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Question 1. AEP and equilibrium statistical mechanics. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of discrete random variables with entropy $H(X)$. Let

$$A_n(t) = \left\{x^n \in \mathcal{X}^n : p(x^n) \geq 2^{-nt}\right\}$$

denote the subset of n-sequences with probabilities $\geq 2^{-nt}$.

a.) Show that $|A_n(t)| \leq 2^{nt}$.

b.) What is $\lim_{n\to\infty} P(X^n \in A_n(t))$? Solve for $t < H$ and $t > H$.

c.) Obtain $\lim_{n\to\infty} P(X^n \in A_n(t))$ for $t = H$.

d.) AEP says that all ergodic stochastic processes can be considered as a uniform distribution over a small set of typical sequences characterized by the entropy rate. Use AEP to justify the equivalence of the Boltzmann entropy and the Gibbs entropy for large systems.

[12 marks: 3/3/3/3]

(a)

x^n has a probability: $p(x^n) \geq 2^{-nt}$

And that: $P(A_n(t)) = \sum\limits_{x^n \in A_n(t)} P(x^n)$

We also know that: $\sum\limits_{x^n \in A_n(t)} P(x^n) \leq 1$

to give a concrete example of elements belonging to x^n:

we have x_1, x_2, x_3 where each of these have probabilities:

$p(x_1) \geq 2^{-nt}$, $P(x_2) \geq 2^{-nt}$, $P(x_3) \geq 2^{-nt}$

thus $x_1 + x_2 + x_3 \geq 3 \cdot 2^{-nt}$

refers to the no. of elements

So we can write:

$$x_1 + x_2 + x_3 \geq |A_n(t)|\, 2^{-nt}$$

$$\sum_{x^n \in A_n(t)} P(x^n) \geq |A_n(t)|\, 2^{-nt}$$

And that:

$$1 \geq \sum_{x^n \in A_n(t)} P(x^n) \geq |A_n(t)|\, 2^{-nt}$$

$$1 \geq |A_n(t)|\, 2^{-nt}$$

Thus,

$$|A_n(t)| \leq 2^{nt}$$

(6) (1) Solve for $t < H$:

Let's recall that $P(Y \in T) = 2^{-nH}$

for an element Y belonging to a typical set T from AEP

When $t < H$: $2^{-nt} > 2^{-nH}$

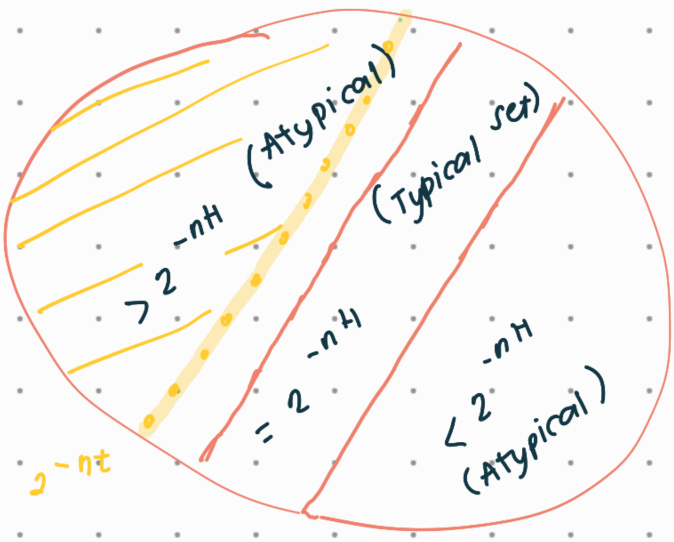


Figure 1: Regions of typical and atypical sets

Since the shaded region (in yellow) does not include the typical set:

then:

$$\lim_{n \to \infty} P(X^n \in A_n(t)) = 0$$

[Since we know that $P(X^n \in T) = 1$ (belonging to a typical set)]

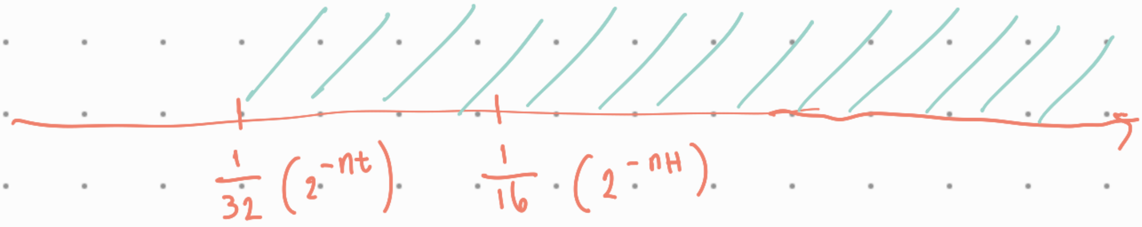
(2) Solve for $t > H$:

when $t > H$: $2^{-nt} < 2^{-nH}$

As an example, let $2^{-nt} = \frac{1}{32}$ where $t = 5$, $n = 1$

$2^{-nH} = \frac{1}{16}$ where $H = 4$, $n = 1$

Since $p(x^n) \geq 2^{-nt}$, then if we have this illustration:



then $T \subseteq A_n(t)$

And

$$p(x^n \in T) \leq p(x^n \in A_n(t))$$
$$1 \leq p(x^n \in A_n(t)) \leq 1$$

then $\lim_{n\to\infty} P(x^n \in A_n(t)) = 1$

(c) Solve for $t = H$:

