

Perform analytical calculation for response function in Fourier space for a 2D non-reciprocal generic model

$$\begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} x_D \\ x_1 \end{bmatrix} + \begin{bmatrix} \zeta_0 \\ \zeta_1 \end{bmatrix}$$

$$\gamma_0 \dot{x}_0 = a_{00} x_0 + a_{01} x_1 + \zeta_0$$

$$\gamma_1 \dot{x}_1 = a_{10} x_0 + a_{11} x_1 + \zeta_1$$

2. Derivation of the response function

To obtain the response function, we perturb the position coordinate with a δ -function in the time domain, resulting in the following equations in Fourier space,

$$-i\omega\gamma\tilde{x}_\delta = -k\tilde{x}_\delta + \tilde{F} + \tilde{\eta}_x + 1 \quad (48)$$

and

$$-i\omega\lambda\tilde{F} = -\bar{k}\tilde{x}_\delta - \tilde{F} + \tilde{\eta}_F. \quad (49)$$

As from the paper, they perturbed the position coordinate w/ a δ -function in the time domain to obtain the response function. Following their procedure, we have:

$$\gamma_0 \dot{x}_0 = a_{00} x_0 + a_{01} x_1 + \zeta_0 + \delta(t) \quad (1)$$

$$\gamma_1 \dot{x}_1 = a_{10} x_0 + a_{11} x_1 + \zeta_1 \quad (2)$$

transforming eq(1) and eq(2) to Fourier space,

$$-i\omega\gamma_0 \tilde{x}_{0\delta} = a_{00} \tilde{x}_0 + a_{01} \tilde{x}_1 + \tilde{\zeta}_0 + \tilde{\delta}(t) \quad (3)$$

$$-i\omega\gamma_1 \tilde{x}_1 = a_{10} \tilde{x}_0 + a_{11} \tilde{x}_1 + \tilde{\zeta}_1 \quad (4)$$

$$\text{Since } \tilde{f}(t) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt = 1$$

then we have:

$$-i\omega \gamma_0 \tilde{x}_0 = a_{00} \tilde{x}_0 + a_{01} \tilde{x}_1 + \tilde{z}_0 + 1 \quad (5)$$

$$-i\omega \gamma_1 \tilde{x}_1 = a_{10} \tilde{x}_0 + a_{11} \tilde{x}_1 + \tilde{z}_1 \quad (6)$$

Solving for the Fourier-transformed position:

Solve for \tilde{x}_1 :

$$-i\omega \gamma_0 \tilde{x}_0 = a_{00} \tilde{x}_0 + a_{01} \tilde{x}_1 + \tilde{z}_0 + 1$$

$$+ \\ -i\omega \gamma_1 \tilde{x}_1 = a_{10} \tilde{x}_0 + a_{11} \tilde{x}_1 + \tilde{z}_1$$

$$-i\omega \gamma_0 \tilde{x}_0 - i\omega \gamma_1 \tilde{x}_1 = (a_{00} + a_{10}) \tilde{x}_0 + (a_{01} + a_{11}) \tilde{x}_1 + \tilde{z}_0 + \tilde{z}_1 + 1$$

$$i\omega \gamma_1 \tilde{x}_1 + (a_{01} + a_{11}) \tilde{x}_1 = -i\omega \gamma_0 \tilde{x}_0 - (a_{00} + a_{10}) \tilde{x}_0 - \tilde{z}_0 - \tilde{z}_1 - 1$$

$$\tilde{x}_1 (i\omega \gamma_1 + a_{01} + a_{11}) = \tilde{x}_0 (-i\omega \gamma_0 - a_{00} - a_{10}) - \tilde{z}_0 - \tilde{z}_1 - 1$$

$$\tilde{x}_1 = \frac{\tilde{x}_0 (-i\omega \gamma_0 - a_{00} - a_{10}) - \tilde{z}_0 - \tilde{z}_1 - 1}{i\omega \gamma_1 + a_{01} + a_{11}}$$

Substitute \tilde{x}_1 in eq (5):

$$-i\omega \gamma_0 \tilde{x}_0 = a_{00} \tilde{x}_0 + a_{01} \tilde{x}_1 + \tilde{z}_0 + 1$$

$$-i\omega \gamma_0 \tilde{x}_0 = a_{00} \tilde{x}_0 + a_{01} \left[\frac{\tilde{x}_0 (-i\omega \gamma_0 - a_{00} - a_{10}) - \tilde{z}_0 - \tilde{z}_1 - 1}{i\omega \gamma_1 + a_{01} + a_{11}} \right] + \tilde{z}_0 + 1$$

