Compute

① $\langle \dot{Q}_x \rangle$

② $\langle \dot{Q}_F \rangle$

③ $\langle \dot{Q} \rangle = \langle \dot{Q}_x \rangle + \langle \dot{Q}_F \rangle$

(eqn 22 and 24 from Prof Sarah's paper)

for equations (7) and (8) found in Berger & Hudspeth's papers using analytical calculation from Prof Sarah's paper, and check if this agrees w/ the result they have in the paper (equation 17)

Linear models of oscillating hair bundles are sufficient to describe experimentally-observed correlation and response functions [20, 21]. Therefore, we will limit our analysis to a linear model. We describe the position x of a hair bundle by

$$\gamma \dot{x} = -kx + F + \eta_x, \tag{7}$$

in which γ is an effective drag coefficient, k a stiffness, and F an active driving force that is generated in the bundle. The bundle is exposed to fluctuations of the thermal environment described by the noise term η_X. Inside the bundle, a molecular machinery generates the active force F that evolves in time according to

$$\lambda \dot{F} = -\bar{k}x - F + \eta_F, \tag{8}$$

with the relaxation time λ, coupling constant $\bar{k}$, and noise η_F that originates from nonquilibrium fluctuations of molecular motors [26–28]. For both noise terms, we assume Gaussian noise that is delta correlated: $\langle \eta_x(t)\eta_x(0)\rangle = D_x\delta(t)$, $\langle \eta_F(t)\eta_F(0)\rangle = D_F\delta(t)$, with amplitude D_x and D_F, respectively. Furthermore, the two noise terms are uncorrelated with each other, $\langle \eta_x \eta_F \rangle = 0$.

$$\tilde{h}(\omega) = \frac{\gamma\omega^2(2k_BT\bar{k}\lambda + D_F)}{2\pi(2k\bar{k} + \gamma^2\omega^2 + (\bar{k} - \gamma\lambda\omega^2)^2 + k^2(1+\lambda^2\omega^2))}. \tag{15}$$

Because $\tilde{h}$ is a symmetric function, we evaluate the integral over $2\tilde{h}$ for positive values and obtain the energy dissipation

$$\langle J \rangle = \int_0^{+\infty} d\omega 2\tilde{h}(\omega) = \frac{2\bar{k}k_BT\lambda + D_F}{2\sqrt{2}\lambda S}\left(\sqrt{B+S} - \sqrt{B-S}\right), \tag{16}$$

in which $S = (\gamma + k\lambda)\sqrt{B - 2\gamma\lambda(k+\bar{k})}$ and $B = \gamma^2 + k^2\lambda^2 - 2\gamma\bar{k}\lambda$. For $4\gamma\bar{k}\lambda \geq (\gamma - k\lambda)^2$, we further simplify the result to

$$\langle J \rangle = \frac{2\bar{k}k_BT\lambda + D_F}{2\lambda b}\left(\sqrt{-B + \sqrt{b^2 + B^2}}\right), \tag{17}$$

in which $b = (\gamma + k\lambda)\sqrt{4\gamma\bar{k}\lambda - (\gamma - k\lambda)^2}$.

Eq. (7) and (8) are non-reciprocal systems:

$$\begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix}\begin{bmatrix} x \\ F \end{bmatrix} + \begin{bmatrix} \eta_x \\ \eta_F \end{bmatrix}$$

Harada-Sasa equality: connects dissipation / entropy production with how much a system does violate FDR

$$\langle J \rangle = \gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\tilde{C}_{vv}(\omega) - 2k_BT\tilde{R}'_v(\omega)\right]$$

$$\parallel$$

$$-\langle \dot{Q} \rangle$$

velocity autocorrelation fn

real part of response fn

Task: Is equation (17)

$$\langle J\rangle = \frac{2\bar{k}k_\mathrm{B}T\lambda + D_F}{2\lambda b}\left(\sqrt{-B+\sqrt{b^2+B^2}}\right), \quad (17)$$

in which $b = (\gamma + k\lambda)\sqrt{4\gamma\bar{k}\lambda - (\gamma - k\lambda)^2}$.

equal to $\langle \dot{Q}_x \rangle$ or $\langle \dot{Q}_F \rangle$ or $\langle \dot{Q} \rangle = \langle \dot{Q}_x \rangle + \langle \dot{Q}_F \rangle$?