When $t = H$: $2^{-nt} = 2^{-nH}$, then $A_n(t)$ is equal to a Typical set, thus:

$$\lim_{n\to\infty} P(x^n \in A_n(t)) = 1$$

(d) Boltzmann entropy : $S_B = k_B \ln \Omega$

where k_B is the Boltzmann's constant

Ω is the no. of microstates

Gibbs entropy : $S_G = -k_B \sum_i P_i \ln P_i$

where P_i is the probabilities of the states

Since $\Omega = |x^n| \rightarrow$ no. of elements in the typical set

Then the probability of each state can be written as:

$$P_i = \frac{1}{|x^n|}$$ (since it is a uniform distribution and $|x^n| \leq 2^{nH}$)

Then for Gibbs entropy:

$$S = -k_B \sum_i \frac{1}{|X^n|} \ln \frac{1}{|X^n|}$$

When n is very large, and performing the summation over all states:

$$S = -k_B \ln \frac{1}{|X^n|}$$

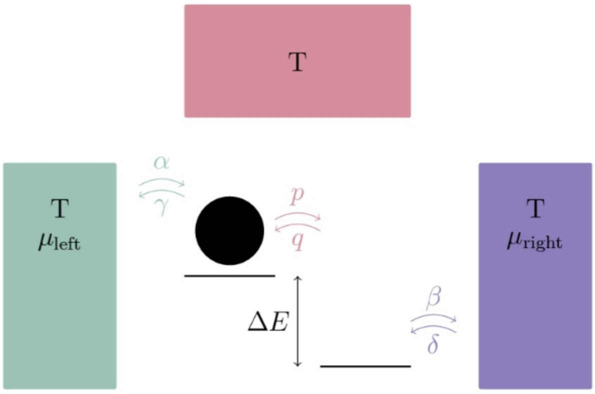
$$S_G = k_B \ln |X^n|$$

Then Gibbs entropy is equal to Boltzmann's entropy.

$$S_B = k_B \ln \Omega$$
$$= k_B \ln |X^n|$$

Question 2:

Question 2. From stochastic dynamics to stochastic thermodynamics. Consider the non-equilibrium system below, modeling a setting which describes the non-equilibrium transport of particles between two particle reservoirs at chemical potentials μ_{left} and μ_{right} with temperature T.



The system consists of two neighbouring sites that differ in energy by ΔE. Each site may be empty or occupied by a single particle. Particles hop either in/out of particle reservoirs or between sites with rates $p, q, \alpha, \beta, \gamma, \delta$.

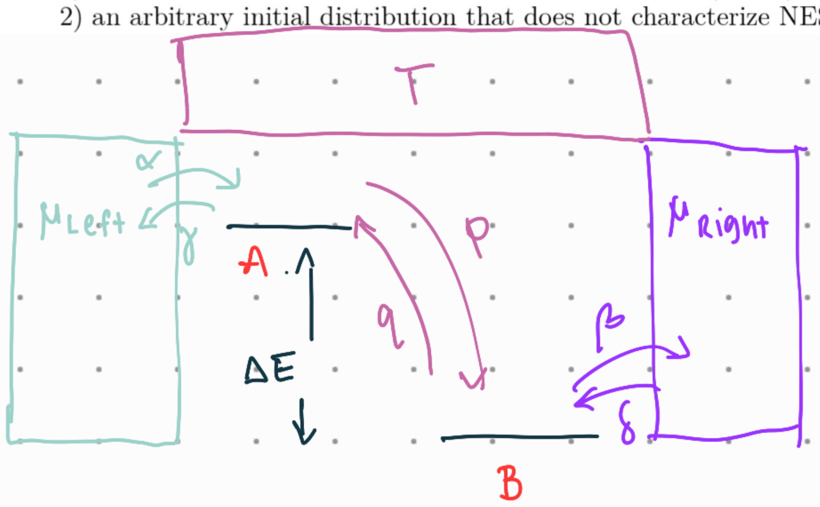
a.) Sketch the dynamics of this model as a random walk on a graph, whose vertices represent configurations and edges allowed state transitions. Do the state transitions conserve particle number or energy?

b.) How likely is it to observe the pink temperature reservoir in a state where its energy has increased by ΔE? Provide an expression in terms of transition rates, and then do the same for particle reservoirs.

c.) Write down the average entropy produced in a time interval τ of dynamical evolution, assuming

