1 Lecture 8

1.1 Determining Motion

Given acceleration $\vec{a}(t)$, can we find $\vec{v}(t)$ and $\vec{s}(t)$? Yes, with some initial conditions given, we can. This is done by integration, consider

$$\vec{v}(t) = \frac{d\vec{s}}{dt}$$

This is a differential equation for $\vec{s}(t)$. To solve it, we integrate both sides in t, from t_1, to t_2. We get

$$\int_{t_0}^{t_1} \vec{v}(t)\, dt = \int_{t_0}^{t_1} \frac{d\vec{s}}{dt} dt$$

The fundamental theorem of calculus gives

$$\vec{s}(t_1) - \vec{s}(t_0) = \int_{t_0}^{t_1} \vec{v}(t)\, dt$$

We can determine $\vec{s}(t)$ for any t if we know $\vec{v}(t)$ and the initial condition $\vec{s}(t_0)$.

Example. Consider an object with acceleration

$$\vec{a}(t) = (1, t) = \vec{i} - j\vec{j}$$

We have the following initial conditions

$$\vec{s}(0) = (2, 0) = 2\vec{i} \quad \wedge \quad \vec{v}(0) = 0$$

We want to determine $\vec{s}(t)$. First, to determine $\vec{v}(t)$, we compute

$$\begin{aligned} \int_0^t \vec{a}(t)\, dt &= \vec{i}\int_0^t 1dt + \vec{j}\int_0^t tdt \\ &= t\vec{i} + \frac{1}{2}t^2\vec{j} \end{aligned}$$

Here $t_0 = 0$, since $\vec{v}(0) = 0$, then

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t)\, dt = t\vec{i} + \frac{1}{2}t^2\vec{j}$$

To determine $\vec{s}$, we compute

$$\begin{aligned} \int_0^t \vec{v}(t)\, dt &= \vec{i}\int_0^t tdt + \vec{j}\int_0^t \frac{1}{2}t^2 dt \\ &= \frac{1}{2}t^2\vec{i} + \frac{1}{6}t^3\vec{j} \end{aligned}$$

Since $\vec{s}(0) = (2, 0) = 2\vec{i}$, we get

$$\vec{s}(t) = \vec{s}(0) + \int_0^t \vec{v}(t)\, dt = \left(\frac{t^2}{2} + 2\right)\vec{i} + \frac{1}{6}t^3\vec{j}$$

1.2 Arc Length

A formula that we can concretely use to compute the length of a curve (using a parametrization). Consider a curve c, with parametrization

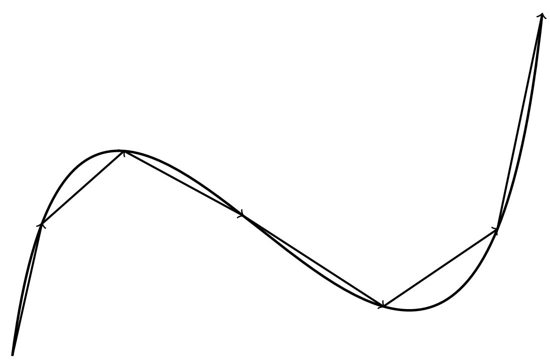
$$\vec{s}\,(t) = (x\,(t)\,, y\,(t))\,, \quad t_0 \leq t \leq t_1$$

Definition 1. The arc length of c is given by

$$S = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

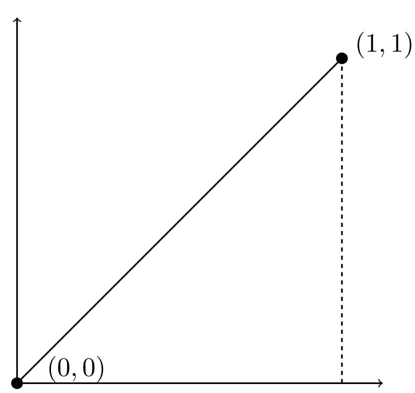
Idea: summing segments of length

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$



To compute S, we need a parametrization of c. Does S depent on this choice? No!

Example. Consider the line segment below



From elementary geometry, its length is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Consider the parametrization

$$\vec{s}(t)\,(t,t)\,, \quad 0 \le t \le 1$$

We have $(x'(t), y'(t)) = (1,1)$. Then

$$S = \int_0^1 \sqrt{1^2+1^2}dt = \sqrt{2}\int_0^1 1dt = \sqrt{2}$$

Instead we choose

$$\vec{s}(t) = (2t, 2t)\,, \quad 0 \le t \le \frac{1}{2}$$

We have $(x'(t), y'(t)) = (2,2)$. Then

$$S = \int_0^{\frac{1}{2}} \sqrt{2^2+2^2}dt = \sqrt{8}\int_0^{\frac{1}{2}} 1dt = \sqrt{8}\cdot\frac{1}{2} = \sqrt{2}$$

Question: What is the distance crossed up to time t?

Definition 2. The arc length parameter is

$$S(t) = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

The difference is that we integrate up to t, not t_1. Special case when $S(t_1) = S$. There is an important relation between $S(t)$ and $\vec{v}(t)$.

