1 Lecture 13

1.1 Line Integrals of Vector Fields

We have seen the following definition earlier

$$\int_C \vec{F} d\vec{r} = \int_{t_0}^{t_1} \vec{F}\left(\vec{r}\left(t\right)\right) \cdot \vec{r}'\left(t\right) dt$$

We can rewrite this to link with scalarfields, lets consider

$$\int_C = \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F} \cdot \vec{v} dt = \int_{t_0}^{t_1} \vec{F} \cdot \frac{\vec{v}}{|\vec{v}|} dt$$

We can see this as the line integral of the field $\vec{F} \cdot \vec{T}$

Example. Consider $\vec{F}$ is constant and directed along the curve, that is $\vec{F} = F\vec{T}$, where $F - \left|\vec{F}\right|$. Then the formula for elementary work $(W = FS)$ gives

$$W = \int_C \vec{F} \cdot \vec{T} ds = \int_C F\vec{T} \cdot \vec{T} ds = F \int_C ds = FS$$

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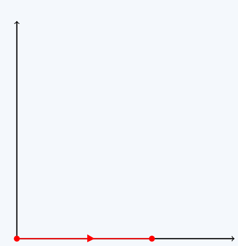
Now lets consider changes of orientation

Proposition 1. We have that

$$\int_C \vec{F} \cdot d\vec{r} = - \int_{-C} \vec{F} \cdot d\vec{r}$$

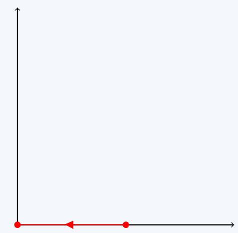
Example. Consider the segment

$$\vec{r}\left(t\right) = \left(t, 0\right), \quad 0 \leq t \leq 1$$



The opposite parametrization is

$$\vec{r}_{\text{opp}}\left(t\right) = \left(-t, 0\right), \quad -1 \leq t \leq 0$$



We have

$$\vec{r}'(t) = (1,0), \quad \vec{r}_{\text{opp}}(t) = (-1,0)$$

Consider $\vec{F}(x,y) = (x,0)$. For C we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (t,0) \cdot (1,0)\, dt = \int_0^1 t dt = \frac{1}{2}$$

For $-C$, we have

$$\int_{-C} \vec{F} \cdot d\vec{r} = \int_{-1}^0 (-t,0) \cdot (-1,0)\, dt = \int_{-1}^0 t dt = -\frac{1}{2}$$

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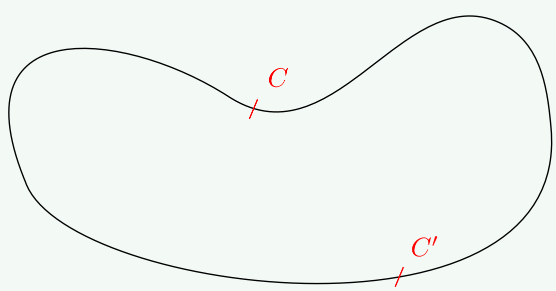
1.2 Conservative Vector Fields

In general, $\int_C \vec{F} \cdot d\vec{r}$ depends on the curve C. However, sometimes it only depends on the endpoints.

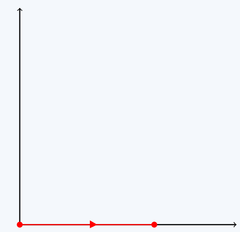
Definition 1. A vector field $\vec{F}$ is conservative if $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of C. That is, if C and C' have the same endpoints, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}$$



Example. Consider $\vec{F} = (1,1)$ and

$$C : x(t) = t, \quad y(t) = 0, \quad 0 \leq t \leq 1$$

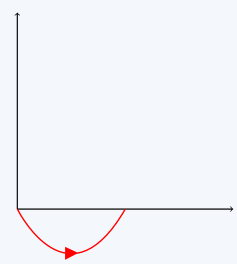


The endpoints are $(0,0)$ and $(1,0)$. We compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 dt = 1$$

Now, lets consider a different curve.

$$C' : x(t) = t, \quad y(t) = t(t-1), \quad 0 \leq t \leq 1$$



We have the same endpoints as C. The velocity is $\vec{r}'(t) = (1, 2t-1)$. Then

$$\begin{aligned}\int_{C'} \vec{F} \cdot d\vec{r} &= \int_0^1 (1,1) \cdot (1, 2t-1)\, dt \\ &= \int_0^1 2tdt - 2 \cdot \frac{1}{2} = 1\end{aligned}$$