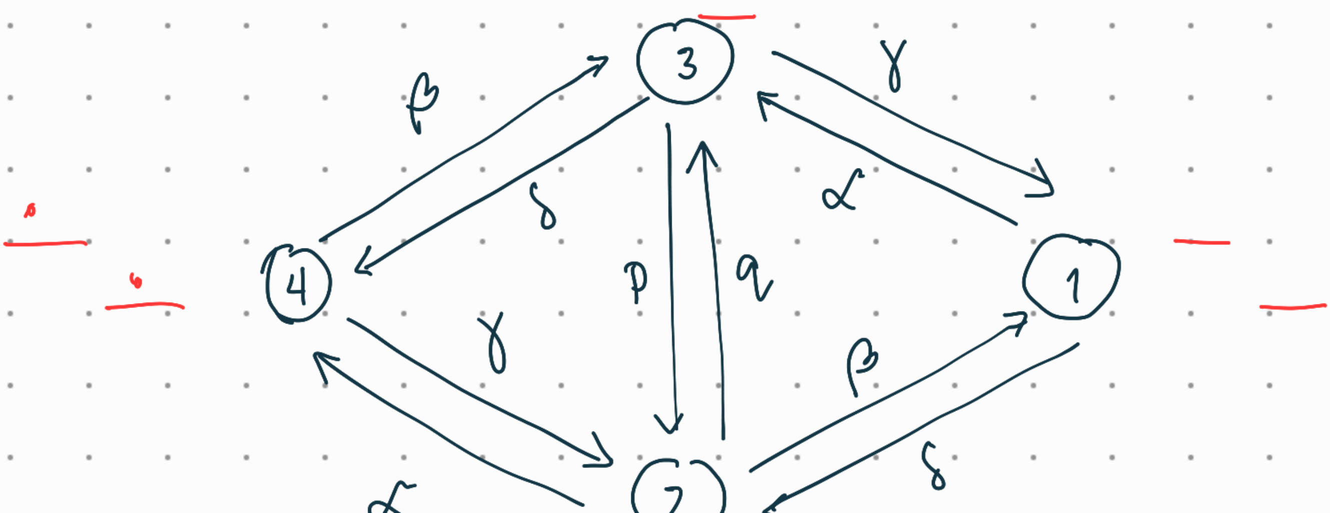
1) the system starts from non-equilibrium steady state (NESS),
2) an arbitrary initial distribution that does not characterize NESS.

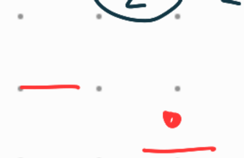
a



There are 4 states:

1. A empty, B empty
2. A empty, B occupied
3. A occupied, B empty
4. A occupied, B occupied





The system above involves an exchange of particles or energy with the reservoirs, thus, the state transitions conserve particle no. or energy

(6) To be able to observe the pink temperature reservoir in a state where its $\underbrace{\text{energy is increased}}_{T}$ by ΔE, this scenario should happen:

because going from state 3 to state 2 will allow the pink temperature reservoir to gain energy

And so:

Probability of hopping from 1 state to the other

$$dP = P_3(t)\, p\, dt$$

p is the rate from state 3 to state 2

solving for P:

$$P = p \int P_3(t)\, dt$$

Now if we consider P_3^{st}, we can solve for this by solving for the systems of eqns in 2.C

for particle reservoirs:

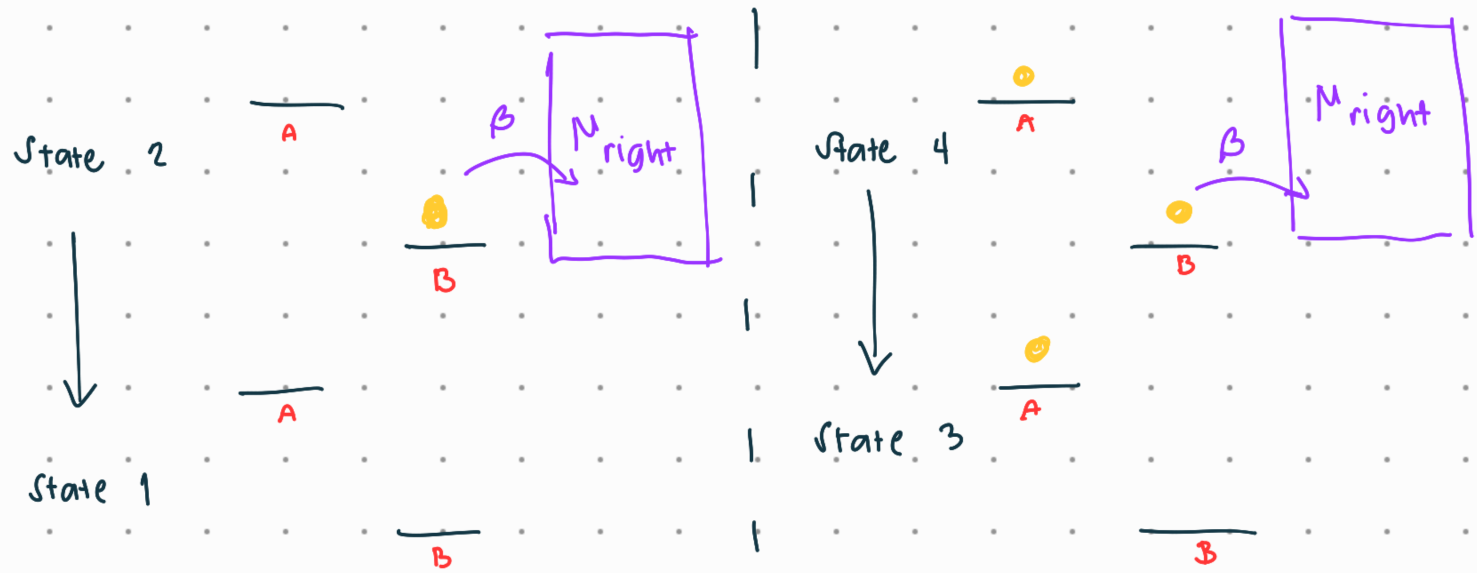
6.1) For N_{Left} to gain energy 2 scenarios can happen:

