1 Lecture 3

1.1 Partial Derivatives

In the case of one variable, we have

$$\frac{df}{dt} = \lim_{n \to 0} \frac{f(x+n) - f(x)}{n}$$

Similarily, for two of more variables, we have the following definition

Definition 1. The partial derivative of $f(x, y)$ with respect to x:

$$\frac{\partial f}{\partial x} = \lim_{n \to 0} \frac{f(x+n, y) - f(x, y)}{n}$$

Also written as f_x, for $\frac{\partial f}{\partial y}$, we have f_y

Note, the expression above is $\frac{\partial f}{\partial x}(x, y)$, which is the value at the point (x, y)

1.2 Higher order derivatives

Given $\frac{\partial f}{\partial x}$, we can take further derivatives. We have

$$\frac{\partial^2 f}{\partial^2 x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Also written as $f_{xx}, f_{yy}, f_{xy}, f_{yx}$. In most cases, f_{xy} and f_{yx} coincide.

Theorem 1. Scwartz theorem: Suppose f_{xy} and f_{yx} exist, and are continous, then

$$f_{xy} = f_{yx}$$

Similar definitions and results for the case of more variables: $x_1, \ldots, x_n$, with n variables.

1.3 Chain Rule

Suppose $f(x) = g(h(x))$, for instance

$$f(x) = (\cos x)^2 \text{ with } g(x) = x^2, h(x) = \cos x$$

then the chain rule is

$$\frac{df}{dt}(x_0) = \frac{dg}{dh}(h(x_0)) \cdot \frac{dh}{dt}(x_0)$$

Generalization to more variables.

Theorem 2. Chain rule: consider $f(x, y)$ x and y depending on a variable t. Then:

$$\frac{df}{dt} t_0 = \frac{\partial f}{\partial x}(x(t_0), y(t_0)) \frac{dx}{dt} t_0 + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \frac{dy}{dt} t_0$$

The "short form" of this result is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example. Consiter $f(x, y) = xy$, where

$$x(t) = \cos t, y(t) = \sin t$$

This cannot be computed directly with $\frac{df}{dt}$.

$$f(t) = f(x(t), y(t)) = f(\cos t, \sin t) = \cos t \cdot \sin t$$

We can compute

$$\frac{df}{dt} = (\cos t)' \sin t + \cos t (\sin t)' = -(\sin t)^2 + (\cos t)^2$$

Usin the chain rule, we get

$$\frac{df}{dt} j = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

1.4 The Gradient

Define an operation that takes a scalar function, and returns a vector function.

Definition 2. The gradient of $f(x, y)$ at (x_0, y_0) is

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$