1 Lecture 15

1.1 General Regions (cont.)

As we saw in the previous lecture, we have the following regions

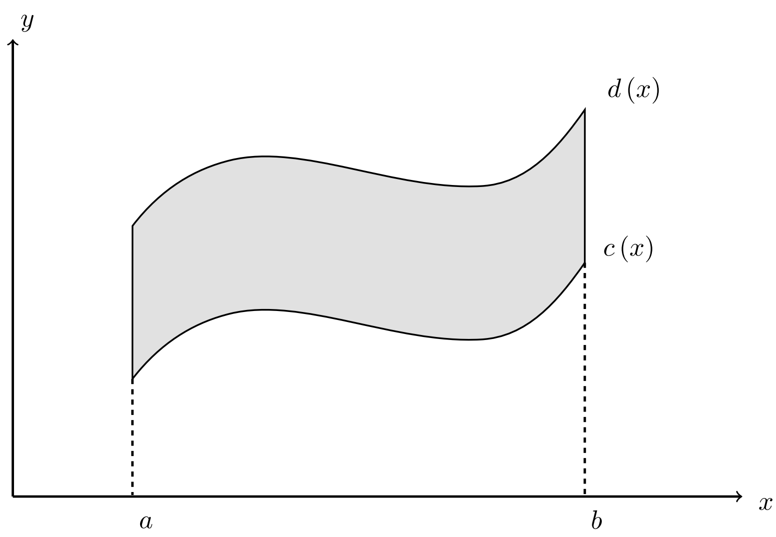


Figure 1: y-simple

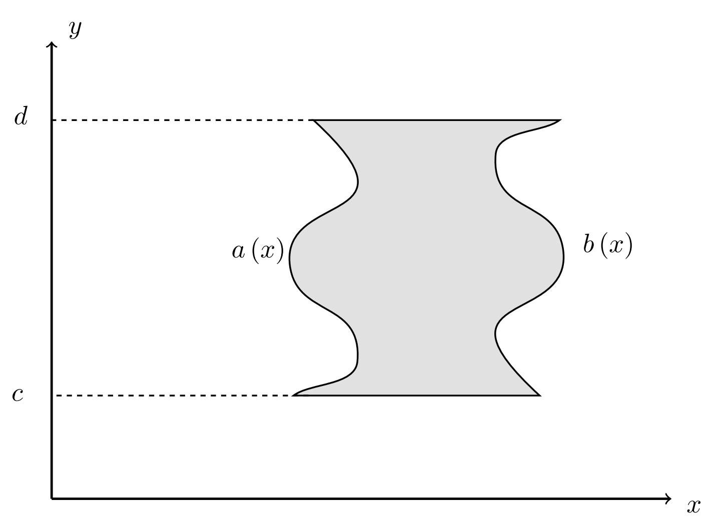
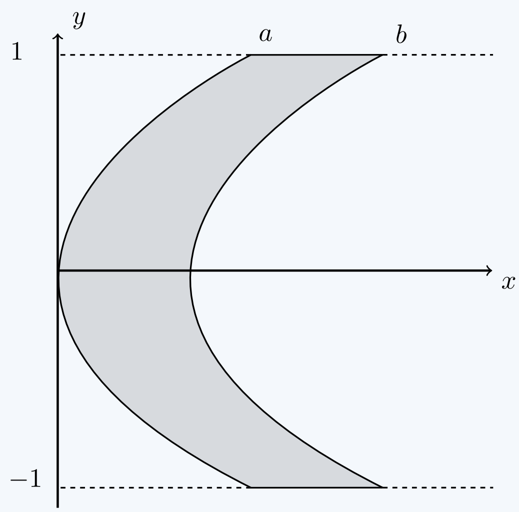


