DAVE3700 - Matte 3000

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2 Domains, Graphs and Level Sets

2.1 Domain of definition

A function may not be defined for all real numbers.

Example. $f(x) = \frac{1}{x}$ is not defined for $x = 0$ ◇

Definition 1. The domain of a function f is the set of numbers for which it is defined. We write the domain of f as D_f.

For instance, for $f(x) = \frac{1}{x}$ we have that

$$D_f = \{x \in \mathbb{R} : x \neq\}$$

This is the lasrgerst possible domain, we can also consider smaller domains. We have the interval from 1 to 2.

$$[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$$

Example. Find the largest domain of $f(x, y) = \frac{1}{y-x}$. The denominator should be non-zero, we get

$$D_f = \left\{(x, y) \in \mathbb{R}^2 : y - x \neq 0\right\}$$

◇

Example. Same excersise with $f(x, y) = \sqrt{y - x^2}$.

Argument: $y - x^2 \geq 0$ (because square root). We will then have $y \geq x^2$ ◇

2.2 Graphs of functions

The plot of a function f descrives its behaviour visually. Mathematically, a plot corresponds to the notion of a graph.

Definition 2. The graph of a function $f(x, y)$ with domain D_f is the set of points (x, y, z) such that:

$$(x, y) \in D_f \text{ and } z = f(x, y)$$

We write G_f for the set

$$G_f = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_f \text{ and } z = f\left(x, y\right)\right\}$$

The graph of a function of two variables will, in general, be a surface.

Example. Lets consider $f(x, y) = 1$, with domain $\mathbb{R}^2$. The graph of f is:

$$G_f = \left\{(x, y, 1) : (x, y)\, \mathbb{R}^2\right\}$$

All points have $z = 1$, this is a plane. More generally, the graph of$f(x, y) = ax + by + c$ is a plane with linear dependence on x and y. ◇

Example. Consider the graph of:

$$f(x,y) = x^2 + y^2, \quad D_F = \mathbb{R}^2$$

This surface is called a parabloids. ◇

Example. A sphere of radius r is defined by

$$x^2 + y^2 + z^2 = r^2$$

All points x, y, z satisfy the equation. ◇

Is this the graph of a function? No!
There is no unique value of z, associated with (x, y) because:

$$z = \pm\sqrt{r^2 - x^2 - y^2}$$

Both satisfy the sphere equation. Lets consider the graph of

$$f(x,y) = \sqrt{r^2 - x^2 - y^2}$$

With the domain $x^2 + y^2 \leq r^2$. The grap is a half sphere.

2.3 Level Sets

Another way to visualize functions.

Definition 3. A level set of a function $f(x, y)$ is constant. Essentialy, this is a topographic map.

Example. Consider the function

$$f(x,y) = x^2 + y^2$$

The level sets for $c > 0$ are circles.

$$f(x,y) = x^2 + y^2 = c = \left(\sqrt{c}\right)^2$$

This is a circle with radius $\sqrt{c}$.
Now consider the case $c < 0$, then:

$$f\left(x,y\right) = x^2 + y^2 = c$$

which doesn't work, because the level sets are empty.
For $c = 0$, we only have the point $(x, y) = (0, 0)$. Generally, level sets of $f(x, y)$ is a curve.

◇

3 Derivatives

3.1 Partial Derivatives

In the case of one variable, we have

$$\frac{df}{dt} = \lim_{n\to 0} \frac{f(x+n) - f(x)}{n}$$

Similarily, for two of more variables, we have the following definition

Definition 4. The partial derivative of $f(x, y)$ with respect to x:

$$\frac{\partial f}{\partial x} = \lim_{n\to 0} \frac{f(x+n, y) - f(x, y)}{n}$$

Also written as f_x, for $\frac{\partial f}{\partial y}$, we have f_y

Note, the expression above is $\frac{\partial f}{\partial x}(x, y)$, which is the value at the point (x, y)

3.2 Higher order derivatives

Given $\frac{\partial f}{\partial x}$, we can take further derivatives. We have

$$\frac{\partial^2 f}{\partial^2 x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Also written as $f_{xx}, f_{yy}, f_{xy}, f_{yx}$. In most cases, f_{xy} and f_{yx} coincide.

Theorem 5. Scwartz theorem: Suppose f_{xy} and f_{yx} exist, and are continous, then

$$f_{xy} = f_{yx}$$

Similar definitions and results for the case of more variables: $x_1, \ldots, x_n$, with n variables.

3.3 Chain Rule

Suppose $f(x) = g(h(x))$, for instance

$$f(x) = (\cos x)^2 \text{ with } g(x) = x^2, h(x) = \cos x$$

then the chain rule is

$$\frac{df}{dt}(x_0) = \frac{dg}{dh}(h(x_0)) \cdot \frac{dh}{dt}(x_0)$$

Generalization to more variables.

Theorem 6. Chain rule: consider $f(x, y)$ x and y depending on a variable t. Then:

$$\frac{df}{dt} t_0 = \frac{\partial f}{\partial x}(x(t_0), y(t_0)) \frac{dx}{dt} t_0 + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \frac{dy}{dt} t_0$$

The "short form" of this result is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example. Consiter $f(x, y) = xy$, where

$$x(t) = \cos t, y(t) = \sin t$$

This cannot be computed directly with $\frac{df}{dt}$.

$$f(t) = f(x(t), y(t)) = f(\cos t, \sin t) = \cos t \cdot \sin t$$

We can compute

$$\frac{df}{dt} = (\cos t)' \sin t + \cos t (\sin t)' = -(\sin t)^2 + (\cos t)^2$$

Usin the chain rule, we get

$$\frac{df}{dt} j = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

◇

3.4 The Gradient

Define an operation that takes a scalar function, and returns a vector function.

