1 Lecture 12

1.1 Identities Between Operations

We have seen three operations defined by ∇.

$$\text{Gradient: } \nabla f, \quad \text{Divergence: } \nabla \cdot \vec{F}, \quad \text{Curl: } \nabla \times \vec{F}$$

There are many identities, we'll now look at one.

Proposition. For any scalar field f, we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$, then

$$\nabla \times (\nabla f) = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix} \right| = (f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{xy})$$

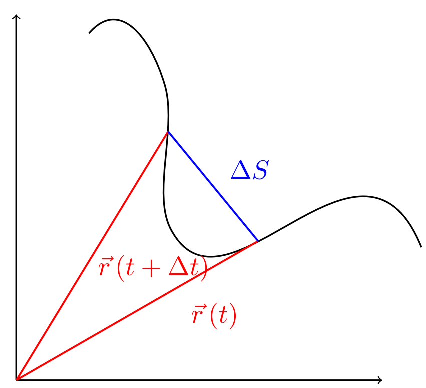
But partial derivatives can be exchanged. Then we find that $\nabla \times (\nabla f) = 0$ □

We are going to use this when we discuss conservative fields.

1.2 Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), t_0 \leq t \leq t_1$$



$$\Delta x = x(t + \Delta t) - x(t), \quad \Delta y = y(t + \Delta t) - y(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x^2) + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt}\Delta t, \quad \Delta y \approx \frac{dy}{dt}\Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt}\Delta t^2\right) + \left(\frac{dy}{dt}\Delta t^2\right)} = \Delta t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Letting $\Delta t \to 0$ leads to the following

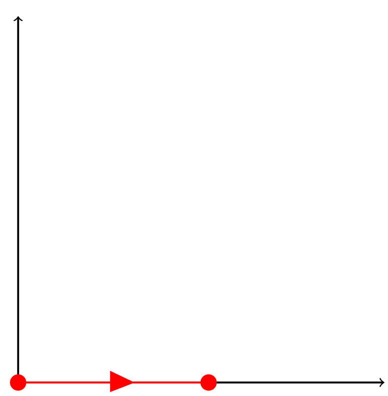
Definition 1. The line integral of $f(x, y)$ along a curve C is defined by

$$\int_C f ds = \int_{t_0}^{t_1} f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Observe. Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$. When $f = 1$, we recover the arc length.

Example. Consider C defined by

$$x(t) = t, \quad y(t) = 0, \quad 0 \leq t \leq 1$$



First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} = 1$$

Now, consider $f(x, y) = x^2 + y$. We want to compute $\int_C f ds$. Restricting f to C gives

$$f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f ds = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

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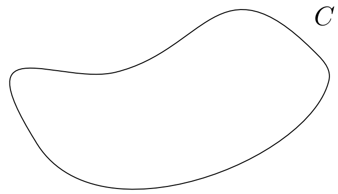
Note. Line integrals can be used to compute the mass of a 1-dimensional object. The curve C describes the object, and the function $\int_C f ds$ is the mass.



1.3 Parametrization and Orientation

The next result is as for the arc length.

Proposition. The integral $\int_C f ds$ does not depent on the parametrization of C.

We will consider a special case

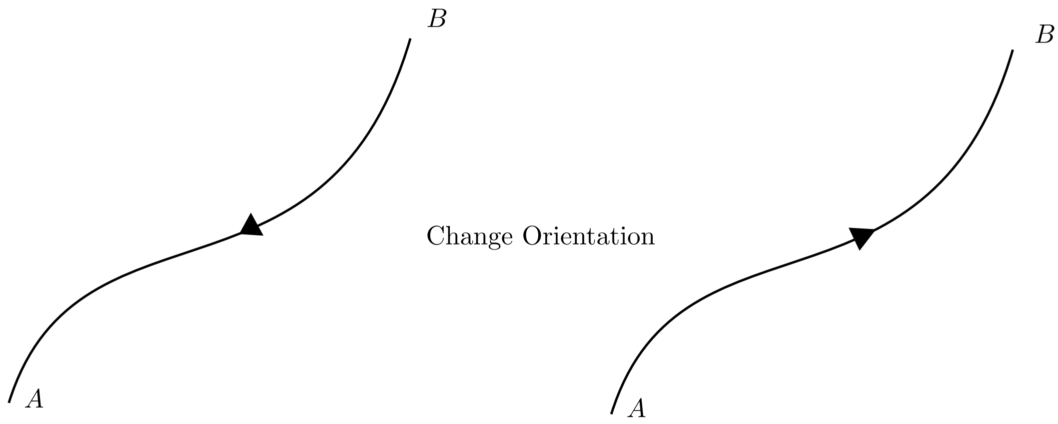
Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), t_0 \leq t \leq t_1$$

