

DAVE3700 - Matte 3000

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Contents

1	Introduction	2
2	Domains, Graphs and Level Sets	3
2.1	Domain of definition	3
2.2	Graphs of functions	3
2.3	Level Sets	4
3	Derivatives	5
3.1	Partial Derivatives	5
3.2	Higher order derivatives	5
3.3	Chain Rule	5
3.4	The Gradient	6
4	Directional Derivatives and Critical Points	7
4.1	Directional Derivatives	7
4.2	Critical Points	8
5	Lecture 5	9
5.1	Hessian Matrix	9
5.2	Global extremal values	9
5.3	Constrained optimization	11
6	Lecture 6	12
6.1	Substitution Method	12
6.2	Lagrange's Method	12
6.3	Method for Langrange	13
7	Lecture 7	16
7.1	Parametrized Curve	16
7.2	Kinematics	19
8	Lecture 8	21
8.1	Determining Motion	21
8.2	Arc Length	22
8.3	Tangent Vectors	24
8.4	Normal Vectors	25
9	Line Integrals, Parametrization and Vector Fields	27
9.1	Identities between operations	27
9.2	Line Integrals	27
9.3	Parametrization and orientation	28
9.4	Case of Vector Fields	29

1 Introduction

2 Domains, Graphs and Level Sets

2.1 Domain of definition

A function may not be defined for all real numbers.

Example. $f(x) = \frac{1}{x}$ is not defined for $x = 0$

Definition 1. The domain of a function f is the set of numbers for which it is defined. We write the domain of f as D_f .

For instance, for $f(x) = \frac{1}{x}$ we have that

$$D_f = \{x \in \mathbb{R} : x \neq 0\}$$

This is the largest possible domain, we can also consider smaller domains. We have the interval from 1 to 2.

$$[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$$

Example. Find the largest domain of $f(x, y) = \frac{1}{y-x}$. The denominator should be non-zero, we get

$$D_f = \{(x, y) \in \mathbb{R}^2 : y - x \neq 0\}$$

Example. Same exercise with $f(x, y) = \sqrt{y - x^2}$.

Argument: $y - x^2 \geq 0$ (because square root). We will then have $y \geq x^2$

2.2 Graphs of functions

The plot of a function f describes its behaviour visually. Mathematically, a plot corresponds to the notion of a graph.

Definition 2. The graph of a function $f(x, y)$ with domain D_f is the set of points (x, y, z) such that:

$$(x, y) \in D_f \text{ and } z = f(x, y)$$

We write G_f for the set

$$G_f = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_f \text{ and } z = f(x, y)\}$$

The graph of a function of two variables will, in general, be a surface.

Example. Let's consider $f(x, y) = 1$, with domain \mathbb{R}^2 . The graph of f is:

$$G_f = \{(x, y, 1) : (x, y) \in \mathbb{R}^2\}$$

All points have $z = 1$, this is a plane. More generally, the graph of $f(x, y) = ax + by + c$ is a plane with linear dependence on x and y .

Example. Consider the graph of:

$$f(x, y) = x^2 + y^2, \quad D_f = \mathbb{R}^2$$

This surface is called a paraboloid.

Example. A sphere of radius r is defined by

$$x^2 + y^2 + z^2 = r^2$$

All points x, y, z satisfy the equation.

Is this the graph of a function? No!

There is no unique value of z , associated with (x, y) because:

$$z = \pm \sqrt{r^2 - x^2 - y^2}$$

Both satisfy the sphere equation. Lets consider the graph of

$$f(x, y) = \sqrt{r^2 - x^2 - y^2}$$

With the domain $x^2 + y^2 \leq r^2$. The grap is a half sphere.

2.3 Level Sets

Another way to visualize functions.

Definition 3. A level set of a function $f(x, y)$ is constant. Essentialy, this is a topographic map.

Example. Consider the function

$$f(x, y) = x^2 + y^2$$

The level sets for $c > 0$ are circles.

$$f(x, y) = x^2 + y^2 = c = (\sqrt{c})^2$$

This is a circle with radius \sqrt{c} .

Now consider the case $c < 0$, then:

$$f(x, y) = x^2 + y^2 = c$$

which doesn't work, because the level sets are empty.

For $c = 0$, we only have the point $(x, y) = (0, 0)$. Generally, level sets of $f(x, y)$ is a curve.

3 Derivatives

3.1 Partial Derivatives

In the case of one variable, we have

$$\frac{df}{dt} = \lim_{n \rightarrow 0} \frac{f(x+n) - f(x)}{n}$$

Similarly, for two or more variables, we have the following definition

Definition 4. The partial derivative of $f(x, y)$ with respect to x :

$$\frac{\partial f}{\partial x} = \lim_{n \rightarrow 0} \frac{f(x+n, y) - f(x, y)}{n}$$

Also written as f_x , for $\frac{\partial f}{\partial y}$, we have f_y

Note, the expression above is $\frac{\partial f}{\partial x}(x, y)$, which is the value at the point (x, y)

3.2 Higher order derivatives

Given $\frac{\partial f}{\partial x}$, we can take further derivatives. We have

$$\frac{\partial^2 f}{\partial^2 x} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

Also written as $f_{xx}, f_{yy}, f_{xy}, f_{yx}$. In most cases, f_{xy} and f_{yx} coincide.

