1 Lecture 14

1.1 Conservative Feilds (cont.)

We will now explore other criteria for conservative fields.

Proposition. Suppose $\vec{F}$ is conservative, then $\nabla \times \vec{F} = 0$

Proof. Since $\vec{F} = \nabla f$, we have

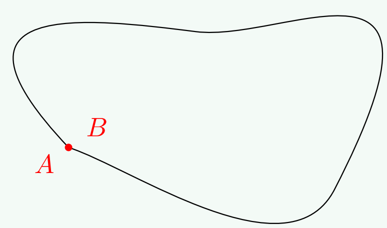
$$\nabla \times \vec{F} = \nabla \times \nabla f = 0$$

This is by an identity previously discussed. □

The converse for this is also true.

Note. It is easy to check if $\vec{F}$ is conservative by computing $\nabla \times \vec{F}$.

Definition 1. A curve is closed if its endpoints coincide.



Notation. The line integral of F along a closed curve is called the circulation. It is written as

$$\oint_C \vec{F} \cdot d\vec{r}$$

Proposition. If $\vec{F}$ is conservative, then

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

,

for any closed curve C.

Proof. By the gradient theorem, we have

$$\oint_C \vec{F} \cdot d\vec{r} = f(\vec{p}) - f(\vec{p}) = 0$$

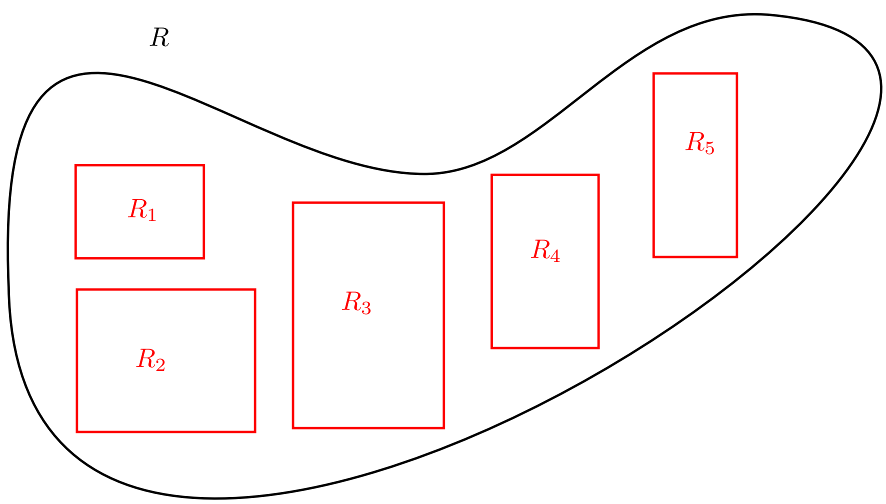
Since the endpoints coincide □

In summary, we have the following equivalent conditions

- $\vec{F}$ is conservative
- $\vec{F} = \nabla f$
- $\nabla \times \vec{F} = 0$
- $oint_C \vec{F} \cdot d\vec{r} = 0$, for any closed curve C.

1.2 Double Integrals

In two dimensions, we have the following method for computing integrals



We approximate a region R by rectangles R_i, with areas ΔA_i

Consider a function $f(x, y)$, pick a sample point (x_i, y_i) in each rectangle R_i. Then we consider the sum

$$\sum_i f(x_i^\star, y_i^\star) \Delta A_i$$

The limit ΔA_i, when it exists, gives the double integral.

Definition 2. The double integral of $f(x, y)$ over the region R is

$$\iint_R f dA = \lim_{\Delta A_i \to 0} \sum_{n=i} f\left(x_i^\star, y_i^\star\right) \Delta A_i$$

When $f = 1$, this gives the area of R, or the size of the region R. When $f > 0$, the integral is also the volume under f.

1.3 Some Properties

We still need concrete forumlas to compute $\iint_R f dA$. First, some general properties.