Definition 7. The gradient of $f(x, y)$ at (x_0, y_0) is

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

4 Directional Derivatives and Critical Points

4.1 Directional Derivatives

We have seen the following

- f_x = the rate of change along the $x-$direction.
- f_y = the rate of change along the $y-$direction.

What about general directions?

Definition 8. Let $\vec{u} = (a, b)$, the directional derivative along $\vec{u}$ at (x, y) is

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h}$$

Note that

$$\vec{u} = (1, 0) \to D_{\vec{u}} = f_x$$
$$\vec{u} = (1, 0) \to D_{\vec{u}} = f_y$$

To compare directions, we take $\left|\vec{t}\right| = 1$. Here $\vec{u}$ is the length of $\vec{u}$, that is

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

Proposition 9. We have the following result

$$D_{\vec{u}} \cdot f = \nabla f \cdot \vec{u}$$

Proof. Consider the following function

$$g(t) = f(x + ta, y + tb)$$

Its derivative at $t = 0$ is

$$\begin{aligned}
\frac{dg}{dt}(0) &= \lim_{h \to 0} \frac{g(h) - g(0)}{h} \\
&= \lim_{h \to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h} \\
&= D_{\vec{u}} \cdot f(x, y)
\end{aligned}$$

On the other hand, using the chain rule, we get

$$\begin{aligned}
\frac{dg}{dt}(0) &= \frac{\partial f}{\partial x} \frac{d(x + ta)}{dt} + \frac{\partial f}{\partial y} \frac{d(y + tb)}{dt} \\
&= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b = \nabla f \cdot \vec{u}
\end{aligned}$$

□

We can now state another property of ∇f. The direction where f changes the most.

Proposition 10. $|D_{\vec{u}} \cdot f|$ is the largest when $\vec{u}$ is paralell to ∇f.

Proof. Recall that, given two vectors, $\vec{v}$ and $\vec{w}$, we have that

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos\alpha$$

When is $|\vec{v} \cdot \vec{w}|$ the largest? We have

$$|\vec{v} \cdot \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot |\cos\alpha|, \quad |\cos\alpha| \leq 1$$

It is the largest when the following condition is true

$$|\cos\alpha| = 1, \quad \alpha = 0 \vee \pi$$

Which means that the vectores are pointing in the same, or opposite direction. Applying this to $D_{\vec{u}} \cdot f$, we get

$$\begin{aligned} |D_{\vec{u}} \cdot f| &= |\nabla f \cdot \vec{u}| \\ &= |\nabla f| \cdot |\vec{u}| \cdot |\cos\alpha| \end{aligned}$$

For fixed values of $|\vec{u}|$, this is the largest when $\alpha = 0 \vee \pi$. That is ∇f and $\vec{u}$ are paralell. □

4.2 Critical Points

How do we find the maxima and minima of $f(x)$? Lets take a look at $f'(x_0) = 0$.

Definition 11. We say that x_0, y_0 is a critical point of f if:

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \quad \wedge \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

Similarily for

5 Lecutre 5

5.1 Hessian Matrix

Example. Consider again the function $f(x, y) = x^2 - y^2$.

$$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{yx} = 0$$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \det H = -4 < 0$$

Hence (x_0, y_0) is a saddle point. ◇

Example. Consider the function $f(x, y) = x^2 + y^2$, we have

$$(f_x, f_y) = (2x, 2y)$$

The only critical point is $(x_0, y_0) = (0, 0)$.

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \det H = 4 > 0$$

Since $f_{xx} = 2 > 0$, we conclude that $(0, 0)$ is a local minima. In this case, it is actually a global minimum, because $f(x, y) = x^2 + y^2 \geq 0$. ◇

5.2 Global extremal values

A function can have many maxima and minimas. Usually, we are intrested in the largest and smallest values.

Definition 12. Let $f(x, y)$ be with domain D_f. Then we have

- (x_0, y_0) is a global maxima if $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in D_f$.
- (x_0, y_0) is a global minima if $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in D_f$.

Trivial example: for $f(x, y) = 1$, all points are global maxima and minima.

Note that global maxima and minima need not be critical points.

Example. We have $f(x) = x$ with $D_f = [-1, 1]$.

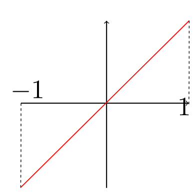


Figure 1: $f(x)$

- Global maxima at $x = 1, f(1) = 1$.
- Global minima at $x = -1, f(-1) = -1$

We have no critical points because $f'(x) = 1 \neq 0$. Also not that maxima and minima depend on the chosen domain.

If we take $D_f = [-2, 3]$, then

$$\text{Max: } x = 3, \quad \text{Min: } x = -2$$

◇

Theorem 13. Let f be continous with domain D_f. Suppose D_f is closed and bounded, then there is at least one global maxima, and one global minima.

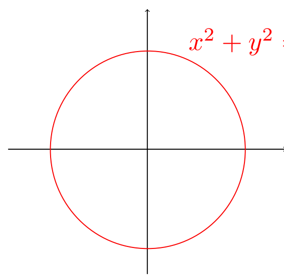


Figure 2: The circle is the boundary.

