Tutorial 3 - 2D & 3D Geometry

Part 1: Circles and ellipses

Exercise 1

Given a circle β in $\mathbb{R}^2$ with radius $r = 4$ and center $C = (5, 1)$:

(a) What is the implicit representation of the circle?
(b) What is the parametric representation of the circle? (i.e. in the polar co-ordinate system)
(c) Give two points that lie on the circle.
(d) For a point on the circle give a line tangent to the circle at that point.
(e) Shooting a ray from the origin in the direction $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where does it intersect the circle?
(f) Represent your answer of (c) in the parametric co-ordinate for the circle.

Exercise 2

Given a circle C_1 in $\mathbb{R}^2$ with radius $r_1 = 5$ and center $c_2 = (4, 6)$:

(a) What is the implicit representation of the circle?
(b) What is the parametric representation of the cycle?
(c) A ray is shot from $(0, 7)$ along the vector $\vec{v} = \begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$. Determine the points P_1 and P_2 where the ray intersects the circle.
(d) If we look at the circle from point $(0, 7)$ along the vector $\vec{v}$, which of the points P_1, P_2 is visible?
(e) What is the length of the segment P that connects P_1 and P_2?
(f) What is the length of the parallel projection of P on the x-axis?

Exercise 3

Given a circle C_2 in $\mathbb{R}^2$ with radius $r = 2$ and center $c = (3, 2)$:

(a) What is the implicit representation of the circle?
(b) What is the parametric representation of the circle? (i.e. with the circular co-ordinate system)
(c) Find the points of intersection P_1 and P_2 between this circle and C_1, the circle of Exercise 2. This was not shown in class, but think about it!

Exercise 4

Given an ellipse in $\mathbb{R}^2$ with center $c = (4, 1)$, semi-major axis $a = 3$, and semi-minor axis $b = 2$.

(a) Draw the ellipse in the plane. What is the implicit representation of the ellipse?
(b) The parametric representation of an ellipse is closely related to that of a circle. What is the parametric representation of the ellipse? (i.e. with the polar coordinate system)

Part 2: Dot and cross products in 3D

Exercise 5

Given coordinates $A = (1, 2, 3)$ and $B = (4, 5, 6)$.

(a) Calculate the equation for the line ℓ going through A and B.
(b) Obtain a unit normal vector $\hat{n}$ to this line. Check your answer by explicitly calculating the distance of line ℓ to point B.
(c) Calculate the distance of line ℓ to point $C = (7, 8, 9)$.

Exercise 6

Given are the following vectors: $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, Calculate and draw[1] the following vectors in a 3D Cartesian coordinate system:

(a) $(\vec{a} \times \vec{b}) \times \vec{c}$
(b) $\vec{a} \times (\vec{b} \times \vec{c})$

Compare the results – have you obtained the same vector? Is the result what you expected?

Exercise 7

Given vectors $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(a) Draw these vectors.
(b) Calculate and draw the dot product of these vectors. What is the angle between vectors $\vec{u}$ and $\vec{v}$?
(c) Calculate and draw the cross product of these vectors. What is the angle between the newly attained vector and $\vec{u}$?

Part 3: Planes and Projections in 3D

Exercise 8

Given coordinates $A = (0, 2, 0)$, $B = (1, 3, 4)$ and $C = (-2, -2, 3)$.

(a) Determine the normalized vector going from A in the direction of B. Call this $\hat{u}$.
(b) Determine the normalized vector going from A in the direction of C. Call this $\hat{v}$.
(c) Determine a normal vector to the plane derived through A, B and C.
(d) Determine the equation of this plane.

Exercise 9

Given the plane $3x + 1y + 6z - 2 = 0$,

(a) Give the normal vector of the plane.
(b) Determine the orthogonal projection of the point $P = (1, 1, 1)$ on the plane.
(c) Determine the distance between point $P = (1, 1, 1)$ and the plane without using the formula for this purpose.
(d) Recalculate using the formula and verify your calculations in item (b).

Exercise 10

Prove that the distance between plane $Ax + By + Cz + D = 0$ and point $P = (x_P, y_P, z_P)$ can be calculated as follows:

$$\frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Exercise 11

Given coordinates $A = (-1, 1, 0)$, $B = (1, -3, 1)$ and $C = (-2, -2, -2)$,

(a) Determine the equation of the plane through A, B and C.
(b) Find an orthonormal basis in this plane.
(c) Consider a point in the plane different from A, B, C and express it as a sum of point A and a linear combination of the vectors of the basis found in (b).