Theorem 1. Schwartz theorem: Suppose f_{xy} and f_{yx} exist, and are continuous, then

$$f_{xy} = f_{yx}$$

Similar definitions and results for the case of more variables: x_1, \dots, x_n , with n variables.

3.3 Chain Rule

Suppose $f(x) = g(h(x))$, for instance

$$f(x) = (\cos x)^2 \text{ with } g(x) = x^2, h(x) = \cos x$$

then the chain rule is

$$\frac{df}{dt}(x_0) = \frac{dg}{dh}(h(x_0)) \cdot \frac{dh}{dt}(x_0)$$

Generalization to more variables.

Theorem 2. Chain rule: consider $f(x, y)$ x and y depending on a variable t . Then:

$$\frac{df}{dt} t_0 = \frac{\partial f}{\partial x}(x(t_0), y(t_0)) \frac{dx}{dt} t_0 + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \frac{dy}{dt} t_0$$

The "short form" of this result is

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Example. Consider $f(x, y) = xy$, where

$$x(t) = \cos t, y(t) = \sin t$$

This cannot be computed directly with $\frac{df}{dt}$.

$$f(t) = f(x(t), y(t)) = f(\cos t, \sin t) = \cos t \cdot \sin t$$

We can compute

$$\frac{df}{dt} = (\cos t)' \sin t + \cos t (\sin t)' = -(\sin t)^2 + (\cos t)^2$$

Using the chain rule, we get

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

3.4 The Gradient

Define an operation that takes a scalar function, and returns a vector function.

Definition 5. The gradient of $f(x, y)$ at (x_0, y_0) is

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

4 Directional Derivatives and Critical Points

4.1 Directional Derivatives

We have seen the following

- f_x = the rate of change along the x -direction.
- f_y = the rate of change along the y -direction.

What about general directions?

Definition 6. Let $\vec{u} = (a, b)$, the directional derivative along \vec{u} at (x, y) is

$$D_{\vec{u}}f(x, y) = \lim_{h \rightarrow 0} \frac{f(x + ha, y + hb) - f(x, y)}{h}$$

Note that

$$\vec{u} = (1, 0) \rightarrow D_{\vec{u}} = f_x$$

$$\vec{u} = (0, 1) \rightarrow D_{\vec{u}} = f_y$$

To compare directions, we take $|\vec{t}| = 1$. Here $|\vec{u}|$ is the length of \vec{u} , that is

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

Proposition 1. We have the following result

$$D_{\vec{u}} \cdot f = \nabla f \cdot \vec{u}$$

Proof. Consider the following function

$$g(t) = f(x + ta, y + tb)$$

Its derivative at $t = 0$ is

$$\begin{aligned} \frac{dg}{dt}(0) &= \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + ha, y + hb) - f(x, y)}{h} \\ &= D_{\vec{u}} \cdot f(x, y) \end{aligned}$$

On the other hand, using the chain rule, we get

$$\begin{aligned} \frac{dg}{dt}(0) &= \frac{\partial f}{\partial x} \frac{d(x + ta)}{dt} + \frac{\partial f}{\partial y} \frac{d(y + tb)}{dt} \\ &= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b = \nabla f \cdot \vec{u} \end{aligned}$$

□

We can now state another property of ∇f . The direction where f changes the most.

Proposition 2. $|D_{\vec{u}} \cdot f|$ is the largest when \vec{u} is parallel to ∇f .

Proof. Recall that, given two vectors, \vec{v} and \vec{w} , we have that

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos \alpha$$

When is $|\vec{v} \cdot \vec{w}|$ the largest? We have

$$|\vec{v} \cdot \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot |\cos \alpha|, \quad |\cos \alpha| \leq 1$$

It is the largest when the following condition is true

$$|\cos \alpha| = 1, \quad \alpha = 0 \vee \pi$$

Which means that the vectors are pointing in the same, or opposite direction. Applying this to $D_{\vec{u}} \cdot f$, we get

$$\begin{aligned} |D_{\vec{u}} \cdot f| &= |\nabla f \cdot \vec{u}| \\ &= |\nabla f| \cdot |\vec{u}| \cdot |\cos \alpha| \end{aligned}$$

For fixed values of $|\vec{u}|$, this is the largest when $\alpha = 0 \vee \pi$. That is ∇f and \vec{u} are parallel. □

4.2 Critical Points

How do we find the maxima and minima of $f(x)$? Lets take a look at $f'(x_0) = 0$.

Definition 7. We say that x_0, y_0 is a critical point of f if:

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \quad \wedge \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

Similarly for

5 Lecutre 5

5.1 Hessian Matrix

Example. Consider again the function $f(x, y) = x^2 - y^2$.

$$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{yx} = 0$$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \det H = -4 < 0$$

Hence (x_0, y_0) is a saddle point.

Example. Consider the function $f(x, y) = x^2 + y^2$, we have

$$(f_x, f_y) = (2x, 2y)$$

The only critical point is $(x_0, y_0) = (0, 0)$.

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \det H = 4 > 0$$

Since $f_{xx} = 2 > 0$, we conclude that $(0, 0)$ is a local minima. In this case, it is actually a global minimum, because $f(x, y) = x^2 + y^2 \geq 0$.

5.2 Global extremal values

A function can have many maxima and minimas. Usually, we are intrested in the largest and smallest values.

