Perform analytical calculation for response function in Fourier space for a 2D non-reciprocal generic model

$$\begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix}$$

$$\gamma_0 \dot{x}_0 = a_{00} x_0 + a_{01} x_1 + \xi_0$$

$$\gamma_1 \dot{x}_1 = a_{10} x_0 + a_{11} x_1 + \xi_1$$

2. Derivation of the response function

To obtain the response function, we perturb the position coordinate with a δ-function in the time domain, resulting in the following equations in Fourier space,

$$-i\omega\gamma\tilde{x}_\delta = -k\tilde{x}_\delta + \tilde{F} + \tilde{\eta}_x + 1 \quad (48)$$

and

$$-i\omega\lambda\tilde{F} = -\bar{k}\tilde{x}_\delta - \tilde{F} + \tilde{\eta}_F. \quad (49)$$

As from the paper, they perturbed the position coordinate w/ a δ-function in the time domain to obtain the responce function. Following their procedure, we have:

$$\gamma_0 \dot{x}_0 = a_{00} x_0 + a_{01} x_1 + \xi_0 + \delta(t) \quad (1)$$

$$\gamma_1 \dot{x}_1 = a_{10} x_0 + a_{11} x_1 + \xi_1 \quad (2)$$

transforming eq(1) and eq (2) to Fourier space,

$$-i\omega\gamma_0 \tilde{x}_{0\delta} = a_{00}\tilde{x}_0 + a_{01}\tilde{x}_1 + \tilde{\xi}_0 + \tilde{\delta}(t) \quad (3)$$

$$-i\omega\gamma_1 \tilde{x}_1 = a_{10}\tilde{x}_0 + a_{11}\tilde{x}_1 + \tilde{\xi}_1 \quad (4)$$

Since $\tilde{\delta}(t) = \int_{-\infty}^{\infty} \delta(t) e^{i\omega t} dt = 1$

then we have:

$$-i\omega\gamma_0 \tilde{x}_{0\delta} = a_{00}\tilde{x}_0 + a_{01}\tilde{x}_1 + \tilde{\xi}_0 + 1 \qquad (5)$$

$$-i\omega\gamma_1 \tilde{x}_1 = a_{10}\tilde{x}_0 + a_{11}\tilde{x}_1 + \tilde{\xi}_1 \qquad (6)$$

solving for the Fourier-transformed position:

solve for $\tilde{x}_1$:

$$-i\omega\gamma_0 \tilde{x}_{0\delta} = a_{00}\tilde{x}_0 + a_{01}\tilde{x}_1 + \tilde{\xi}_0 + 1$$

$$+ \quad -i\omega\gamma_1 \tilde{x}_1 = a_{10}\tilde{x}_0 + a_{11}\tilde{x}_1 + \tilde{\xi}_1$$

$$-i\omega\gamma_0 \tilde{x}_0 - i\omega\gamma_1\tilde{x}_1 = (a_{00}+a_{10})\tilde{x}_0 + (a_{01}+a_{11})\tilde{x}_1 + \tilde{\xi}_0 + \tilde{\xi}_1 + 1$$

$$i\omega\gamma_1\tilde{x}_1 + (a_{01}+a_{11})\tilde{x}_1 = -i\omega\gamma_0\tilde{x}_0 - (a_{00}+a_{10})\tilde{x}_0 - \tilde{\xi}_0 - \tilde{\xi}_1 - 1$$

$$\tilde{x}_1(i\omega\gamma_1 + a_{01} + a_{11}) = \tilde{x}_0(-i\omega\gamma_0 - a_{00} - a_{10}) - \tilde{\xi}_0 - \tilde{\xi}_1 - 1$$

$$\tilde{x}_1 = \frac{\tilde{x}_0(-i\omega\gamma_0 - a_{00} - a_{10}) - \tilde{\xi}_0 - \tilde{\xi}_1 - 1}{i\omega\gamma_1 + a_{01} + a_{11}}$$