Some terminology:

$$\begin{aligned} \text{open } &= \left\{x^2 + y^2 < 1\right\} \\ \text{closed } &= \left\{x^2 + y^2 \leq 1\right\} \end{aligned}$$

The method for finding maxima and minima is as follows:

1. Find critical points of f in D_f, and characterize them.
2. Study the points that are on the boundary.
3. Compare them.

Example. Consider $f(x, y) = x^2 + y^2$ with domain

$$D = \left\{(x, y) \in \mathbb{R} : x^2 + y^2 \leq 1\right\}$$

The domain in this case is a disc. The red circle is the boundary.

We compute $f_x = 2x, f_y = 2y$. The only critical point is the origin at $(x_0, y_0) = (0, 0)$. This is a global minium since $f(0, 0)$ and $f(x, y) \geq 0$.

Now, lets consider the boundary

$$C = \left\{x^2 + y^2 = 1\right\}$$

For any point (x_0, y_0) on the circle C, we have

$$f(x_0, y_0) = x_0^2 + y_0^2 = 1$$

We claim that this point is a global maximum. For any (x, y) in domain D_f, we have

$$f(x, y) = x^2 + y^2 \leq 1$$

The value $f(x, y) = 1$ is obtain only at the boundary C. Any point on the circle is a global maxima. ◇

5.3 Constrained optimization

In this section, we will discuss how to find maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. Think of $g = 0$ ad a budget, or a geometrical constraint.

Example. We want to minimize $f(x, y) = x^2 + y^2$ with the constraint $g(x, y) = xy - 1 = 0$.

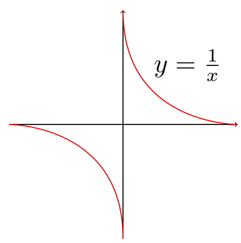


Figure 3: $y = \frac{1}{x}$

We have the following strategy

- Solve $g = 0$ for one variable. For instance $y = \frac{1}{x}$.
- Consider: $h(x) = f\left(x, \frac{1}{x}\right) = x^2 + x^{-2}$

Now we can study this function of one variable with no constraints. We can proceed as usual.

$$\frac{dh}{dt} = 2x - 2x^{-3} = 0$$

This is equivalent to $x^4 = 1$. The real solutions are $x = \pm 1$. Since $y = \frac{1}{x}$, we get the critical points:

$$(x, y) = (1, 1), \quad (x, y) = (-1, -1)$$

◇

6 Lecture 6

6.1 Substitution Method

We want to maximize / minimize $f(x,y)$ with constraint $g(x,y)=0$. We can solve $g(x,y)=0$ for one variable $y=y(x)$.

Example. Consider $f(x,y)=x^2+y^2$ and $g(x,y)=xy-1=0$. In this case, we have $f=\frac{1}{x}$ from $g=0$, we get

$$h(x)=f\left(x,x^{-1}\right)=x^2+x^{-2}$$

We have found the minima at $(1,1)$ and $(-1,-1)$. ◇

This method isn't always feasible, so lets look at some alternatives.

6.2 Lagrange's Method

Example. Lets look at the level curves, which are circles. We have the that $f(x,y)=x^2+y^2=c$.

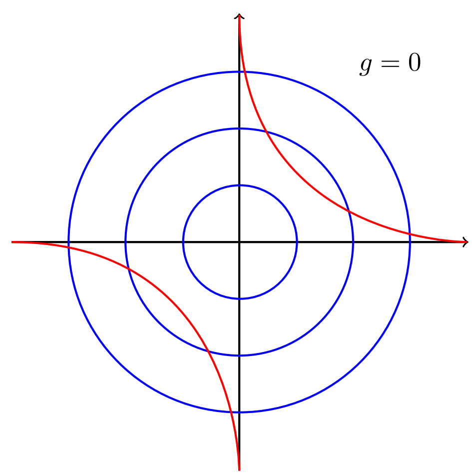
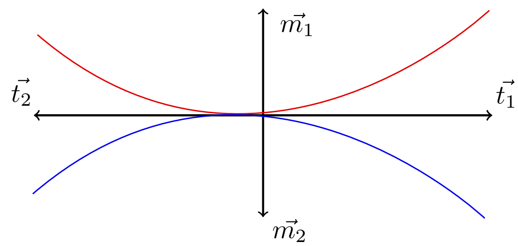


Figure 4: $f(x,y)$

Smaller circles correspont to smaller values of $f(x,y)$, but we must also satisfy $g(x,y)=0$. In the best case, $f(x,y)=c$ is just touching $g(x,y)=0$. If this is worked out geometrically, we get $(1,1)$ and $(-1,-1)$. ◇

This idea is used in Lagranges method. We want $f(x,y)=c$ to be parallell to $g(x,y)=0$. More precisely: Their vectors should be parallell.



Equivalently, their normal vectors are also paralell. Recall that a normal vector to $g = 0$ is given by ∇f. Similarly, ∇f is normal to $f = c$.