thus $dP = P_4(t)\gamma\, dt + P_3(t)\,\gamma\, dt$

$$dP = \left(P_4(t) + P_3(t)\right)\gamma\, dt$$

$$P = \gamma \int \left(P_4(t) + P_3(t)\right) dt$$

6.2) For Nright to gain energy, 2 scenarios can happen:



thus $dP = P_2(t)\,\beta\, dt + P_4(t)\,\beta\, dt$

$$dP = \left(P_2(t) + P_4(t)\right)\beta\, dt$$

$$P = \beta \int \left(P_2(t) + P_4(t)\right) dt$$

c)

1) For a system that starts from NESS, the ff. system of equations characterizes the whole system

$$\dot{P}_1(t) = -P_1(t)(\alpha + \delta) + \beta P_2(t) + \gamma P_3(t)$$

$$\dot{P}_2(t) = \delta P_1(t) - P_2(t)(q + \beta + \alpha) + p P_3(t) + \gamma P_4(t)$$

$$\dot{P}_3(t) = \alpha P_1(t) + q P_2(t) - P_3(t)(\delta + \gamma + p) + \beta P_4(t)$$

$$\dot{P}_4(t) = \alpha \cdot P_2(t) + \delta \cdot P_3(t) - P_4(t)(\gamma + \beta)$$

then P_1^{st}, P_2^{st}, P_3^{st}, P_4^{st} can be solved by solving for the system of equations including the normalization equation:

$$P_1^{st} + P_2^{st} + P_3^{st} + P_4^{st} = 1$$

then average entropy is:

$$\langle \sigma \rangle = \int_0^{\tau} \frac{k_B}{2} \sum_{mm'=1}^{4} \left(W_{mm'} P_{m'}^{st} - W_{m'm} P_m^{st} \right) \ln \frac{W_{mm'} P_{m'}^{st}}{W_{m'm} P_m^{st}} \, dt$$

Exercise 3:

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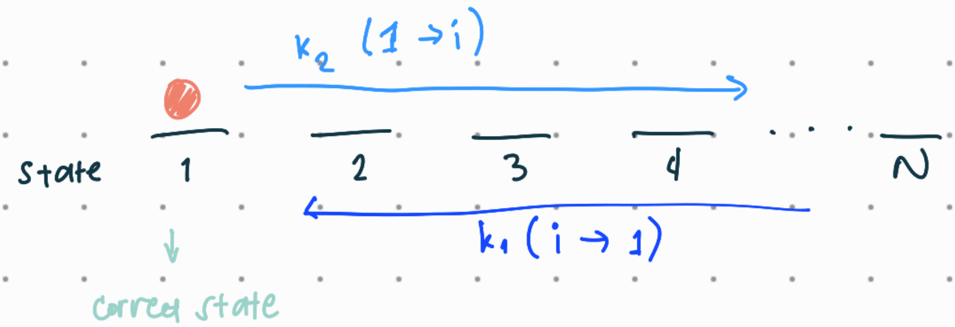
Question 3. Minimal model of error correction. There is a probability distribution over N states of a system, which evolves according to some rate matrix W. We denote the probability to be in state i as $p_i(t)$, and let `State 1` be a special, correct state. The system is coupled to two reservoirs. Let the first reservoir cause diffusion, i.e., errors. It contributes 1 to the rate of each possible state transition $i \to j$. Let the second reservoir be partially responsible from error correction, it contributes k_1 to an error correction transition $i \to 1$ and k_2 to an error inducing transition $1 \to i$. The total rate matrix (for $i \neq j$) is $W(i \to j) = 1 + \delta(j, 1)k_1 + \delta(i, 1)k_2$.

a.) Write down the master equation in terms of 1) rate matrix, and 2) associated currents. Write down the entropy production (EP).

a.) For simplicity, consider a setting with $N = 2$ first. Solve for the stationary state as a function of k_1 and k_2.

b.) Calculate the EP in stationary phase, as a function of k_1 and k_2. Give an interpretation of your result regarding EP in terms of physical work.

