1 Lecture 13

1.1 Line Integrals of Vector Fields

We have seen the following definition earlier

$$\int_C \vec{F} d\vec{r} = \int_{t_0}^{t_1} \vec{F}\left(\vec{r}\,(t)\right) \cdot \vec{r}'\,(t)\, dt$$

We can rewrite this to link with scalarfields, lets consider

$$\int_C = \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F} \cdot \vec{v} dt = \int_{t_0}^{t_1} \vec{F} \cdot \frac{\vec{v}}{|\vec{v}|} dt$$

We can see this as the line integral of the field $\vec{F} \cdot \vec{T}$

Example. Consider $\vec{F}$ is constant and directed along the curve, that is $\vec{F} = F\vec{T}$, where $F - \left|\vec{F}\right|$. Then the formula for elementary work ($W = FS$) gives

$$W = \int_C \vec{F} \cdot \vec{T} ds = \int_C F\vec{T} \cdot \vec{T} ds = F \int_C ds = FS$$

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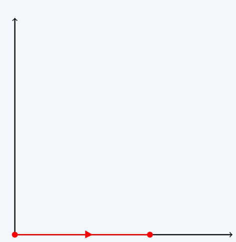
Now lets consider changes of orientation

Proposition 1. We have that

$$\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$$

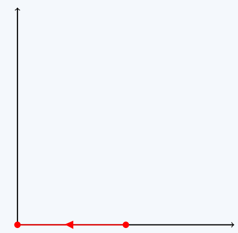
Example. Consider the segment

$$\vec{r}\,(t) = (t, 0)\,, \quad 0 \leq t \leq 1$$



The opposite parametrization is

$$\vec{r}_{\text{opp}}\,(t) = (-t, 0)\,, \quad -1 \leq t \leq 0$$



We have

$$\vec{r}'(t) = (1,0), \quad \vec{r}_{\text{opp}}(t) = (-1,0)$$

Consider $\vec{F}(x,y) = (x,0)$. For C we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (t,0) \cdot (1,0)\, dt = \int_0^1 t dt = \frac{1}{2}$$

For $-C$, we have

$$\int_{-C} \vec{F} \cdot d\vec{r} = \int_{-1}^0 (-t,0) \cdot (-1,0)\, dt = \int_{-1}^0 t dt = -\frac{1}{2}$$

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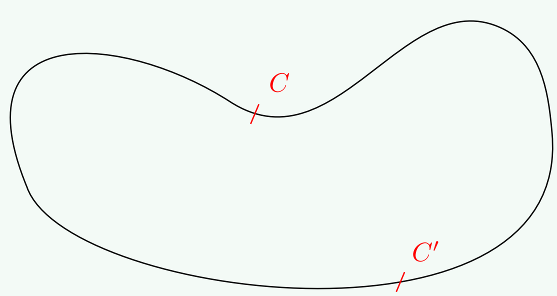
1.2 Conservative Vector Fields

In general, $\int_C \vec{F} \cdot d\vec{r}$ depends on the curve C. However, sometimes it only depends on the endpoints.

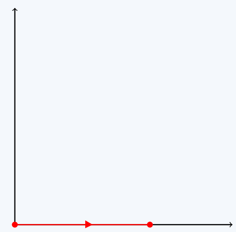
Definition 1. A vector field $\vec{F}$ is conservative if $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of C. That is, if C and C' have the same endpoints, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}$$



Example. Consider $\vec{F} = (1,1)$ and

$$C : x(t) = t, \quad y(t) = 0, \quad 0 \le t \le 1$$

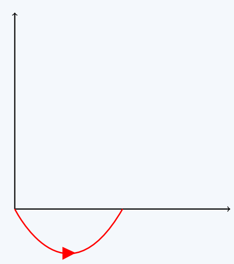


The endpoints are $(0,0)$ and $(1,0)$. We compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 dt = 1$$

Now, lets consider a different curve.

$$C' : x(t) = t, \quad y(t) = t(t-1), \quad 0 \le t \le 1$$



We have the same endpoints as C. The velocity is $\vec{r}'(t) = (1, 2t-1)$. Then

$$\begin{aligned} \int_{C'} \vec{F} \cdot d\vec{r} &= \int_0^1 (1,1) \cdot (1, 2t-1)\, dt \\ &= \int_0^1 2t dt - 2 \cdot \frac{1}{2} = 1 \end{aligned}$$

At this stage, we cannot conclude $\vec{F}$ is consrvative (although it is). Note that

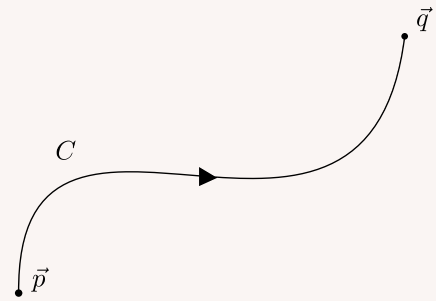
$$\vec{F} = \nabla f \quad \wedge \quad f(x,y) = x + y$$

We will prove that being a gradient field is the condition we want. ◇

Theorem 2. The Gradient Theorem.

Suppose $\vec{F} = \nabla f$, consider a curve C starting at $\vec{p}$, and ending at $\vec{q}$, then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{q}) - f(\vec{p})$$



Proof. Pick a parametrization

$$\vec{r}(t), \quad t_0 \leq t \leq t_1$$

Note that: $\vec{r}(t_0) = \vec{p}$ and $r(t_1) = \vec{q}$. Using $\vec{F} = \nabla f$, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$\square$