DAVE3700 - Matte 3000

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July 1, 2022

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2 Domains, Graphs and Level Sets

2.1 Domain of definition

A function may not be defined for all real numbers.

Example. $f(x) = \frac{1}{x}$ is not defined for $x = 0$ ◇

Definition 1. The domain of a function f is the set of numbers for which it is defined. We write the domain of f as D_f.

For instance, for $f(x) = \frac{1}{x}$ we have that

$$D_f = \{x \in \mathbb{R} : x \neq\}$$

This is the lasrgerst possible domain, we can also consider smaller domains. We have the interval from 1 to 2.

$$[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$$

Example. Find the largest domain of $f(x, y) = \frac{1}{y-x}$. The denominator should be non-zero, we get

$$D_f = \left\{(x, y) \in \mathbb{R}^2 : y - x \neq 0\right\}$$

◇

Example. Same excersise with $f(x, y) = \sqrt{y - x^2}$.
Argument: $y - x^2 \geq 0$ (because square root). We will then have $y \geq x^2$ ◇

2.2 Graphs of functions

The plot of a function f descrives its behaviour visually. Mathematically, a plot corresponds to the notion of a graph.

Definition 2. The graph of a function $f(x, y)$ with domain D_f is the set of points (x, y, z) such that:

$$(x, y) \in D_f \text{ and } z = f(x, y)$$

We write G_f for the set

$$G_f = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_f \text{ and } z = f(x, y)\right\}$$

The graph of a function of two variables will, in general, be a surface.

Example. Lets consider $f(x,y) = 1$, with domain $\mathbb{R}^2$. The graph of f is:

$$G_f = \left\{(x, y, 1) : (x, y)\, \mathbb{R}^2\right\}$$

All points have $z = 1$, this is a plane. More generally, the graph of$f(x,y) = ax + by + c$ is a plane with linear dependence on x and y. ◇

Example. Consider the graph of:

$$f(x,y) = x^2 + y^2, \quad D_F = \mathbb{R}^2$$

This surface is called a paraboloids. ◇

Example. A sphere of radius r is defined by

$$x^2 + y^2 + z^2 = r^2$$

All points x, y, z satisfy the equation. ◇

Is this the graph of a function? No!
There is no unique value of z, associated with (x, y) because:

$$z = \pm\sqrt{r^2 - x^2 - y^2}$$

Both satisfy the sphere equation. Lets consider the graph of

$$f(x,y) = \sqrt{r^2 - x^2 - y^2}$$

With the domain $x^2 + y^2 \leq r^2$. The grap is a half sphere.

2.3 Level Sets

Another way to visualize functions.

Definition 3. A level set of a function $f(x,y)$ is constant. Essentialy, this is a topographic map.

Example. Consider the function

$$f(x,y) = x^2 + y^2$$

The level sets for $c > 0$ are circles.

$$f(x,y) = x^2 + y^2 = c = \left(\sqrt{c}\right)^2$$

This is a circle with radius $\sqrt{c}$.
Now consider the case $c < 0$, then:

$$f(x, y) = x^2 + y^2 = c$$

which doesn't work, because the level sets are empty.
For $c = 0$, we only have the point $(x, y) = (0, 0)$. Generally, level sets of $f(x, y)$ is a curve.

$\diamond$

3 Derivatives

3.1 Partial Derivatives

In the case of one variable, we have

$$\frac{df}{dt} = \lim_{n \to 0} \frac{f(x+n) - f(x)}{n}$$

Similarily, for two of more variables, we have the following definition

Definition 4. The partial derivative of $f(x, y)$ with respect to x:

$$\frac{\partial f}{\partial x} = \lim_{n \to 0} \frac{f(x+n, y) - f(x, y)}{n}$$

Also written as f_x, for $\frac{\partial f}{\partial y}$, we have f_y

Note, the expression above is $\frac{\partial f}{\partial x}(x, y)$, which is the value at the point (x, y)

3.2 Higher order derivatives

Given $\frac{\partial f}{\partial x}$, we can take further derivatives. We have

$$\frac{\partial^2 f}{\partial^2 x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Also written as $f_{xx}, f_{yy}, f_{xy}, f_{yx}$. In most cases, f_{xy} and f_{yx} coincide.

Theorem 5. Scwartz theorem: Suppose f_{xy} and f_{yx} exist, and are continous, then

$$f_{xy} = f_{yx}$$

Similar definitions and results for the case of more variables: $x_1, \ldots, x_n$, with n variables.

3.3 Chain Rule

Suppose $f(x) = g(h(x))$, for instance

$$f(x) = (\cos x)^2 \text{ with } g(x) = x^2, h(x) = \cos x$$

then the chain rule is

$$\frac{df}{dt}(x_0) = \frac{dg}{dh}(h(x_0)) \cdot \frac{dh}{dt}(x_0)$$

Generalization to more variables.

Theorem 6. Chain rule: consider $f(x, y)$ x and y depending on a variable t. Then:

$$\frac{df}{dt} t_0 = \frac{\partial f}{\partial x}(x(t_0), y(t_0)) \frac{dx}{dt} t_0 + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \frac{dy}{dt} t_0$$

The "short form" of this result is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example. Consiter $f(x,y) = xy$, where

$$x(t) = \cos t, y(t) = \sin t$$

This cannot be computed directly with $\frac{df}{dt}$.

$$f(t) = f(x(t), y(t)) = f(\cos t, \sin t) = \cos t \cdot \sin t$$

We can compute

$$\frac{df}{dt} = (\cos t)' \sin t + \cos t(\sin t)' = -(\sin t)^2 + (\cos t)^2$$

Usin the chain rule, we get

$$\frac{df}{dt}j = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

◇

3.4 The Gradient

Define an operation that takes a scalar function, and returns a vector function.

Definition 7. The gradient of $f(x,y)$ at (x_0, y_0) is

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

4 Directional Derivatives and Critical Points

4.1 Directional Derivatives

We have seen the following

- f_x = the rate of change along the $x-$direction.
- f_y = the rate of change along the $y-$direction.

What about general directions?

Definition 8. Let $\vec{u} = (a, b)$, the directional derivative along $\vec{u}$ at (x, y) is

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h}$$

Note that

$$\vec{u} = (1, 0) \to D_{\vec{u}} = f_x$$
$$\vec{u} = (1, 0) \to D_{\vec{u}} = f_y$$

To compare directions, we take $\left|\vec{t}\right| = 1$. Here $\vec{u}$ is the length of $\vec{u}$, that is

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

Proposition. We have the following result

$$D_{\vec{u}} \cdot f = \nabla f \cdot \vec{u}$$

Proof. Consider the following function

$$g(t) = f(x + ta, y + tb)$$

Its derivative at $t = 0$ is

$$\begin{aligned} \frac{dg}{dt}(0) &= \lim_{h \to 0} \frac{g(h) - g(0)}{h} \\ &= \lim_{h \to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h} \\ &= D_{\vec{u}} \cdot f(x, y) \end{aligned}$$

On the other hand, using the chain rule, we get

$$\begin{aligned}\frac{dg}{dt}(0) &= \frac{\partial f}{\partial x}\frac{d(x+ta)}{dt} + \frac{\partial f}{\partial y}\frac{d(y+tb)}{dt} \\ &= \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b = \nabla f \cdot \vec{u}\end{aligned}$$

□

We can now state another property of ∇f. The direction where f changes the most.

Proposition. $|D_{\vec{u}} \cdot f|$ is the largest when $\vec{u}$ is paralell to ∇f.

Proof. Recall that, given two vectors, $\vec{v}$ and $\vec{w}$, we have that

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos\alpha$$

When is $|\vec{v} \cdot \vec{w}|$ the largest? We have

$$|\vec{v} \cdot \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot |\cos\alpha|, \quad |\cos\alpha| \leq 1$$

It is the largest when the following condition is true

$$|\cos\alpha| = 1, \quad \alpha = 0 \vee \pi$$

Which means that the vectores are pointing in the same, or opposite direction. Applying this to $D_{\vec{u}} \cdot f$, we get

$$\begin{aligned}|D_{\vec{u}} \cdot f| &= |\nabla f \cdot \vec{u}| \\ &= |\nabla f| \cdot |\vec{u}| \cdot |\cos\alpha|\end{aligned}$$

For fixed values of $|\vec{u}|$, this is the largest when $\alpha = 0 \vee \pi$. That is ∇f and $\vec{u}$ are paralell. □

4.2 Critical Points

How do we find the maxima and minima of $f(x)$? Lets take a look at $f'(x_0) = 0$.

Definition 9. We say that x_0, y_0 is a critical point of f if:

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \quad \wedge \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

Similarily for

5 Lecutre 5

5.1 Hessian Matrix

Example. Consider again the function $f(x, y) = x^2 - y^2$.

$$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{yx} = 0$$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \det H = -4 < 0$$

Hence (x_0, y_0) is a saddle point. ◇

Example. Consider the function $f(x, y) = x^2 + y^2$, we have

$$(f_x, f_y) = (2x, 2y)$$

The only critical point is $(x_0, y_0) = (0, 0)$.