Proposition 1. We have

$$|\vec{v}(t)| = \frac{dS}{dt}$$

Proof. The fundamental theorem of calculus states that if

$$F(x) = \int_a^x f(x)dt \to F'(x) = f(x)$$

Applying this to $S(t)$, then

$$S'(t) = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

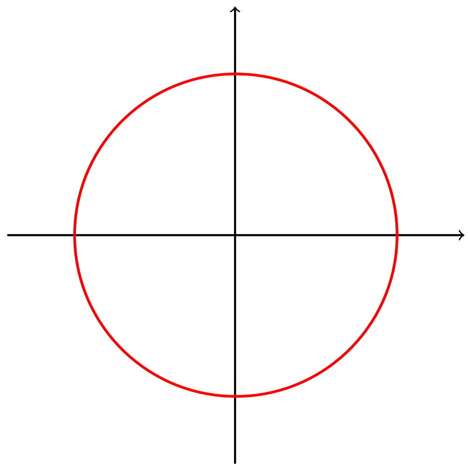
On the other hand, we have that

$$\vec{v}(t) = (x'(t), y'(t)) \quad \wedge \quad |\vec{v}(t)| = (t) = \sqrt{x'(t)^2 + y'(t)^2}$$

The two expressions coincide. □

Example. Consider a circle of radius R, with

$$\vec{S}\,(t) = (R\cos t, R\sin t)\,, \quad 0 \le t \le 2\pi$$



We want to compute $S\,(t)$, we have

$$\sqrt{x'\,(t)^2 + y'\,(t)^2} = \sqrt{R^2\,(\sin t)^2 + R^2\,(\cos t)^2} = R$$

We want to check that $\frac{dS}{dt} = |\vec{v}\,(t)|$. We have

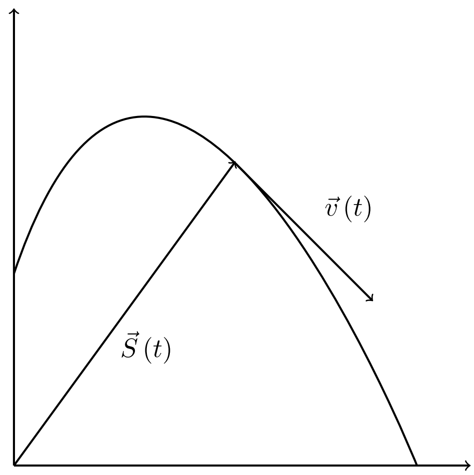
$$\vec{v}\,(t) = \frac{d\vec{S}}{dt} = (-R\sin t, R\cos t)$$

Its length is equal to $|\vec{v}\,(t)| = R$. Since $S\,(t) = R\,(t)$, we see that

$$\frac{dS}{dt} = |\vec{v}\,(t)|$$

1.3 Tangent Vectors

Geometrically,, the velocity $\vec{v}\,(t)$ is tanget to a curve. It is useful to define a tangent vector of length 1.



Definition 3. The unit tangent vector is

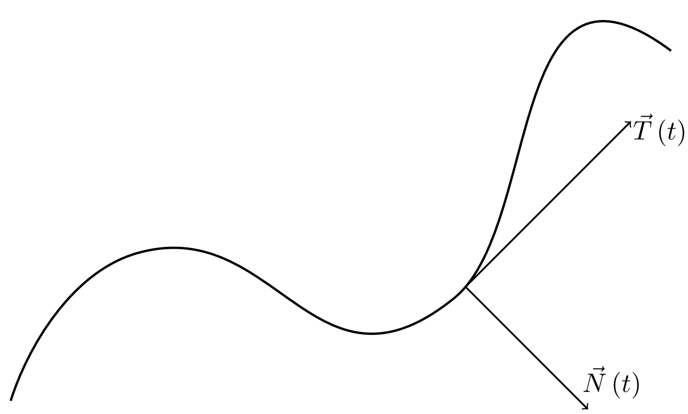
$$\vec{T}\,(t) = \frac{\vec{v}\,(t)}{|\vec{v}\,(t)|}$$

Note that $\vec{T}$ has length 1 since

$$\vec{T}\,(t) \cdot \vec{T}\,(t) = \frac{\vec{v}\,(t) \cdot \vec{v}\,(t)}{|\vec{v}\,(t)|^2} = 1$$

1.4 Normal Vectors

Normal vectors are normal to the curve, or in other words, they are orthogonal. Recall that for implicit curves $f\,(x, y) = 0$, a normal vector is given by ∇f.



Lets now consider parametrizised curves. We have

$$S(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Definition 4. A unit normal vector to the curve is defined by

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

We need to check that $\vec{N}$ is orthogonal to $\vec{T}$, that is: $\vec{N}(t) \cdot \vec{T}(t) = 0$ for all t.

Proposition 2. We have that

$$\vec{N}(t) \cdot \vec{T}(t) = 0$$

Proof. Since $\vec{T}$ is a unit vector, we have that, for all t

$$\vec{T}(t) \cdot \vec{T}(t) = 1$$

Take the time derivative, the $\left(\vec{T} \cdot \vec{T}\right) = 0$. We also have

$$\frac{d}{dt}\left(\vec{T} \cdot \vec{T}\right) = \frac{d\vec{T}}{dt} \cdot \vec{T} + \vec{T}\frac{d\vec{T}}{dt} = 2\frac{d\vec{T}}{dt} \cdot \vec{T}$$

Since $\left(\vec{T} \cdot \vec{T}\right) = 0$, we get $\frac{d\vec{T}}{dt} \cdot \vec{T} = 0$. Dividing by $\left|\frac{d\vec{T}}{dt}\right|$, we get $\vec{N}(t) \cdot \vec{T}(t) = 0$ □