

Compute

$$\textcircled{1} \quad \langle \dot{Q}_x \rangle$$

$$\textcircled{2} \quad \langle \dot{Q}_F \rangle$$

$$\textcircled{3} \quad \langle \dot{Q} \rangle = \langle \dot{Q}_x \rangle + \langle \dot{Q}_F \rangle$$

(eqn 22 and 24 from Prof Sarah's paper)

for equations (7) and (8) found in Berger & Hudspeth's paper using analytical calculation from Prof Sarah's paper, and check if this agrees w/ the result they have in the paper (equation 17)

Linear models of oscillating hair bundles are sufficient to describe experimentally-observed correlation and response functions [20, 21]. Therefore, we will limit our analysis to a linear model. We describe the position x of a hair bundle by

$$\gamma \dot{x} = -kx + F + \eta_x, \quad (7)$$

in which γ is an effective drag coefficient, k a stiffness, and F an active driving force that is generated in the bundle. The bundle is exposed to fluctuations of the thermal environment described by the noise term η_X . Inside the bundle, a molecular machinery generates the active force F that evolves in time according to

$$\lambda \dot{F} = -\bar{k}x - F + \eta_F, \quad (8)$$

with the relaxation time λ , coupling constant \bar{k} , and noise η_F that originates from non-equilibrium fluctuations of molecular motors [26–28]. For both noise terms, we assume Gaussian noise that is delta correlated: $\langle \eta_x(t)\eta_x(0) \rangle = D_x \delta(t)$, $\langle \eta_F(t)\eta_F(0) \rangle = D_F \delta(t)$, with amplitude D_x and D_F , respectively. Furthermore, the two noise terms are uncorrelated with each other, $\langle \eta_x \eta_F \rangle = 0$.

$$\tilde{h}(\omega) = \dots \quad (15)$$

$$\frac{\gamma \omega^2 (2k_B T \bar{k} \lambda + D_F)}{2\pi(2kk + \gamma^2 \omega^2 + (k - \gamma \lambda \omega^2)^2 + k^2(1 + \lambda^2 \omega^2))}.$$

Because \tilde{h} is a symmetric function, we evaluate the integral over $2\tilde{h}$ for positive values and obtain the energy dissipation

$$\langle J \rangle = \int_0^{+\infty} d\omega 2\tilde{h}(\omega) = \frac{2\bar{k}k_B T \lambda + D_F}{2\sqrt{2}\lambda S} (\sqrt{B+S} - \sqrt{B-S}), \quad (16)$$

in which $S = (\gamma + k\lambda)\sqrt{B - 2\gamma\lambda(k + \bar{k})}$ and $B = \gamma^2 + k^2\lambda^2 - 2\gamma\bar{k}\lambda$. For $4\gamma\bar{k}\lambda \geq (\gamma - k\lambda)^2$, we further simplify the result to

$$\langle J \rangle = \frac{2\bar{k}k_B T \lambda + D_F}{2\lambda b} \left(\sqrt{-B + \sqrt{b^2 + B^2}} \right), \quad (17)$$

in which $b = (\gamma + k\lambda)\sqrt{4\gamma\bar{k}\lambda - (\gamma - k\lambda)^2}$.

Eq. (7) and (8) are non-reciprocal systems:

$$\begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ F \end{bmatrix} + \begin{bmatrix} \eta_x \\ \eta_F \end{bmatrix}$$

Harada-Sasa equality : connects dissipation / entropy production with how much a system does violate FDR

$$\langle J \rangle = \gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{C}_{vv}(\omega) - 2k_B T \tilde{R}'_v(\omega)$$

↑
velocity autocorrelation fn

↑
real part of response fn

Task: Is equation (17)

$$\langle J \rangle = \frac{2\bar{k}k_B T \lambda + D_F}{2\lambda b} \left(\sqrt{-B + \sqrt{b^2 + B^2}} \right), \quad (17)$$

in which $b = (\gamma + k\lambda)\sqrt{4\gamma\bar{k}\lambda - (\gamma - k\lambda)^2}$.

equal to $\langle \dot{Q}_x \rangle$ or $\langle \dot{Q}_F \rangle$ or $\langle \dot{Q} \rangle = \langle \dot{Q}_x \rangle + \langle \dot{Q}_F \rangle$?

Analytical calculation from Prof Sarah's paper

$$\dot{Q}_0 = \sum_{j>0}^n a_{0j} \langle X_j \dot{X}_0 \rangle = \sum_{i=0}^n \sum_{j>0}^n \frac{a_{0j} a_{0i}}{\gamma_0} \langle X_j X_i \rangle = \dot{W}_0. \quad (22)$$

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Likewise, one can calculate the heat flows of the other d.o.f. $\dot{Q}_l = \sum_{i=0}^n \sum_{j \neq l} \frac{a_{lj} a_{li}}{\gamma_l} \langle X_j X_i \rangle$. It should be noted that by writing down this expression for the dissipation of $X_{j>0}$ and the total EP (21), we implicitly assume that all X_j are even under time-reversal, that means, position-like variables. In contrast, odd variables would not contribute to the total EP, see [35]. We note that for active matter models the parity of $X_{j>0}$ is in fact a nontrivial aspect, and subject of an ongoing debate, see e.g., [35, 37, 85, 86].

