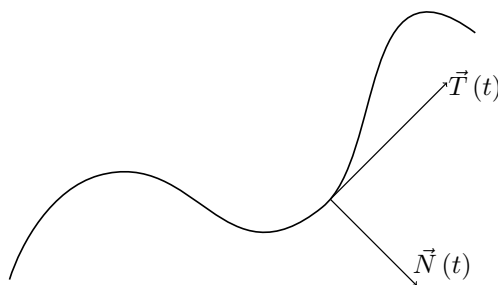


1 Lecture 9

1.1 Normal Vectors - Continued

Yesterday we saw that

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left| \frac{d\vec{T}}{dt} \right|}$$



Example. Consider the circle

$$\vec{r}(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$$

The velocity is

$$\vec{v}(t) = \vec{r}' = (-\sin t, \cos t)$$

We have $|\vec{v}(t)| = 1$, since

$$\vec{v}(t) \cdot \vec{v}(t) = (\sin^2 t + \cos^2 t) = 1$$

We find that $\vec{T}(t) = \vec{v}(t)$. To find \vec{N} , we need first

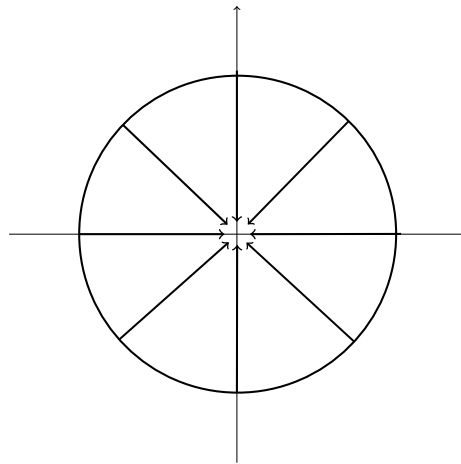
$$\frac{d\vec{T}}{dt} = \frac{d\vec{v}}{dt} = (-\cos t, -\sin t)$$

We check that $\left| \frac{d\vec{T}}{dt} \right| = 1$, then

$$\vec{N}(t) = (-\cos t, -\sin t) = -\vec{r}(t)$$

We compute more explicitly:

$$\vec{N}(t) \cdot \vec{T}(t) = (-\cos t, -\sin t) \cdot (-\sin t, \cos t) = \cos t \cdot \sin t - \sin t \cdot \cos t = 0$$



◇

We revisit the implicit case $f(x, y) = 0$.

Proposition 1. Let C be defined by $f(x, y) = 0$. A unit normal to C is given by $\vec{n} = \frac{\nabla f}{|\nabla f|}$

Proof. Suppose C is parametrized by

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

By definition of C , we have

$$f(\vec{r}(t)) = f(x(t), y(t)), \quad t_0 \leq t \leq t_1$$

We have that $\frac{d}{dt}f(\vec{r}(t)) = 0$, but using the chain rule, we get

$$\frac{df(\vec{r}(t))}{dt} = \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

Hence, we get

$$\nabla f(\vec{r}(t)) \cdot \vec{v}(t) = 0$$

Therefore $\nabla f(\vec{r}(t))$ is normal to the curve C , or orthogonal to the tangent $\vec{v}(t)$ □

Example. The circle of radius 1 can be described implicitly by $f(x, y) = 0$ with

$$f(x, y) = x^2 + y^2 - 1$$

The gradient is $\nabla f(x, y) = (2x, 2y)$, not of length 1, since

$$|\nabla f(x, y)| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2}$$

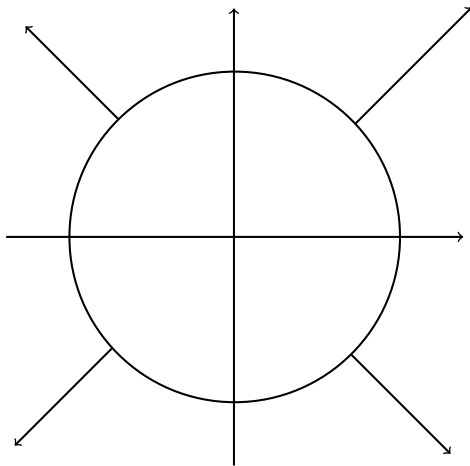
We are only interested in (x, y) such that $f(x, y) = 0$, that is $x^2 + y^2 = 1$, then