Definition 8. Let $f(x, y)$ be with domain D_f . Then we have

- (x_0, y_0) is a global maxima if $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in D_f$.
- (x_0, y_0) is a global minima if $f(x_0, y_0) \leq f(x, y)$ for all $(x, y) \in D_f$.

Trivial example: for $f(x, y) = 1$, all points are global maxima and minima.

Note that global maxima and minima need not be critical points.

Example. We have $f(x) = x$ with $D_f = [-1, 1]$.

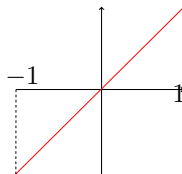


Figure 1: $f(x)$

- Global maxima at $x = 1, f(1) = 1$.

- Global minima at $x = -1, f(-1) = -1$

We have no critical points because $f'(x) = 1 \neq 0$. Also note that maxima and minima depend on the chosen domain.

If we take $D_f = [-2, 3]$, then

$$\text{Max: } x = 3, \quad \text{Min: } x = -2$$

Theorem 3. Let f be continuous with domain D_f . Suppose D_f is closed and bounded, then there is at least one global maxima, and one global minima.

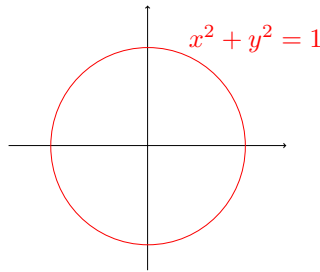


Figure 2: The circle is the boundary.

Some terminology:

$$\text{open} = \{x^2 + y^2 < 1\}$$

$$\text{closed} = \{x^2 + y^2 \leq 1\}$$

The method for finding maxima and minima is as follows:

1. Find critical points of f in D_f , and characterize them.
2. Study the points that are on the boundary.
3. Compare them.

Example. Consider $f(x, y) = x^2 + y^2$ with domain

$$D = \{(x, y) \in \mathbb{R} : x^2 + y^2 \leq 1\}$$

The domain in this case is a disc. The red circle is the boundary.

We compute $f_x = 2x, f_y = 2y$. The only critical point is the origin at $(x_0, y_0) = (0, 0)$. This is a global minimum since $f(0, 0) = 0$ and $f(x, y) \geq 0$.

Now, let's consider the boundary

$$C = \{x^2 + y^2 = 1\}$$

For any point (x_0, y_0) on the circle C , we have

$$f(x_0, y_0) = x_0^2 + y_0^2 = 1$$

We claim that this point is a global maximum. For any (x, y) in domain D_f , we have

$$f(x, y) = x^2 + y^2 \leq 1$$

The value $f(x, y) = 1$ is obtain only at the boundary C . Any point on the circle is a global maxima.

5.3 Constrained optimization

In this section, we will discuss how to find maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. Think of $g = 0$ as a budget, or a geometrical constraint.

Example. We want to minimize $f(x, y) = x^2 + y^2$ with the constraint $g(x, y) = xy - 1 = 0$.

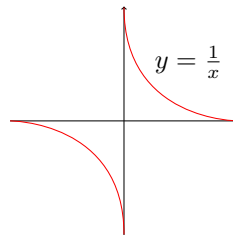


Figure 3: $y = \frac{1}{x}$

We have the following strategy

- Solve $g = 0$ for one variable. For instance $y = \frac{1}{x}$.
- Consider: $h(x) = f\left(x, \frac{1}{x}\right) = x^2 + x^{-2}$

Now we can study this function of one variable with no constraints. We can proceed as usual.

$$\frac{dh}{dx} = 2x - 2x^{-3} = 0$$

This is equivalent to $x^4 = 1$. The real solutions are $x = \pm 1$. Since $y = \frac{1}{x}$, we get the critical points:

$$(x, y) = (1, 1), \quad (x, y) = (-1, -1)$$

6 Lecture 6

6.1 Substitution Method

We want to maximize / minimize $f(x, y)$ with constraint $g(x, y) = 0$. We can solve $g(x, y) = 0$ for one variable $y = y(x)$.

Example. Consider $f(x, y) = x^2 + y^2$ and $g(x, y) = xy - 1 = 0$. In this case, we have $f = \frac{1}{x}$ from $g = 0$, we get

$$h(x) = f(x, x^{-1}) = x^2 + x^{-2}$$

We have found the minima at $(1, 1)$ and $(-1, -1)$.

This method isn't always feasible, so let's look at some alternatives.

6.2 Lagrange's Method

Example. Let's look at the level curves, which are circles. We have that $f(x, y) = x^2 + y^2 = c$.

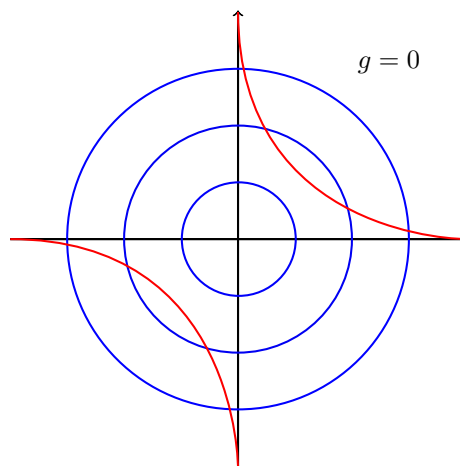
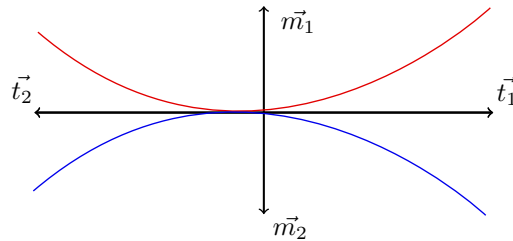


Figure 4: $f(x, y)$

Smaller circles correspond to smaller values of $f(x, y)$, but we must also satisfy $g(x, y) = 0$. In the best case, $f(x, y) = c$ is just touching $g(x, y) = 0$. If this is worked out geometrically, we get $(1, 1)$ and $(-1, -1)$.