c.) Extend your results for $N = 2$ from any $N > 1$. [7 marks: 2/2/2/1]



ⓐ Master Eqn: $\dfrac{dP_i}{dt} = \sum_{j=1}^{N} W_{ji} P_i = J_i$

1.) rate matrix:

$$W_{ji} = 1 + \delta(j,1)k_1 + \delta(i,1)k_2 \qquad (i \to j)$$

$$W = \begin{bmatrix} W_{11} & 1+k_1 & 1+k_1 & \cdots & 1+k_1 \\ 1+k_2 & W_{22} & 1 & \cdots & 1 \\ 1+k_2 & 1 & W_{33} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \\ 1+k_2 & 1 & 1 & & W_{NN} \end{bmatrix}$$

Since the columns must sum to 0 for a continuous-time Markov process

Then, we can just replace $W_{11}, W_{22}, \ldots, W_{NN}$, as:

$$W = \begin{bmatrix} -(N-1)(1+k_2) & 1+k_1 & 1+k_1 & \cdots & 1+k_1 \\ 1+k_2 & -(N-1)-k_1 & 1 & \cdots & 1 \\ 1+k_2 & 1 & -(N-1)-k_1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \\ 1+k_2 & 1 & 1 & & -(N-1)-k_1 \end{bmatrix}$$

Currents :

$$J_1 = -(N-1)(1+k_2)P_1(t) + (1+k_1)\sum_{i=2}^{N} P_i(t)$$

$$J_2 = (1+k_2)P_1(t) - (N-1+k_1)P_2(t) + \sum_{i=3}^{N} P_i(t)$$

$$\vdots$$

Entropy production :

$$\dot{S}_{EP} = \sum_{i,j\,(i\neq j)} W_{ji}\, P_i \ln \frac{W_{ji}\, P_i}{W_{ij}\, P_j}$$

$$= \sum_{i,j\,(i\neq j)} (1 + \delta(j,1)k_1 + \delta(i,1)k_2)\, P_i \ln \frac{W_{ji}\, P_i}{W_{ij}\, P_j}$$

(b) for $N=2$, we have the following matrix:

$$W = \begin{bmatrix} -(1+k_2) & 1+k_1 \\ 1+k_2 & -(1+k_1) \end{bmatrix}$$

$$\frac{d\vec{P}}{dt} = \vec{0} \quad \text{for a stationary state}$$

$$\begin{bmatrix} -(1+k_2) & 1+k_1 \\ 1+k_2 & -(1+k_1) \end{bmatrix} \begin{bmatrix} P_1^{st} \\ P_2^{st} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-P_1^{st}(1+k_2) + P_2^{st}(1+k_1) = 0$$

$$P_1^{st}(1+k_2) - P_2^{st}(1+k_1) = 0$$

Then we have 3 equations:

$$P_1^{st}(1+k_2) = P_2^{st}(1+k_1) \qquad (1)$$

$$P_1^{st}(1+k_2) = P_2^{st}(1+k_1) \qquad (2)$$

$$P_1^{st} + P_2^{st} = 1 \quad \text{(for normalization)} \qquad (3)$$

Solving for the 3 equations:

Step 1: We can express P_1^{st} as:

$$P_1^{st} = \frac{1+k_1}{1+k_2} P_2^{st} \qquad (4)$$

Step 2: Using eq (3), we have:

$$P_1^{st} + P_2^{st} = 1$$

$$\frac{1+k_1}{1+k_2} P_2^{st} + P_2^{st} = 1$$

$$\left(\frac{1+k_1}{1+k_2} + 1\right) P_2^{st} = 1$$

$$P_2^{st} = \frac{1}{\left(\frac{1+k_1}{1+k_2} + 1\right)}$$

$$\boxed{P_2^{st} = \frac{1+k_2}{2+k_1+k_2}}$$

Then P_1^{st} from eq (4) is:

$$P_1^{st} = \frac{1+k_1}{1+k_2}\left(\frac{1+k_2}{2+k_1+k_2}\right)$$

$$\boxed{P_1^{st} = \frac{1+k_1}{2+k_1+k_2}}$$

c) Entropy production for $N=2$

$$\dot{S}_{EP} = W_{12} P_2 \ln \frac{W_{12} P_2}{W_{21} P_1} + W_{21} P_1 \ln \frac{W_{21} P_1}{W_{12} P_2}$$