6.3 Method for Langrange

Suppose we want to find a local maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. We proceed as follows

1. Find all possible solutions to the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0$$

2. Plug in all solutions from 1. into $f(x, y)$ and identify the largest and smallest.

- The number λ is called the *Lagrange Multiplier*.
- Easy extension to n variables.
- Can also be generalized to multiple constraints, $g_1, \ldots, g_n$

Example. We have the following functions

$$f(x, y) = x^2 + y^2, \quad g(x, y) = xy - 1 = 0$$

We compute the gradients

$$\nabla f(x, y) = (2x, 2y), \quad \nabla g(x, y) = (x, y)$$

The Lagrange equiations are

$$2x = \lambda y, \quad 2y = \lambda x, \quad xy - 1 = 0$$

Observe that $(x, y, \lambda) = (0, 0, 0)$ is not a solution. For $x \neq 0$, we get $y = \frac{1}{x}$, from the third equation.

$$
\begin{aligned}
&2x = \lambda y \Rightarrow 2x = \lambda \frac{1}{x} \Rightarrow \lambda = 2x^2 \\
&2y = \lambda x \Rightarrow \frac{2}{x} = 2xs^2 \cdot x \Rightarrow x^4 = 1 \\
&x^4 = 1 \Rightarrow x \pm 1, \quad y = \frac{1}{x}
\end{aligned}
$$

$$(x, y) = (1, 1) \quad \wedge \quad (x, y) = (-1, -1)$$

Are theses points the minima? We have

$$f(1, 1) = f(-1, -1) = 2$$

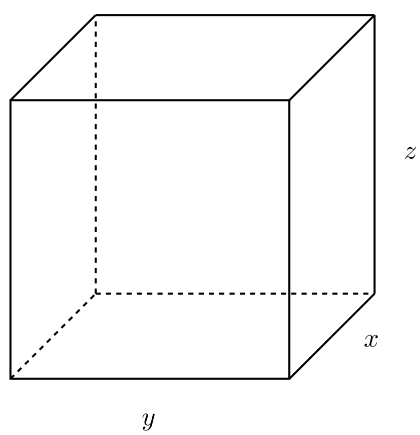
Lets Compare this to some other points, such that $g(x, y) = 0$. For instance, $(x, y) = \left(2, \frac{1}{2}\right)$

$$g\left(2, \frac{1}{2}\right) = 2 \cdot \frac{1}{2} - 1 = 0$$

$$f\left(2, \frac{1}{2}\right) = 2^2 + \left(\frac{1}{2}\right)^2 > 2$$

This tells us that $(x, y) = (\pm 1, \pm 1)$ are local minima. ◇

Example. Consider a box of surface area 24 cm^2.



Determine the dimensions (x, y, z) such that the volume is max.

We have the surface area $2xy + 2xz + 2yz$. Our constraint is

$$g(x, y, z) = 2xy + 2xz + 2yz - 24 = 0$$

The goal is to maximize $f(x, y, z) = 0$, with the consstraint $g(x, y, z) = 0$.

The equation $\nabla f = \lambda \nabla g$ gives

$$yz = 2\lambda\,(y+2)\,,\quad xz = 2\lambda\,(x+2)\,,\quad xy = 2\lambda\,(x+y)$$

Obsereve that $\lambda \neq$, since $x, y, z > 0$. To solve the equations, we can multiply by x, y and z, respectively, then we get

$$x\,(y+z) = y\,(x+z) = x\,(y+x)$$

Consider $x\,(y+z) = y\,(x+z)$.

$$x\,(y+z) = y\,(x+z) \Rightarrow (x-y)\,z = 0$$

$$z \neq 0,\quad x = y$$

Its the same for the other equations, so $x = y = z$. We also need to use $g = 0$. Setting $x = y = z$j, we get

$$\begin{aligned} g\,(x,x,x) &= 2x^2 + 2x^2 + 2x^2 - 24 = 0 \\ x^2 &= 4 \\ x &= 2 \end{aligned}$$

Lagranges method gives:

$$(x,y,z) = (2,2,2) \Rightarrow V = 8$$

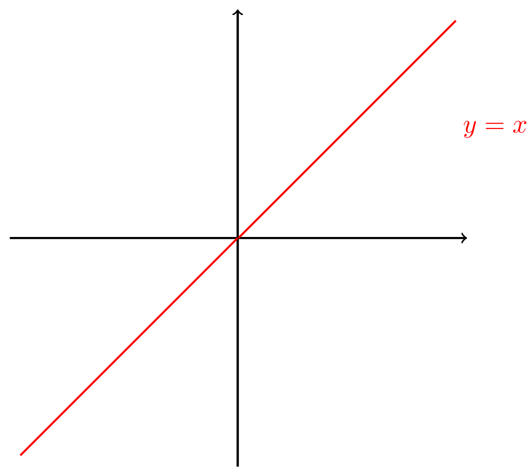
◇

7 Lecture 7

7.1 Parametrized Curve

A curve is described as a sset of points in $\mathbb{R}^2$ and $\mathbb{R}^3$. For instance a line is described by

$$f = \{(x, y) \in \mathbb{R}^2 : x = y\}$$



This is a static picture. But how do we give a dynamical picture? We'll use parametrized curves.

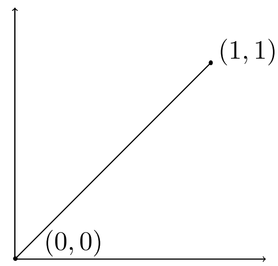
Definition 14. A parametrization of a curve c in $\mathbb{R}^2$, is given by

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Such that $\vec{r}(t) \in c$ for all time t.

