Perform analytical calculation of autocorrelation function and response function in Fourier space for a 2D non-reciprocal generic model.

$$\begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix}$$

$$\gamma_0 \dot{x}_0 = a_{00} x_0 + a_{01} x_1 + \xi_0 \qquad (1)$$
$$\gamma_1 \dot{x}_1 = a_{10} x_0 + a_{11} x_1 + \xi_1 \qquad (2)$$

Transform eq(1) and eq(2) to Fourier space $\quad A(\omega) = \int_{-\infty}^{\infty} dt\, a(t) e^{i\omega t}$

$$-i\omega\gamma_0 \tilde{x}_0 = a_{00}\tilde{x}_0 + a_{01}\tilde{x}_1 + \tilde{\xi}_0 \qquad (3)$$

$$-i\omega\gamma_1 \tilde{x}_1 = a_{10}\tilde{x}_0 + a_{11}\tilde{x}_1 + \tilde{\xi}_1 \qquad (4)$$

Solve for the Fourier transformed position:

Solve for $\tilde{x}_1$:

$$-i\omega\gamma_0 \tilde{x}_0 = a_{00}\tilde{x}_0 + a_{01}\tilde{x}_1 + \tilde{\xi}_0$$
$$+ \quad -i\omega\gamma_1 \tilde{x}_1 = a_{10}\tilde{x}_0 + a_{11}\tilde{x}_1 + \tilde{\xi}_1$$

$$-i\omega\gamma_0\tilde{x}_0 - i\omega\gamma_1\tilde{x}_1 = (a_{00} + a_{10})\tilde{x}_0 + (a_{01} + a_{11})\tilde{x}_1 + \tilde{\xi}_0 + \tilde{\xi}_1$$

$$i\omega\gamma_1\tilde{x}_1 + (a_{01} + a_{11})\tilde{x}_1 = -i\omega\gamma_0\tilde{x}_0 - (a_{00} + a_{10})\tilde{x}_0 - \tilde{\xi}_0 - \tilde{\xi}_1$$
$$\tilde{x}_1(i\omega\gamma_1 + a_{01} + a_{11}) = \tilde{x}_0(-i\omega\gamma_0 - a_{00} - a_{10}) - \tilde{\xi}_0 - \tilde{\xi}_1$$

$$(5) \qquad \tilde{x}_1 = \frac{\tilde{x}_0(-i\omega\gamma_0 - a_{00} - a_{10}) - \tilde{\xi}_0 - \tilde{\xi}_1}{i\omega\gamma_1 + a_{01} + a_{11}}$$

Substitute $\tilde{x}_1$ in eq (3):

$$-i\omega\gamma_0\tilde{x}_0 = a_{00}\tilde{x}_0 + a_{01}\tilde{x}_1 + \tilde{\xi}_0 \qquad (3)$$

$$-i\omega\gamma_0\tilde{x}_0 = a_{00}\tilde{x}_0 + a_{01}\left[\frac{\tilde{x}_0(-i\omega\gamma_0 - a_{00} - a_{10}) - \tilde{\xi}_0 - \tilde{\xi}_1}{i\omega\gamma_1 + a_{01} + a_{11}}\right] + \tilde{\xi}_0$$

Multiply $i\omega\gamma_1 + a_{01} + a_{11}$ on both sides of the equation:

$$-i\omega\gamma_0\tilde{x}_0(i\omega\gamma_1 + a_{01} + a_{11}) = a_{00}\tilde{x}_0(i\omega\gamma_1 + a_{01} + a_{11})$$

$$+ \tilde{x}_0(-i\omega\gamma_0 a_{01} - a_{00}a_{01} - a_{10}a_{01}) - \tilde{\xi}_0 a_{01} - \tilde{\xi}_1 a_{01}$$

$$+ \tilde{\xi}_0(i\omega\gamma_1 + a_{01} + a_{11})$$

Group terms w/ $\tilde{x}_0$ in one side of the equation:

$$\tilde{\xi}_0 a_{01} + \tilde{\xi}_1 a_{01} - \tilde{\xi}_0(i\omega\gamma_1 + a_{01} + a_{11}) =$$

$$-\omega^2\gamma_0\gamma_1\tilde{x}_0 + \cancel{i\omega\gamma_0 a_{01}\tilde{x}_0} + i\omega\gamma_0 a_{11}\tilde{x}_0$$

$$+ i\omega\gamma_1 a_{00}\tilde{x}_0 + \cancel{a_{01}a_{00}\tilde{x}_0} + a_{11}a_{00}\tilde{x}_0$$

$$- \cancel{i\omega\gamma_0 a_{01}\tilde{x}_0} - \cancel{a_{00}a_{01}\tilde{x}_0} - a_{10}a_{01}\tilde{x}_0$$