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \det H = 4 > 0$$

Since $f_{xx} = 2 > 0$, we conclude that $(0, 0)$ is a local minima. In this case, it is actually a global minimum, because $f(x, y) = x^2 + y^2 \geq 0$. ◇

5.2 Global extremal values

A function can have many maxima and minimas. Usually, we are intrested in the largest and smallest values.

Definition 10. Let $f(x, y)$ be with domain D_f. Then we have

- (x_0, y_0) is a global maxima if $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in D_f$.
- (x_0, y_0) is a global minima if $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in D_f$.

Trivial example: for $f(x, y) = 1$, all points are global maxima and minima.
Note that global maxima and minima need not be critical points.

Example. We have $f(x) = x$ with $D_f = [-1, 1]$.

- Global maxima at $x = 1, f(1) = 1$.
- Global minima at $x = -1, f(-1) = -1$

We have no critical points because $f'(x) = 1 \neq 0$. Also not that maxima and minima depend on the chosen domain.
If we take $D_f = [-2, 3]$, then

$$\text{Max: } x = 3, \quad \text{Min: } x = -2$$

◇

Theorem 11. Let f be continous with domain D_f. Suppose D_f is closed and bounded, then there is at least one global maxima, and one global minima.

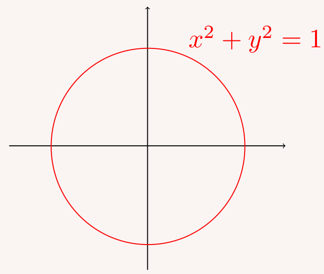


Figure 1: The circle is the boundary.

Some terminology:

$$\begin{aligned} \text{open } &= \{x^2 + y^2 < 1\} \\ \text{closed } &= \{x^2 + y^2 \leq 1\} \end{aligned}$$

The method for finding maxima and minima is as follows:

1. Find critical points of f in D_f, and characterize them.
2. Study the points that are on the boundary.
3. Compare them.

Example. Consider $f(x, y) = x^2 + y^2$ with domain

$$D = \{(x, y) \in \mathbb{R} : x^2 + y^2 \leq 1\}$$

The domain in this case is a disc. The red circle is the boundary.

We compute $f_x = 2x, f_y = 2y$. The only critical point is the origin at $(x_0, y_0) = (0, 0)$. This is a global minium since $f(0, 0)$ and $f(x, y) \geq 0$.

Now, lets consider the boundary

$$C = \{x^2 + y^2 = 1\}$$

For any point (x_0, y_0) on the circle C, we have

$$f(x_0, y_0) = x_0^2 + y_0^2 = 1$$

We claim that this point is a global maximum. For any (x, y) in domain D_f, we have

$$f(x, y) = x^2 + y^2 \leq 1$$

The value $f(x, y) = 1$ is obtain only at the boundary C. Any point on the circle is a global maxima. ◇

5.3 Constrained optimization

In this section, we will discuss how to find maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. Think of $g = 0$ ad a budget, or a geometrical constraint.

Example. We want to minimize $f(x, y) = x^2 + y^2$ with the constraint $g(x, y) = xy - 1 = 0$.

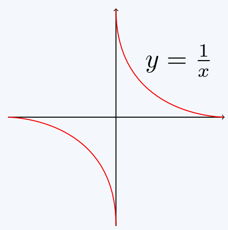


Figure 2: $y = \frac{1}{x}$

We have the following strategy

- Solve $g = 0$ for one variable. For instance $y = \frac{1}{x}$.
- Consider: $h(x) = f\left(x, \frac{1}{x}\right) = x^2 + x^{-2}$

Now we can study this function of one variable with no constraints. We can proceed as usual.

$$\frac{dh}{dt} = 2x - 2x^{-3} = 0$$

This is equivalent to $x^4 = 1$. The real solutions are $x = \pm 1$. Since $y = \frac{1}{x}$, we get the critical points:

$$(x, y) = (1, 1), \quad (x, y) = (-1, -1)$$

◇

6 Lecture 6

6.1 Substitution Method

We want to maximize / minimize $f(x,y)$ with constraint $g(x,y) = 0$. We can solve $g(x,y) = 0$ for one variable $y = y(x)$.

Example. Consider $f(x,y) = x^2 + y^2$ and $g(x,y) = xy - 1 = 0$. In this case, we have $f = \frac{1}{x}$ from $g = 0$, we get

$$h(x) = f\left(x, x^{-1}\right) = x^2 + x^{-2}$$

We have found the minima at $(1,1)$ and $(-1,-1)$. ◇

This method isn't always feasible, so lets look at some alternatives.

6.2 Lagrange's Method

Example. Lets look at the level curves, which are circles. We have the that $f(x,y) = x^2+y^2 = c$.

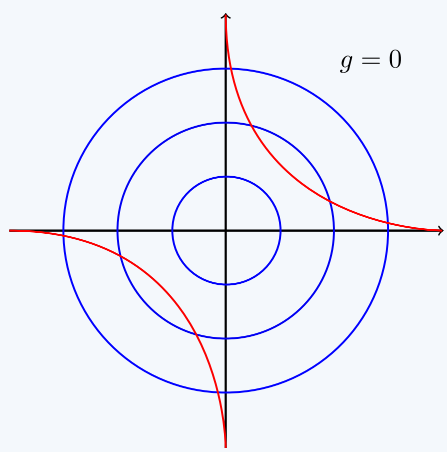
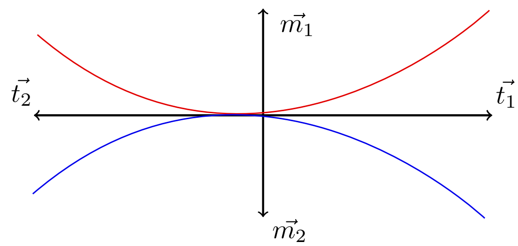


Figure 3: $f(x,y)$

Smaller circles correspont to smaller values of $f(x,y)$, but we must also satisfy $g(x,y) = 0$. In the best case, $f(x,y) = c$ is just touching $g(x,y) = 0$. If this is worked out geometrically, we get $(1,1)$ and $(-1,-1)$. ◇

This idea is used in Lagranges method. We want $f(x,y) = c$ to be parallell to $g(x,y) = 0$. More precisely: Their vectors should be parallell.



Equivalently, their normal vectors are also paralell. Recall that a normal vector to $g = 0$ is given by ∇f. Similarly, ∇f is normal to $f = c$.

6.3 Method for Langrange

Suppose we want to find a local maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. We proceed as follows

1. Find all possible solutions to the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0$$

2. Plug in all solutions from 1. into $f(x, y)$ and identify the largest and smallest.

   - The number λ is called the *Lagrange Multiplier*.
   - Easy extension to n variables.
   - Can also be generalized to multiple constraints, $g_1, \ldots, g_n$

Example. We have the following functions

$$f(x, y) = x^2 + y^2, \quad g(x, y) = xy - 1 = 0$$

We compute the gradients

$$\nabla f(x, y) = (2x, 2y), \quad \nabla g(x, y) = (x, y)$$

The Lagrange equiations are

$$2x = \lambda y, \quad 2y = \lambda x, \quad xy - 1 = 0$$

Observe that $(x, y, \lambda) = (0, 0, 0)$ is not a solution. For $x \neq 0$, we get $y = \frac{1}{x}$, from the third equation.

$$
\begin{aligned}
&2x = \lambda y \Rightarrow 2x = \lambda \frac{1}{x} \Rightarrow \lambda = 2x^2 \\
&2y = \lambda x \Rightarrow \frac{2}{x} = 2xs^2 \cdot x \Rightarrow x^4 = 1 \\
&x^4 = 1 \Rightarrow x \pm 1, \quad y = \frac{1}{x}
\end{aligned}
$$

$$(x, y) = (1, 1) \quad \wedge \quad (x, y) = (-1, -1)$$

Are theses points the minima? We have

$$f(1, 1) = f(-1, -1) = 2$$

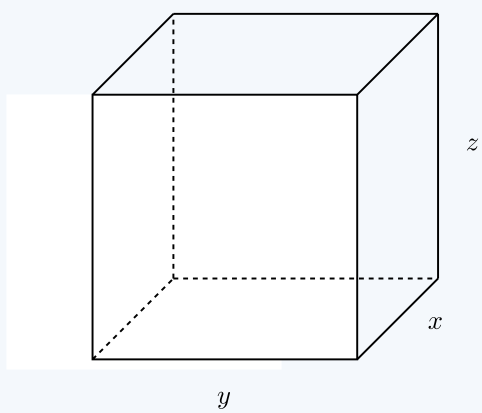
Lets Compare this to some other points, such that $g(x, y) = 0$. For instance, $(x, y) = \left(2, \frac{1}{2}\right)$

$$g\left(2, \frac{1}{2}\right) = 2 \cdot \frac{1}{2} - 1 = 0$$

$$f\left(2, \frac{1}{2}\right) = 2^2 + \left(\frac{1}{2}\right)^2 > 2$$

This tells us that $(x, y) = (\pm 1, \pm 1)$ are local minima. ◇

Example. Consider a box of surface area 24 cm^2.