$$|\nabla f| = 2$$

Hence we get

$$\vec{n} \frac{\nabla f}{|\nabla f|} = \frac{(2x, 2y)}{2} = (x, y)$$

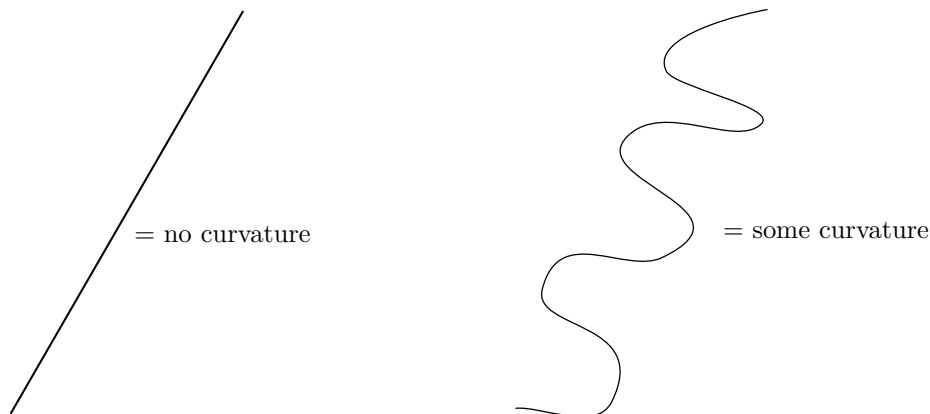
The normal vectors point outside of the circle, note: $\vec{n} = -\vec{N}$, compare with the parameter.



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1.2 Curvature

We want to compute how much a curve "curves".



How do we quantify this?

Definition 2. Let C be a parametrized curve, and \vec{T} its unit tangent vector. The curvature is then defined by

$$K = \left| \frac{d\vec{T}}{dS} \right|$$

Where S is the arc length parameter of C . We also define the radius of curvature as

$$\rho = \frac{1}{K}$$

Note: We consider S , not t . Reason: K does not depend on parametrization.

Example. Consider the following

$$\vec{r}(t) = (0, 0), \quad 0 \leq t \leq 1$$

We expect K to be zero. We can easily compute

$$S(t) = \int_0^t \sqrt{1^2 - 0^2} dt = t$$

Here we have $S(t) = t$. Now we compute \vec{T} . We have $\vec{v}(t) = (1, 0)$, and $|\vec{v}(t)| = 1$. Then $\vec{T}(t) = \vec{v}(t) = (1, 0)$. Furthermore, we get that

$$\frac{d\vec{T}}{dS} = 0 \quad \wedge \quad K = 0$$

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Example. Consider a circle of radius R with

$$\vec{r}(t) = (R \cos t, R \sin t), \quad 0 \leq t \leq 2\pi$$

The velocity is

$$\vec{v}(t) = (-R \sin t, R \cos t)$$

We get that $|\vec{v}(t)| = R$, then

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = (-\sin t, \cos t)$$

To compute K , we need to express \vec{T} in terms of $\vec{S}(t)$. We saw that

$$S(t) = R \cdot t \Rightarrow t = \frac{S}{R}$$

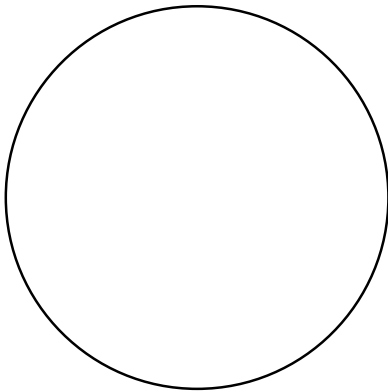
Now we compute

$$\frac{d\vec{T}}{dS} = \left(-\frac{1}{R} \cos \frac{S}{R}, -\frac{1}{R} \sin \frac{S}{R} \right)$$

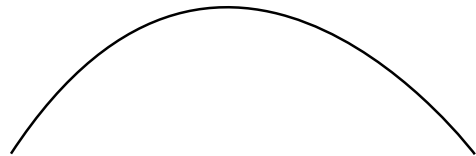
Finally, we have

$$K = \left| \frac{d\vec{T}}{dS} \right| = \sqrt{\frac{1}{R^2} \left(\cos \frac{S}{R} \right)^2 + \frac{1}{R^2} \left(\sin \frac{S}{R} \right)^2} = \frac{1}{R}$$

We have non-zero, constant curvature, we also have that $\rho = R$. Note that $K \rightarrow 0$, as $r \rightarrow \infty$



Large Curvature



Small Curvature

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Proposition 3. We have that

$$K = \frac{1}{|\vec{v}|} = \left| \frac{d\vec{T}}{dt} \right|$$

Example. Consider again

$$\vec{r}(t) = (R \cos t, R \sin t)$$

We have seen that

$$|\vec{v}(t)|, \vec{T}(t) = (-\sin t, \cos t)$$

To get K , we compute

$$\frac{d\vec{T}}{dt} = (-\cos t, -\sin t)$$

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