This idea is used in Lagrange's method. We want $f(x, y) = c$ to be parallel to $g(x, y) = 0$. More precisely: Their vectors should be parallel.



Equivalently, their normal vectors are also parallel. Recall that a normal vector to $g = 0$ is given by ∇g . Similarly, ∇f is normal to $f = c$.

6.3 Method for Lagrange

Suppose we want to find a local maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. We proceed as follows

1. Find all possible solutions to the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0$$

2. Plug in all solutions from 1. into $f(x, y)$ and identify the largest and smallest.

- The number λ is called the *Lagrange Multiplier*.
- Easy extension to n variables.
- Can also be generalized to multiple constraints, g_1, \dots, g_n

Example. We have the following functions

$$f(x, y) = x^2 + y^2, \quad g(x, y) = xy - 1 = 0$$

We compute the gradients

$$\nabla f(x, y) = (2x, 2y), \quad \nabla g(x, y) = (y, x)$$

The Lagrange equations are

$$2x = \lambda y, \quad 2y = \lambda x, \quad xy - 1 = 0$$

Observe that $(x, y, \lambda) = (0, 0, 0)$ is not a solution. For $x \neq 0$, we get $y = \frac{1}{x}$, from the third equation.

$$\begin{aligned}
 2x &= \lambda y \Rightarrow 2x = \lambda \frac{1}{x} \Rightarrow \lambda = 2x^2 \\
 2y &= \lambda x \Rightarrow \frac{2}{x} = 2x^2 \cdot x \Rightarrow x^4 = 1 \\
 x^4 &= 1 \Rightarrow x = \pm 1, \quad y = \frac{1}{x}
 \end{aligned}$$

$$(x, y) = (1, 1) \quad \wedge \quad (x, y) = (-1, -1)$$

Are these points the minima? We have

$$f(1, 1) = f(-1, -1) = 2$$

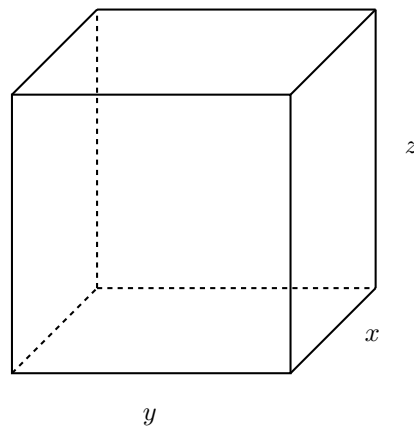
Lets Compare this to some other points, such that $g(x, y) = 0$. For instance, $(x, y) = (2, \frac{1}{2})$

$$g\left(2, \frac{1}{2}\right) = 2 \cdot \frac{1}{2} - 1 = 0$$

$$f\left(2, \frac{1}{2}\right) = 2^2 + \left(\frac{1}{2}\right)^2 > 2$$

This tells us that $(x, y) = (\pm 1, \pm 1)$ are local minima.

Example. Consider a box of surface area 24 cm^2 .



Determine the dimensions (x, y, z) such that the volume is max.

We have the surface area $2xy + 2xz + 2yz$. Our constraint is

$$g(x, y, z) = 2xy + 2xz + 2yz - 24 = 0$$

The goal is to maximize $f(x, y, z) = xyz$, with the constraint $g(x, y, z) = 0$.

The equation $\nabla f = \lambda \nabla g$ gives

$$yz = 2\lambda(y + 2), \quad xz = 2\lambda(x + 2), \quad xy = 2\lambda(x + y)$$

Observe that $\lambda \neq 0$, since $x, y, z > 0$. To solve the equations, we can multiply by x, y and z , respectively, then we get

$$x(y + z) = y(x + z) = x(y + x)$$

Consider $x(y + z) = y(x + z)$.

$$x(y + z) = y(x + z) \Rightarrow (x - y)z = 0$$

$$z \neq 0, \quad x = y$$

It's the same for the other equations, so $x = y = z$. We also need to use $g = 0$. Setting $x = y = z$, we get

$$g(x, x, x) = 2x^2 + 2x^2 + 2x^2 - 24 = 0$$

$$x^2 = 4$$

$$x = 2$$

Lagrange's method gives:

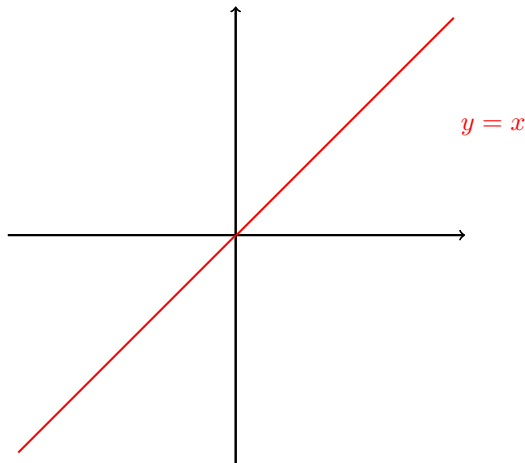
$$(x, y, z) = (2, 2, 2) \Rightarrow V = 8$$

7 Lecture 7

7.1 Parametrized Curve

A curve is described as a sset of points in \mathbb{R}^2 and \mathbb{R}^3 . For instance a line is described by

$$f = \{(x, y) \in \mathbb{R}^2 : x = y\}$$



This is a static picture. But how do we give a dynamical picture? We'll use parametrized curves.