- A parametrization describes motion. Think of t as the time.
- A parametrization is not unique.
- Various natural assumptions, such as continuity and differentiability.

Example. Consider the funtion $\vec{r}(t)$ with $0 \leq t \leq 1$.



$$\vec{r}(0) = (0,0), \quad \vec{r}(1) = (1,1)$$

We have the portion of the line where $y = x$. Notce that here, $x(t) = t$, $y(t) = t$ and $y(t) = x(t) = t$ for all t.

Lets consider a different function,

$$\vec{r}(t) = (2t, 2t), \quad 0 \leq t \leq \frac{1}{2}$$

When we parametrize, we get

$$\vec{r}(0) = (0,0), \quad \vec{r}\left(\frac{1}{2}\right) = (1,1)$$

We are moving along the curve at twice the speed. ◇

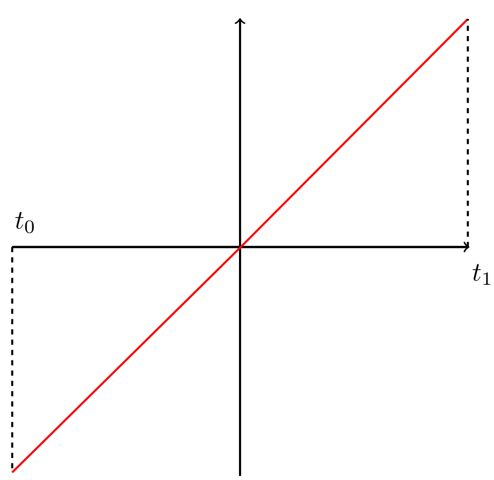
Example. Given $f(x)$, we consider

$$\vec{r}(t) = (t, f(t)), \quad t_0 \leq t \leq t_1$$

This describes a portion of the graph f, with

$$\text{Start: } (t_0, f(t_0)), \quad \text{End: } (t_1, f(t_1))$$

For instance, consider the line $y = mx + c$, we have



We have that

$$\vec{r}(t) = (t, mt + c), \quad t_0 \leq t \leq t_1$$

$\diamond$

Example. We want to describe a line with

$$\text{Start: } A = (x_0, y_0), \quad \text{End: } B = (x_1, y_1)$$

Then we take the parametrization

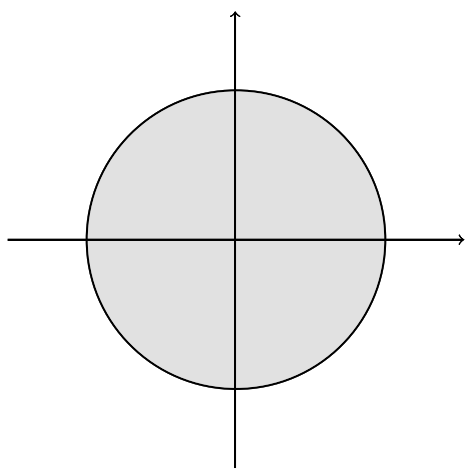
$$\vec{r}(t) = (1 - t) A + tB, \quad 0 \leq t \leq 1$$

More explicitly, we have

$$\vec{r}(t) = ((1 - t) x_0 + tx, (1 - t) y_0 + ty)$$

Note that $\vec{r}(0) = A$ and $\vec{r}(1) = B$. $\diamond$

Example. Consider $\vec{r}(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$. What curve does this describe?



It describes a circle.

$$x(t)^2 + y(t)^2 = \left(\cos(t)^2 + \sin(t)^2\right) = 1$$

We start at $(1, 0)$ and move counter-clockwise. $\diamond$

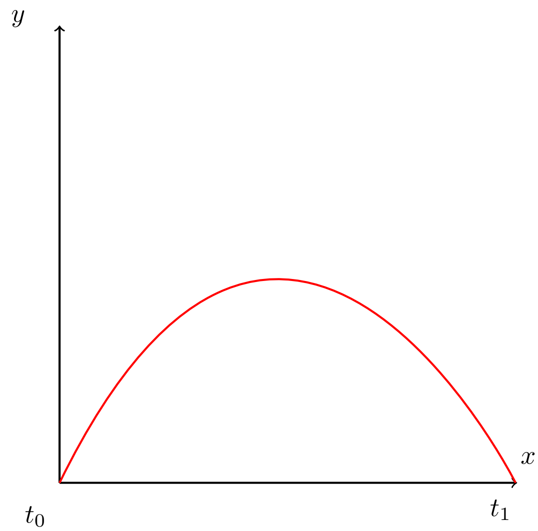
Example. Here is an example from physics. Consider

$$x(t) = v_x t, \quad y(t) = v_y t - \frac{1}{2} g t^2, \quad 0 \leq t \leq \frac{2v_y}{g}$$

This describes the motion of an object with initial velocity $\vec{v} = (v_x, v_y)$, under gravity. We write $t_0 = 0$ and $t_1 = \frac{2v_y}{g}$. Note that

$$\vec{r}\,(t_0) = (0,0)\,, \quad \vec{r}\,(t_1) = \left(\frac{2v_xv_y}{g}, 0\right)$$

The object falls back to the ground at time t_1.



Well known fact: This motion is parabolic, we will rederive this.