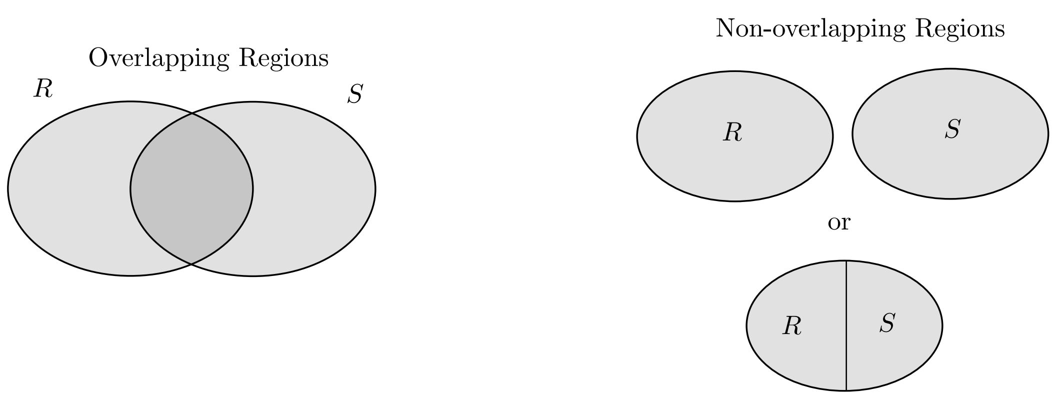
Proposition. Linearity.

Let a and b be two constants, then

$$\int_R (af + bg)\, dA = a \int_R f dA + b \int_R g dA$$

Proof. This follows the linearity of limits. $\square$

The next property is related to portions of the region of integrations.



Proposition. Partitions

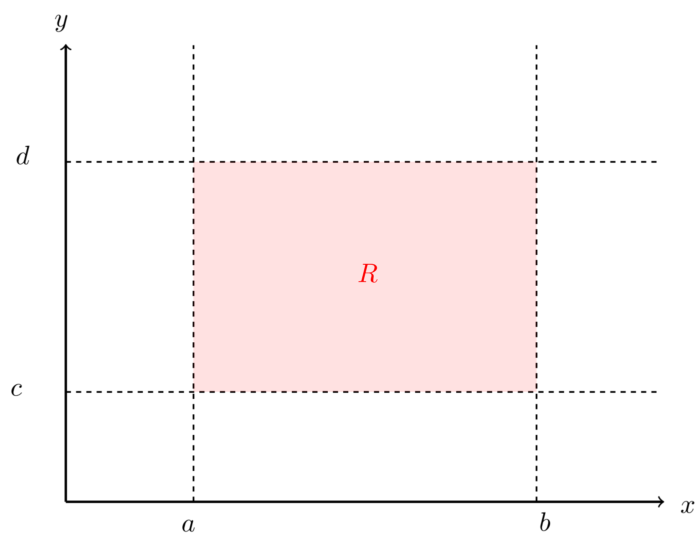
Let R and S be non-overlapping regions. Then we have

$$\int_{R \cup B} f dA = \int_R f dA + \int_S f dA$$

Idea. The total area is the sum of the areas.

1.4 Integrations Over Rectangles

Integrations over a rectangle is the easiest case of a double integral.



General rectangle:

$$R = (a, b) \times (c, d)$$

Proposition. Let $R = (a, b) \times (c, d)$, then

$$\iint_R f dA = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$$

We reduce to the case of two ordinary integrals.

Example. By simple geometry, the area of a rectangle is $(b - a) \cdot (d - c)$. The double integral gives

$$\iint_R 1 dA = \int_a^b \left(\int_c^d 1 dy \right) dx = \int_a^b (d - c)\, dx = (b - a) \cdot (d - c)$$

$\diamond$

Example. Compute $\iint_R f dA$ with

$$f(x,y) = xy, \quad R = [0,1] \times [0,2]$$

We compute

$$\begin{aligned}
\iint_R xy dA &= \int_{y=0}^{2} \left(\int_{x=0}^{1} xy dx \right) dy \\
&= \frac{1}{2} \int_{y=0}^{2} y dy \\
&= \frac{1}{2} \cdot \frac{2^2}{2} \\
&= 1
\end{aligned}$$

We can also use

$$\begin{aligned}
\iint_R xy dA &= \int_{x=0}^{1} \left(\int_{y=0}^{2} xy dy \right) dx \\
&= \frac{2^2}{2} \int_{x=0}^{1} x dx \\
&= 2 \cdot \frac{1}{2} \\
&= 1
\end{aligned}$$