Since $W_{12} = 1 + k_1$ $\qquad W_{21} = 1 + k_2$

$$P_1 = \frac{1+k_1}{2+k_1+k_2} \qquad P_2 = \frac{1+k_2}{2+k_1+k_2}$$

then $$\frac{P_2}{P_1} = \frac{1+k_2}{2+k_1\cancel{+k_2}} \cdot \frac{2+\cancel{k_1}+k_2}{1+k_1} = \frac{1+k_2}{1+k_1}$$

$$\frac{P_1}{P_2} = \frac{1+k_1}{2+\cancel{k_1}+k_2} \cdot \frac{2+k_1\cancel{+k_2}}{1+k_2} = \frac{1+k_1}{1+k_2}$$

thus $\dot{S}_{EP}$ is:

$$\dot{S}_{EP} = (1+k_1)\left(\frac{1+k_2}{2+k_1+k_2}\right)\ln\left(\overbrace{\cancel{\frac{1+k_1}{1+k_2}\cdot\frac{1+k_2}{1+k_1}}}^{1}\right)$$

$$+\ (1+k_2)\left(\frac{1+k_1}{2+k_1+k_2}\right)\ln\left(\overbrace{\cancel{\frac{1+k_2}{1+k_1}\cdot\frac{1+k_1}{1+k_2}}}^{1}\right)$$

thus, $\dot{S}_{EP} = 0$

when the system is in equilibrium, the entropy production is equal to zero

Question 4:

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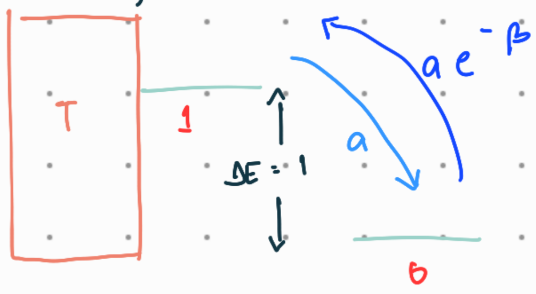
Question 4. Full counting statistics and thermodynamic uncertainty relations. Suppose there is a two-level system with states 0 and 1, coupled to a single heat reservoir with fixed inverse temperature β. The energy gap between the two states is set to $\Delta = 1$ and the state 0 is ascribed lower energy. Dynamical evolution starts from an arbitrary initial state $\mathbf{P}(0) = [p, 1-p]^{\mathbb{T}}$. The current is chosen to be the net flow from 1 to 0 . Local detailed balance imposes that two transition rates $r(0,1)$ and $r(1,0)$ satisfy $r(0,1) = e^{-\beta} r(1,0)$. For this setup, analytically calculate all the quantities in the TUR and GTUR by utilizing full counting statistics.

(Hint: Ensemble-averaged accumulated current and its variance can be calculated by obtaining the characteristic function first.)

[10 marks]

2 level - system



$r(0,1) = e^{-\beta} r(1,0)$

let's define $r(1,0) = a$

then: $r(0,1) = ae^{-\beta}$

With initial state $\vec{p}(0) = \begin{bmatrix} p \\ 1-p \end{bmatrix}$.

The master equation is: $\frac{d\vec{p}}{dt} = R\vec{p}(t)$

To solve for $\vec{P}(T)$, then we have

$$\vec{P}(T) = e^{RT}\vec{p}(0)$$

With transition rate matrix R as:

$$R = \begin{bmatrix} -ae^{-\beta} & a \\ ae^{-\beta} & -a \end{bmatrix}$$

where $ae^{-\beta}$ is the rate of going from zero to 1, and a is the opposite

To compute for the quantities in the TUR and GTUR:

$$\langle J \rangle, \quad Var[J], \quad \langle \sigma \rangle$$

The characteristic function is:

$$Z(\chi) = [1, 1]\, e^{R(\chi)T}\vec{p}(0)$$

where $R(\chi) = \begin{bmatrix} -ae^{-\beta} & ae^{-i\chi} \\ ae^{-\beta}e^{i\chi} & -a \end{bmatrix}$

And we can compute $\langle J \rangle$ and $Var[J]$ by:

① $$\langle J \rangle = \left. \frac{\partial \ln Z(x)}{\partial (ix)} \right|_{x=0}$$

② $$Var[J] = \left. \frac{\partial^2 \ln Z(x)}{\partial (ix)^2} \right|_{x=0}$$

③ The total entropy production σ involves 2 contributions:

$\sigma_S = S(T) - S(0)$ (Shannon entropy)

$\sigma_B = -\langle J \rangle$ (entropy production of heat bath)

First we solve for $Z(x)$:

Step 1: Solve for $R(x)T$

$$R(x)T = \begin{bmatrix} -aTe^{-\beta} & aTe^{-ix} \\ aTe^{-\beta+ix} & -aT \end{bmatrix}$$

