1 Lecture 8

1.1 Determining Motion

Given acceleration $\vec{a}(t)$, can we find $\vec{v}(t)$ and $\vec{s}(t)$? Yes, with some initial conditions given, we can. This is done by integration, consider

$$\vec{v}(t) = \frac{d\vec{s}}{dt}$$

This is a differential equation for $\vec{s}(t)$. To solve it, we integrate both sides in t, from t_1, to t_2. We get

$$\int_{t_0}^{t_1} \vec{v}(t)\, dt = \int_{t_0}^{t_1} \frac{d\vec{s}}{dt} dt$$

The fundamental theorem of calculus gives

$$\vec{s}(t_1) - \vec{s}(t_0) = \int_{t_0}^{t_1} \vec{v}(t)\, dt$$

We can determine $\vec{s}(t)$ for any t if we know $\vec{v}(t)$ and the initial condition $\vec{s}(t_0)$.
Example. Consider an object with acceleration

$$\vec{a}(t) = (1, t) = \vec{i} - j\vec{j}$$

We have the following initial conditions

$$\vec{s}(0) = (2, 0) = 2\vec{i} \quad \wedge \quad \vec{v}(0) = 0$$

We want to determine $\vec{s}(t)$. First, to determine $\vec{v}(t)$, we compute

$$\begin{aligned} \int_0^t \vec{a}(t)\, dt &= \vec{i}\int_0^t 1dt + \vec{j}\int_0^t tdt \\ &= t\vec{i} + \frac{1}{2}t^2\vec{j} \end{aligned}$$

Here $t_0 = 0$, since $\vec{v}(0) = 0$, then

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t)\, dt = t\vec{i} + \frac{1}{2}t^2\vec{j}$$

To determine $\vec{s}$, we compute

$$\begin{aligned} \int_0^t \vec{v}(t)\, dt &= \vec{i}\int_0^t tdt + \vec{j}\int_0^t \frac{1}{2}t^2 dt \\ &= \frac{1}{2}t^2\vec{i} + \frac{1}{6}t^3\vec{j} \end{aligned}$$

Since $\vec{s}(0) = (2, 0) = 2\vec{i}$, we get

$$\vec{s}(t) = \vec{s}(0) + \int_0^t \vec{v}(t)\, dt = \left(\frac{t^2}{2} + 2\right)\vec{i} + \frac{1}{6}t^3\vec{j}$$

1.2 Arc Length

A formula that we can concretely use to compute the length of a curve (using a parametrization). Consider a curve c, with parametrization

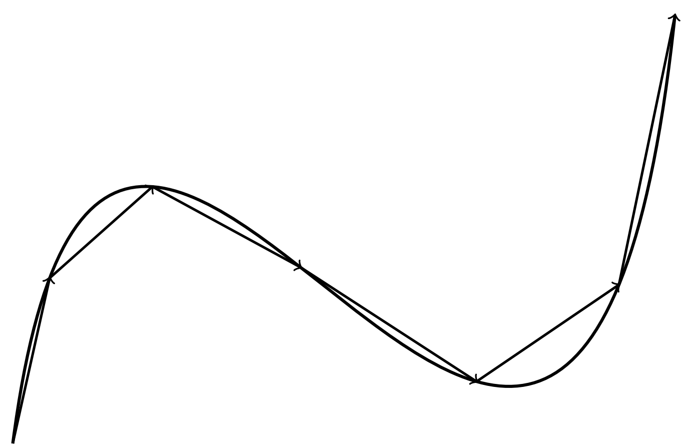
$$\vec{s}\,(t) = (x\,(t)\,, y\,(t))\,, \quad t_0 \le t \le t_1$$

Definition 1. The arc length of c is given by

$$S = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

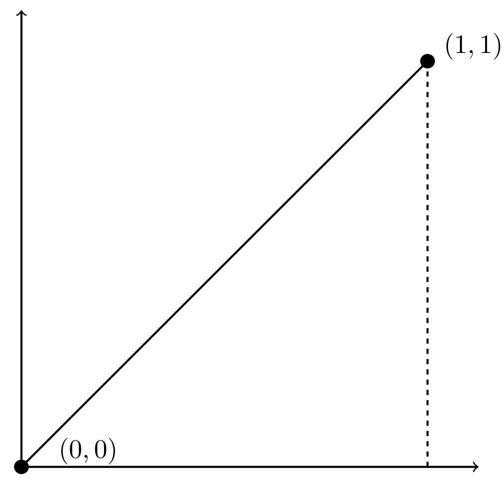
Idea: summing segments of length

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$



To compute S, we need a parametrization of c. Does S depent on this choice? No!

Example. Consider the line segment below



From elementary geometry, its length is $\sqrt{1^2 + 1^2} = \sqrt{2}$