Substitute $\tilde{x}_1$ in eq (5):

$$-i\omega\gamma_0\tilde{x}_0 = a_{00}\tilde{x}_0 + a_{01}\tilde{x}_1 + \tilde{\xi}_0 + 1$$

$$-i\omega\gamma_0\tilde{x}_0 = a_{00}\tilde{x}_0 + a_{01}\left[\frac{\tilde{x}_0(-i\omega\gamma_0 - a_{00} - a_{10}) - \tilde{\xi}_0 - \tilde{\xi}_1 - 1}{i\omega\gamma_1 + a_{01} + a_{11}}\right] + \tilde{\xi}_0 + 1$$

Multiply $i\omega\gamma_1 + a_{01} + a_{11}$ on both sides of the equation

$$-i\omega\gamma_0 \tilde{x}_0 (i\omega\gamma_1 + a_{01} + a_{11}) = a_{00}\tilde{x}_0 (i\omega\gamma_1 + a_{01} + a_{11})$$
$$+ \tilde{x}_0 (-i\omega\gamma_0 a_{01} - a_{00} a_{01} - a_{10} a_{01}) - \tilde{\xi}_0 a_{01} - \tilde{\xi}_1 a_{01}$$
$$-(1) a_{01} + \tilde{\xi}_0 (i\omega\gamma_1 + a_{01} + a_{11}) + 1(i\omega\gamma_1 + a_{01} + a_{11})$$

Group terms w/ $\tilde{x}_0$ in one side of the equation:

$$\tilde{\xi}_0 a_{01} + \tilde{\xi}_1 a_{01} + (1) a_{01} - \tilde{\xi}_0 (i\omega\gamma_1 + a_{01} + a_{11}) - 1(i\omega\gamma_1 + a_{01} + a_{11}) =$$

$$-\omega^2\gamma_0\gamma_1\tilde{x}_0 + \cancel{i\omega\gamma_0 a_{01} \tilde{x}_0} + i\omega\gamma_0 a_{11}\tilde{x}_0$$
$$+ i\omega\gamma_1 a_{00}\tilde{x}_0 + \cancel{a_{01} a_{00} \tilde{x}_0} + a_{11} a_{00} \tilde{x}_0$$
$$- \cancel{i\omega\gamma_0 a_{01} \tilde{x}_0} - \cancel{a_{01} a_{00} \tilde{x}_0} - a_{10} a_{01} \tilde{x}_0$$

Simplify terms:

$$\tilde{\xi}_0 a_{01} + \tilde{\xi}_1 a_{01} + a_{01} - \tilde{\xi}_0 (i\omega\gamma_1 + a_{01} + a_{11}) - 1(i\omega\gamma_1 + a_{01} + a_{11}) =$$

$$\tilde{x}_0 (-\omega^2\gamma_0\gamma_1 + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) + a_{11} a_{00} - a_{10} a_{01})$$

Finally:

$$\tilde{x}_{0\delta} = \frac{\tilde{\xi}_0 a_{01} + \tilde{\xi}_1 a_{01} + a_{01} - \tilde{\xi}_0 (i\omega\gamma_1 + a_{01} + a_{11}) - 1(i\omega\gamma_1 + a_{01} + a_{11})}{a_{11} a_{00} - a_{10} a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1}$$

Averaged position:

$$\langle \tilde{x}_{0\delta} \rangle = \left\langle \frac{\tilde{\xi}_0 a_{01} + \tilde{\xi}_1 a_{01} + a_{01} - \tilde{\xi}_0 (i\omega\gamma_1 + a_{01} + a_{11}) - 1(i\omega\gamma_1 + a_{01} + a_{11})}{a_{11} a_{00} - a_{10} a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1} \right\rangle$$