Determine the dimensions (x, y, z) such that the volume is max.
We have the surface area $2xy + 2xz + 2yz$. Our constraint is

$$g(x, y, z) = 2xy + 2xz + 2yz - 24 = 0$$

The goal is to maximize $f(x,y,z) = 0$, with the consstraint $g(x,y,z) = 0$.
The equation $\nabla f = \lambda \nabla g$ gives

$$yz = 2\lambda (y+2), \quad xz = 2\lambda (x+2), \quad xy = 2\lambda (x+y)$$

Obsereve that $\lambda \neq$, since $x, y, z > 0$. To solve the equations, we can multiply by x, y and z, respectively, then we get

$$x(y+z) = y(x+z) = x(y+x)$$

Consider $x(y+z) = y(x+z)$.

$$x(y+z) = y(x+z) \Rightarrow (x-y)z = 0$$

$$z \neq 0, \quad x = y$$

Its the same for the other equations, so $x = y = z$. We also need to use $g = 0$. Setting $x = y = z$j, we get

$$\begin{aligned} g(x,x,x) &= 2x^2 + 2x^2 + 2x^2 - 24 = 0 \\ x^2 &= 4 \\ x &= 2 \end{aligned}$$

Lagranges method gives:

$$(x,y,z) = (2,2,2) \Rightarrow V = 8$$

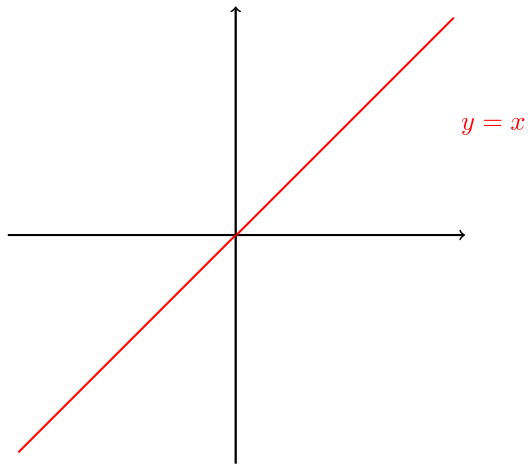
◇

7 Lecture 7

7.1 Parametrized Curve

A curve is described as a sset of points in $\mathbb{R}^2$ and $\mathbb{R}^3$. For instance a line is described by

$$f = \left\{(x, y) \in \mathbb{R}^2 : x = y\right\}$$



This is a static picture. But how do we give a dynamical picture? We'll use parametrized curves.

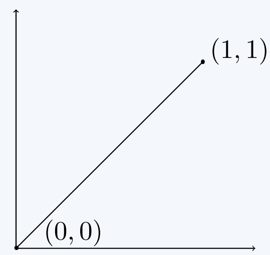
Definition 12. A parametrization of a curve c in $\mathbb{R}^2$, is given by

$$\vec{r}\,(t) = (x\,(t)\,, y\,(t))\,, \quad t_0 \le t \le t_1$$

Such that $\vec{r}\,(t) \in c$ for all time t.

- A parametrization describes motion. Think of t as the time.
- A parametrization is not unique.
- Various natural assumptions, such as continuity and differentiability.

Example. Consider the funtion $\vec{r}\,(t)$ with $0 \le t \le 1$.



$$\vec{r}(0) = (0, 0), \quad \vec{r}(1) = (1, 1)$$

We have the portion of the line where $y = x$. Notce that here, $x(t) = t$, $y(t) = t$ and $y(t) = x(t) = t$ for all t.

Lets consider a different function,

$$\vec{r}(t) = (2t, 2t), \quad 0 \leq t \leq \frac{1}{2}$$

When we parametrize, we get

$$\vec{r}(0) = (0, 0), \quad \vec{r}\left(\frac{1}{2}\right) = (1, 1)$$

We are moving along the curve at twice the speed. ◇

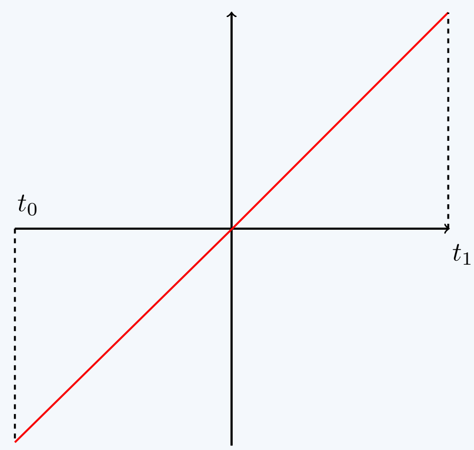
Example. Given $f(x)$, we consider

$$\vec{r}(t) = (t, f(t)), \quad t_0 \leq t \leq t_1$$

This describes a portion of the graph f, with

$$\text{Start: } (t_0, f(t_0)), \quad \text{End: } (t_1, f(t_1))$$

For instance, consider the line $y = mx + c$, we have



We have that

$$\vec{r}(t) = (t, mt + c), \quad t_0 \leq t \leq t_1$$

◇

Example. We want to describe a line with

$$\text{Start: } A = (x_0, y_0), \quad \text{End: } B = (x_1, y_1)$$

Then we take the parametrization

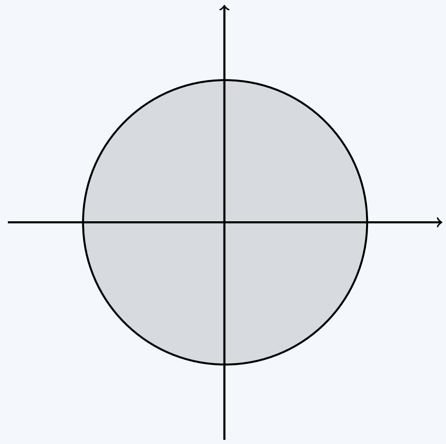
$$\vec{r}(t) = (1 - t) A + tB, \quad 0 \leq t \leq 1$$

More explicitly, we have

$$\vec{r}(t) = ((1 - t) x_0 + tx, (1 - t) y_0 + ty)$$

Note that $\vec{r}(0) = A$ and $\vec{r}(1) = B$. ◇

Example. Consider $\vec{r}(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$. What curve does this describe?



It describes a circle.

$$x(t)^2 + y(t)^2 = \left(\cos(t)^2 + \sin(t)^2\right) = 1$$

We start at $(1, 0)$ and move counter-clockwise. ◇

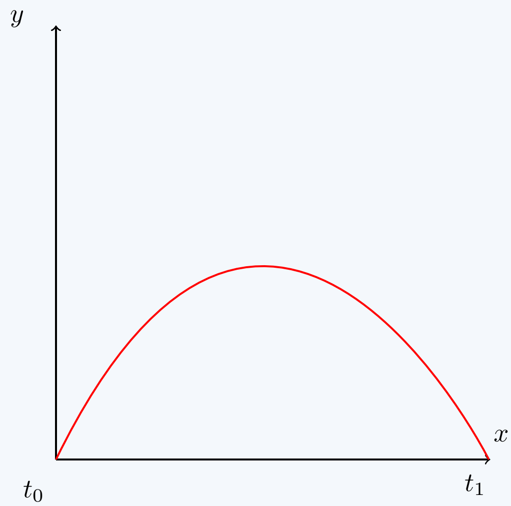
Example. Here is an example from physics. Consider

$$x(t) = v_x t, \quad y(t) = v_y t - \frac{1}{2} g t^2, \quad 0 \leq t \leq \frac{2v_y}{g}$$

This describes the motion of an object with initial velocity $\vec{v} = (v_x, v_y)$, under gravity. We write $t_0 = 0$ and $t_1 = \frac{2v_y}{g}$. Note that

$$\vec{r}(t_0) = (0, 0), \quad \vec{r}(t_1) = \left(\frac{2v_x v_y}{g}, 0\right)$$

The object falls back to the ground at time t_1.



Well known fact: This motion is parabolic, we will rederive this.
From $x(t) = v_x t$, we get $t = \frac{x(t)}{v_x}$. Then

$$y(t) = v_y t - \frac{1}{2} g t^2 \Rightarrow \frac{v_x}{v_y} x(t) - \frac{1}{2} \frac{g}{v_x^2} x(t)^2$$

This is the expression of a parabola

$$y = ax^2 + bx + c, \quad a \neq 0$$

It can also be written as

$$y(t) = -\frac{1}{2} \frac{g}{v_x^2} \left(x(t) - \frac{v_y}{g} \right)^2 + \frac{1}{2} \frac{v_y}{g} \frac{v_y}{v_x}$$

◇

7.2 Kinematics

Kinematics describes position, velocity and acceleration of an object.

Definition 13. The position vector is $\vec{r}(t)$. The velocity vector is $\vec{v}(t) \frac{d\vec{r}}{dt}$. The acceleration vector is $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2}$.