From $x\,(t) = v_x t$, we get $t = \frac{x(t)}{v_x}$. Then

$$y\,(t) = v_y t - \frac{1}{2} g t^2 \Rightarrow \frac{v_x}{v_y} x\,(t) - \frac{1}{2}\frac{g}{v_x^2} x\,(t)^2$$

This is the expression of a parabola

$$y = ax^2 + bx + c, \quad a \neq 0$$

It can also be written as

$$y\,(t) = -\frac{1}{2}\frac{g}{v_x^2}\left(x\,(t) - \frac{v_y}{g}\right)^2 + \frac{1}{2}\frac{v_y}{g}\frac{v_y}{v_x}$$

◇

7.2 Kinematics

Kinematics describes position, velocity and acceleration of an object.

Definition 15. The position vector is $\vec{r}\,(t)$. The velocity vector is $\vec{v}\,(t)\,\frac{d\vec{r}}{dt}$. The acceleration vector is $\vec{a}\,(t) = \frac{d^2\vec{r}}{dt^2}$.

If we write $\vec{r}\,(t) = (x\,(t)\,, y\,(t))$, then

$$\vec{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(x'(t), y(t)\right)$$

Similarily

$$\vec{a}(t) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right) = \left(x''(t), y''(t)\right)$$

Example. Consider again the gravity example. Here we have

$$\vec{r}(t) = \left(v_x t, v_y t - \frac{1}{2}gt^2\right)$$

The velocity is

$$\vec{v}(t) = (v_x, v_y - gt)$$

Note that $v(0) = (v_x, v_y)$ is the initial velocity of the object. For acceleration, we get

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (0, -g)$$

◇

8 Lecture 8

8.1 Determining Motion

Given acceleration $\vec{a}(t)$, can we find $\vec{v}(t)$ and $\vec{s}(t)$? Yes, with some initial conditions given, we can. This is done by integration, consider

$$\vec{v}(t) = \frac{d\vec{s}}{dt}$$

This is a differential equation for $\vec{s}(t)$. To solve it, we integrate both sides in t, from t_1, to t_2. We get

$$\int_{t_0}^{t_1} \vec{v}(t)\,dt = \int_{t_0}^{t_1} \frac{d\vec{s}}{dt}dt$$

The fundamental theorem of calculus gives

$$\vec{s}(t_1) - \vec{s}(t_0) = \int_{t_0}^{t_1} \vec{v}(t)\,dt$$

We can determine $\vec{s}(t)$ for any t if we know $\vec{v}(t)$ and the initial condition $\vec{s}(t_0)$.

Example. Consider an object with acceleration

$$\vec{a}(t) = (1, t) = \vec{i} - j\vec{j}$$

We have the following initial conditions

$$\vec{s}(0) = (2, 0) = 2\vec{i} \quad \wedge \quad \vec{v}(0) = 0$$

We want to determine $\vec{s}(t)$. First, to determine $\vec{v}(t)$, we compute

$$\begin{aligned}\int_0^t \vec{a}(t)\,dt &= \vec{i}\int_0^t 1dt + \vec{j}\int_0^t tdt \\ &= t\vec{i} + \frac{1}{2}t^2\vec{j}\end{aligned}$$

Here $t_0 = 0$, since $\vec{v}(0) = 0$, then

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t)\,dt = t\vec{i} + \frac{1}{2}t^2\vec{j}$$

To determine $\vec{s}$, we compute

$$\begin{aligned}\int_0^t \vec{v}(t)\,dt &= \vec{i}\int_0^t tdt + \vec{j}\int_0^t \frac{1}{2}t^2dt \\ &= \frac{1}{2}t^2\vec{i} + \frac{1}{6}t^3\vec{j}\end{aligned}$$

Since $\vec{s}(0) = (2, 0) = 2\vec{i}$, we get

$$\vec{s}(t) = \vec{s}(0) + \int_0^t \vec{v}(t)\,dt = \left(\frac{t^2}{2} + 2\right)\vec{i} + \frac{1}{6}t^3\vec{j}$$

◇

8.2 Arc Length

A formula that we can concretely use to compute the length of a curve (using a parametrization). Consider a curve c, with parametrization

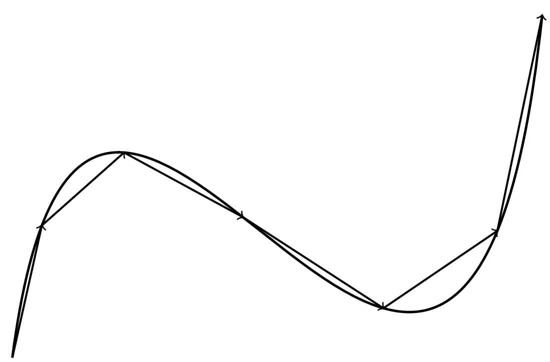
$$\vec{s}\,(t) = (x\,(t)\,, y\,(t))\,, \quad t_0 \leq t \leq t_1$$

Definition 16. The arc length of c is given by

$$S = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

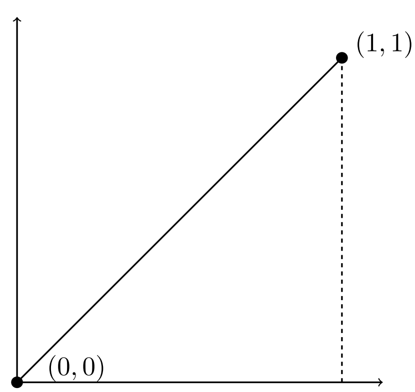
Idea: summing segments of length

$\sqrt{(\Delta x)^2 + (\Delta y)^2}$



To compute S, we need a parametrization of c. Does S depent on this choice? No!