Figure 2: x-simple

Example. Consider the region with $-1 \leq x \leq 1$ and bounder by

$$a(y) = y^2, \quad b(y) = y^2 + \frac{1}{2}$$

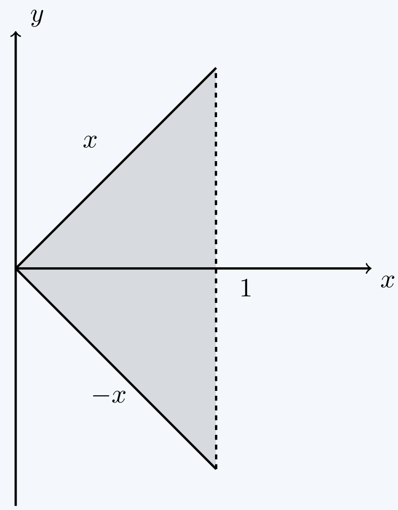


This is an x-simple region. ◇

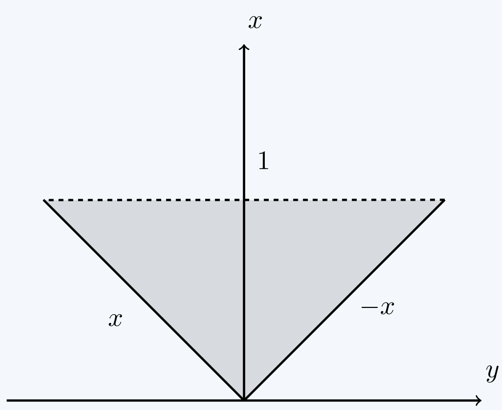
More general regions can be partitioned into x-simple and y-simple regions. A region can be described in many different ways.

Example. Consider the region

$$D = \{(x, y) : 0 \leq x \leq 1, \quad -x \leq y \leq x\}$$



This is a y-simple region. We can also describe this triangle using x-simple regions.



We have two regions:

$$D_1 = \{(x, y) : y \leq x \leq 1, \quad 0 \leq y \leq 1\}$$
$$D_2 = \{(x, y) : -y \leq x \leq 1, \quad -1 \leq y \leq 0\}$$

We have that $D = D_1 \cup D_2$, and

$$\iint_D f dA = \iint_{D_1} f dA + \iint_{D_2} f dA$$

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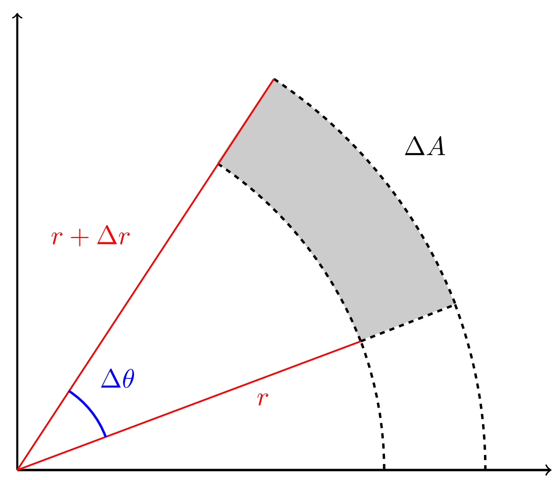
1.2 Integrations in polar coordinates

We want to compute $\iint_D f dA$ using the polar coordinates (r, θ), recall that

$$x = r\cos\theta, \quad y = r\sin\theta$$

In cartesian coordinates, we use the "infinitesimal area $dA = dxdy$". We will now consider small polar regions.



Recall that area is equal to $\frac{1}{2}\theta r^2$.

$$\begin{aligned}
\Delta A &= \text{"Large region"} - \text{"Small Region"} \\
&= \frac{1}{2}\Delta\theta\,(r + \Delta r)^2 - \frac{1}{2}\Delta\theta r^2 \\
&= r\Delta\theta r + \frac{1}{2}\Delta\theta\,(\Delta r)^2
\end{aligned}$$

Neglecting th $(\Delta r)^2$ term, we get

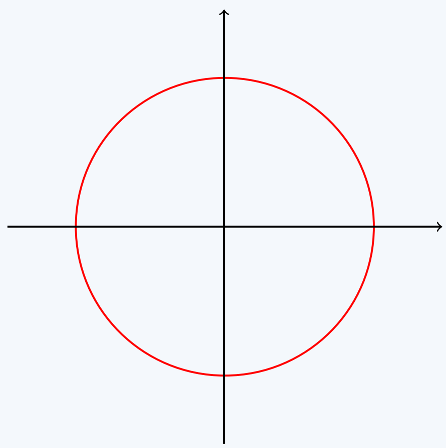
$$\Delta A \approx r\Delta\theta\Delta r$$

Proposition. Suppose R is described in polar coordinates by $a \le r \le b$ and $\alpha \le \theta \le \beta$, then

$$\iint_R f dA = \int_{r=a}^{b} \left[\int_{\theta=\alpha}^{\beta} f\left(r\cos\theta, r\sin\theta\right) r d\theta \right] dr$$

Here, $dA = rd\theta dr$ is the infinitesimal area. Also, $f\left(r\cos\theta, r\sin\theta\right)$ is simply $f\left(x, y\right)$ in polar coordinates.