If we write $\vec{r}(t) = (x(t), y(t))$, then

$$\vec{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (x'(t), y(t))$$

Similarily

$$\vec{a}(t) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right) = (x''(t), y''(t))$$

Example. Consider again the gravity example. Here we have

$$\vec{r}\,(t) = \left(v_x t, v_y t - \frac{1}{2} g t^2\right)$$

The velocity is

$$\vec{v}\,(t) = (v_x, v_y - gt)$$

Note that $v\,(0) = (v_x, v_y)$ is the initial velocity of the object. For acceleration, we get

$$\vec{a}\,(t) = \frac{d\vec{v}}{dt} = (0, -g)$$

◇

8 Lecture 8

8.1 Determining Motion

Given acceleration $\vec{a}(t)$, can we find $\vec{v}(t)$ and $\vec{s}(t)$? Yes, with some initial conditions given, we can. This is done by integration, consider

$$\vec{v}(t) = \frac{d\vec{s}}{dt}$$

This is a differential equation for $\vec{s}(t)$. To solve it, we integrate both sides in t, from t_1, to t_2. We get

$$\int_{t_0}^{t_1} \vec{v}(t)\,dt = \int_{t_0}^{t_1} \frac{d\vec{s}}{dt}dt$$

The fundamental theorem of calculus gives

$$\vec{s}(t_1) - \vec{s}(t_0) = \int_{t_0}^{t_1} \vec{v}(t)\,dt$$

We can determine $\vec{s}(t)$ for any t if we know $\vec{v}(t)$ and the initial condition $\vec{s}(t_0)$.

Example. Consider an object with acceleration

$$\vec{a}(t) = (1, t) = \vec{i} - j\vec{j}$$

We have the following initial conditions

$$\vec{s}(0) = (2, 0) = 2\vec{i} \quad \wedge \quad \vec{v}(0) = 0$$

We want to determine $\vec{s}(t)$. First, to determine $\vec{v}(t)$, we compute

$$\begin{aligned}\int_0^t \vec{a}(t)\,dt &= \vec{i}\int_0^t 1dt + \vec{j}\int_0^t tdt \\ &= t\vec{i} + \frac{1}{2}t^2\vec{j}\end{aligned}$$

Here $t_0 = 0$, since $\vec{v}(0) = 0$, then

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t)\,dt = t\vec{i} + \frac{1}{2}t^2\vec{j}$$

To determine $\vec{s}$, we compute

$$\begin{aligned}\int_0^t \vec{v}(t)\,dt &= \vec{i}\int_0^t tdt + \vec{j}\int_0^t \frac{1}{2}t^2dt \\ &= \frac{1}{2}t^2\vec{i} + \frac{1}{6}t^3\vec{j}\end{aligned}$$

Since $\vec{s}(0) = (2, 0) = 2\vec{i}$, we get

$$\vec{s}\,(t) = \vec{s}\,(0) + \int_0^t \vec{v}\,(t)\,dt = \left(\frac{t^2}{2} + 2\right)\vec{i} + \frac{1}{6}t^3\vec{j}$$

◇

8.2 Arc Length

A formula that we can concretely use to compute the length of a curve (using a parametrization). Consider a curve c, with parametrization

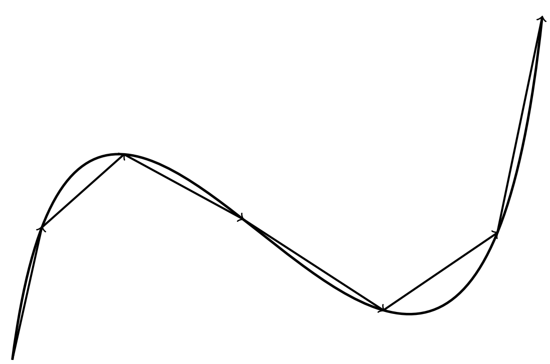
$$\vec{s}\,(t) = (x\,(t)\,, y\,(t))\,, \quad t_0 \leq t \leq t_1$$

Definition 14. The arc length of c is given by

$$S = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

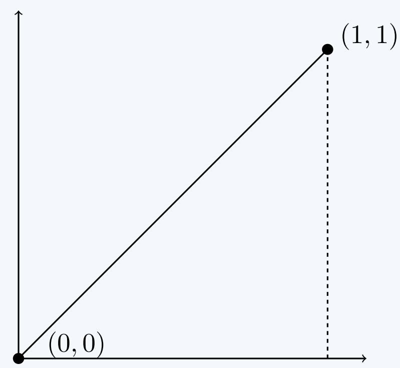
Idea: summing segments of length

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$



To compute S, we need a parametrization of c. Does S depent on this choice? No!

Example. Consider the line segment below



From elementary geometry, its length is $\sqrt{1^2+1^2}=\sqrt{2}$. Consider the parametrization

$$\vec{s}\,(t)\,(t,t)\,,\quad 0\leq t\leq 1$$

We have $(x'(t)\,, y'(t)) = (1,1)$. Then

$$S=\int_0^1\sqrt{1^2+1^2}dt=\sqrt{2}\int_0^1 1dt=\sqrt{2}$$

Instead we choose

$$\vec{s}\,(t)=(2t,2t)\,,\quad 0\leq t\leq\frac{1}{2}$$

We have $(x'(t)\,, y'(t)) = (2,2)$. Then

$$S=\int_0^{\frac{1}{2}}\sqrt{2^2+2^2}dt=\sqrt{8}\int_0^{\frac{1}{2}} 1dt=\sqrt{8}\cdot\frac{1}{2}=\sqrt{2}$$

$\diamond$

Question: What is the distance crossed up to time t?

Definition 15. The arc length parameter is

$$S\,(t)=\int_{t_0}^{t_1}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$$

The difference is that we integrate up to t, not t_1. Special case when $S\,(t_1)=S$. There is an important relation between $S\,(t)$ and $\vec{v}\,(t)$.

Proposition. We have

$$|\vec{v}\,(t)|=\frac{dS}{dt}$$

Proof. The fundamental theorem of calculus states that if

$$F(x) = \int_a^x f(x)dt \to F'(x) = f(x)$$

Applying this to $S(t)$, then

$$S'(t) = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

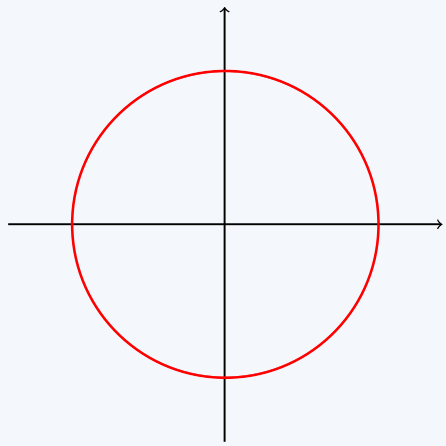
On the other hand, we have that

$$\vec{v}(t) = (x'(t), y'(t)) \quad \wedge \quad |\vec{v}(t)| = (t) = \sqrt{x'(t)^2 + y'(t)^2}$$

The two expressions coincide. $\square$

Example. Consider a circle of radius R, with

$$\vec{S}(t) = (R\cos t, R\sin t), \quad 0 \leq t \leq 2\pi$$



We want to compute $S(t)$, we have

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{R^2(\sin t)^2 + R^2(\cos t)^2} = R$$

We want to check that $\frac{dS}{dt} = |\vec{v}(t)|$. We have

$$\vec{v}(t) = \frac{d\vec{S}}{dt} = (-R\sin t, R\cos t)$$

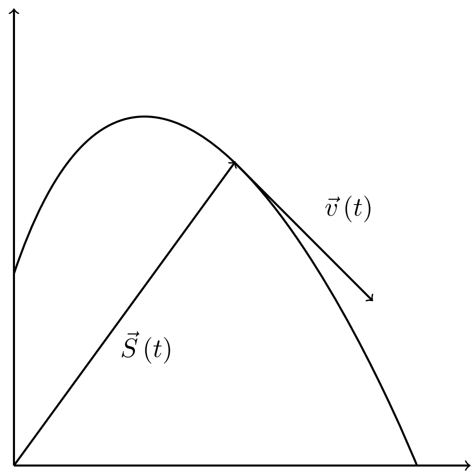
Its length is equal to $|\vec{v}(t)| = R$. Since $S(t) = R(t)$, we see that

$$\frac{dS}{dt} = |\vec{v}(t)|$$

$\diamond$

8.3 Tangent Vectors

Geometrically,, the velocity $\vec{v}(t)$ is tanget to a curve. It is useful to define a tangent vector of length 1.