Example. Consider the line segment below



From elementary geometry, its length is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Consider the parametrization

$$\vec{s}(t)\,(t,t)\,,\quad 0\leq t\leq 1$$

We have $(x'(t), y'(t)) = (1,1)$. Then

$$S=\int_0^1\sqrt{1^2+1^2}dt=\sqrt{2}\int_0^1 1dt=\sqrt{2}$$

Instead we choose

$$\vec{s}(t)=(2t,2t)\,,\quad 0\leq t\leq \frac{1}{2}$$

We have $(x'(t), y'(t)) = (2,2)$. Then

$$S=\int_0^{\frac{1}{2}}\sqrt{2^2+2^2}dt=\sqrt{8}\int_0^{\frac{1}{2}} 1dt=\sqrt{8}\cdot\frac{1}{2}=\sqrt{2}$$

◇

Question: What is the distance crossed up to time t?

Definition 17. The arc length parameter is

$$S(t)=\int_{t_0}^{t_1}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$$

The difference is that we integrate up to t, not t_1. Special case when $S(t_1)=S$. There is an important relation between $S(t)$ and $\vec{v}(t)$.

Proposition 18. We have

$$|\vec{v}(t)|=\frac{dS}{dt}$$

Proof. The fundamental theorem of calculus states that if

$$F(x)=\int_a^x f(x)dt\to F'(x)=f(x)$$

Applying this to $S(t)$, then

$$S'(t)=\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

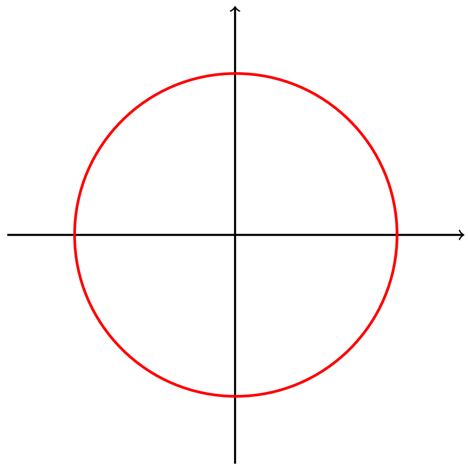
On the other hand, we have that

$$\vec{v}(t)=(x'(t),y'(t))\quad\wedge\quad|\vec{v}(t)|=(t)=\sqrt{x'(t)^2+y'(t)^2}$$

The two expressions coincide. □

Example. Consider a circle of radius R, with

$$\vec{S}(t) = (R\cos t, R\sin t), \quad 0 \leq t \leq 2\pi$$



We want to compute $S(t)$, we have

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{R^2(\sin t)^2 + R^2(\cos t)^2} = R$$

We want to check that $\frac{dS}{dt} = |\vec{v}(t)|$. We have

$$\vec{v}(t) = \frac{d\vec{S}}{dt} = (-R\sin t, R\cos t)$$

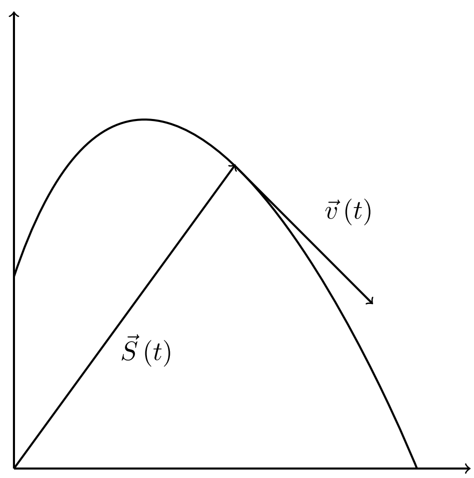
Its length is equal to $|\vec{v}(t)| = R$. Since $S(t) = R(t)$, we see that

$$\frac{dS}{dt} = |\vec{v}(t)|$$

$\diamond$

8.3 Tangent Vectors

Geometrically,, the velocity $\vec{v}(t)$ is tanget to a curve. It is useful to define a tangent vector of length 1.



Definition 19. The unit tangent vector is

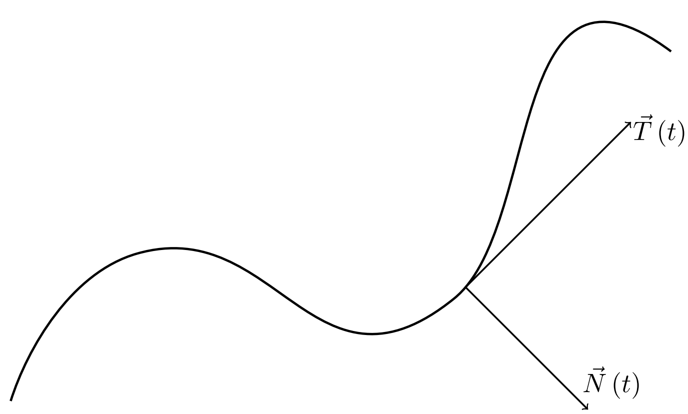
$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

Note that $\vec{T}$ has length 1 since

$$\vec{T}(t) \cdot \vec{T}(t) = \frac{\vec{v}(t) \cdot \vec{v}(t)}{|\vec{v}(t)|^2} = 1$$

8.4 Normal Vectors

Normal vectors are normal to the curve, or in other words, they are orthogonal. Recall that for implicit curves $f(x,y) = 0$, a normal vector is given by ∇f.



