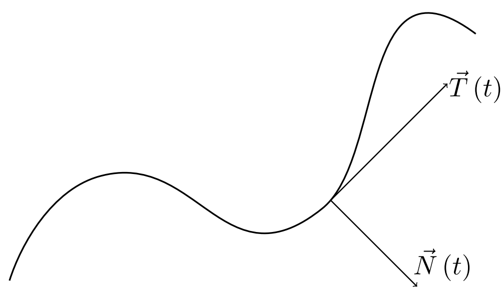
1 Lecture 9

1.1 Normal Vectors - Continued

Yesterday we saw that

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$



Example. Consider the circle

$$\vec{r}(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$$

The velocity is

$$\vec{v}(t) = \vec{r}' = (-\sin t, \cos t)$$

We have $|\vec{v}(t)| = 1$, since

$$\vec{v}(t) \cdot \vec{v}(t) = \left(\sin t^2 = \cos t^2\right) = 1$$

We find that $\vec{T}(t) = \vec{v}(t)$. To find $\vec{N}$, we need first

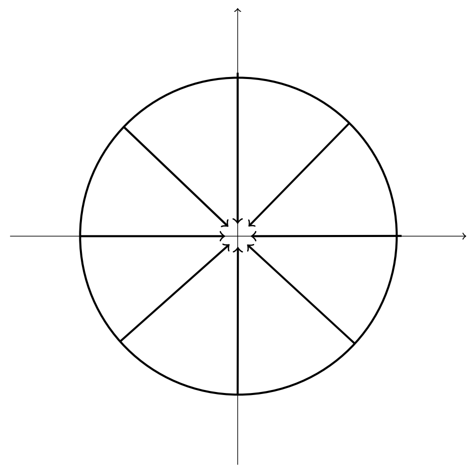
$$\frac{d\vec{T}}{dt} = \frac{d\vec{v}}{dt} = (-\cos t, -\sin t)$$

We check that $\left|\frac{d\vec{T}}{dt}\right| = 1$, then

$$\vec{N}(t) = (-\cos t, -\sin t) = -\vec{r}(t)$$

We compute more explicitly:

$$\vec{N}(t) \cdot \vec{T}(t) = (-\cos t, -\sin t) \cdot (-\sin t, \cos t) = \cos t \cdot \sin t - \sin t \cdot \cos t = 0$$



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We revisit the implicit case $f(x,y)=0$.

Proposition 1. Let C be defined by $f(x,y)=0$. A unit normal to C is given by $\vec{n}=\frac{\nabla f}{|\nabla f|}$

Proof. Suppose C is parametrized by

$$\vec{r}(t)=(x(t),y(t)),\quad t_0\le t\le t_1$$

By definition of C, we have

$$f(\vec{r}(t))=f(x(t)\,y(t)),t_0\le t\le t_1$$

We have that $\frac{d}{dt}f(\vec{r}(t))=0$, but using the chain rule, we get

$$\frac{df(\vec{r}(t))}{dt}=\nabla f(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}$$

Hence, we get

$$\nabla f(\vec{r}(t))\cdot\vec{v}(t)=0$$

Therefore $\nabla f(\vec{r}(t))$ is normal to the curve C, or orthogonal to the tangent $\vec{v}(t)$ □

Example. The circle of radius 1 can be described implicitly by $f(x,y)=0$ with

$$f(x,y)=x^2+y^2-1$$

The gradient is $\nabla f(x,y)=(2x,2y)$, not of length 1, since

$$|\nabla f(x,y)|=\sqrt{4x^2+4y^2}=2\sqrt{x^2+y^2}$$

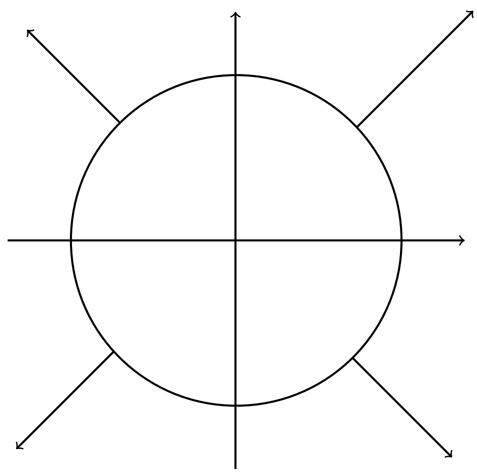
We are only interested in (x, y) such that $f(x, y) = 0$, that is $x^2 + y^2 = 1$, then

$$|\nabla f| = 2$$

Hence we get

$$\vec{n}\frac{\nabla f}{|\nabla f|} = \frac{(2x, 2y)}{2} = (x, y)$$

The normal vectors point outside of the circle, note: $\vec{n} = -\vec{N}$, compare with the parameter.



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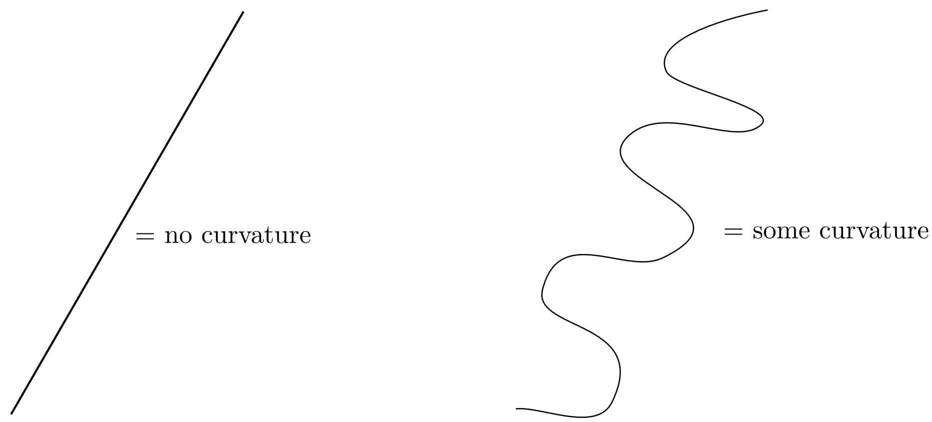
1.2 Curvature

We want to compute how much a curve "curves".



How do we quantify this?

Definition 2. Let C be a parametrized curve, ant $\vec{T}$ its unit tangent vector. The curvature is then defined by

$$K = \left| \frac{d\vec{T}}{dS} \right|$$

Where S is the arc length parameter of C. We also define the radius of curvature as

$$\rho = \frac{1}{K}$$

Notation. We consider S, not t.