Definition 16. The unit tangent vector is

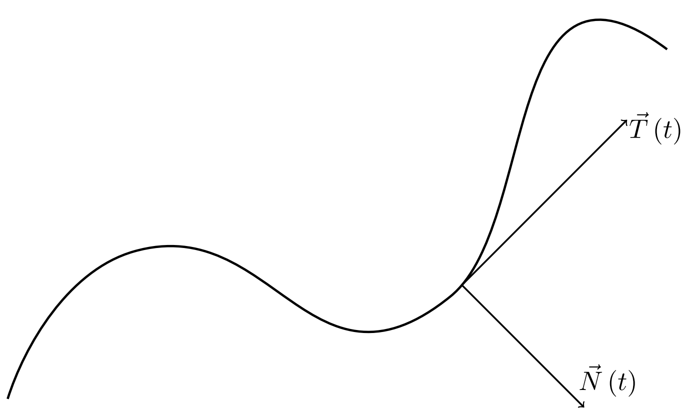
$$\vec{T}\,(t) = \frac{\vec{v}\,(t)}{|\vec{v}\,(t)|}$$

Note that $\vec{T}$ has length 1 since

$$\vec{T}\,(t) \cdot \vec{T}\,(t) = \frac{\vec{v}\,(t) \cdot \vec{v}\,(t)}{|\vec{v}\,(t)|^2} = 1$$

8.4 Normal Vectors

Normal vectors are normal to the curve, or in other words, they are orthogonal. Recall that for implicit curves $f\,(x, y) = 0$, a normal vector is given by ∇f.



Lets now consider parametrizised curves. We have

$$S(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Definition 17. A unit normal vector to the curve is defined by

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

We need to check that $\vec{N}$ is orthogonal to $\vec{T}$, that is: $\vec{N}(t) \cdot \vec{T}(t) = 0$ for all t.

Proposition. We have that

$$\vec{N}(t) \cdot \vec{T}(t) = 0$$

Proof. Since $\vec{T}$ is a unit vector, we have that, for all t

$$\vec{T}(t) \cdot \vec{T}(t) = 1$$

Take the time derivative, the $\left(\vec{T} \cdot \vec{T}\right) = 0$. We also have

$$\frac{d}{dt}\left(\vec{T} \cdot \vec{T}\right) = \frac{d\vec{T}}{dt} \cdot \vec{T} + \vec{T}\frac{d\vec{T}}{dt} = 2\frac{d\vec{T}}{dt} \cdot \vec{T}$$

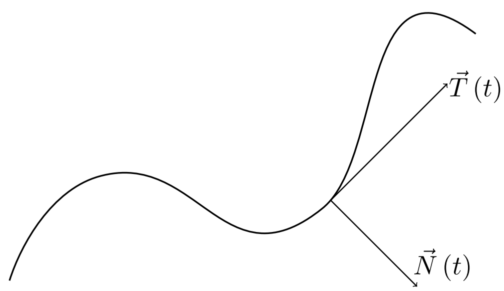
Since $\left(\vec{T} \cdot \vec{T}\right) = 0$, we get $\frac{d\vec{T}}{dt} \cdot \vec{T} = 0$. Dividing by $\left|\frac{d\vec{T}}{dt}\right|$, we get $\vec{N}(t) \cdot \vec{T}(t) = 0$ □

9 Lecture 9

9.1 Normal Vectors - Continued

Yesterday we saw that

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$



Example. Consider the circle

$$\vec{r}(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$$

The velocity is

$$\vec{v}(t) = \vec{r}' = (-\sin t, \cos t)$$

We have $|\vec{v}(t)| = 1$, since

$$\vec{v}(t) \cdot \vec{v}(t) = \left(\sin t^2 = \cos t^2\right) = 1$$

We find that $\vec{T}(t) = \vec{v}(t)$. To find $\vec{N}$, we need first

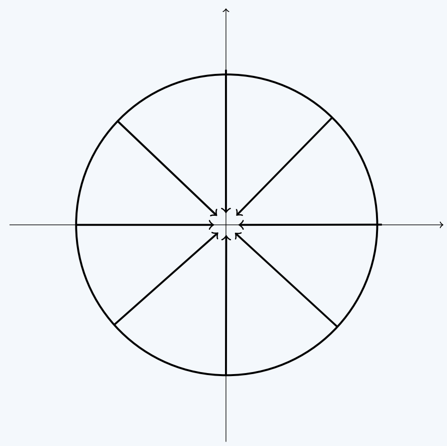
$$\frac{d\vec{T}}{dt} = \frac{d\vec{v}}{dt} = (-\cos t, -\sin t)$$

We check that $\left|\frac{d\vec{T}}{dt}\right| = 1$, then

$$\vec{N}(t) = (-\cos t, -\sin t) = -\vec{r}(t)$$

We compute more explicitly:

$$\vec{N}(t) \cdot \vec{T}(t) = (-\cos t, -\sin t) \cdot (-\sin t, \cos t) = \cos t \cdot \sin t - \sin t \cdot \cos t = 0$$



◇

We revisit the implicit case $f(x,y) = 0$.

Proposition. Let C be defined by $f(x,y) = 0$. A unit normal to C is given by $\vec{n} = \frac{\nabla f}{|\nabla f|}$

Proof. Suppose C is parametrized by

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

By definition of C, we have

$$f(\vec{r}(t)) = f(x(t)\, y(t)), t_0 \leq t \leq t_1$$

We have that $\frac{d}{dt} f(\vec{r}(t)) = 0$, but using the chain rule, we get

$$\frac{df(\vec{r}(t))}{dt} = \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

Hence, we get

$$\nabla f(\vec{r}(t)) \cdot \vec{v}(t) = 0$$

Therefore $\nabla f(\vec{r}(t))$ is normal to the curve C, or orthogonal to the tangent $\vec{v}(t)$ □

Example. The circle of radius 1 can be described implicitly by $f(x,y) = 0$ with

$$f(x,y) = x^2 + y^2 - 1$$

The gradient is $\nabla f(x,y) = (2x, 2y)$, not of length 1, since

$$|\nabla f(x,y)| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2}$$

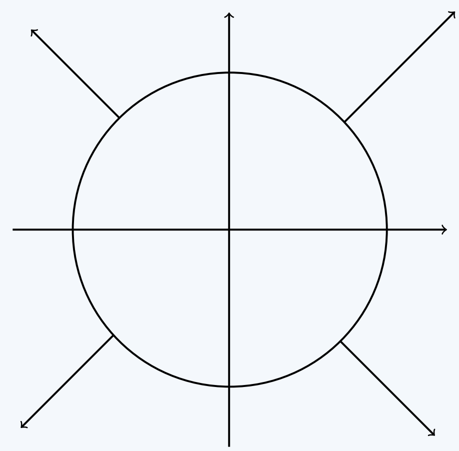
We are only interested in (x,y) such that $f(x,y)=0$, that is $x^2+y^2=1$, then

$$|\nabla f| = 2$$

Hence we get

$$\vec{n}\frac{\nabla f}{|\nabla f|} = \frac{(2x,2y)}{2} = (x,y)$$

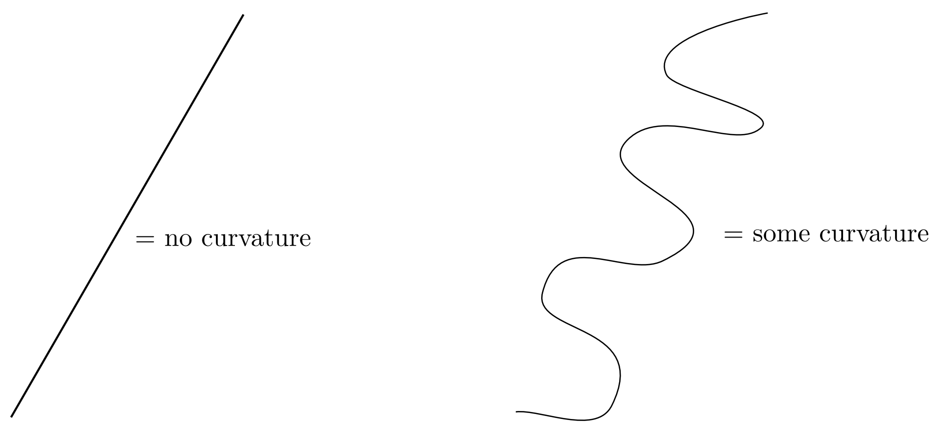
The normal vectors point outside of the circle, note: $\vec{n} = -\vec{N}$, compare with the parameter.



$\diamond$

9.2 Curvature

We want to compute how much a curve "curves".



How do we quantify this?

Definition 18. Let C be a parametrized curve, ant $\vec{T}$ its unit tangent vector. The curvature is then defined by

$$K = \left|\frac{d\vec{T}}{dS}\right|$$

Where S is the arc length parameter of C. We also define the radius of curvature as

$$\rho = \frac{1}{K}$$

Note: We consider S, not t. Reason: K does not depend on parametrization.