Simplify terms:

$$\tilde{\xi}_0 a_{01} + \tilde{\xi}_1 a_{01} - \tilde{\xi}_0(i\omega\gamma_1 + a_{01} + a_{11}) =$$

$$\tilde{x}_0\left(-\omega^2\gamma_0\gamma_1 + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) + a_{11}a_{00} - a_{10}a_{01}\right)$$

Finally:

$$\tilde{x}_0 = \frac{\tilde{\xi}_0 a_{01} + \tilde{\xi}_1 a_{01} - \tilde{\xi}_0(i\omega\gamma_1 + a_{01} + a_{11})}{a_{11}a_{00} - a_{10}a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1} \quad (6)$$

Sanity check if agrees w/ eq (45) in Appendix C:

$$\begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ F \end{bmatrix} + \begin{bmatrix} \eta_x \\ \eta_F \end{bmatrix}$$

$$\tilde{x}_0 = \frac{\tilde{\eta}_x(1) + \tilde{\eta}_F(1) - \tilde{\eta}_x(i\omega\lambda + \cancel{1-1})}{k + \bar{k} + i\omega(\gamma(-1) + \lambda(-k)) - \omega^2\gamma\lambda}$$

$$= \frac{\tilde{\eta}_x + \tilde{\eta}_F - \tilde{\eta}_x(i\omega\lambda)}{k + \bar{k} - i\omega(\gamma + \lambda k) - \omega^2\gamma\lambda}$$

C. Solutions of the linear model of hair bundle oscillations

We transform the model equations eq. (7) and eq. (8) to Fourier-space,

$$-i\omega\gamma\tilde{x} = -k\tilde{x} + \tilde{F} + \tilde{\eta}_x \tag{43}$$

and

$$-i\omega\lambda\tilde{F} = -\bar{k}\tilde{x} - \tilde{F} + \tilde{\eta}_F. \tag{44}$$

Now, we can solve for the Fourier-transformed position

$$\tilde{x} = \frac{\tilde{\eta}_F + \tilde{\eta}_x - i\lambda\tilde{\eta}_x\omega}{k + \bar{k} - i(\gamma + k\lambda)\omega - \gamma\lambda\omega^2}. \tag{45}$$

1. Derivation of the autocorrelation function

The averaged autocorrelation function $\tilde{C}_{xx}(\omega) = \langle\tilde{x}\tilde{x}^*\rangle$ is given by the averaged product of $\tilde{x}$ from eq. (45) with its complex-conjugate $\tilde{x}^*$,

$$\tilde{C}_{xx}(\omega) = \left\langle \frac{(\tilde{\eta}_F+\tilde{\eta}_x)^2 + \lambda^2\tilde{\eta}_x^2\omega^2}{2k\bar{k} + \gamma^2\omega^2 + (\bar{k} - \gamma\lambda\omega^2)^2 + k^2(1+\lambda^2\omega^2)} \right\rangle. \tag{46}$$

Averaged autocorrelation function

$$C_{xx}(\omega) = \langle \tilde{x}\,\tilde{x}^* \rangle$$

$$C_{xx}(\omega) = \langle \tilde{x}_0\,\tilde{x}_0^* \rangle \; ?$$

how about x_1?

Further simplify the Fourier-transformed position eq. (6)

$$\tilde{x}_0 = \frac{\tilde{\xi}_0 a_{01} + \tilde{\xi}_1 a_{01} - \tilde{\xi}_0(i\omega\gamma_1 + a_{01} + a_{11})}{a_{11}a_{00} - a_{10}a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1} \tag{6}$$

$$\tilde{x}_0 = \frac{\tilde{\xi}_0(\cancel{a_{01} - a_{01}} - a_{11}) + \tilde{\xi}_1 a_{01} - i\omega\gamma_1\tilde{\xi}_0}{a_{11}a_{00} - a_{10}a_{01} + i\omega\gamma_0 a_{11} + i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1}$$

Solve for complex conjugate:

$$\tilde{x}_0^* = \frac{-a_{11}\tilde{\xi}_0 + \tilde{\xi}_1 a_{01} + i\omega\gamma_1\tilde{\xi}_0}{a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0 a_{11} - i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1} \tag{7}$$

then:

$\langle \tilde{x}_0 \tilde{x}_0^* \rangle$:

Numerator:

$$\left(-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 - i\omega\gamma_1\tilde{\xi}_0\right)\left(-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 + i\omega\gamma_1\tilde{\xi}_0\right)$$

$$= -a_{11}\tilde{\xi}_0\left(-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 + i\omega\gamma_1\tilde{\xi}_0\right)$$
$$+ a_{01}\tilde{\xi}_1\left(-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 + i\omega\gamma_1\tilde{\xi}_0\right)$$
$$- i\omega\gamma_1\tilde{\xi}_0\left(-a_{11}\tilde{\xi}_0 + a_{01}\tilde{\xi}_1 + i\omega\gamma_1\tilde{\xi}_0\right)$$

$$= a_{11}^2\tilde{\xi}_0^2 - a_{01}a_{11}\tilde{\xi}_0\tilde{\xi}_1 - \cancel{ia_{11}\omega\gamma_1\tilde{\xi}_0^2}$$
$$- a_{01}a_{11}\tilde{\xi}_1\tilde{\xi}_0 + a_{01}^2\tilde{\xi}_1^2 + \cancel{ia_{01}\omega\gamma_1\tilde{\xi}_0\tilde{\xi}_1}$$
$$+ \cancel{ia_{11}\omega\gamma_1\tilde{\xi}_0^2} - \cancel{a_{01}i\omega\gamma_1\tilde{\xi}_0\tilde{\xi}_1} + \omega^2\gamma_1^2\tilde{\xi}_0^2$$

$$= a_{11}^2\tilde{\xi}_0^2 + a_{01}^2\tilde{\xi}_1^2 - 2a_{01}a_{11}\tilde{\xi}_1\tilde{\xi}_0 + \omega^2\gamma_1^2\tilde{\xi}_0^2$$

$$= \left(a_{11}\tilde{\xi}_0 - a_{01}\tilde{\xi}_1\right)^2 + \omega^2\gamma_1^2\tilde{\xi}_0^2$$

Sanity check numerator:

$$= \left((-1)\tilde{\eta}_x - \tilde{\eta}_F\right)^2 + \omega^2\lambda^2\tilde{\eta}_x^2 = \left(-\eta_x - \eta_F\right)^2 + \omega^2\lambda^2\tilde{\eta}_x^2$$

✓

Denominator:

$$\tilde{x}_0 = \frac{\tilde{\xi}_0\left(\cancel{a_{01}} - a_{01} - a_{11}\right) + \tilde{\xi}_1 a_{01} - i\omega\gamma_1\tilde{\xi}_0}{a_{11}a_{00} - a_{10}a_{01} + i\omega\gamma_0 a_{11} + i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1}$$

Solve for complex conjugate:

$$\tilde{x}_0^* = \frac{-a_{11}\tilde{\xi}_0 + \tilde{\xi}_1 a_{01} + i\omega\gamma_1\tilde{\xi}_0}{a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0 a_{11} - i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1} \quad (7)$$

$$(a_{11}a_{00} - a_{10}a_{01} + i\omega\gamma_0 a_{11} + i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1)$$
$$(a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0 a_{11} - i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1)$$

$$
\begin{aligned}
= \; & a_{11}a_{00}(a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0 a_{11} - i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1) \\
& - a_{10}a_{01}(a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0 a_{11} - i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1) \\
& + i\omega\gamma_0 a_{11}(a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0 a_{11} - i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1) \\
& + i\omega\gamma_1 a_{00}(a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0 a_{11} - i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1) \\
& - \omega^2\gamma_0\gamma_1(a_{11}a_{00} - a_{10}a_{01} - i\omega\gamma_0 a_{11} - i\omega\gamma_1 a_{00} - \omega^2\gamma_0\gamma_1)
\end{aligned}
$$

$$
\begin{aligned}
= \; & a_{11}^2 a_{00}^2 - a_{11}a_{00}a_{10}a_{01} \overset{1}{\cancel{- i a_{11}^2 a_{00}\omega\gamma_0}} \overset{2}{\cancel{- i a_{00}^2 a_{11}\omega\gamma_1}} - a_{11}a_{00}\omega^2\gamma_0\gamma_1 \\
& - a_{11}a_{00}a_{10}a_{01} + a_{10}^2 a_{01}^2 \overset{3}{\cancel{+ i a_{10}a_{01}a_{11}\omega\gamma_0}} \overset{4}{\cancel{+ i a_{10}a_{01}a_{00}\omega\gamma_1}} + a_{10}a_{01}\omega^2\gamma_0\gamma_1 \\
& \overset{1}{\cancel{+ i a_{11}^2 a_{00}\omega\gamma_0}} \overset{3}{\cancel{- i a_{10}a_{01}a_{11}\omega\gamma_0}} + \omega^2\gamma_0^2 a_{11}^2 + \omega^2\gamma_1\gamma_0 a_{00}a_{11} \overset{6}{\cancel{- i\omega^3\gamma_0^2\gamma_1 a_{11}}} \\
& \overset{2}{\cancel{+ i a_{00}^2 a_{11}\omega\gamma_1}} \overset{4}{\cancel{- i a_{00}a_{10}a_{01}\omega\gamma_1}} + \omega^2\gamma_0\gamma_1 a_{00}a_{11} + \omega^2\gamma_1^2 a_{00}^2 \overset{7}{\cancel{- i\omega^3\gamma_0\gamma_1^2 a_{00}}} \\
& - a_{11}a_{00}\omega^2\gamma_0\gamma_1 + a_{10}a_{01}\omega^2\gamma_0\gamma_1 \overset{6}{\cancel{+ i\omega^3\gamma_0^2\gamma_1 a_{11}}} \overset{7}{\cancel{+ i\omega^3\gamma_0\gamma_1^2 a_{00}}} + \omega^4\gamma_0^2\gamma_1^2
\end{aligned}
$$

