2 a_{11} a_{00} w^2 \gamma_0 \gamma_1} + \cancel{w^2 \gamma_0^2 a_{11}^2} + \cancel{2 w^2 \gamma_1 \gamma_0 a_{00} a_{11}} + \cancel{w^2 \gamma_1^2 a_{00}^2} \\ + \cancel{w^4 \gamma_0^2 \gamma_1^2}$$

$$= - \cancel{2 a_{11} a_{00} a_{10} a_{01}} + \cancel{w^2 \gamma_0^2 a_{11}^2} + (a_{10} a_{01} - \gamma_0 \gamma_1 w^2)^2 \\ + a_{11}^2 a_{00}^2 - 2 a_{11} a_{00} w^2 \gamma_0 \gamma_1 + \cancel{2 a_{00} a_{11} w^2 \gamma_0 \gamma_1} \\ + w^2 \gamma_1^2 a_{00}^2$$

$$= - 2 a_{11} a_{00} a_{10} a_{01} + w^2 \gamma_0^2 a_{11}^2 + (a_{10} a_{01} - \gamma_0 \gamma_1 w^2)^2 \\ + a_{00}^2 (a_{11}^2 + w^2 \gamma_1^2)$$

Sanity check (for denominator)

$$= -2(-1)(-\bar{k})(-\bar{k})(1) + w^2 \gamma^2 (-1)^2 \\ + (-\bar{k} - \gamma \lambda w^2)^2 + (-\bar{k}^2 (-1)^2 + w^2 \gamma^2)$$

$$= 2k\bar{k} + \gamma^2 w^2 + (-k)^2 - 2\bar{k}\gamma \lambda w^2 + \gamma^2 \lambda^2 w^4 \\ + k^2 (1 + w^2 \lambda^2)$$

$$= 2k\bar{k} + \gamma^2 w^2 + k^2 - 2\bar{k}\gamma \lambda w^2 + \gamma^2 \lambda^2 w^4 \\ + k^2 (1 + w^2 \lambda^2)$$

✓ =  $2k\bar{k} + \gamma^2 w^2 + (\bar{k} - \gamma \lambda w^2)^2 + k^2 (1 + w^2 \lambda^2)$

Finally:

$$\tilde{C}_{xx}(w) = \frac{(a_{11}\tilde{\xi}_0 - a_{01}\tilde{\xi}_1)^2 + w^2 \gamma_1^2 \tilde{\xi}_0^2}{-2a_{11}a_{00}a_{10}a_{01} + w^2 \gamma_0^2 a_{11}^2 + (a_{10}a_{01} - \gamma_0 \gamma_1 w^2)^2 + a_{00}^2 (a_{11}^2 + w^2 \gamma_1^2)}$$