Example. Consider the following

$$\vec{r}(t) = (0,0)\,, \quad 0 \leq t \leq 1$$

We expect K to be zero. We can easily compute

$$S(t) = \int_0^t \sqrt{1^2 - 0^2} dt = t$$

Here we have $S(t) = t$. Now we compute $\vec{T}$. We have $\vec{v}(t) = (1,0)$, and $|\vec{v}(t)| = 1$. Then $\vec{T}(t) = \vec{v}(t) = (1,0)$. Furthermore, we get that

$$\frac{d\vec{T}}{dS} = 0 \quad \wedge \quad K = 0$$

◇

Example. Consider a circle of radius R with

$$\vec{r}\,(t) = (R\cos t, R\sin t)\,, \quad 0 \leq t \leq 2\pi$$

The velocity is

$$\vec{v}\,(t) = (-R\sin t, R\cos t)$$

We get that $|\vec{v}\,(t)| = R$, then

$$\vec{T}\,(t) = \frac{\vec{v}\,(t)}{|\vec{v}\,(t)|} = (-\sin t, \cos t)$$

To compute K, we need to express $\vec{T}$ in terms of $\vec{S}\,(t)$. We saw that

$$S\,(t) = R \cdot t \Rightarrow t = \frac{S}{R}$$

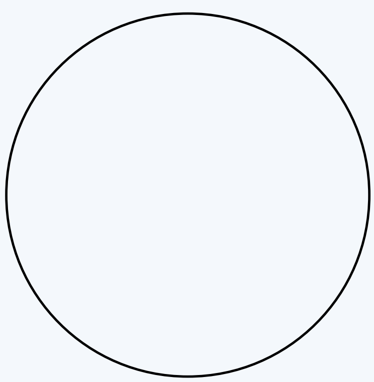
Now we compute

$$\frac{d\vec{T}}{dS} = \left(-\frac{1}{R}\cos\frac{S}{R}, -\frac{1}{R}\cos\frac{S}{R}\right)$$

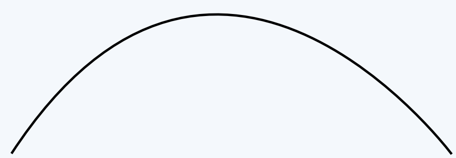
Finally, we have

$$K = \left|\frac{d\vec{T}}{dS}\right| = \sqrt{\frac{1}{R^2}\left(\cos\frac{S}{R}\right)^2 + \frac{1}{R^2}\left(\sin\frac{S}{R}\right)^2} = \frac{1}{R}$$

We have non-zero, constant curvature, we also have that $\rho = R$. Note that $K \to 0$, as $r \to \infty$



Large Curvature



Small Curvature

◇

Proposition. We have that

$$K = \frac{1}{|\vec{v}|} = \left| \frac{d\vec{T}}{dt} \right|$$

Example. Consider again

$$\vec{r}\,(t) = (R\cos t, R\sin t)$$

We have seen that

$$|\vec{v}\,(t)|\,, \vec{T}\,(t) = (-\sin t, \cos t)$$

To get K, we compute

$$\frac{d\vec{T}}{dt} = (-\cos t, -\sin t)$$

◇

10 Lecture 10

10.1 Curves in polar form

Polar coordinates are an alternative description to cartesian coordinates (x, y).

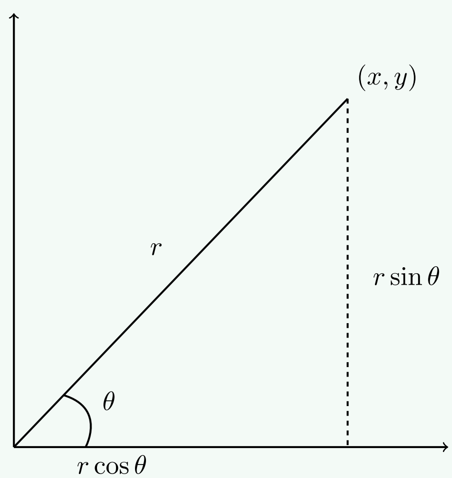
Definition 19. The polar coordinates r, θ are defined by

$$x = r\cos\theta, \quad y = r\sin\theta$$

Their range is respectively

$$0 \le r \le \infty, \quad 0 \le \theta \le 2\pi$$

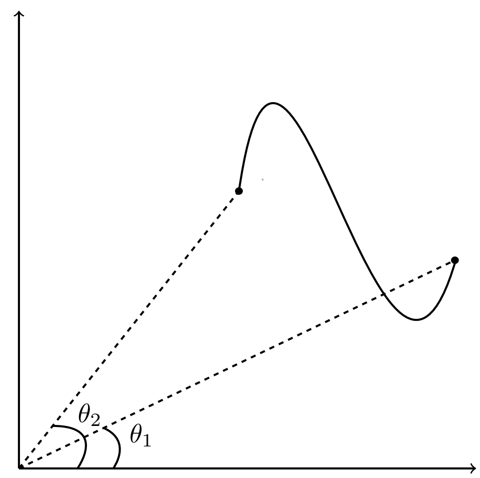
Geometrical meaning is that r is the distance from the origin, and θ is the angle.



From (x, y) to (r, θ), we can use

$$r^2 = x^2 + y^2$$

We can describe curves using (r, θ). The idea is to give r as a function of θ. The curve will be "traced" as we vary θ. It is an analogue of $y = f(x)$.



Example. Consider the curve

$$r(\theta) = 1, \quad 0 \leq \theta \leq 2\pi$$

What curve is it? All points have distance 1 from origin ($r = 1$)
Using $r^2 = x^2 + y^2$, we find that $x^2 + y^2 = 1$. We have a circle of radius 1. ◇

Example. The next curve is called the cardioid, it is defined by

$$r(\theta) = 1 - \cos\theta, \quad 0 \leq \theta \leq 2\pi$$

It is described in cartesian coordinates by

$$\left(x^2 + y^2 + x\right)^2 = x^2 + y^2$$

Polar coordinates work best in the prescence of spherical symmetry. The length of C can be computed using polar coordinates. ◇

Proposition. Let C be given in polar form by

$$r = r(\theta), \quad \alpha \leq \theta \leq \beta$$

Then its arc length can be computed by

$$S = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 d\theta}$$

Proof. Parametrize C by

$$\vec{r}(\theta) = (x(\theta), y(\theta)), \quad \alpha \leq \theta \leq \beta$$

where we set

$$x(\theta) = r(\theta)\cos\theta, \quad y(\theta) = r(\theta)\sin\theta$$

The derivates are, with $r' = \frac{dr}{d\theta}$

$$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = r'\sin\theta + r\cos\theta$$

After some computation we get

$$(x')^2 + (y')^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

Therefore we get

$$\begin{aligned} S &= \int_\alpha^\beta \sqrt{(x')^2 + (y')^2} d\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \end{aligned}$$

□

Example. We have a circle given by

$$r(\theta) = R, \quad 0 \leq \theta \leq 2\pi$$

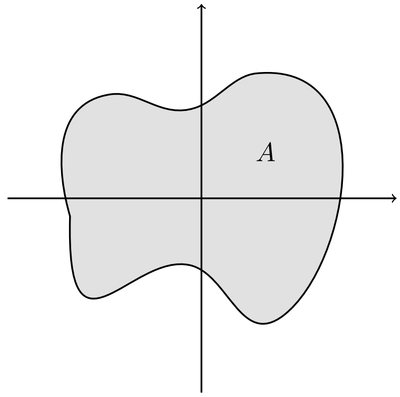
We have $\frac{dr}{d\theta} = 0$, then

$$S = \int_0^{2\pi} \sqrt{R^2 + 0^2} d\theta = R\int_0^{2\pi} d\theta = 2\pi R$$

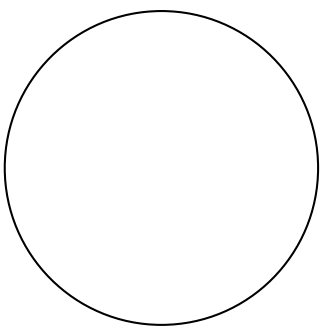
This gives us the circumference of the circle. ◇

10.2 Areas In Polar Form

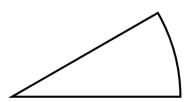
We want to compute the area inside a closed curve in polar form.



Basic observation:



Has area πr^2



Has area $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$

Add small regions with angle $\Delta\theta$ and area $\frac{1}{2} r^2 \Delta\theta$.