Exercise 12

Given vector $\hat{n} = (1, 1, 2)$ and point $A = (1, 1, 1)$,

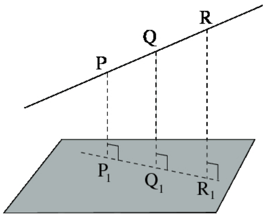
(a) Find an equation of the plane π that has $\hat{n}$ as a normal vector and contains point A.
(b) Find point P that is symmetric to point $(5, 6, 2)$ with respect to the plane π.
(c) Assume that we place an ideal mirror at point $M = (2, 0, 1)$ in π and a point light at $L = (0, 0, 3)$. Find the ray representing reflection of the light at point M.

[1] You can use GeoGebra to draw the vectors `https://www.geogebra.org`

Exercise 13

Given two points $P = (3, 4, 5)$ and $Q = (5, 8, 9)$.

(a) Calculate the line l going through P and Q in parametric form.
(b) Then find the orthographic the piece of line between P and Q on the xy-plane (see figure; their projections are P_1 and Q_1). Calculate the length of this projected piece of line.
(c) Now consider another point R on line l. Say that $QR = t$, calculate the length of Q_1R_1 in terms of t.



Exercise 14

Given two points $P = (3, 4, 5)$ and $R = (5, 8, 9)$, and camera at point $E = (4, 4, -5)$. The xy-plane is the screen.

(a) Project PR on the screen as seen by the camera (see figure) - this is perspective projection. Obtain the coordinates of P_1 and R_1.
(b) Given $PQ = l$, calculate the coordinates of point Q_1 in the xy-plane.

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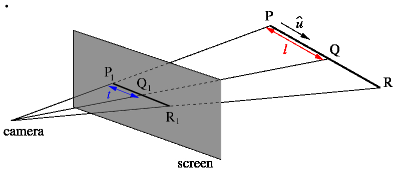

Part 4: Spheres

Exercise 15

Given sphere σ in $\mathbb{R}^3$ with radius $r = 3.14$ and center $C = (3, 5, 1)$

(a) What is the implicit representation of this sphere?
(b) What is the parametric representation of this sphere?
(c) Using the parametric representation, calculate the coordinates two opposing points on the sphere.
(d) Calculate the distance between these opposing points, and verify this using the diameter of the sphere.
(e) For one of the points give the equation of the tangent plane to the sphere.
(f) Which point on the sphere is visible from the point $(1, 1, 1)$ in the direction $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?

Exercise 16

Assume that the sphere from the previous exercise represents a mirror ball,

(a) Which point on the sphere is visible from the point $L = (1, 1, 1)$ in the direction $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$? Call this point P.
(b) Give the equation of the tangent plane to the sphere in point P.
(c) Assume that point L is a point light source, find the equation of the ray describing the direction of the light after being reflected at P.

Exercise 17

Given a sphere in $\mathbb{R}^3$ with center $C = (3, 3, 3)$ and radius 3.

(a) Find the range of parametric angles (θ, ϕ) that represents the part of the sphere viewed from $(3, 3, 3 - 3\sqrt{2})$
(b) Choose a point P on the sphere with $3 < z < 6$ and determine the the parametric form of the line that passes through this point and the origin of the sphere. Call this line l.
(c) Calculate the parametric angle (θ, ϕ) of the chosen point P with respect to the sphere.
(d) Calculate the intersection of l with the xy-plane.

Exercise 18

Note: At the moment you get the exercise sheet you will not be able to solve this exercise. Try it later, when you learn about ray tracing.

The scene consists of the eye $E = (0, 0, 0)$, the screen with corners $A = (1000, 1000, 10)$, $B = (1000, -1000, 10)$, $C = (-1000, -1000, 10)$, $D = (-1000, 1000, 10)$, the light source $L = (100, 100, 12)$, and two spheres, the blue $S_b : (x - 2)^2 + (y - 100)^2 + (z - 107)^2 = 1$ and the red $S_r : (x - 3)^2 + y^2 + (z - 100)^2 = 25$. Find what the color is of the pixel in the middle of the screen according to the ray-tracing algorithm.

4