Lets now consider parametrizised curves. We have

$$S(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Definition 20. A unit normal vector to the curve is defined by

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

We need to check that $\vec{N}$ is orthogonal to $\vec{T}$, that is: $\vec{N}(t) \cdot \vec{T}(t) = 0$ for all t.

Proposition 21. We have that

$$\vec{N}(t) \cdot \vec{T}(t) = 0$$

Proof. Since $\vec{T}$ is a unit vector, we have that, for all t

$$\vec{T}(t) \cdot \vec{T}(t) = 1$$

Take the time derivative, the $\left(\vec{T} \cdot \vec{T}\right) = 0$. We also have

$$\frac{d}{dt}\left(\vec{T} \cdot \vec{T}\right) = \frac{d\vec{T}}{dt} \cdot \vec{T} + \vec{T}\frac{d\vec{T}}{dt} = 2\frac{d\vec{T}}{dt} \cdot \vec{T}$$

Since $\left(\vec{T} \cdot \vec{T}\right) = 0$, we get $\frac{d\vec{T}}{dt} \cdot \vec{T} = 0$. Dividing by $\left|\frac{d\vec{T}}{dt}\right|$, we get $\vec{N}(t) \cdot \vec{T}(t) = 0$ $\square$

9 Line Integrals, Parametrization and Vector Fields

9.1 Identities between operations

We have seen three operations defined by ∇.

Gradient: ∇f Divergence: $\nabla \cdot \vec{F}$ Curl: $\nabla \times \vec{F}$

There are many of them, we look at only one.

Proposition 22. For any scalar field f we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$. Then

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix} \right| = \vec{i}\,(f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{cy})$$

But partial derivatives can be exhanged, $f_{xy} = f_{yx}$. Then we find that

$$\nabla \times (\nabla f) = 0$$

□

We are going to use this when we discuss conservative fields.

9.2 Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Consider the points $\vec{r}(t)$ and $\vec{r}(t + \Delta t)$ where Δt is small.

$$\Delta x = x\,(t + \Delta t) - x\,(t)\,, \quad \Delta y = y\,(t + \Delta t) - y\,(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt}\Delta t, \quad \Delta y \approx \frac{dy}{dt}\Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt}\Delta t\right)^2 + \left(\frac{dy}{dt}\Delta t\right)^2} = \Delta t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Definition 23. The line integral of $f(x,y)$ along a curve C is defined by

$$\int_C f dS = \int_{t_0}^{t_1} f(x(y), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Some observations:

- Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$.
- When $f = 1$, we recover the arc length.

Example. Consider C definet by

$$x(t), y(t) = 0, \quad 0 \leq t \leq 1$$

First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} = 1$$

◇

Now, consider $f(x,y) = x^2 + y$. Compute $\int_C f dS$. Restricting f to C gives

$$f(x(t), y(t)) = x(t)^2 + y(t) = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f dS = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

- Line integrals can be used to compute the mass of an objects (1 - dimensional)
- The curve C describes the object
- The function $\int_C f dS$ is the mass.

9.3 Parametrization and orientation

Next results as for the arc legnth.

Proposition 24. The integral $\int_C f dS$ does not depend on the parametrization of C.

We will consider a special case.

Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\text{Start: A} = (x(t_0), y(t_0)), \quad \text{End: B} = (x(t0), y(t_1))$$

We want to go from B to A, which can be done in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\vec{r}_{\text{opp}}(-t_1) = (x(t_1), y(t_1)) = \text{B}$$

$$\vec{r}_{\text{opp}}(-t_0) = (x(t_0), y(t_0)) = \text{A}$$

◇

Example. Consider C = segement from (0,0), to (1,0), take $f(x,y) = x$. Show that $\int_C f dS = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$. ◇

If C parametrized by $\vec{r}(t)$, we will use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have that

$$\int_C f dS = \int_{-C} f dS$$

The situation will be different for vector fields.

9.4 Case of Vector Fields

Consider the curve with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\vec{r'}(t) = (x'(t), y'(t)) \quad \textit{Velocity Vector}$$

Definition 25. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r'}(t)\, dt$$

Here $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$

- In physics, we have that $\vec{F}$ = Force and $\int_C \vec{F} d\vec{r}$ = Work done by $\vec{F}$ along C.
- The elementary case $W = FS$, where W is work, F is Force and S is the displacement.

More explicitly, we consider

$$\vec{F}(x,y) = (P(x,y), Q(x,y))$$

Then we have that

$$\vec{F}\left(\vec{r}(t) \cdot \vec{r'}(t)\right) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t)\, dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t)\, dt$$

We will write the expression as

$$\int_C \vec{F} d\vec{r} = \int_C P\ dx + \int_C Q\ dy$$

Example. Consider a curve C with

$$\vec{r}(t) = \left(t, t^2\right), \quad 0 \leq t \leq 1$$

We have $x(t) = t$ and $y(t) = t^2$, its derivative is

$$\vec{r'}(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is $P = x + y$ and $Q = x$. When restricted to C, we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = \left(t + t^2, t\right)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r'}(t) = \left(t + t^2, t\right) \cdot (1, 2t) = \left(3t^2 + t\right)$$

We then obation the following

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(3t^2 + t\right) dt = \frac{3}{2}$$

$\diamond$