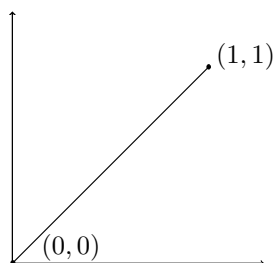
Definition 9. A parametrization of a curve c in \mathbb{R}^2 , is given by

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Such that $\vec{r}(t) \in c$ for all time t .

- A parametrization describes motion. Think of t as the time.
- A parametrization is not unique.
- Various natural assumptions, such as continuity and differentiability.

Example. Consider the funtion $\vec{r}(t)$ with $0 \leq t \leq 1$.



$$\vec{r}(0) = (0, 0), \quad \vec{r}(1) = (1, 1)$$

We have the portion of the line where $y = x$. Notice that here, $x(t) = t$, $y(t) = t$ and $y(t) = x(t) = t$ for all t .

Lets consider a different function,

$$\vec{r}(t) = (2t, 2t), \quad 0 \leq t \leq \frac{1}{2}$$

When we parametrize, we get

$$\vec{r}(0) = (0, 0), \quad \vec{r}\left(\frac{1}{2}\right) = (1, 1)$$

We are moving along the curve at twice the speed.

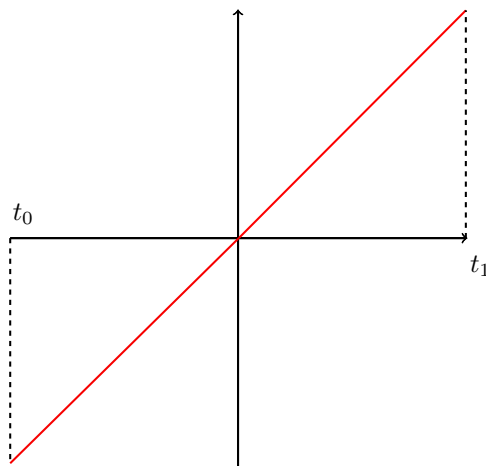
Example. Given $f(x)$, we consider

$$\vec{r}(t) = (t, f(t)), \quad t_0 \leq t \leq t_1$$

This describes a portion of the graph f , with

$$\text{Start: } (t_0, f(t_0)), \quad \text{End: } (t_1, f(t_1))$$

For instance, consider the line $y = mx + c$, we have



We have that

$$\vec{r}(t) = (t, mt + c), \quad t_0 \leq t \leq t_1$$

Example. We want to describe a line with

$$\text{Start: } A = (x_0, y_0), \quad \text{End: } B = (x_1, y_1)$$

Then we take the parametrization

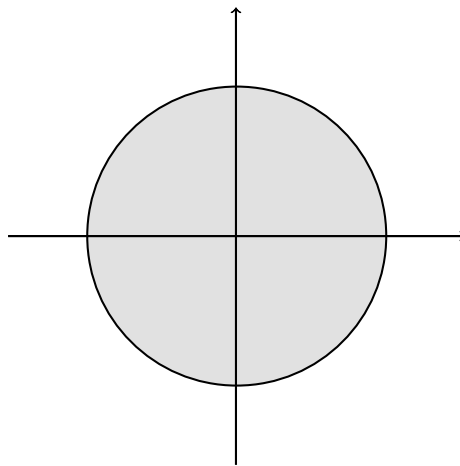
$$\vec{r}(t) = (1-t)A + tB, \quad 0 \leq t \leq 1$$

More explicitly, we have

$$\vec{r}(t) = ((1-t)x_0 + tx, (1-t)y_0 + ty)$$

Note that $\vec{r}(0) = A$ and $\vec{r}(1) = B$.

Example. Consider $\vec{r}(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$. What curve does this describe?



It describes a circle.

$$x(t)^2 + y(t)^2 = (\cos(t)^2 + \sin(t)^2) = 1$$

We start at $(1, 0)$ and move counter-clockwise.