Sanity check if agrees w/ eq (50) in Appendix C

$$\begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ F \end{bmatrix} + \begin{bmatrix} \eta_x \\ \eta_F \end{bmatrix} \qquad \begin{matrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{matrix}$$

$$\langle \tilde{x}_{0\delta} \rangle = \left\langle \frac{\tilde{\eta}_x (1) + \tilde{\eta}_F (1) + 1 - \tilde{\eta}_x (i\omega\lambda + \cancel{1} - 1) - 1 (i\omega\lambda + \cancel{1 - 1})}{k - (-\bar{k})(1) + i\omega(\gamma(-1) + \lambda(-k)) - \omega^2 \gamma\lambda} \right\rangle$$

$$= \left\langle \frac{\tilde{\eta}_x + \tilde{\eta}_F + 1 - \tilde{\eta}_x (i\omega\lambda) - i\omega\lambda}{k + \bar{k} - i\omega(\gamma + \lambda k) - \omega^2 \gamma\lambda} \right\rangle$$

$$= \left\langle \frac{\tilde{\eta}_x + \tilde{\eta}_F + 1 - i\omega\lambda(\tilde{\eta}_x + 1)}{k + \bar{k} - i\omega(\gamma + \lambda k) - \omega^2 \gamma\lambda} \right\rangle$$

Finally:

$$\tilde{R}_x = \langle \tilde{x}_{0\delta} \rangle = \left\langle \frac{\tilde{\xi}_0 a_{01} + \tilde{\xi}_1 a_{01} + a_{01} - \tilde{\xi}_0 (i\omega\gamma_1 + a_{01} + a_{11}) - 1 (i\omega\gamma_1 + a_{01} + a_{11})}{a_{11} a_{00} - a_{10} a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2 \gamma_0 \gamma_1} \right\rangle$$

$$\tilde{R}_x = \frac{\cancel{a_{01}} - i\omega\gamma_1 - \cancel{a_{01}} - a_{11}}{a_{11} a_{00} - a_{10} a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2 \gamma_0 \gamma_1}$$

Sanity check if similar to eqn (51) in paper:

$$\begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ F \end{bmatrix} + \begin{bmatrix} \eta_x \\ \eta_F \end{bmatrix} \qquad \begin{matrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{matrix}$$

$$\tilde{R}_x = \frac{-i\omega\lambda + 1}{(-1)(-k) - (-\bar{k})(1) + i\omega(\gamma(-1) + \lambda(-k)) - \omega^2 \gamma\lambda}$$

$$= \frac{1 - i\omega\lambda}{k + \bar{k} + i\omega(-\gamma - \lambda k) - \omega^2 \gamma\lambda}$$

$$= \frac{1 - i\omega\lambda}{k + \bar{k} - i\omega(\gamma + \lambda k) - \omega^2\gamma\lambda}$$

To get real and imaginary part, multiply $\tilde{R}_x$ by its complex conjugate (for both numerator and denominator)

$$\tilde{R}_x = \frac{-i\omega\gamma_1 - a_{11}}{a_{11}a_{00} - a_{10}a_{01} + i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1} \cdot \frac{(a_{11}a_{00} - a_{10}a_{01} - i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1)}{(a_{11}a_{00} - a_{10}a_{01} - i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1)}$$

Numerator:

$$= -i\omega\gamma_1(a_{11}a_{00} - a_{10}a_{01} - i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1)$$
$$- a_{11}(a_{11}a_{00} - a_{10}a_{01} - i\omega(\gamma_0 a_{11} + \gamma_1 a_{00}) - \omega^2\gamma_0\gamma_1)$$

$$= -i\omega\gamma_1 a_{11}a_{00} + i\omega\gamma_1 a_{10}a_{01} - \omega^2\gamma_1\gamma_0 a_{11} - \omega^2\gamma_1^2 a_{00} + i\omega^3\gamma_0\gamma_1^2$$
$$- a_{11}^2 a_{00} + a_{10}a_{01}a_{11} + i\omega\gamma_0 a_{11}^2 + i\omega\gamma_1 a_{00}a_{11} + \omega^2\gamma_0\gamma_1 a_{11}$$