Proposition. Consider a closed curve described by

$$r = r(\theta), \quad \alpha \leq \theta \leq \beta$$

Then its area is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r(\theta)^2 \, d\theta$$

Example. We have a circle of radius R

$$r(\theta) = R, \quad 0 \leq \theta \leq 2\pi$$

we get

$$A = \frac{1}{2}\int_0^{2\pi} R^2 d\theta = \frac{1}{2}R^2 \int_0^{2\pi} d\theta = \pi R^2$$

◇

Example. Consider the cardioid

$$r(\theta) = 1 - \cos\theta, \quad 0 \leq \theta \leq 2\pi$$

The area is given by:

$$\begin{aligned} A &= \int_0^{2\pi} (1-\cos\theta)^2 \, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(1 - 2\cos\theta + (\cos\theta)^2\right) d\theta \end{aligned}$$

To compute this, we use

$$\int \cos\theta d\theta = \sin\theta + C, \qquad \int (\cos\theta)^2 \, d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

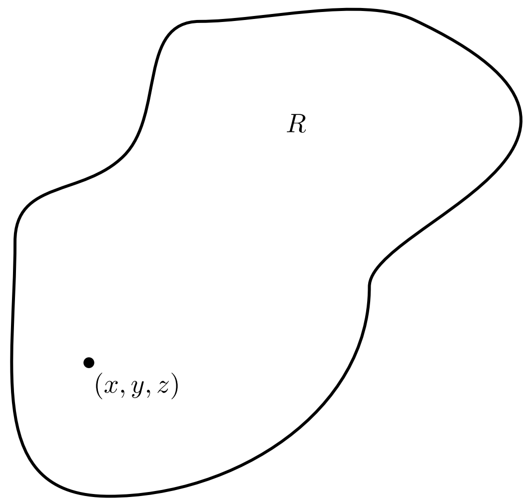
Finally, we obtain

$$A = \frac{1}{2}\cdot 2\pi + 0 + \frac{1}{2}\cdot\frac{1}{2}2\pi = \frac{3}{2}\pi$$

◇

11 Scalar and Vector Fields

Idea. A field describes a property of a region R



Mathematically described by

$$f : \mathbb{R}^m \to \mathbb{R}^n, \quad m \text{ inputs}, n \text{ outputs}$$

For scalar fields the output is a scalar. For vector fields the output is a vector.

Example. The temperature is a scalar field.

$$T : (x, y, z) \to T\,(x, y, z)$$

The wind velocity is a vector field

$$\vec{W} : (x, y, z) \to \vec{W}\,(x, y, z)$$

◇

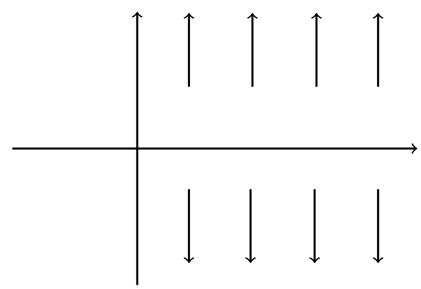
Notation. The notation for vector fields:

$$\begin{aligned}\vec{F}\,(x, y, z) &= (P\,(x, y, z)\,, Q\,(x, y, z)\,, R\,(x, y, z)) \\ &= P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}\end{aligned}$$

P, Q, R are componente functions.

In 2D we visualize vector fields by vector plots. For instance, take

$$\vec{F}\,(x, y) = (0, y)$$



A vector field $\vec{F}$ and a scalar field f can be related as follows

Definition 20. If $\vec{F} = \Delta f$, we say that $\vec{F}$ is a gradient field, and f is a potential.

11.1 Gradient, Divergence and Curl

These are operation defined in terms of the formal vector

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Definition 21. The gradient of a scalar field f is

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

The output is a vector.

The divergence of a vector field $\vec{F}\,(P, Q, R)$ is

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The output is a scalar field.

The curl of a vector field $\vec{F}$ is

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left|\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{bmatrix}\right|$$

These operations can all be obtained from ∇.

Operation	Input	Output	Symbol
Gradient	Scalar	Vector	∇f
Divergence	Vector	Scalar	$\nabla \cdot f$
Curl	Vector	Vector	$\nabla \times f$

12 Lecture 11

12.1 Operations on fields

Example. Consider $\vec{F}(x,y,z) = (x^2, y^2, z^2)$, that is

$$P = x^2, \quad Q = y^2, \quad R = z^2$$

Its divergence is

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2x + 2y + 2z$$

We can check that $\vec{F}$ is a gradient field. We have that $\vec{F} = \nabla f$, with the potential

$$f(x,y,z) = \frac{1}{3}\left(x^2 + y^2 + z^2\right)$$

◇

$\nabla \times \vec{F}$ is computed as a determinant. We can use the cofactor, or the Laplace expansion.

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{bmatrix} \right| = \begin{bmatrix} \partial_y & \partial_z \\ Q & R \end{bmatrix} \vec{i} - \begin{bmatrix} \partial_x & \partial_z \\ P & R \end{bmatrix} \vec{j} + \begin{bmatrix} \partial_x & \partial_y \\ P & Q \end{bmatrix} \vec{k}$$

$$= \vec{i}\left(\partial_y R - \partial_z Q\right) - \vec{j}\left(\partial_x R - \partial_z P\right) + \vec{k}\left(\partial_x - Q\partial_y P\right)$$

This is a concrete formula for $\nabla \times \vec{F}$.

Example. Consider $\vec{F} = xy\vec{i} + (x+z)\vec{j} + yz\vec{k}$. We compute

$$\nabla \times \vec{F} = \vec{i}\left(\partial_y R - \partial_z Q\right) - \vec{j}\left(\partial_x R - \partial_z P\right) + \vec{k}\left(\partial_x - Q\partial_y P\right)$$

where

$$P = xy, \qquad Q = x + z, \qquad R = yz$$

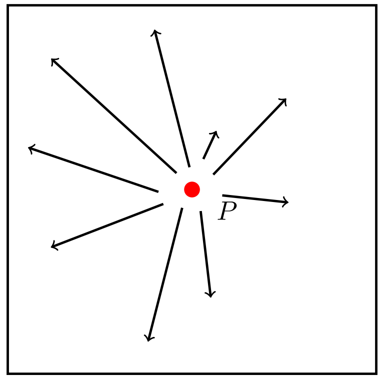
$$\nabla \times \vec{F} = (2-1)\,\vec{i} + (1-x)\,\vec{k}$$

◇

Note. Gradient and divergence can be defined in any dimension. The curl is only defined in up to three dimensions.

12.2 Intepretation of Divergence

Think of $\vec{F}$ as the velocity of a fluid. $\nabla \cdot \vec{F}$ at a point P is the amount of fluid entering / leaving a small region around P.



Example. Consider $\vec{F_1}(x, y) = (x, y)$, then

$$\nabla \cdot \vec{F_1} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + 2$$

This corresponds to fluid leaving the region. Similarily for $\vec{F_2} = (-x, -y)$, then

$$\nabla \cdot \vec{F_2} = \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -2$$

This corresponds to fluid entering the region. Finally consider $\vec{F_3} = (0, 1)$, then

$$\nabla \cdot \vec{F_3} = 0$$

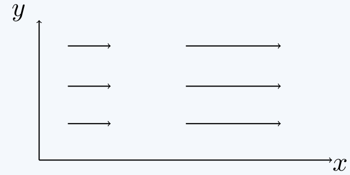
This is an equillibrium situation. ◇

12.3 Interpretation of Curl

$\nabla \times \vec{F}$ measures the "rotation" of $\vec{F}$.

Example. Consider $\vec{F}(x, y, z) = \left(x^2, 0, 0\right)$. We expect no rotation, we compute

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 0 & 0 \end{bmatrix} \right| = 0$$



◇

Example. Consider $\vec{F}(x,y,z) = (-\omega y, \omega x, 0)$, where ω is a non-zero constant. We compute

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -\omega y & \omega x & 0 \end{bmatrix} \right| = \begin{bmatrix} \partial_x & \partial_y \\ -\omega y & -\omega x \end{bmatrix} \vec{k} = 2\omega \vec{k}$$

$\nabla \times \vec{F} \neq 0$ gives non-zero rotation. Also $\nabla \times \vec{F} =$ const gives that that there is some rotation everywhere.

For a physical intepretation of this, we can write

$$\vec{v} = \vec{F}, \quad \vec{\omega} = (0, 0, \omega), \quad \vec{r} = (x, y, z)$$

Then we can check that $\vec{v} = \vec{\omega} \cdot \vec{r}$. This is the velocity corresponding to the angular velocity $\vec{\omega}$

◇

12.4 Scalar Field from Gradient

Suppose that we know ∇f. Can we recover f? Yes, up to the initial conditions. The strategy is the following, first we write

$$\vec{F} = \nabla f = (P, Q, R)$$

By the definition of the gradient field, we have

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

Comparing, we get

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q, \quad , \quad \frac{\partial f}{\partial x} = R$$

Example. Suppose we are given the gradient field

$$\vec{F} = \nabla f = \left(2x + 2y, \frac{1}{2}x^2 + 3y \right)$$

We want to find f, we must have

$$\frac{\partial f}{\partial x} = 2x + 2y, \quad \frac{\partial f}{\partial y} = \frac{1}{2}x^2 + 3y$$

Integrating the first equation in x gives us

$$\begin{aligned} f(x,y) &= \int (2x + xy)\, dx \\ &= x^2 + \frac{1}{2}x^2 y + g(y) \end{aligned}$$

Where $g(y)$ is the integration constant. It can depend on y.
Now we compute $\frac{\partial f}{\partial y}$. We get

$$\frac{\partial f}{\partial y} = \frac{1}{2}x^2 + \frac{dg}{dy}$$

But we also have that $\frac{\partial f}{\partial y} = \frac{1}{2}x^2 + 3y$.
Comparing them, we get

$$\cancel{\frac{1}{2}x^2} + \frac{dg}{dy} = \cancel{\frac{1}{2}x^2} + 3y$$

Then $\frac{dg}{dy} = 3y$. Integrating in y we get

$$g(y) = \int 3y dy = \frac{3}{2}y^3 + C$$

Here, C is a constant. Finally, inserting it, we get

$$\begin{aligned} f(x,y) &= x^2 + \frac{1}{2}x^2 y + g(k) \\ &= x^2 + \frac{1}{2}x^2 y + \frac{3}{2}y^3 + C \end{aligned}$$

The constant C is usually not important. It can be fixed by an inital condition. For instance $f(0,0)$ implies that $C = 0$

◇

13 Lecture 12

13.1 Identities Between Operations

We have seen three operations defined by ∇.

$$\text{Gradient: } \nabla f, \quad \text{Divergence: } \nabla \cdot \vec{F}, \quad \text{Curl: } \nabla \times \vec{F}$$

There are many identities, we'll now look at one.