We have

$$\text{Start: } A = (x(t_0), y(t_0)), \quad \text{End: } B = (x(t_1), y(t_1))$$

We want to go from B to A.



We can do this in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \le t \le -t_0$$

Note that

$$\vec{r}_{\text{opp}}(-t_1) = (x(t1), y(t_1)) = B$$
$$\vec{r}_{\text{opp}}(-t_0) = (x(t0), y(t_0)) = A$$

$\diamond$

Example. Consider the segment C from $(0,0)$ to $(1,0)$. Take $f(x,y) = x$. Show that $\int_C f ds = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$.

If C parametrized by $\vec{r}(t)$, we use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have

$$\int_C f ds = \int_{-C} f ds$$

The situation will be different for vector fields. $\diamond$

1.4 Case of Vector Fields

Consider the curve with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \le t \le t_1$$
$$\vec{r'}(t) = (x'(t), y'(t)) \quad \text{(Velocity vector)}$$

Definition 2. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}\left(\vec{r}\left(t\right)\right) \cdot \vec{r}' dt$$

Here $\vec{F}\left(\vec{r}\left(t\right)\right) = \vec{F}\left(x\left(t\right), y\left(t\right)\right)$

In physics, we have that $\vec{F}$ is the force, and $\int_C \vec{F} d\vec{r}$ is the work done by $\vec{F}$ along C. The elementary case is given by $W = FS$, or work = force · displacement. More explicitly, we consider

$$\vec{F}\left(x, y\right) = \left(P\left(x, y\right), Q\left(x, y\right)\right)$$

Then we have

$$\vec{F}\left(\vec{r}\left(t\right)\right) \cdot \vec{r}'\left(t\right) = P\left(x\left(t\right), y\left(t\right)\right) \cdot x'\left(t\right) + Q\left(x\left(t\right), y\left(t\right)\right) \cdot y'\left(t\right)$$

The line integral is then

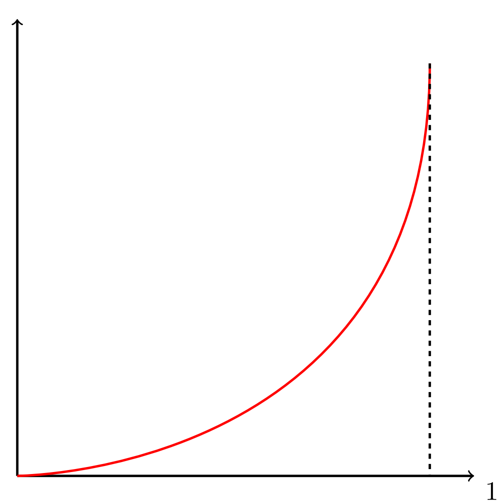
$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P\left(x\left(t\right), y\left(t\right)\right) \cdot x'\left(t\right) dt + \int_{t_0}^{t_1} Q\left(x\left(t\right), y\left(t\right)\right) \cdot y'\left(t\right) dt$$

We will write the expression as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + \int_C Q dy$$

Example. Consider the curve C with

$$\vec{r}\left(t\right) = \left(t, t^2\right), \quad 0 \le t \le 1$$



We have $x\left(t\right) = t$ and $y\left(t\right) = t^2$, its derivative is

$$\vec{r}'(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is $P = x + y$ and $Q = x$, when this is restricted to C, we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = \left(t + t^2, t\right)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}' = \left(t + t^2, t\right) \cdot (1, 2t) = \left(3t^2 + t\right)$$

We finally obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(3t^2 + t\right) dt = \frac{3}{2}$$

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