Denominator: see autocorr calculation of $\tilde{X}_0 \tilde{X}_0^*$

then we have:

$$-2a_{11}a_{00}a_{10}a_{01} + \omega^2\gamma_0^2 a_{11}^2 + (a_{10}a_{01} + \gamma_0\gamma_1\omega^2)^2 + a_{00}^2(a_{11}^2 + \omega^2\gamma_1^2)$$

Obtain real part and imaginary part of numerator:

Divide terms w/ i and w/out i

Real part:

$$= -\omega^2 \cancel{\gamma_1\gamma_0 a_{11}} - \omega^2\gamma_1^2 a_{00} - a_{11}^2 a_{00} + a_{10}a_{01}a_{11} + \cancel{\omega^2\gamma_0\gamma_1 a_{11}}$$
$$= -\omega^2\gamma_1^2 a_{00} - a_{11}^2 a_{00} + a_{10}a_{01}a_{11}$$

Imaginary part:

isolate i:

$$= i(\cancel{-\omega\gamma_1 a_{11}a_{00}} + \omega\gamma_1 a_{10}a_{01} + \omega^3\gamma_0\gamma_1^2 + \omega\gamma_0 a_{11}^2 \cancel{+ \omega\gamma_1 a_{00}a_{11}})$$
$$= \omega(\gamma_1 a_{10}a_{01} + \gamma_0\gamma_1^2\omega^2 + a_{11}^2\gamma_0)$$

Perform sanity check for real part:

Sanity check:

$$\begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix} \begin{bmatrix} x \\ F \end{bmatrix} + \begin{bmatrix} \eta_x \\ \eta_F \end{bmatrix} \qquad \begin{matrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{matrix}$$

$$= -w^2\lambda\gamma(-1) - w^2\lambda^2(-k) - (-1)^2(-k) + (-\bar{k})(1)(-1) + w^2\gamma\lambda(-1)$$

$$= \cancel{w^2\lambda\gamma} + w^2\lambda^2 k + k + \bar{k} - \cancel{w^2\gamma\lambda}$$

$$= k + \bar{k} + w^2\lambda^2 k \checkmark$$

Perform sanity check for imaginary part:

Sanity check:

$$\begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -k & 1 \\ -\bar{k} & -1 \end{bmatrix}\begin{bmatrix} x \\ F \end{bmatrix} + \begin{bmatrix} \eta_x \\ \eta_F \end{bmatrix} \qquad \begin{matrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{matrix}$$

Imaginary part is:

$$= w(\gamma_1 a_{10} a_{01} + \gamma_0 \gamma_1^2 w^2 + a_{11}^2 \gamma_0)$$

so we have:

$$= w(\lambda(-\bar{k})(1) + \gamma\lambda^2 w^2 + (-1)^2\gamma)$$

$$= w(\gamma - \lambda\bar{k} + \gamma\lambda^2 w^2)$$

Then, the averaged complex response is:

$$\vec{R}_x = \frac{-iw\gamma_1 - a_{11}}{a_{11}a_{00} - a_{10}a_{01} + iw(\gamma_0 a_{11} + \gamma_1 a_{00}) - w^2\gamma_0\gamma_1}$$

with real part:

$$\tilde{R}'_x(w) = \frac{-w^2\gamma_1^2 a_{00} - a_{11}^2 a_{00} + a_{10}a_{01}a_{11}}{-2a_{11}a_{00}a_{10}a_{01} + w^2\gamma_0^2 a_{11}^2 + (a_{10}a_{01} + \gamma_0\gamma_1 w^2)^2 + a_{00}^2(a_{11}^2 + w^2\gamma_1^2)}$$

and imaginary part:

$$\tilde{R}''_x(w) = \frac{w(\gamma_1 a_{10} a_{01} + \gamma_0\gamma_1^2 w^2 + a_{11}^2\gamma_0)}{-2a_{11}a_{00}a_{10}a_{01} + w^2\gamma_0^2 a_{11}^2 + (a_{10}a_{01} - \gamma_0\gamma_1 w^2)^2 + a_{00}^2(a_{11}^2 + w^2\gamma_1^2)}$$