1 Lecture 11

1.1 Operations on fields

Example. Consider $\vec{F}(x, y, z) = (x^2, y^2, z^2)$, that is

$$P = x^2, \quad Q = y^2, \quad R = z^2$$

Its divergence is

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2x + 2y + 2z$$

We can check that $\vec{F}$ is a gradient field. We have that $\vec{F} = \nabla f$, with the potential

$$f(x, y, z) = \frac{1}{3}\left(x^2 + y^2 + z^2\right)$$

$\diamond$

$\nabla \times \vec{F}$ is computed as a determinant. We can use the cofactor, or the Laplace expansion.

$$\nabla \times \vec{F} = \left\| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{bmatrix} \right\| = \begin{bmatrix} \partial_y & \partial_z \\ Q & R \end{bmatrix} \vec{i} - \begin{bmatrix} \partial_x & \partial_z \\ P & R \end{bmatrix} \vec{j} + \begin{bmatrix} \partial_x & \partial_y \\ P & Q \end{bmatrix} \vec{k}$$

$$= \vec{i}\left(\partial_y R - \partial_z Q\right) - \vec{j}\left(\partial_x R - \partial_z P\right) + \vec{k}\left(\partial_x - Q\partial_y P\right)$$

This is a concrete formula for $\nabla \times \vec{F}$.

Example. Consider $\vec{F} = xy\vec{i} + (x + z)\vec{j} + yz\vec{k}$. We compute

$$\nabla \times \vec{F} = \vec{i}\left(\partial_y R - \partial_z Q\right) - \vec{j}\left(\partial_x R - \partial_z P\right) + \vec{k}\left(\partial_x - Q\partial_y P\right)$$

where

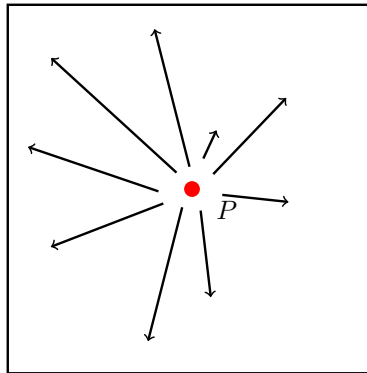$$P = xy, \qquad\qquad Q = x + z, \qquad\qquad R = yz$$

$$\nabla \times \vec{F} = (2 - 1)\vec{i} + (1 - x)\vec{k}$$

$\diamond$

Note. Gradient and divergence can be defined in any dimension. The curl is only defined in up to three dimensions.

1.2 Intepretation of Divergence

Think of $\vec{F}$ as the velocity of a fluid. $\nabla \cdot \vec{F}$ at a point P is the amount of fluid entering / leaving a small region around P.

Example. Consider $\vec{F_1}(x, y) = (x, y)$, then

$$\nabla \cdot \vec{F_1} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + 2$$

This corresponds to fluid leaving the region. Similarily for $\vec{F_2} = (-x, -y)$, then

$$\nabla \cdot \vec{F_2} = \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -2$$

This corresponds to fluid entering the region. Finally consider $\vec{F_3} = (0, 1)$, then
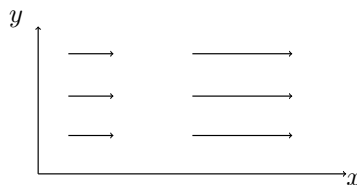
$$\nabla \cdot \vec{F_3} = 0$$

This is an equillibrium situation. $\diamond$

1.3   Interpretation of Curl

$\nabla \times \vec{F}$ measures the "rotation" of $\vec{F}$.

Example. Consider $\vec{F}(x, y, z) = (x^2, 0, 0)$. We expect no rotation, we compute

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 0 & 0 \end{bmatrix} \right| = 0$$

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Example. Consider $\vec{F}(x, y, z) = (-\omega y, \omega x, 0)$, where ω is a non-zero constant. We compute

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -\omega y & \omega x & 0 \end{bmatrix} \right| = \begin{bmatrix} \partial_x & \partial_y \\ -\omega y & -\omega x \end{bmatrix} \vec{k} = 2\omega \vec{k}$$

$\nabla \times \vec{F} \neq 0$ gives non-zero rotation. Also $\nabla \times \vec{F} = $ const gives that that there is some rotation everywhere.

For a physical intepretation of this, we can write

$$\vec{v} = \vec{F}, \quad \vec{\omega} = (0, 0, \omega), \quad \vec{r} = (x, y, z)$$

Then we can check that $\vec{v} = \vec{\omega} \cdot \vec{r}$. This is the velocity corresponding to the angular velocity $\vec{\omega}$

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1.4   Scalar Field from Gradient

Suppose that we know ∇f. Can we recover f? Yes, up to the initial conditions. The strategy is the following, first we write

$$\vec{F} = \nabla f = (P, Q, R)$$

By the definition of the gradient field, we have

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

Comparing, we get

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q, \quad , \quad \frac{\partial f}{\partial x} = R$$

Example. Suppose we are given the gradient field

$$\vec{F} = \nabla f = \left(2x + 2y, \frac{1}{2}x^2 + 3y \right)$$

We want to find f, we must have

$$\frac{\partial f}{\partial x} = 2x + 2y, \quad \frac{\partial f}{\partial y} = \frac{1}{2}x^2 + 3y$$

Integrating the first equation in x gives us

$$f(x, y) = \int (2x + xy) \, dx$$
$$= x^2 + \frac{1}{2}x^2 y + g(y)$$

Where $g(y)$ is the integration constant. It can depend on y.
Now we compute $\frac{\partial f}{\partial y}$. We get

$$\frac{\partial f}{\partial y} = \frac{1}{2}x^2 + \frac{dg}{dy}$$

But we also have that $\frac{\partial f}{\partial y} = \frac{1}{2}x^2 + 3y$.

Comparing them, we get

$$\frac{1}{2}\cancel{x^2} + \frac{dg}{dy} = \frac{1}{2}\cancel{x^2} + 3y$$

Then $\frac{dg}{dy} = 3y$. Integrating in y we get

$$g(y) = \int 3y\, dy = \frac{3}{2}y^3 + C$$

Here, C is a constant. Finally, inserting it, we get

$$f(x,y) = x^2 + \frac{1}{2}x^2 y + g(k)$$
$$= x^2 + \frac{1}{2}x^2 y + \frac{3}{2}y^3 + C$$

The constant C is usually not important. It can be fixed by an inital condition. For instance $f(0,0)$ implies that $C = 0$

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