$$
\begin{aligned}
= \; & a_{11}^2 a_{00}^2 - 2a_{11}a_{00}a_{10}a_{01} + a_{10}^2 a_{01}^2 + 2a_{10}a_{01}\omega^2\gamma_0\gamma_1 \\
& - 2a_{11}a_{00}\omega^2\gamma_0\gamma_1 + \omega^2\gamma_0^2 a_{11}^2 + 2\omega^2\gamma_1\gamma_0 a_{00}a_{11} + \omega^2\gamma_1^2 a_{00}^2 \\
& + \omega^4\gamma_0^2\gamma_1^2
\end{aligned}
$$

$$
\begin{aligned}
= \; & -2a_{11}a_{00}a_{10}a_{01} + \omega^2\gamma_0^2 a_{11}^2 + (a_{10}a_{01} + \gamma_0\gamma_1\omega^2)^2 \\
& + a_{11}^2 a_{00}^2 \cancel{- 2a_{11}a_{00}\omega^2\gamma_0\gamma_1} \cancel{+ 2a_{00}a_{11}\omega^2\gamma_0\gamma_1} \\
& + \omega^2\gamma_1^2 a_{00}^2
\end{aligned}
$$

$$
\begin{aligned}
= \; & -2a_{11}a_{00}a_{10}a_{01} + \omega^2\gamma_0^2 a_{11}^2 + (a_{10}a_{01} + \gamma_0\gamma_1\omega^2)^2 \\
& + a_{00}^2(a_{11}^2 + \omega^2\gamma_1^2)
\end{aligned}
$$

Sanity check (for denominator)

$$= -2(-1)(-k)(-\bar{k})(1) + \omega^2\gamma^2(-1)^2 + (-\bar{k} - \gamma\lambda\omega^2)^2 + (-k)^2\left((-1)^2 + \omega^2\lambda^2\right)$$

$$= 2k\bar{k} + \gamma^2\omega^2 + (-k)^2 - 2\bar{k}\gamma\lambda\omega^2 + \gamma^2\lambda^2\omega^4 + k^2(1+\omega^2\lambda^2)$$

$$= 2k\bar{k} + \gamma^2\omega^2 + k^2 - 2\bar{k}\gamma\lambda\omega^2 + \gamma^2\lambda^2\omega^4 + k^2(1+\omega^2\lambda^2)$$

✓ $$= 2k\bar{k} + \gamma^2\omega^2 + (\bar{k} - \gamma\lambda\omega^2)^2 + k^2(1+\omega^2\lambda^2)$$

Finally:

$$\tilde{C}_{xx}(\omega) = \frac{(a_{11}\tilde{\xi}_0 - a_{01}\tilde{\xi}_1)^2 + \omega^2\gamma_1^2\tilde{\xi}_0^2}{-2a_{11}a_{00}a_{10}a_{01} + \omega^2\gamma_0^2 a_{11}^2 + (a_{10}a_{01} + \gamma_0\gamma_1\omega^2)^2 + a_{00}^2(a_{11}^2 + \omega^2\gamma_1^2)}$$

Note w/ Prof Edgar:

$$\langle \bar{\xi}(t)\,\xi(0)\rangle = 2k_BT\gamma\,\delta(t-t')$$

Fourier:

$$|\tilde{\xi}(\omega)|^2 = 2k_BT\gamma$$

$$\int e^{i\omega t}\,\delta(t-t')\,dt = 1$$

- no coupling $a_{01} = a_{10} = 0 \Rightarrow \tilde{h}(\omega) = 0$
- coupling reciprocal

$$a_{01} = a_{10} \neq 0 \Rightarrow \tilde{h}(\omega) = ?$$

- nonreciprocal coupling $a_{01} \neq a_{10}$

$$a_{01} \in \mathbb{R}, \quad a_{10} \in \mathbb{R}$$

$$\Rightarrow \tilde{h}(\omega) \neq 0$$