Example. Consider a disc of radius t. Its area is πt^2



It is described by $R = \{(1, \theta) : 0 \leq r \leq t, 0 \leq \theta \leq 2\pi\}$. We compute

$$\iint_R dA = \int_0^t \left[\int_0^{2\pi} r d\theta \right] dr$$

$$\begin{aligned} \iint_R dA &= \int_0^t \left[\int_0^{2\pi} r d\theta \right] dr \\ &= 2\pi \int_0^t r dr \\ &= 2\pi \left[\frac{r^2}{2} \right]_0^t \\ &= \pi t^2 \end{aligned}$$

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Example. We have R as before, we want to compute $\iint_R f dA$ where

$$f(x,y) = x^2 + y^2$$

Using polar coordinates, we get

$$f(r\cos\theta, r\sin\theta) = r^2(\cos\theta) + r^2 + (\sin\theta)^2 = r^2$$

Our double integral is

$$\begin{aligned}\iint_R f\,dA &= \int_0^t \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, r\,d\theta\,dr \\ &= \int_0^t \int_0^{2\pi} r^3\,d\theta\,dr \\ &= 2\pi \int_0^t r^3\,dr \\ &= 2\pi \frac{1}{4} t^4 \\ &= \frac{\pi}{2} t^4\end{aligned}$$

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1.3 Change in variables

We want to consider general coordinates (u,v) defined by

$$x = x(u,v), \quad y = y(u,v)$$

How to integrate with (u,v)?

Definition 1. The *jacobian determinant* is defined by

$$J(u,v) = \left| \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right| = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

Example. Consider the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

We want to compute $J(r,\theta)$, we have

$$J(r,\theta) = \left| \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right| = \left| \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \right|$$

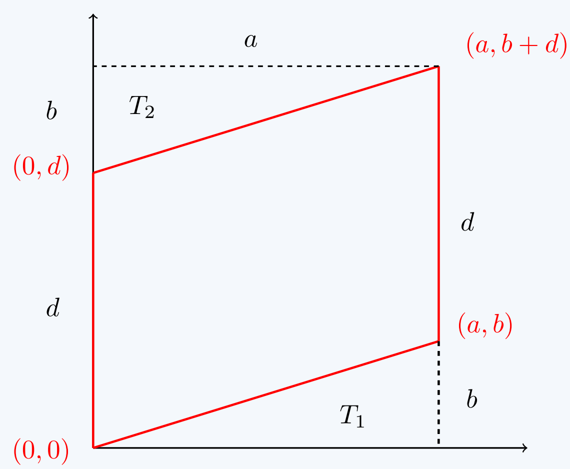
We obtain

$$\begin{aligned}J(r,\theta) &= \cos\theta \cdot r\cos\theta - \sin\theta \cdot (-r\sin\theta) \\ &= r\left((\cos\theta)^2 + (\sin\theta)^2\right) \\ &= r\end{aligned}$$

This correspons to r in $dA = rd\theta dr$. ◇

The determinant should be intepreted as an area (up to certain signs).

Example. Given $\vec{v} = (a, b)$ and $\vec{u} = (0, d)$, consider $(0,0)$, $\vec{v} = (a,b)$, $\vec{w} = (0,d)$, $\vec{v} + \vec{w} = (a, b+d)$. These four points define a paralellogram.