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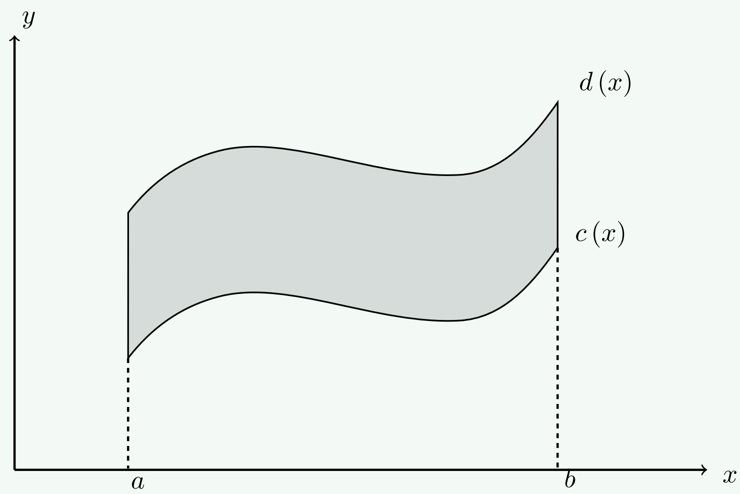
1.5 General Regions

In this section we will consider simple regions.

Definition 3. A region D is called *y-simple* if it is of the following form.

$$D = \{(x,y) : a \leq x \leq b, \quad c(x) \leq y \leq d(x)\}$$



This is the region below $d(x)$, and above $c(x)$. When $c(x) = \text{c}$ and $d(x) = d$, we get a rectangle.

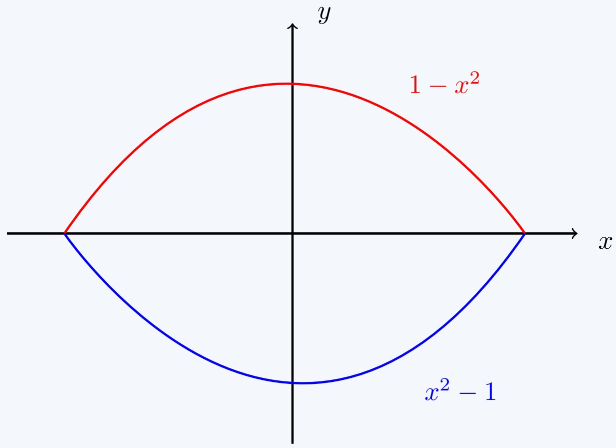
Proposition. Suppose D is a y-simple region. Then:

$$\iint_D f dA = \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y)\, dy \right) dx$$

Note. We first integrate in y, and then in x.

Example. Consider

$$D = \left\{ (x, y) : -1 \leq x \leq 1, \quad x^2 - 1 \leq y \leq 1 - x^2 \right\}$$



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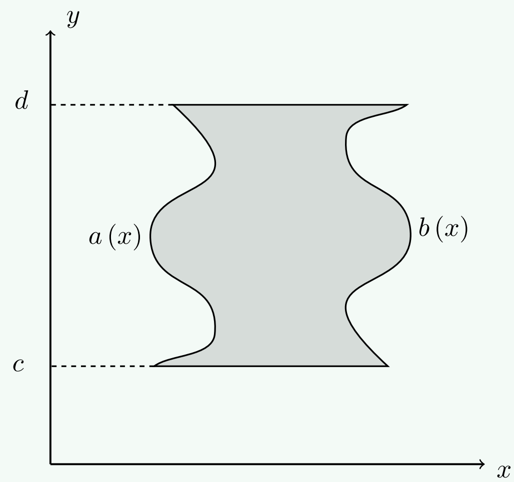
D is y-simple with $c(x) = x^2 - 1$ and $d(x) = x^2 - 1$. We compute the area of D, we have

$$\begin{aligned}
\iint_D dA &= \int_{-1}^{1} \left(\int_{x^2-1}^{1-x^2} 1 dy \right) dx \\
&= \int_{-1}^{1} \left(\left(1 - x^2\right) - \left(x^2 - 1\right) \right) dx \\
&= 2 \int_{-1}^{1} \left(1 - x^2\right) dx \\
&= 2 \left(x - \frac{1}{3} x^3 \right)_{-1}^{1} \\
&= \frac{8}{3}
\end{aligned}$$

We can exchange the role of x and y.

Definition 4. A region D is *x-simple* if it is of the form

$$D = \{(x, y) : a(y) \leq x \leq b(y), \quad c \leq y \leq d\}$$



Proposition. If D is x-simple, then

$$\iint_D f\,dA = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\,dx \right) dy$$