Physical interactions between mutually coupled particles are typically reciprocal, and fulfills Newton’s third law action = reaction. An example of this in a macroscopic scale is a book resting on the ground. However, the idea of reciprocal coupling breaks down in many complex living systems. Birds in flock can easily demonstrate this since birds have vision cones. The red bird sees the blue bird and responds to the movement of the blue bird, but the blue bird doesn’t react to the red bird, simply because it does not see the red bird. And so the interactions between the blue bird and the red bird are not reciprocal.

In small-scale systems which is different from the birds who are macroscopic, these non-reciprocal interactions are also present. An example is hair bundles. These hair bundles oscillate spontaneously . These oscillations can be described by the interaction between the position of the hair bundle and the active force generated by the molecular motors which are non-reciprocal. Another example is a sensor, which in this case is a cell receptor (in violet) that measures an external stochastic signal like ligands (in green). The sensor depends on the state of the external ligand, but not the other way around. We will talk about these models in details later. Our goal is to understand what are the thermodynamic implications of systems that exhibit non-reciprocal interactions.

We now proceed to the theory.

These small-scale systems are described by overdamped Langevin equations. In this work, we only consider two degrees of freedom x0 and x1. The simplest case that we have as an example are two particles, each is trapped in a potential having their own independent heat baths. These 2 systems are uncoupled. The second case is when these 2 systems are coupled by a spring. In this case, the coupling between them is reciprocal. The third case and the more general case is when the coupling between the 2 particles are non-reciprocal – forces a01x1 and a10x0 are acting on particles x0 and x1 respectively.

When a10 = a01, the coupling is reciprocal and we can express equation 4 as gamma x dot equal to derivate of a potential + noise

However, when the coupling is non-reciprocal, then equation 4 cannot be expressed as derivatives of potential plus noise. In this case, a01x1 and a10x0 appear as non-conservative forces. Thus, the dynamics of x0 and x1 is not conservative in the presence of nonreciprocal coupling. We remember that the work done by a conservative force on a “closed path” is zero. And is non-zero for a non conservative force. When external non-conservative forces act on the particle, they prevent the system to reach equilibrium, because they continuously pump energy into the system. From the perspective of thermodynamics, ultimately, this energy is dissipated in the form of heat. Thus, a finite energy dissipation or heat flow is a signature of non-equilibrium.

We studied 2 measures of non-equilibrium. The heat rate and the violation of the fluctuation-dissipation theorem. We start with the heat rate. Ken Sekimoto extended the notions of heat and work that we learned in thermodynamics to systems described by Langevin equation. Stochastic heat is the energy exchanged between the system and the thermal environment. In Equation 9, We consider an overdamped langevin equation in 1D where the sum of all forces acting on the particle is 0. The forces exerted by the thermal environment on the particle, are the friction force and the noise term. Therefore, the energy transfer from the environment to the particle is given in equation 10. However, the gaussian white noise cannot be measured experimentally in the laboratory. Instead we write the stochastic heat as shown in Equation 11.

Considering our 2-dimensional generic model, the stochastic heat dissipated for particle x0 and x1 in the small interval t, t+dt is shown in equation 12.

We compute for the heat rate numerically by taking the slope of its average cumulative stochastic heat. We can also compute for the heat rate analytically as shown in Equation 14.

The second measure of non-equilibrium is the violation of the FDT. One can embed microscopic probe particles or beads (this color yellow) in the image in soft viscoelastic environments like cells and gels to study the micromechanical properties of these viscoelastic materials. In the absence of an applied force, the fluctuations of the bead can be characterized by the autocorrelation function as the average of the product over many measurements as shown in Equation 15. This same bead can also be used to extract the linear response function by measuring the average displacement induced by a small force. In systems at thermal equilibrium, these two quantities are related through the Fluctuation-Dissipation theorem (FDT). That is the fourier transformed position autocorrelation function, and the imaginary part of the fourier transformed response function multiplied by a factor of 2kbt/w. And so when an active force is present on the bath that drives the system away from equilibrium, then fdt is broken. Harada and Sasa found that for a class of Langevin equations, the rate of energy dissipation from the system to the environment is related to the extent of the violation of FDT. Looking at the equation we can see that when the quantity inside the parenthesis is zero, then energy dissipation is zero and the system is in equilibrium. For a non-equilibrium system, the energy dissipation is nonzero.