Step 2: Solve for $e^{R(x)T}$

$$e^{R(x)T} = P e^{D} P^{-1}$$

Step 3: Find for P, D by solving for the eigenvalues and eigenvectors

where $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ $\quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Finding for eigenvalues, we solve for:

$$\det(RT - \lambda I) = \begin{vmatrix} -aTe^{-\beta} - \lambda & aTe^{-ix} \\ aTe^{ix-\beta} & -aT - \lambda \end{vmatrix} = 0$$

$$(\lambda + aT)(\lambda + aTe^{-\beta}) - a^2T^2e^{-\beta} = 0$$

$$\lambda^2 + \lambda aTe^{-\beta} + \lambda aT + \cancel{a^2T^2e^{-\beta}} - \cancel{a^2T^2e^{-\beta}} = 0$$

$$\lambda^2 + \lambda aT(e^{-\beta} + 1) = 0$$

Thus the eigenvalues are:

$$\lambda_1 = 0$$

$$\lambda_2 = -aT\left(e^{-\beta}+1\right)$$

To solve for eigenvectors:

$$A\vec{v} = \lambda\vec{v}$$

For $\vec{v}_1$: $R(x)T\,\vec{v}_1 = 0$

$$R(x)T\begin{pmatrix} y \\ z \end{pmatrix} = 0$$

$$\begin{bmatrix} -aTe^{-\beta} & aTe^{-ix} \\ aTe^{-\beta+ix} & -aT \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-aTe^{-\beta}y + aTe^{-ix}z = 0 \qquad (1)$$

$$aTe^{-\beta+ix}y - aTz = 0 \qquad (2)$$

From eq (2) we have:

$$z = \frac{aTe^{-\beta+ix}y}{aT}$$

$$z = ye^{-\beta+ix}$$

Thus :

$$\vec{v}_1 = \begin{bmatrix} 1 \\ e^{-\beta+ix} \end{bmatrix}$$

For $\vec{v}_2$: $R(x)T\,\vec{v}_2 = \lambda_2\vec{v}_2$

$$\lambda_2 = -aT\left(1+e^{-\beta}\right)$$

Then we have:

$$R(x)T\begin{pmatrix} y \\ z \end{pmatrix} = -aT\left(1+e^{-\beta}\right)\begin{pmatrix} y \\ z \end{pmatrix}$$

$$\begin{bmatrix} -\cancel{aTe^{-\beta}} + aT + \cancel{aTe^{-\beta}} & aTe^{-ix} \\ aTe^{-\beta+ix} & -\cancel{aT} + \cancel{aT} + aTe^{-\beta} \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} aT & aTe^{-ix} \end{bmatrix}\begin{bmatrix} y \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \cdots \\ aTe^{-\beta+ix} & aTe^{-\beta} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$aTy + aTe^{-ix} z = 0 \qquad (1)$$
$$aTe^{-\beta+ix} y + aTe^{-\beta} z = 0 \qquad (2)$$

using eq (2) we have :

$$z = \frac{-aTe^{-\beta+ix}\, y}{aTe^{-\beta}}$$
$$z = -ye^{ix}$$

Thus $\vec{v}_2 = \begin{bmatrix} 1 \\ -e^{ix} \end{bmatrix}$

Now we have :

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -aT(1+e^{-\beta}) \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ e^{ix-\beta} & -e^{ix} \end{pmatrix}$$

To solve for P^{-1} :

$$\det P = -e^{ix} - e^{ix-\beta}$$
$$P^{-1} = -\frac{1}{e^{ix}+e^{ix-\beta}} \begin{pmatrix} -e^{ix} & -1 \\ -e^{ix-\beta} & 1 \end{pmatrix}$$

Finally, we can solve :

$$e^{R(x)T} = Pe^{D}P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ e^{ix-\beta} & -e^{ix} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-aT(e^{-\beta}+1)} \end{bmatrix} \begin{bmatrix} \frac{e^{ix}}{e^{ix}+e^{ix-\beta}} & \frac{1}{e^{ix}+e^{ix-\beta}} \\ \frac{e^{ix-\beta}}{\cdots} & \frac{-1}{\cdots} \end{bmatrix}$$

$$\cdots \; e \qquad \left[\; \frac{\cdots}{e^{ix}+e^{ix-\beta}} \qquad \frac{\cdots}{e^{ix}+e^{ix-\beta}} \;\right]$$

$$= \begin{bmatrix} 1 & 1 \\ e^{ix-\beta} & -e^{ix} \end{bmatrix} \begin{bmatrix} \dfrac{e^{ix}}{e^{ix}+e^{ix-\beta}} & \dfrac{1}{e^{ix}+e^{ix-\beta}} \\[2ex] \dfrac{e^{ix-\beta-aT(e^{-\beta}+1)}}{e^{ix}+e^{ix-\beta}} & \dfrac{-e^{-aT(e^{-\beta}+1)}}{e^{ix}+e^{ix-\beta}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{e^{ix}+e^{ix-\beta-aT(e^{-\beta}+1)}}{e^{ix}+e^{ix-\beta}} & \dfrac{1-e^{-aT(e^{-\beta}+1)}}{e^{ix}+e^{ix-\beta}} \\[2ex] \dfrac{e^{2ix-\beta}-e^{2ix-\beta-aT(e^{-\beta}+1)}}{e^{ix}+e^{ix-\beta}} & \dfrac{e^{ix-\beta}+e^{ix-aT(e^{-\beta}+1)}}{e^{ix}+e^{ix-\beta}} \end{bmatrix}$$