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we claim that

$$a = \left| \det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right| = |ad|$$

we consider $a, b, d > 0$, then

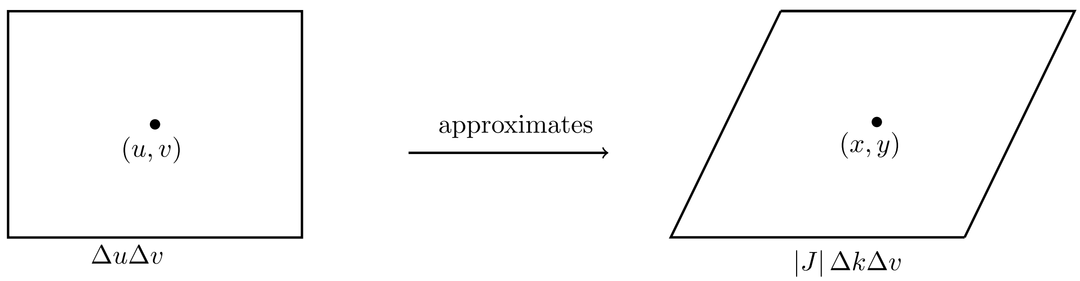
$$a = a\left(b + d\right) - \frac{1}{2}ab = \frac{1}{2}ab = ad$$

this is the same as the determinant. more generally, with

$$\left(0,0\right), \vec{v} = \left(a,b\right), \quad \vec{w} = \left(c,d\right), \quad \vec{v} + \vec{w} = \left(a+c, b+d\right)$$

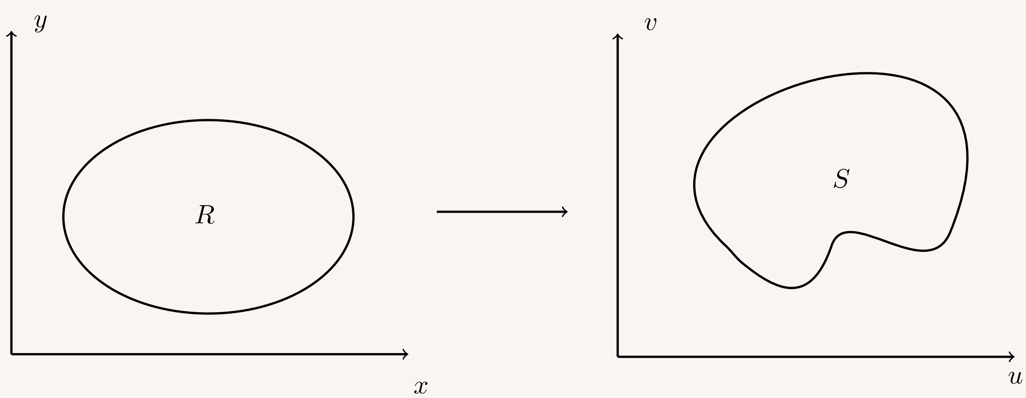
$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ab - bc|$$

the idea of change of variables can be described by the following figure.



Theorem 2. Change of variables.

Let R be a region in cartesian coordinates

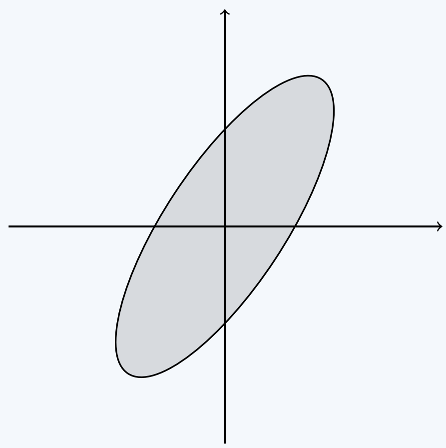


Let S be the corresponding region in (u, v)-coordinates, then

$$\iint_R f dA = \iint_S f\left(x\left(u,v\right), y\left(u,v\right)\right) \left|J\left(u,v\right)\right| dudv$$

Example. Consider the elliptical region

$$R : x^2 - xy + y^2 \leq 2, \quad f(x, y) = x^2 - xy + y^2$$



We want to compute $\iint_R f dA$. Consider (u, v) defined by

$$x = \sqrt{2} - \sqrt{\frac{2}{3}}u, \quad y = \sqrt{2} + \sqrt{\frac{2}{3}}v$$

$$x^2 - xy + y^2 = 2u^2 + 2v^2$$

In (u, v) coordinates, we get

$$\begin{aligned} S &= \left\{(u, v) : 2u^2 + 2v^2 \leq 2\right\} \\ &= \left\{(u, v) : u^2 + v^2 \leq 1\right\} \end{aligned}$$

This equals a circle. ◇