At this stage, we cannot conclude $\vec{F}$ is consrvative (although it is). Note that

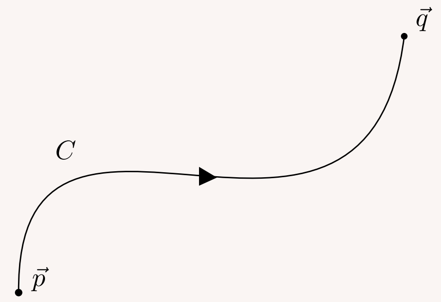
$$\vec{F} = \nabla f \quad \wedge \quad f(x,y) = x + y$$

We will prove that being a gradient field is the condition we want. $\diamond$

Theorem 2. The Gradient Theorem.

Suppose $\vec{F} = \nabla f$, consider a curve C starting at $\vec{p}$, and ending at $\vec{q}$, then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{q}) - f(\vec{p})$$



Proof. Pick a parametrization

$$\vec{r}(t), \quad t_0 \le t \le t_1$$

Note that: $\vec{r}(t_0) = \vec{p}$ and $r(t_1) = \vec{q}$. Using $\vec{F} = \nabla f$, we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \\ &= \int_{t_0}^{t_1} \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \end{aligned}$$

From the chain rule, we get

$$\frac{df(\vec{r}(t))}{dt} = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$

Using this, we obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \frac{df(\vec{r}(t))}{dt} dt$$

Using the fundamental theorem of calculus, we get

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= f(\vec{r}(t_1)) - f(\vec{r}(t_0)) \\ &= f(\vec{q}) - f(\vec{p}) \end{aligned}$$

□

If $\vec{F} = \nabla f$, then $\vec{F}$ is conservative, since $\int_C \vec{F} \cdot d\vec{r} = f(\vec{q}) - f(\vec{p})$ only depends on $\vec{p}$ and $\vec{q}$

Example. Consider the previous example with $\vec{F} = (1, 1)$. We saw $\vec{F} = \nabla f$ with $f(x, y) = x + y$.

For any curve starting at $\vec{p} = (0,0)$, and ending at $\vec{q} = (1,0)$, we have

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(\vec{q}) - f(\vec{p}) \\ &= f(1,0) - f(0,0) \\ &= 1 - 0 = 1\end{aligned}$$

Note that we could take

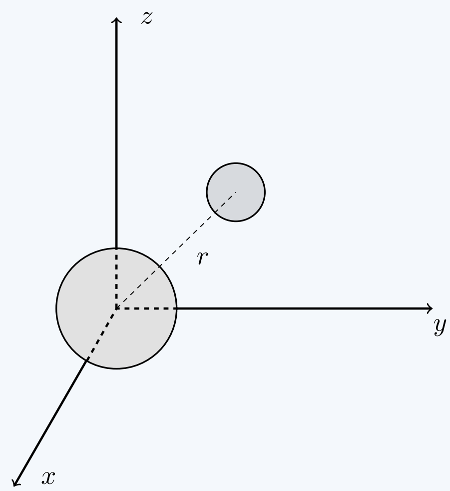
$$\tilde{f}(x,y) = x + y + c$$

With c being a constant. This gives the same result.

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \tilde{f}(1,0) - \tilde{f}(0,0) \\ &= 1 + c - c = 0\end{aligned}$$

$\diamond$

Example. Consider two objects A and B of mass M and m, as in the picutre



The force exerted by A on B is

$$\vec{F}(x,y,z) = -G\frac{Mm}{r^2}\hat{r}$$

Here we have $r = \sqrt{x^2+y^2+z^2}$ ant $\hat{r} = \frac{(x,y,z)}{r}$. It is a unit vector pointing at B, from A. $\vec{F}$, can be rewritten as

$$\vec{F}(x,y,z) = -GMm\frac{(x,y,z)\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}{}$$

Consider the function

$$V = \frac{GMm}{r} = GMm\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

We compute

$$\frac{\partial V}{\partial x} - GMm\frac{1}{2}\cdot\frac{2x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

Similarily for y and z, then

$$\nabla V = -GMm\frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$\diamond$