Since $z(x) = [1, 1]\, e^{R(x)T} \begin{bmatrix} p \\ 1-p \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{e^{ix}+e^{ix-\beta-aT(e^{-\beta}+1)}+e^{2ix-\beta}-e^{2ix-\beta-aT(e^{-\beta}+1)}}{e^{ix}+e^{ix-\beta}} \\[2ex] \dfrac{1-e^{-aT(e^{-\beta}+1)}+e^{ix-\beta}+e^{ix-aT(e^{-\beta}+1)}}{e^{ix}+e^{ix-\beta}} \end{bmatrix}^T \begin{bmatrix} p \\ 1-p \end{bmatrix}$$

Finally, we have: $z(x) =$

$$\frac{pe^{ix}+pe^{ix-\beta-aT(e^{-\beta}+1)}+pe^{2ix-\beta}-pe^{2ix-\beta-aT(e^{-\beta}+1)}}{e^{ix}+e^{ix-\beta}}$$

$$(1-p) - (1-p)e^{-aT(e^{-\beta}+1)} + (1-p)e^{ix-\beta} + (1-p)e^{ix-aT(e^{-\beta}+1)}$$

$$(1-p) \quad (1+\ldots)e \quad \ldots \quad (\ldots+p) \quad \ldots \quad (1+\ldots)e$$

$$+ \frac{\qquad\qquad}{e^{ix} + e^{ix-\beta}}$$

Thus the ff quantities in TUR and GTUR are:

1. Solving for $\langle J \rangle = \left. \dfrac{\partial \ln z(x)}{\partial (ix)} \right|_{x=0}$

then:

$$\langle J \rangle = \frac{a(-1+p) + ae^{-\beta}p}{ae^{-\beta} + a} \left(1 - e^{-aT(e^{-\beta}+1)}\right)$$

2. Solving for : $\quad Var[J] = \left. \dfrac{\partial^2 \ln z(x)}{\partial (ix)^2} \right|_{x=0}$

then,

$$Var[J] = \left[\frac{e^{-2aT(e^{-\beta}+1)}}{(ae^{-\beta}+a)^2} \right] \cdot$$

$$\Big[-\left[a(-1+p) + ae^{-\beta}p\right]^2$$
$$+ e^{2aT(e^{-\beta}+1)} \left[a^2 e^{-\beta} + (ae^{-\beta}+a)^2 p - (ae^{-\beta}+a)^2 p^2\right]$$
$$+ e^{aT(e^{-\beta}+1)} \left[-a^2 e^{-\beta} + a^2 - (a+ae^{-\beta})(ae^{-\beta}+3a)p + 2(a+ae^{-\beta})^2 p^2\right] \Big]$$

3. Solving for σ : $\sigma_{bath} + \sigma_s$

a) $\sigma_{bath} = -\langle J \rangle$

$$\langle \sigma \rangle = -\frac{a(-1+p) + ae^{-\beta}p}{ae^{-\beta} + a} \left(1 - e^{-aT(e^{-\beta}+1)}\right)$$

b) $\sigma_s = s(T) - s(0)$

$$s(T) = -\left[P_0(t) \ln P_0(t) + P_1(t) \ln P_1(t)\right]$$
$$s(0) = -\left[p \ln p + (1-p) \ln p\right]$$

where $$P_0(t) = \frac{e^{-aT(e^{-\beta}+1)}}{a+ae^{-\beta}}\left[ae^{-\beta}p + b\left(-1+p+e^{aT(e^{-\beta}+1)}\right)\right]$$

$$P_1(t) = 1 - P_0(t)$$

And then taking $\langle \sigma \rangle$ afterwards

Question 6:

Browsing through a few pages of Pelitti's and Pigolotti's book on Stochastic Thermodynamics: An Introduction (a few pages available on Google Books), they have defined Stochastic Thermodynamics as a thermodynamic theory for mesoscopic, nonequilibrium physical systems interacting with equilibrium heat reservoirs. Going through this course for the past week, I slowly understood the meaning of the following terms: mesoscopic, nonequilibrium physical systems, interacting, equilibrium heat reservoirs. But the most interesting thing that I've learnt so far is that, taking a step back, I come to realize that stochastic thermodynamics is built on the theories that I have learned from information theory, stochastic process, thermodynamics, and statistical mechanics. And I find this interesting because stochastic thermodynamics completes the picture – I have been wondering how and in what scenarios or systems can I be able to apply all the things that I have learned so far in the diploma program, and here comes stochastic thermodynamics stitching all of these concepts together. In addition, at first glance, you may think that these concepts are not relevant to processes involved in studying biological systems, but what I've learnt so far in the past week is that, through using the tools we have in stochastic thermodynamics, we can better understand the dynamics present in cells found in our body, biopolymers (proteins, DNA, etc), molecular motors, etc. This then leads me to the question that I am most curious about (but may have been studied by others already), how can we advance our knowledge in medicine, or solve the most relevant questions in biology or medicine through stochastic thermodynamics? Through the years, there have been recent advancements in treating diseases, such as modifying certain cells to treat neurological conditions, or certain forms of cancer. Have researchers been exploring how the techniques we have in stochastic thermodynamics can be used to advance these methods further?