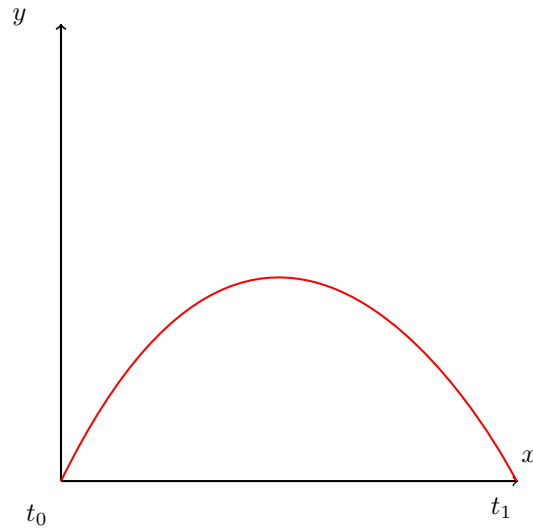
Example. Here is an example from physics. Consider

$$x(t) = v_x t, \quad y(t) = v_y t - \frac{1}{2}gt^2, \quad 0 \leq t \leq \frac{2v_y}{g}$$

This describes the motion of an object with initial velocity $\vec{v} = (v_x, v_y)$, under gravity. We write $t_0 = 0$ and $t_1 = \frac{2v_y}{g}$. Note that

$$\vec{r}(t_0) = (0, 0), \quad \vec{r}(t_1) = \left(\frac{2v_x v_y}{g}, 0 \right)$$

The object falls back to the ground at time t_1 .



Well known fact: This motion is parabolic, we will rederive this.

From $x(t) = v_x t$, we get $t = \frac{x(t)}{v_x}$. Then

$$y(t) = v_y t - \frac{1}{2} g t^2 \Rightarrow \frac{v_x}{v_y} x(t) - \frac{1}{2} \frac{g}{v_x^2} x(t)^2$$

This is the expression of a parabola

$$y = ax^2 + bx + c, \quad a \neq 0$$

It can also be written as

$$y(t) = -\frac{1}{2} \frac{g}{v_x^2} \left(x(t) - \frac{v_y}{g} \right)^2 + \frac{1}{2} \frac{v_y}{g} \frac{v_y}{v_x}$$

7.2 Kinematics

Kinematics describes position, velocity and acceleration of an object.

Definition 10. The position vector is $\vec{r}(t)$. The velocity vector is $\vec{v}(t) = \frac{d\vec{r}}{dt}$. The acceleration vector is $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2}$.

If we write $\vec{r}(t) = (x(t), y(t))$, then

$$\vec{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (x'(t), y'(t))$$

Similarly

$$\vec{a}(t) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right) = (x''(t), y''(t))$$

Example. Consider again the gravity example. Here we have

$$\vec{r}(t) = \left(v_x t, v_y t - \frac{1}{2} g t^2 \right)$$

The velocity is

$$\vec{v}(t) = (v_x, v_y - g t)$$

Note that $v(0) = (v_x, v_y)$ is the initial velocity of the object. For acceleration, we get

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (0, -g)$$

8 Lecture 8

8.1 Determining Motion

Given acceleration $\vec{a}(t)$, can we find $\vec{v}(t)$ and $\vec{s}(t)$? Yes, with some initial conditions given, we can. This is done by integration, consider

$$\vec{v}(t) = \frac{d\vec{s}}{dt}$$

This is a differential equation for $\vec{s}(t)$. To solve it, we integrate both sides in t , from t_1 , to t_2 . We get

$$\int_{t_0}^{t_1} \vec{v}(t) dt = \int_{t_0}^{t_1} \frac{d\vec{s}}{dt} dt$$

The fundamental theorem of calculus gives

$$\vec{s}(t_1) - \vec{s}(t_0) = \int_{t_0}^{t_1} \vec{v}(t) dt$$

We can determine $\vec{s}(t)$ for any t if we know $\vec{v}(t)$ and the initial condition $\vec{s}(t_0)$.

Example. Consider an object with acceleration

$$\vec{a}(t) = (1, t) = \vec{i} - j\vec{j}$$

We have the following initial conditions

$$\vec{s}(0) = (2, 0) = 2\vec{i} \quad \wedge \quad \vec{v}(0) = 0$$

We want to determine $\vec{s}(t)$. First, to determine $\vec{v}(t)$, we compute

$$\begin{aligned} \int_0^t \vec{a}(t) dt &= \vec{i} \int_0^t 1 dt + \vec{j} \int_0^t t dt \\ &= t\vec{i} + \frac{1}{2}t^2\vec{j} \end{aligned}$$

Here $t_0 = 0$, since $\vec{v}(0) = 0$, then

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t) dt = t\vec{i} + \frac{1}{2}t^2\vec{j}$$

To determine \vec{s} , we compute

$$\begin{aligned} \int_0^t \vec{v}(t) dt &= \vec{i} \int_0^t t dt + \vec{j} \int_0^t \frac{1}{2}t^2 dt \\ &= \frac{1}{2}t^2\vec{i} + \frac{1}{6}t^3\vec{j} \end{aligned}$$

Since $\vec{s}(0) = (2, 0) = 2\vec{i}$, we get

$$\vec{s}(t) = \vec{s}(0) + \int_0^t \vec{v}(t) dt = \left(\frac{t^2}{2} + 2\right)\vec{i} + \frac{1}{6}t^3\vec{j}$$

8.2 Arc Length

A formula that we can concretely use to compute the length of a curve (using a parametrization). Consider a curve c , with parametrization

$$\vec{s}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Definition 11. The arc length of c is given by

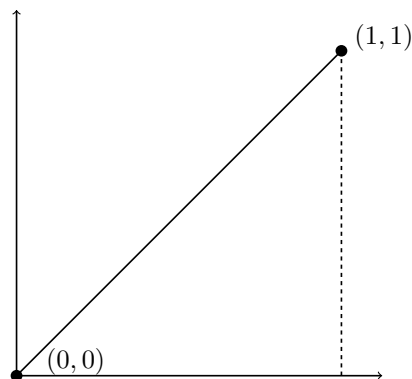
$$S = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Idea: summing segments of length



To compute S , we need a parametrization of c . Does S depend on this choice? No!