We now discuss our results. We start with our 1st biophysical model which is the hair bundle oscillations. Previous researchers studied the oscillations of hair bundles experimentally by tracking the dynamics of hair bundle. In order to do this, a flexible glass fiber was attached to the tip of the bundle to measure the position autocorrelation function and the corresponding reaction to periodic external stimuli (as shown in figure a). These hair bundles are made up of several stereocilia and there is a tip link that connects two stereocilias ashown in figure b. When sound enters the ear the stereocilia bend at their base and the hair bundle is deflected. Due to the increased tip link tension caused by the deflection, a transduction channel is pulled open at each end. To restore the resting tension, molecular motors slip or climb - a process called adaptation. These researchers have found that there is an internal energy source keeping the system away from equilibrium which is the force generated by the molecular motors.

From this experimental work, Berger, et. al [8] introduced a model that can adequately describe the hair bundle oscillations.

X is the position of the hair bundle, where γ is an effective drag coefficient, k a stiffness, and F an active driving force that is generated in the bundle. Molecular motors generate the active force F inside the bundle where λ is the relaxation time, ¯k is the coupling constant, the etas are noise terms.

We immediately see that the interaction between x and F are non-reciprocal. We also compare this same system to the uncoupled case which is the equilibrium limit where we set a10 and a01 as zero. The parameters that we used here are from the paper of Berger. First we show the results of the autocorrelation function that we have derived analytically and also computed numerically. We observed that for the equilibrium case, a peak is observed at 0 Hz, in contrast to the non-equilibrium case at 8 Hz. This result agrees with Berger, et.al who has shown in their paper the results from their experimental data and fit. For the response function, we were only able to do the analytical calculation and we showed that the real part of the response function shows a peak at 8 Hz and a sign change at 8 Hz. Which is also in agreement with Berger’s results.

Next we compare the heat rate by taking the slope of the average cumulative stochastic heat. As we would expect, there is a non-zero heat rate for the non-equilibrium case. Next when we plot the position autocorrelation function and the imaginary part of the response function x 2kbt/omega, we observed that for the equilibrium case, these two plots overlap completely. Whereas for the non-equilibrium case, the autocorrelation function is way larger than the response function. And so the energy dissipation is computed by taking the area between these two curves. This is zero for the equilibrium case as we would expect and non-zero for the non-equilibrium case. We also point out that the values of the heat rate for x and the energy dissipation are in close agreement with each other. Now we would like to ask if the energy dissipation from the violation of FDT is equal to the heat rate of the degree of freedom that we have perturbed, or is energy dissipation J equal to the sum of the heat rate of both degrees of freedom.

The second biophysical model is the cellular sensor model. The sensor r is related to the number of bound receptors where ωr indicates the rate of binding to a ligand. The sensor measures the signal x which is the external ligand concentration, where ωx indicates the rate of ligand concentration. the interaction between x0 and x1 is unidirectionally coupled, with the receptor sensing the ligand , but not the other way around, hence a10 = 0, and also therefore at non-equilibrium. The etas are noise terms. In the interest of time, I just wanted to show that heat rate for r is non-zero, and the energy dissipation J matches the heat rate for r.

The last thing that we did is to vary the degree of reciprocity between the 2 systems and to check the behavior of the heat rate of x0 and energy dissipation. In this case we use our 2D generic model. We fixed the value of a10 to -0.5, and we vary a01 from -0.5 to 0. We are in the isothermal condition where the two heat baths have the same temperature.

First we observe that when a01 = -0.5, the system has reciprocal coupling and so there is no energy dissipation. When a01 = 0, x0 becomes uncoupled to x1 and so there is no energy dissipation. The third case is when there is non-reciprocal coupling and so we observe energy dissipation. We also observe that heat rate x0 (the blue line) is in excellent agreement with the energy dissipation J. We show this next figure to further support this claim.

Now we vary the degree of reciprocity between the 2 systems, but we also vary the temperature of one heat bath. There are two main observations. First we observe that thermal equilibrium exist in the sense that there is no energy dissipation or heat rate as long as equation 19 holds. We see this clearly in the last 2 panels in the right wherein energy dissipation is zero when a01 is around 4.3 for T1 at 350K, and energy dissipation is zero when a01 is around 3.7 for T1 at 400k. Another observation is that the plot is symmetric with respect to a01 for equation 20.

Conclusions. We have shown that for a system with non-reciprocal interaction, the value of ⟨J⟩ matches the heat flow of the degree of freedom that we have perturbed. Hair bundle oscillations show that the system is out of equilibrium. The frequency that contributes the highest to the energy dissipation is at 8 Hz, the bundle’s frequency of spontaneous oscillation. This is clearly shown when we plot the violation function h tilde. Cellular sensor model shows that the system is out of equilibrium. The frequency that contributes the highest to the energy dissipation is at 0 Hz