

1 Lecture 12

1.1 Identities Between Operations

We have seen three operations defined by ∇ .

Gradient: ∇f , Divergence: $\nabla \cdot \vec{F}$, Curl: $\nabla \times \vec{F}$

There are many identities, we'll now look at one.

Proposition 1. For any scalar field f , we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$, then

$$\nabla \times (\nabla f) = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix} \right| = (f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{xy})$$

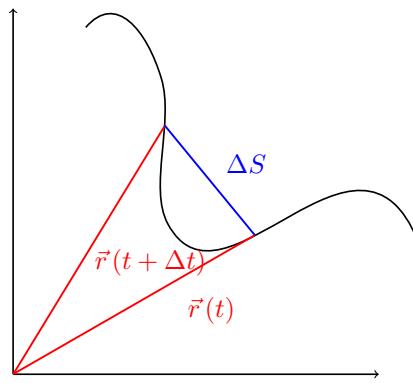
But partial derivatives can be exchanged. Then we find that $\nabla \times (\nabla f) = 0$ □

We are going to use this when we discuss conservative fields.

1.2 Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), t_0 \leq t \leq t_1$$



$$\Delta x = x(t + \Delta t) - x(t), \quad \Delta y = y(t + \Delta t) - y(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x^2) + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt} \Delta t, \quad \Delta y \approx \frac{dy}{dt} \Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt} \Delta t^2\right) + \left(\frac{dy}{dt} \Delta t^2\right)} = \Delta t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Letting $\Delta t \rightarrow 0$ leads to the following

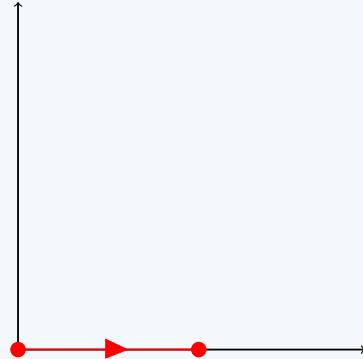
Definition 1. The line integral of $f(x, y)$ along a curve C is defined by

$$\int_C f ds = \int_{t_0}^{t_1} f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Observe. Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$. When $f = 1$, we recover the arc length.

Example. Consider C defined by

$$x(t) = t, \quad y(t) = 0, \quad 0 \leq t \leq 1$$



First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} dt = 1$$

Now, consider $f(x, y) = x^2 + y$. We want to compute $\int_C f ds$. Restricting f to C gives

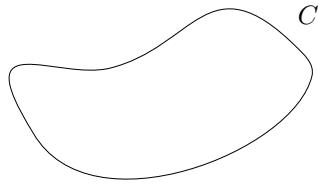
$$f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f ds = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

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Note. Line integrals can be used to compute the mass of a 1-dimensional object. The curve C describes the object, and the function $\int_C f ds$ is the mass.



1.3 Parametrization and Orientation

The next result is as for the arc length.

Proposition 2. The integral $\int_C f ds$ does not depend on the parametrization of C .

We will consider a special case

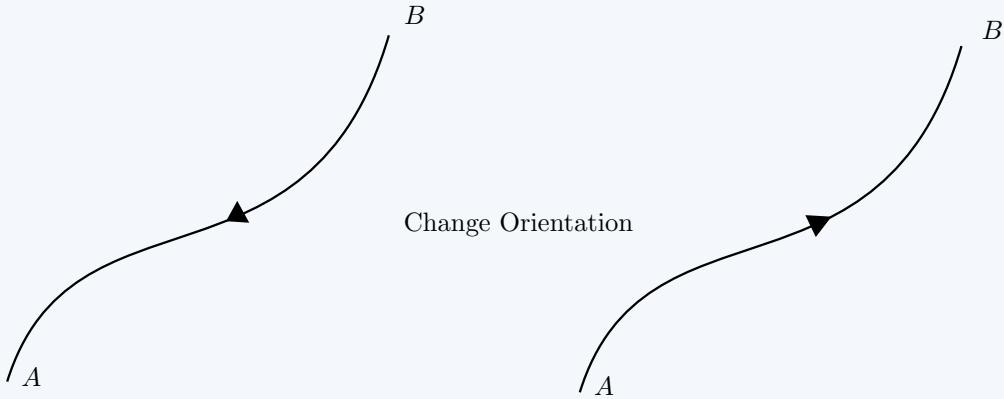
Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), t_0 \leq t \leq t_1$$

We have

$$\text{Start: } A = (x(t_0), y(t_0)), \quad \text{End: } B = (x(t_1), y(t_1))$$

We want to go from B to A .



We can do this in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\begin{aligned}\vec{r}_{\text{opp}}(-t_1) &= (x(t_1), y(t_1)) = B \\ \vec{r}_{\text{opp}}(-t_0) &= (x(t_0), y(t_0)) = A\end{aligned}$$

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Example. Consider the segment C from $(0, 0)$ to $(1, 0)$. Take $f(x, y) = x$. Show that $\int_C f ds = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$.

If C parametrized by $\vec{r}(t)$, we use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have

$$\int_C f ds = \int_{-C} f ds$$

The situation will be different for vector fields. ◊

1.4 Case of Vector Fields

Consider the curve with

$$\begin{aligned}\vec{r}(t) &= (x(t), y(t)), \quad t_0 \leq t \leq t_1 \\ \vec{r}'(t) &= (x'(t), y'(t)) \quad (\text{Velocity vector})\end{aligned}$$

Definition 2. The line integral of \vec{F} along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}' dt$$

Here $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$

In physics, we have that \vec{F} is the force, and $\int_C \vec{F} d\vec{r}$ is the work done by \vec{F} along C . The elementary case is given by $W = FS$, or work = force \cdot displacement. More explicitly, we consider

$$\vec{F}(x, y) = (P(x, y), Q(x, y))$$

Then we have

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

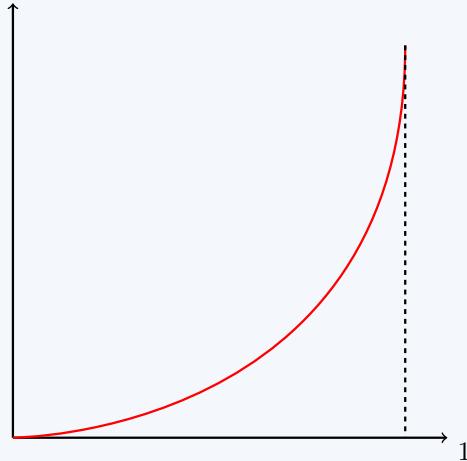
$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t) dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t) dt$$

We will write the expression as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + \int_C Q dy$$

Example. Consider the curve C with

$$\vec{r}(t) = (t, t^2), \quad 0 \leq t \leq 1$$



We have $x(t) = t$ and $y(t) = t^2$, its derivative is

$$\vec{r}(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is $P = x + y$ and $Q = x$, when this is restricted to C , we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = (t + t^2, t)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}' = (t + t^2, t) \cdot (1, 2t) = (3t^2 + t)$$

We finally obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + t) dt = \frac{3}{2}$$

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