Example. Consider the line segment below



From elementary geometry, its length is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Consider the parametrization

$$\vec{s}(t) = (t, t), \quad 0 \leq t \leq 1$$

We have $(x'(t), y'(t)) = (1, 1)$. Then

$$S = \int_0^1 \sqrt{1^2 + 1^2} dt = \sqrt{2} \int_0^1 1 dt = \sqrt{2}$$

Instead we choose

$$\vec{s}(t) = (2t, 2t), \quad 0 \leq t \leq \frac{1}{2}$$

We have $(x'(t), y'(t)) = (2, 2)$. Then

$$S = \int_0^{\frac{1}{2}} \sqrt{2^2 + 2^2} dt = \sqrt{8} \int_0^{\frac{1}{2}} 1 dt = \sqrt{8} \cdot \frac{1}{2} = \sqrt{2}$$

Question: What is the distance crossed up to time t ?

Definition 12. The arc length parameter is

$$S(t) = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The difference is that we integrate up to t , not t_1 . Special case when $S(t_1) = S$. There is an important relation between $S(t)$ and $\vec{v}(t)$.

Proposition 3. We have

$$|\vec{v}(t)| = \frac{dS}{dt}$$

Proof. The fundamental theorem of calculus states that if

$$F(x) = \int_a^x f(x) dt \rightarrow F'(x) = f(x)$$

Applying this to $S(t)$, then

$$S'(t) = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

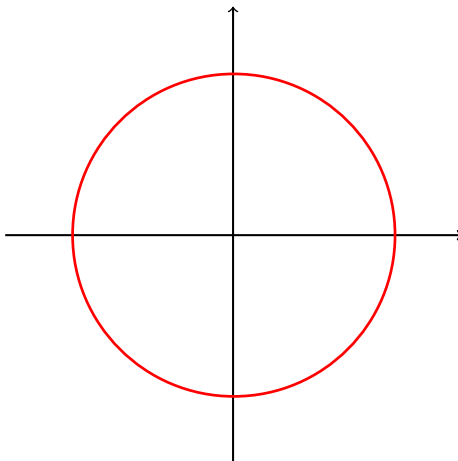
On the other hand, we have that

$$\vec{v}(t) = (x'(t), y'(t)) \quad \wedge \quad |\vec{v}(t)| = (t) = \sqrt{x'(t)^2 + y'(t)^2}$$

The two expressions coincide. □

Example. Consider a circle of radius R , with

$$\vec{S}(t) = (R \cos t, R \sin t), \quad 0 \leq t \leq 2\pi$$



We want to compute $S(t)$, we have

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{R^2 (\sin t)^2 + R^2 (\cos t)^2} = R$$

We want to check that $\frac{dS}{dt} = |\vec{v}(t)|$. We have

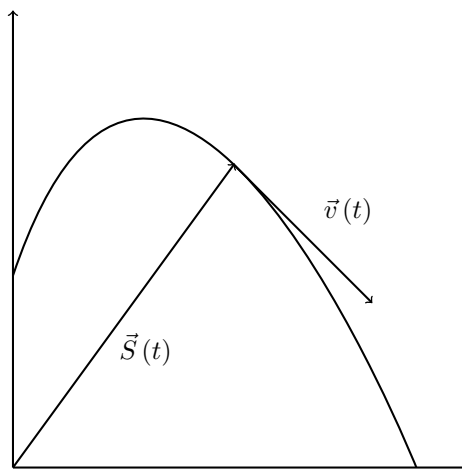
$$\vec{v}(t) = \frac{d\vec{S}}{dt} = (-R \sin t, R \cos t)$$

Its length is equal to $|\vec{v}(t)| = R$. Since $S(t) = R(t)$, we see that

$$\frac{dS}{dt} = |\vec{v}(t)|$$

8.3 Tangent Vectors

Geometrically, the velocity $\vec{v}(t)$ is tangent to a curve. It is useful to define a tangent vector of length 1.



Definition 13. The unit tangent vector is

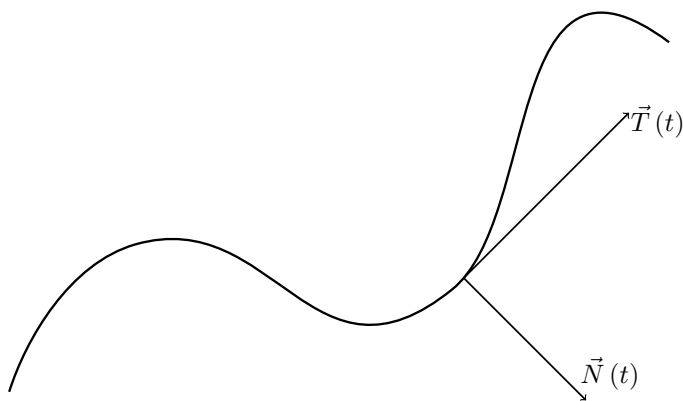
$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

Note that \vec{T} has length 1 since

$$\vec{T}(t) \cdot \vec{T}(t) = \frac{\vec{v}(t) \cdot \vec{v}(t)}{|\vec{v}(t)|^2} = 1$$

8.4 Normal Vectors

Normal vectors are normal to the curve, or in other words, they are orthogonal. Recall that for implicit curves $f(x, y) = 0$, a normal vector is given by ∇f .