Multiply $i\omega \gamma_1 + a_{01} + a_{11}$ on both sides of the equation

$$\begin{aligned}
 -i\omega\gamma_0 \tilde{x}_0 (i\omega\gamma_1 + a_{01} + a_{11}) &= a_{00} \tilde{x}_0 (i\omega\gamma_1 + a_{01} + a_{11}) \\
 + \tilde{x}_0 (-i\omega\gamma_0 a_{01} - a_{00} a_{01} - a_{10} a_{01}) - \tilde{\zeta}_0 a_{01} - \tilde{\zeta}_0 a_{01} \\
 - (i)a_{01} + \tilde{\zeta}_0 (i\omega\gamma_1 + a_{01} + a_{11}) + i(i\omega\gamma_1 + a_{01} + a_{11})
 \end{aligned}$$

Group terms w/ \tilde{x}_0 in one side of the equation:

$$\tilde{\zeta}_0 a_{01} + \tilde{\zeta}_1 a_{01} + (i)a_{01} - \tilde{\zeta}_0 (i\omega\gamma_1 + a_{01} + a_{11}) - i(i\omega\gamma_1 + a_{01} + a_{11}) =$$

$$\begin{aligned}
 -\omega^2 \gamma_0 \gamma_1 \tilde{x}_0 + \cancel{i\omega\gamma_0 a_{01} \tilde{x}_0} + i\omega\gamma_0 a_{11} \tilde{x}_0 \\
 + i\omega \gamma_1 a_{00} \tilde{x}_0 + \cancel{a_{01} a_{00} \tilde{x}_0} + a_{11} a_{00} \tilde{x}_0 \\
 - \cancel{i\omega \gamma_0 a_{01} \tilde{x}_0} - \cancel{a_{01} a_{00} \tilde{x}_0} - a_{10} a_{01} \tilde{x}_0
 \end{aligned}$$

Simplify terms:

$$\tilde{\zeta}_0 a_{01} + \tilde{\zeta}_1 a_{01} + a_{01} - \tilde{\zeta}_0 (i\omega\gamma_1 + a_{01} + a_{11}) - i(i\omega\gamma_1 + a_{01} + a_{11}) =$$

$$\tilde{x}_0 (-\omega^2 \gamma_0 \gamma_1 + i\omega (\gamma_0 a_{11} + \gamma_1 a_{00}) + a_{11} a_{00} - a_{10} a_{01})$$

Finally:

$$\tilde{x}_{0g} = \tilde{\zeta}_0 a_{01} + \tilde{\zeta}_1 a_{01} + a_{01} - \tilde{\zeta}_0 (i\omega\gamma_1 + a_{01} + a_{11}) - i(i\omega\gamma_1 + a_{01} + a_{11})$$

$$a_{11} a_{00} - a_{10} a_{01} + i\omega (\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2 \gamma_0 \gamma_1$$

Averaged position:

$$\langle \tilde{x}_{0g} \rangle = \left\{ \frac{\tilde{\zeta}_0 a_{01} + \tilde{\zeta}_1 a_{01} + a_{01} - \tilde{\zeta}_0 (i\omega\gamma_1 + a_{01} + a_{11}) - i(i\omega\gamma_1 + a_{01} + a_{11})}{a_{11} a_{00} - a_{10} a_{01} + i\omega (\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2 \gamma_0 \gamma_1} \right\}$$

Sanity check if agrees w/ eq(50) in Appendix C

$$\begin{bmatrix} \gamma_0 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{f} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} + \begin{bmatrix} n_x \\ n_f \end{bmatrix} \quad \begin{array}{l} a_{00} \ a_{01} \\ a_{10} \ a_{11} \end{array}$$

$$\langle \tilde{x}_{0s} \rangle = \left\langle \frac{\tilde{n}_x(1) + \tilde{n}_f(1) + 1 - \tilde{n}_x(i\omega\lambda + 1 - 1) - 1(i\omega\lambda + 1 - 1)}{k - (-\bar{k})(1) + i\omega(\gamma(-1) + \lambda(-k)) - \omega^2\gamma\lambda} \right\rangle$$

$$= \left\langle \frac{\tilde{n}_x + \tilde{n}_f + 1 - \tilde{n}_x(i\omega\lambda) - i\omega\lambda}{k + \bar{k} - i\omega(\gamma + \lambda k) - \omega^2\gamma\lambda} \right\rangle$$

$$= \left\langle \frac{\tilde{n}_x + \tilde{n}_f + 1 - i\omega\lambda(\tilde{n}_x + 1)}{k + \bar{k} - i\omega(\gamma + \lambda k) - \omega^2\gamma\lambda} \right\rangle$$

