
1 Lecutre 5

1.1 Hessian Matrix

Example. Consider again the function $f(x, y) = x^2 - y^2$.

$$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{yx} = 0$$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \det H = -4 < 0$$

Hence (x_0, y_0) is a saddle point.

Example. Consider the function $f(x, y) = x^2 + y^2$, we have

$$(f_x, f_y) = (2x, 2y)$$

The only critical point is $(x_0, y_0) = (0, 0)$.

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \det H = 4 > 0$$

Since $f_{xx} = 2 > 0$, we conclude that $(0, 0)$ is a local minima. In this case, it is actually a global minimum, because $f(x, y) = x^2 + y^2 \geq 0$.

1.2 Global extremal values

A function can have many maxima and minimas. Usually, we are intrested in the largest and smallest values.

Definition 1. Let $f(x, y)$ be with domain D_f . Then we have

- (x_0, y_0) is a global maxima if $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in D_f$.
- (x_0, y_0) is a global minima if $f(x_0, y_0) \leq f(x, y)$ for all $(x, y) \in D_f$.

Trivial example: for $f(x, y) = 1$, all points are global maxima and minima.

Note that global maxima and minima need not be critical points.

Example. We have $f(x) = x$ with $D_f = [-1, 1]$.

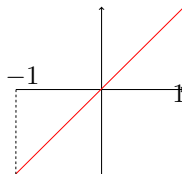


Figure 1: $f(x)$

- Global maxima at $x = 1, f(1) = 1$.

- Global minima at $x = -1, f(-1) = -1$

We have no critical points because $f'(x) = 1 \neq 0$. Also not that maxima and minima depend on the chosen domain.

If we take $D_f = [-2, 3]$, then

$$\text{Max: } x = 3, \quad \text{Min: } x = -2$$

Theorem 1. Let f be continuous with domain D_f . Suppose D_f is closed and bounded, then there is at least one global maxima, and one global minima.

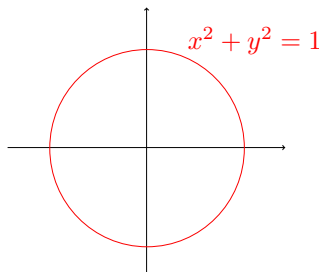


Figure 2: The circle is the boundary.

Some terminology:

$$\text{open} = \{x^2 + y^2 < 1\}$$

$$\text{closed} = \{x^2 + y^2 \leq 1\}$$

The method for finding maxima and minima is as follows:

1. Find critical points of f in D_f , and characterize them.
2. Study the points that are on the boundary.
3. Compare them.

Example. Consider $f(x, y) = x^2 + y^2$ with domain

$$D = \{(x, y) \in \mathbb{R} : x^2 + y^2 \leq 1\}$$

The domain in this case is a disc. The red circle is the boundary.

We compute $f_x = 2x, f_y = 2y$. The only critical point is the origin at $(x_0, y_0) = (0, 0)$. This is a global minimum since $f(0, 0) = 0$ and $f(x, y) \geq 0$.

Now, let's consider the boundary

$$C = \{x^2 + y^2 = 1\}$$

For any point (x_0, y_0) on the circle C , we have

$$f(x_0, y_0) = x_0^2 + y_0^2 = 1$$

We claim that this point is a global maximum. For any (x, y) in domain D_f , we have

$$f(x, y) = x^2 + y^2 \leq 1$$

The value $f(x, y) = 1$ is obtained only at the boundary C . Any point on the circle is a global maxima.

1.3 Constrained optimization

In this section, we will discuss how to find maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. Think of $g = 0$ as a budget, or a geometrical constraint.

Example. We want to minimize $f(x, y) = x^2 + y^2$ with the constraint $g(x, y) = xy - 1 = 0$.

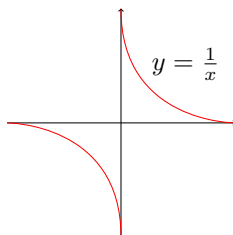


Figure 3: $y = \frac{1}{x}$

We have the following strategy

- Solve $g = 0$ for one variable. For instance $y = \frac{1}{x}$.
- Consider: $h(x) = f(x, \frac{1}{x}) = x^2 + x^{-2}$

Now we can study this function of one variable with no constraints. We can proceed as usual.

$$\frac{dh}{dx} = 2x - 2x^{-3} = 0$$

This is equivalent to $x^4 = 1$. The real solutions are $x = \pm 1$. Since $y = \frac{1}{x}$, we get the critical points:

$$(x, y) = (1, 1), \quad (x, y) = (-1, -1)$$