Lets now consider parametrised curves. We have

$$S(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Definition 14. A unit normal vector to the curve is defined by

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left| \frac{d\vec{T}}{dt} \right|}$$

We need to check that \vec{N} is orthogonal to \vec{T} , that is: $\vec{N}(t) \cdot \vec{T}(t) = 0$ for all t .

Proposition 4. We have that

$$\vec{N}(t) \cdot \vec{T}(t) = 0$$

Proof. Since \vec{T} is a unit vector, we have that, for all t

$$\vec{T}(t) \cdot \vec{T}(t) = 1$$

Take the time derivative, the $\left(\vec{T} \cdot \vec{T} \right) = 0$. We also have

$$\frac{d}{dt} \left(\vec{T} \cdot \vec{T} \right) = \frac{d\vec{T}}{dt} \cdot \vec{T} + \vec{T} \frac{d\vec{T}}{dt} = 2 \frac{d\vec{T}}{dt} \cdot \vec{T}$$

Since $\left(\vec{T} \cdot \vec{T} \right) = 0$, we get $\frac{d\vec{T}}{dt} \cdot \vec{T} = 0$. Dividing by $\left| \frac{d\vec{T}}{dt} \right|$, we get $\vec{N}(t) \cdot \vec{T}(t) = 0$ □

9 Line Integrals, Parametrization and Vector Fields

9.1 Identities between operations

We have seen three operations defined by ∇ .

Gradient: ∇f

Divergence: $\nabla \cdot \vec{F}$

Curl: $\nabla \times \vec{F}$

There are many of them, we look at only one.

Proposition 5. For any scalar field f we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$. Then

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \vec{i}(f_{zy} - f_{yz}) - f_{zx} + f_{xz} + f_{yx} - f_{cy}$$

But partial derivatives can be exchanged, $f_{xy} = f_{yx}$. Then we find that

$$\nabla \times (\nabla f) = 0$$

□

We are going to use this when we discuss conservative fields.

9.2 Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Consider the points $\vec{r}(t)$ and $\vec{r}(t + \Delta t)$ where Δt is small.

$$\Delta x = x(t + \Delta t) - x(t), \quad \Delta y = y(t + \Delta t) - y(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt} \Delta t, \quad \Delta y \approx \frac{dy}{dt} \Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt} \Delta t\right)^2 + \left(\frac{dy}{dt} \Delta t\right)^2} = \Delta t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Definition 15. The line integral of $f(x, y)$ along a curve C is defined by

$$\int_C f dS = \int_{t_0}^{t_1} f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Some observations:

- Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$.
- When $f = 1$, we recover the arc length.

Example. Consider C defined by

$$x(t), y(t) = 0, \quad 0 \leq t \leq 1$$

First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} = 1$$

Now, consider $f(x, y) = x^2 + y$. Compute $\int_C f dS$. Restricting f to C gives

$$f(x(t), y(t)) = x(t)^2 + y(t) = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f dS = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

- Line integrals can be used to compute the mass of an object (1 - dimensional)
- The curve C describes the object
- The function $\int_C f dS$ is the mass.

9.3 Parametrization and orientation

Next results as for the arc length.

Proposition 6. The integral $\int_C f dS$ does not depend on the parametrization of C .

We will consider a special case.

Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\text{Start: } A = (x(t_0), y(t_0)), \quad \text{End: } B = (x(t_1), y(t_1))$$

We want to go from B to A, which can be done in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\vec{r}_{\text{opp}}(-t_1) = (x(t_1), y(t_1)) = B$$

$$\vec{r}_{\text{opp}}(-t_0) = (x(t_0), y(t_0)) = A$$

Example. Consider C = segment from $(0,0)$, to $(1,0)$, take $f(x,y) = x$. Show that $\int_C f dS = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$.

If C parametrized by $\vec{r}(t)$, we will use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have that

$$\int_C f dS = \int_{-C} f dS$$

The situation will be different for vector fields.

9.4 Case of Vector Fields

Consider the curve with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\vec{r}'(t) = (x'(t), y'(t)) \quad \text{Velocity Vector}$$

Definition 16. The line integral of \vec{F} along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

Here $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$

- In physics, we have that \vec{F} = Force and $\int_C \vec{F} d\vec{r}$ = Work done by \vec{F} along C .
- The elementary case $W = FS$, where W is work, F is Force and S is the displacement.

More explicitly, we consider

$$\vec{F}(x, y) = (P(x, y), Q(x, y))$$

Then we have that

$$\vec{F}(\vec{r}(t) \cdot \vec{r}'(t)) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t) dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t) dt$$

We will write the expression as

$$\int_C \vec{F} d\vec{r} = \int_C P dx + \int_C Q dy$$

Example. Consider a curve C with

$$\vec{r}(t) = (t, t^2), \quad 0 \leq t \leq 1$$

We have $x(t) = t$ and $y(t) = t^2$, its derivative is

$$\vec{r}'(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is $P = x + y$ and $Q = x$. When restricted to C , we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = (t + t^2, t)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = (t + t^2, t) \cdot (1, 2t) = (3t^2 + t)$$

We then obtain the following

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + t) dt = \frac{3}{2}$$