Finally:

$$\tilde{R}_x = \langle \tilde{x}_{0s} \rangle = \left\langle \frac{\tilde{\gamma}_0 a_{01} + \tilde{\gamma}_1 a_{01} + a_{01} - \tilde{\gamma}_0(i\omega\gamma_1 + a_{01} + a_{11}) - 1(i\omega\gamma_1 + a_{01} + a_{11})}{a_{11}a_{00} - a_{10}a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1} \right\rangle$$

$$\tilde{R}_x = \frac{a_0 - i\omega\gamma_1 - a_{01} - a_{11}}{a_{11}a_{00} - a_{10}a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1}$$

Sanity check if similar to eqn (51) in paper:

$$\begin{bmatrix} \gamma_0 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{f} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} + \begin{bmatrix} n_x \\ n_f \end{bmatrix} \quad \begin{array}{l} a_{00} \ a_{01} \\ a_{10} \ a_{11} \end{array}$$

$$\tilde{R}_x = \frac{-i\omega\lambda + 1}{(-1)(-k) - (-\bar{k})(1) + i\omega(\gamma(-1) + \lambda(-k)) - \omega^2\gamma\lambda}$$

$$= \frac{1 - i\omega\lambda}{k + \bar{k} + i\omega(-\gamma - \lambda k) - \omega^2\gamma\lambda}$$

$$= \frac{1 - i\omega\gamma}{k + \bar{k} - i\omega(\gamma + \bar{\gamma}k) - \omega^2\gamma}$$

To get real and imaginary part, multiply \tilde{R}_x by its complex conjugate (for both numerator and denominator)

$$\tilde{R}_x = \frac{-i\omega\gamma_1 - a_{11}}{a_{11}a_{00} - a_{10}a_{01} + i\omega(\gamma_0a_{11} + \gamma_1a_{00}) - \omega^2\gamma_0\gamma_1}$$

$$\circ \frac{(a_{11}a_{00} - a_{10}a_{01} - i\omega(\gamma_0a_{11} + \gamma_1a_{00}) - \omega^2\gamma_0\gamma_1)}{(a_{11}a_{00} - a_{10}a_{01} - i\omega(\gamma_0a_{11} + \gamma_1a_{00}) - \omega^2\gamma_0\gamma_1)}$$

Numerator:

$$\begin{aligned} &= -i\omega\gamma_1(a_{11}a_{00} - a_{10}a_{01} - i\omega(\gamma_0a_{11} + \gamma_1a_{00}) - \omega^2\gamma_0\gamma_1) \\ &\quad - a_{11}(a_{11}a_{00} - a_{10}a_{01} - i\omega(\gamma_0a_{11} + \gamma_1a_{00}) - \omega^2\gamma_0\gamma_1) \\ &= -i\omega\gamma_1a_{11}a_{00} + i\omega\gamma_1a_{10}a_{01} - \omega^2\gamma_1\gamma_0a_{11} - \omega^2\gamma_1^2a_{00} + i\omega^3\gamma_0\gamma_1^2 \\ &\quad - a_{11}^2a_{00} + a_{10}a_{01}a_{11} + i\omega\gamma_0a_{11}^2 + i\omega\gamma_1a_{00}a_{11} + \omega^2\gamma_0\gamma_1a_{11} \end{aligned}$$

Denominator: see autocorr calculation of $\tilde{x}_0 \tilde{x}_0^*$

then we have:

$$-2a_{11}a_{00}a_{10}a_{01} + \omega^2\gamma_0^2a_{11}^2 + (a_{10}a_{01} - \gamma_0\omega^2)^2 + a_{00}^2(a_{11}^2 + \omega^2\gamma_1^2)$$

Obtain real part and imaginary part of numerator:

Real part: Divide terms w/ i and without i

$$= -\omega^2\gamma_1\gamma_0a_{11} - \omega^2\gamma_1^2a_{00} - a_{11}^2a_{00} + a_{10}a_{01}a_{11} + \omega^2\gamma_0\gamma_1a_{11}$$

Imaginary part:

isolate i :

$$\begin{aligned} &= i(-\omega\gamma_1a_{11}a_{00} + \omega\gamma_1a_{10}a_{01} + \omega^3\gamma_0\gamma_1^2 + \omega\gamma_0a_{11}^2 + \cancel{\omega\gamma_1a_{00}a_{11}}) \\ &= \omega(\gamma_1a_{10}a_{01} + \gamma_0\gamma_1^2\omega^2 + a_{11}^2\gamma_0) \end{aligned}$$

Perform sanity check for real part:

Sanity check:

$$\begin{bmatrix} \gamma_0 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ f \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} + \begin{bmatrix} n_x \\ n_f \end{bmatrix} \quad \begin{matrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{matrix}$$

$$\begin{aligned}
 &= -\omega^2 \gamma \gamma (-1) - \omega^2 \gamma^2 (-k) - (-1)^2 (-k) + (-\bar{k})(1)(-1) + \omega^2 \gamma \gamma (-1) \\
 &= \cancel{\omega^2 \gamma \gamma} + \omega^2 \gamma^2 k + k + \bar{k} - \cancel{\omega^2 \gamma \gamma} \\
 &= k + \bar{k} + \omega^2 \gamma^2 k \quad \checkmark
 \end{aligned}$$

Perform sanity check for imaginary part:

Sanity check:

$$\left[\begin{array}{c} \gamma_0 \\ 0 \end{array} \right] \left[\begin{array}{c} \dot{x} \\ \dot{f} \end{array} \right] = \left[\begin{array}{cc} -k & 1 \\ -\bar{k} & -1 \end{array} \right] \left[\begin{array}{c} x \\ f \end{array} \right] + \left[\begin{array}{c} n_x \\ n_f \end{array} \right]$$

$\alpha_{00} \quad \alpha_{01}$
 $\alpha_{10} \quad \alpha_{11}$

Imaginary part is:
 $= \omega (\gamma_1 \alpha_{10} \alpha_{01} + \gamma_0 \gamma_1^2 \omega^2 + \alpha_{11}^2 \gamma_0)$
so we have:

$$\begin{aligned}
 &= \omega (\gamma \gamma (-\bar{k})(1) + \gamma \gamma^2 \omega^2 + (-1)^2 \gamma) \\
 &= \omega (\gamma - \gamma \bar{k} + \gamma \gamma^2 \omega^2)
 \end{aligned}$$

Then, the averaged complex response is:

$$\tilde{R}_x = \frac{-i\omega \gamma_1 - \alpha_{11}}{\alpha_{11} \alpha_{00} - \alpha_{10} \alpha_{01} + i\omega (\gamma_0 \alpha_{11} + \gamma_1 \alpha_{00}) - \omega^2 \gamma_0 \gamma_1}$$

with real part:

$$\tilde{R}'_x(\omega) = \frac{-\omega^2 \gamma_1 \gamma_0 \alpha_{11} - \omega^2 \gamma_1^2 \alpha_{00} - \alpha_{11}^2 \alpha_{00} + \alpha_{10} \alpha_{01} \alpha_{11} + \omega^2 \gamma_0 \gamma_1 \alpha_{11}}{-2\alpha_{11} \alpha_{00} \alpha_{10} \alpha_{01} + \omega^2 \gamma_0^2 \alpha_{11}^2 + (\alpha_{10} \alpha_{01} - \gamma_0 \gamma_1 \omega^2)^2 + \alpha_{00}^2 (\alpha_{11}^2 + \omega^2 \gamma_1^2)}$$

and imaginary part:

$$\tilde{R}''_x(\omega) = \frac{\omega (\gamma_1 \alpha_{10} \alpha_{01} + \gamma_0 \gamma_1^2 \omega^2 + \alpha_{11}^2 \gamma_0)}{-2\alpha_{11} \alpha_{00} \alpha_{10} \alpha_{01} + \omega^2 \gamma_0^2 \alpha_{11}^2 + (\alpha_{10} \alpha_{01} - \gamma_0 \gamma_1 \omega^2)^2 + \alpha_{00}^2 (\alpha_{11}^2 + \omega^2 \gamma_1^2)}$$