Together with the correlations $\langle X_i X_j \rangle$ that are derived in appendix C, equations (21), (22) represent analytical expressions for heat flow and entropy production for any n . For example, in the case $n = 1$ (which was also discussed in [57]), where the expression significantly simplifies, we find from (22), (21) in combination with (C5)

$$\dot{S}_{\text{tot}} = \frac{k_B}{T_0 T_1} \frac{(a_{10} T_0 - a_{01} T_1)^2}{-a_{00} \gamma_1 - a_{11} \gamma_0} \geq 0,$$

$$\dot{Q}_0 = k_B \frac{a_{01} (a_{10} T_0 - a_{01} T_1)}{(a_{00} \gamma_1 + a_{11} \gamma_0)}. \quad (23)$$

$$\dot{Q}_1 = k_B \frac{a_{10} (a_{01} T_1 - a_{10} T_0)}{a_{00} \gamma_1 + a_{11} \gamma_0}. \quad (24)$$



System:

$$\begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \begin{bmatrix} \zeta_0 \\ \zeta_1 \end{bmatrix}$$

$$\dot{Q}_0 = \sum_{i=0}^n \sum_{j>0} \frac{a_{0j} a_{0i}}{\gamma_0} \langle X_j X_i \rangle \quad \text{for } n=1$$

$$= \sum_{i=0}^1 \sum_{j=1}^1 \frac{a_{0j} a_{0i}}{\gamma_0} \langle X_j X_i \rangle$$

$$= \frac{a_{01} a_{00}}{\gamma_0} \langle x_1 x_0 \rangle$$

$$\text{Since } \langle x_1 x_0 \rangle \equiv \begin{pmatrix} \langle X_0^2 \rangle & \langle X_0 X_1 \rangle \\ \langle X_1 X_0 \rangle & \langle X_1^2 \rangle \end{pmatrix}$$

$$= k_B \begin{pmatrix} \frac{T_1 a_{01}^2 - T_0 a_{01} a_{10} + T_0 a_{11} (a_{00} + a_{11} \gamma_0 / \gamma_1)}{(a_{00} + a_{11} \gamma_0 / \gamma_1)(a_{01} a_{10} - a_{00} a_{11})} & \frac{-T_1 a_{00} a_{01} - T_0 a_{11} a_{10} \gamma_0 / \gamma_1}{(a_{00} + a_{11} \gamma_0 / \gamma_1)(a_{01} a_{10} - a_{00} a_{11})} \\ \frac{-T_1 a_{00} a_{01} - T_0 a_{11} a_{10} \gamma_0 / \gamma_1}{(a_{00} + a_{11} \gamma_0 / \gamma_1)(a_{01} a_{10} - a_{00} a_{11})} & \frac{T_0 a_{10}^2 \gamma_0 / \gamma_1 - T_1 a_{01} a_{10} \gamma_0 / \gamma_1 + T_1 a_{00} (a_{00} + a_{11} \gamma_0 / \gamma_1)}{(a_{00} + a_{11} \gamma_0 / \gamma_1)(a_{01} a_{10} - a_{00} a_{11})} \end{pmatrix}.$$

(C5)

$$\dot{Q}_0 = \frac{a_{01} a_{00}}{\gamma_0} \left[k_B \frac{-T_1 a_{00} a_{01} - T_0 a_{11} a_{10} \frac{\gamma_0}{\gamma_1}}{(a_{00} + a_{11} \frac{\gamma_0}{\gamma_1}) (a_{01} a_{10} - a_{00} a_{11})} \right]$$

From eqn (24)

$$\dot{Q}_0 = k_B \frac{a_{01} (a_{10} T_0 - a_{01} T_1)}{a_{00} \gamma_1 + a_{11} \gamma_0} \quad (1)$$

From Berger & Hudspeth:

$$\begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{f} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} + \begin{bmatrix} n_x \\ n_f \end{bmatrix}$$

Using equation (1):

$$(a) \dot{Q}_x = k_B \frac{(-\bar{k} T - T)}{-k \gamma - \lambda} = \frac{k_B T (-\bar{k} - 1)}{-k \gamma - \lambda}$$

So $\langle \dot{Q}_x \rangle = \langle \dot{Q}_x \rangle$ (from Prof Sarah's paper) $\dot{Q} = \langle \frac{\delta q}{\delta t} \rangle$
notation used

(b)

$$\langle \dot{Q}_F \rangle =$$

From Prof Sarah's paper:

$$\dot{Q}_l = \sum_{i=0}^l \sum_{j \neq l} \frac{a_{lj} a_{li}}{\gamma_l} \langle x_j x_i \rangle$$

$$\begin{aligned} \dot{Q}_1 &= \sum_{i=0}^1 \sum_{j \neq i} \frac{a_{ij} a_{ii}}{\gamma_1} \langle x_j x_i \rangle \\ &= \frac{a_{10} a_{11}}{\gamma_1} \langle x_0 x_1 \rangle \end{aligned}$$

$$\dot{Q}_1 = \dot{Q}_F$$

$$\dot{Q}_F = k_B \frac{a_{10} (a_{01} T_1 - a_{10} T_0)}{a_{00} \gamma_1 + a_{11} \gamma_0}$$

$$= k_B \frac{-\bar{k} ((1)T + \bar{k} T)}{-k \gamma - \lambda}$$

Since symmetric,
use \dot{Q}_0 in eq. 24
from Prof Saran's
paper and switch
 $0 \rightarrow 1$ in numerator

$$\dot{Q}_F = k_B \frac{-\bar{k} (T + \bar{k} T)}{-k \gamma - \lambda}$$

$$\dot{Q}_X = k_B \frac{(-k T - T)}{-k \gamma - \lambda}$$

Now check if $\dot{Q}_F + \dot{Q}_X = \langle S \rangle$