Analytical calculation from Prof Sarah's paper

$$\dot{Q}_0 = \sum_{j>0}^{n} a_{0j}\langle X_j \dot{X}_0\rangle = \sum_{i=0}^{n}\sum_{j>0}^{n}\frac{a_{0j}a_{0i}}{\gamma_0}\langle X_j X_i\rangle = \dot{W}_0. \quad (22)$$

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Likewise, one can calculate the heat flows of the other d.o.f. $\dot{Q}_l = \sum_{i=0}^{n}\sum_{j\neq l}\frac{a_{lj}a_{li}}{\gamma_l}\langle X_j X_i\rangle$. It should be noted that by writing down this expression for the dissipation of $X_{j>0}$ and the total EP (21), we implicitly assume that all X_j are *even* under time-reversal, that means, position-like variables. In contrast, odd variables would not contribute to the total EP, see [35]. We note that for active matter models the parity of $X_{j>0}$ is in fact a nontrivial aspect, and subject of an ongoing debate, see e.g., [35, 37, 85, 86].

Together with the correlations $\langle X_i X_j\rangle$ that are derived in appendix C, equations (21), (22) represent analytical expressions for heat flow and entropy production for any n. For example, in the case $n = 1$ (which was also discussed in [57]), where the expression significantly simplifies, we find from (22), (21) in combination with (C5)

$$\dot{S}_\mathrm{tot} = \frac{k_\mathrm{B}}{T_0 T_1}\frac{(a_{10}T_0 - a_{01}T_1)^2}{-a_{00}\gamma_1 - a_{11}\gamma_0} \geqslant 0, \quad (23)$$

$$\dot{Q}_0 = k_\mathrm{B}\frac{a_{01}(a_{10}T_0 - a_{01}T_1)}{(a_{00}\gamma_1 + a_{11}\gamma_0)}. \quad (24)$$

$$\dot{Q}_1 = k_B \frac{a_{10}(a_{01}T_1 - a_{10}T_0)}{a_{00}\gamma_1 + a_{11}\gamma_0}$$

System:

$$\begin{bmatrix}\gamma_0 & 0\\ 0 & \gamma_1\end{bmatrix}\begin{bmatrix}\dot{x}_0\\ \dot{x}_1\end{bmatrix} = \begin{bmatrix}a_{00} & a_{01}\\ a_{10} & a_{11}\end{bmatrix}\begin{bmatrix}x_0\\ x_1\end{bmatrix} + \begin{bmatrix}\xi_0\\ \xi_1\end{bmatrix}$$

$$\dot{Q}_0 = \sum_{i=0}^{n}\sum_{j>0}^{n}\frac{a_{0j}a_{0i}}{\gamma_0}\langle x_j x_i\rangle \quad \text{for } n = 1$$

$$= \sum_{i=0}^{1}\sum_{j=1}^{1}\frac{a_{0j}a_{0i}}{\gamma_0}\langle x_j x_i\rangle$$

$$= \frac{a_{01}a_{00}}{\gamma_0}\langle x_1 x_0\rangle$$

Since $\langle x_1 x_0\rangle$

$$\underline{\underline{\Sigma}} = \begin{pmatrix}\langle X_0^2\rangle & \langle X_0X_1\rangle\\ \langle X_1X_0\rangle & \langle X_1^2\rangle\end{pmatrix}$$

$$= k_\mathrm{B}\begin{pmatrix}\frac{T_1a_{01}^2 - T_0a_{01}a_{10} + T_0a_{11}(a_{00}+a_{11}\gamma_0/\gamma_1)}{(a_{00}+a_{11}\gamma_0/\gamma_1)(a_{01}a_{10}-a_{00}a_{11})} & \frac{-T_1a_{00}a_{01} - T_0a_{11}a_{10}\gamma_0/\gamma_1}{(a_{00}+a_{11}\gamma_0/\gamma_1)(a_{01}a_{10}-a_{00}a_{11})}\\ \frac{-T_1a_{00}a_{01} - T_0a_{11}a_{10}\gamma_0/\gamma_1}{(a_{00}+a_{11}\gamma_0/\gamma_1)(a_{01}a_{10}-a_{00}a_{11})} & \frac{T_0a_{10}^2\gamma_0/\gamma_1 - T_1a_{01}a_{10}\gamma_0/\gamma_1 + T_1a_{00}(a_{00}+a_{11}\gamma_0/\gamma_1)}{(a_{00}+a_{11}\gamma_0/\gamma_1)(a_{01}a_{10}-a_{00}a_{11})}\end{pmatrix}. \quad (C5)$$

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$$\dot{Q}_0 = \frac{a_{01}\, a_{00}}{\gamma_0}\left[k_B \frac{-T_1 a_{00} a_{01} - T_0 a_{11} a_{10} \frac{\gamma_0}{\gamma_1}}{\left(a_{00} + a_{11}\frac{\gamma_0}{\gamma_1}\right)\left(a_{01} a_{10} - a_{00} a_{11}\right)}\right]$$

From eqn (24)

$$\dot{Q}_0 = k_B \frac{a_{01}(a_{10} T_0 - a_{01} T_1)}{a_{00}\gamma_1 + a_{11}\gamma_0} \quad (1)$$

From Berger & Hudspeth:

$$\begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix}\begin{bmatrix} x \\ F \end{bmatrix} + \begin{bmatrix} \eta_x \\ \eta_F \end{bmatrix}$$

Using equation (1):

(a) $$\dot{Q}_x = k_B \frac{(-\bar{k}T - T)}{-k\gamma - \lambda} = \frac{k_B T(-\bar{k} - 1)}{-k\gamma - \lambda}$$

So $\langle \dot{Q}_x \rangle = \dot{Q}_x$ (from Prof Sarah's paper $\dot{Q} = \langle \frac{\delta q_x}{dt} \rangle$)

notation used

(b) $\langle \dot{Q}_F \rangle =$

From Prof Sarah's paper:

$$\dot{Q}_\ell = \sum_{i=0}^{n}\sum_{j\neq \ell} \frac{a_{\ell j}\, a_{\ell i}}{\gamma_\ell}\langle x_j x_i \rangle$$

$$\dot{Q}_1 = \sum_{i=0}^{1}\sum_{j\neq 1} \frac{a_{1j}\, a_{1i}}{\gamma_1}\langle x_j x_i \rangle$$

$$= \frac{a_{10}\, a_{11}}{\gamma_1}\langle x_0 x_1 \rangle$$

$$\dot{Q}_1 = \dot{Q}_F$$

$$\dot{Q}_F = k_B \frac{a_{10}(a_{01}T_1 - a_{10}t_0)}{a_{00}\gamma_1 + a_{11}\gamma_0}$$

Since symmetric, use $\dot{Q}_0$ in eq. 24 from Prof Saran's paper and switch $0 \to 1$ in numerator

$$= k_B \frac{-\bar{k}((1)T + \bar{k}T)}{-k\gamma - \lambda}$$

$$\langle \dot{Q}_F \rangle = k_B \frac{-\bar{k}(T + \bar{k}T)}{-k\gamma - \lambda}$$

$$\langle \dot{Q}_x \rangle = k_B \frac{(-kT - T)}{-k\gamma - \lambda}$$

Now check if $\langle \dot{Q}_F \rangle + \langle \dot{Q}_x \rangle = \langle J \rangle$ in eq. (17)