Proposition. For any scalar field f, we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$, then

$$\nabla \times (\nabla f) = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix} \right| = (f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{xy})$$

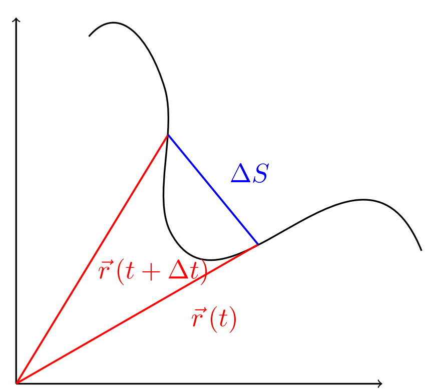
But partial derivatives can be exchanged. Then we find that $\nabla \times (\nabla f) = 0$ □

We are going to use this when we discuss conservative fields.

13.2 Line Integrals

Consider the curve C with

$$\vec{r}\,(t) = (x\,(t)\,, y\,(t))\,, t_0 \le t \le t_1$$



$$\Delta x = x\,(t + \Delta t) - x\,(t)\,, \quad \Delta y = y\,(t + \Delta t) - y\,(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x^2) + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt}\Delta t, \quad \Delta y \approx \frac{dy}{dt}\Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt}\Delta t^2\right) + \left(\frac{dy}{dt}\Delta t^2\right)} = \Delta t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Letting $\Delta t \to 0$ leads to the following

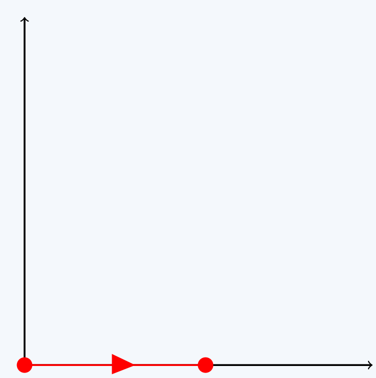
Definition 22. The line integral of $f(x, y)$ along a curve C is defined by

$$\int_C f ds = \int_{t_0}^{t_1} f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}dt$$

Observe. Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$. When $f = 1$, we recover the arc length.

Example. Consider C defined by

$$x(t) = t, \quad y(t) = 0, \quad 0 \le t \le 1$$



First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2}dt = \sqrt{1^2 + 0^2} = 1$$

Now, consider $f(x, y) = x^2 + y$. We want to compute $\int_C f ds$. Restricting f to C gives

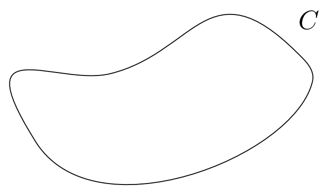
$$f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f ds = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

◇

Note. Line integrals can be used to compute the mass of a 1-dimensional object. The curve C describes the object, and the function $\int_C f ds$ is the mass.



13.3 Parametrization and Orientation

The next result is as for the arc length.

Proposition. The integral $\int_C f ds$ does not depent on the parametrization of C.

We will consider a special case

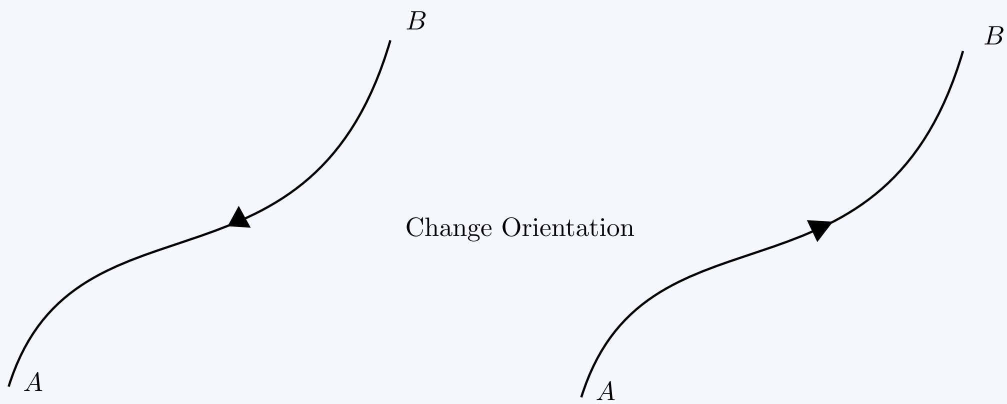
Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), t_0 \leq t \leq t_1$$

We have

$$\text{Start: } A = (x(t_0), y(t_0)), \quad \text{End: } B = (x(t_1), y(t_1))$$

We want to go from B to A.



We can do this in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\begin{aligned}\vec{r}_{\text{opp}}(-t_1) &= (x(t1), y(t_1)) = B \\ \vec{r}_{\text{opp}}(-t_0) &= (x(t0), y(t_0)) = A\end{aligned}$$

◇

Example. Consider the segment C from $(0,0)$ to $(1,0)$. Take $f(x,y) = x$. Show that $\int_C f ds = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$.

If C parametrized by $\vec{r}(t)$, we use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have

$$\int_C f ds = \int_{-C} f ds$$

The situation will be different for vector fields. ◇

13.4 Case of Vector Fields

Consider the curve with

$$\begin{aligned}\vec{r}(t) &= (x(t), y(t)), \quad t_0 \leq t \leq t_1 \\ \vec{r'}(t) &= (x'(t), y'(t)) \quad \text{(Velocity vector)}\end{aligned}$$

Definition 23. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}' dt$$

Here $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$

In physics, we have that $\vec{F}$ is the force, and $\int_C \vec{F} d\vec{r}$ is the work done by $\vec{F}$ along C. The elementary case is given by $W = FS$, or work = force · displacement. More explicitly, we consider

$$\vec{F}(x, y) = (P(x, y), Q(x, y))$$

Then we have

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

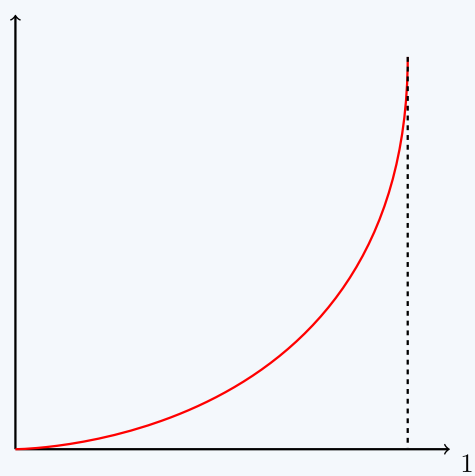
$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t)\, dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t)\, dt$$

We will write the expression as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + \int_C Q dy$$

Example. Consider the curve C with

$$\vec{r}(t) = \left(t, t^2\right), \quad 0 \le t \le 1$$



We have $x(t) = t$ and $y(t) = t^2$, its derivative is

$$\vec{r}(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is $P = x + y$ and $Q = x$, when this is restricted to C, we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = \left(t + t^2, t\right)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}' = \left(t + t^2, t\right) \cdot (1, 2t) = \left(3t^2 + t\right)$$

We finally obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(3t^2 + t\right) dt = \frac{3}{2}$$

◇

14 Lecture 13

14.1 Line Integrals of Vector Fields

We have seen the following definition earlier

$$\int_C \vec{F} d\vec{r} = \int_{t_0}^{t_1} \vec{F}\left(\vec{r}\left(t\right)\right) \cdot \vec{r}'\left(t\right) dt$$

We can rewrite this to link with scalarfields, lets consider

$$\int_C = \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F} \cdot \vec{v} dt = \int_{t_0}^{t_1} \vec{F} \cdot \frac{\vec{v}}{|\vec{v}|} dt$$

We can see this as the line integral of the field $\vec{F} \cdot \vec{T}$

Example. Consider $\vec{F}$ is constant and directed along the curve, that is $\vec{F} = F\vec{T}$, where $F - \left|\vec{F}\right|$. Then the formula for elementary work $(W = FS)$ gives

$$W = \int_C \vec{F} \cdot \vec{T} ds = \int_C F\vec{T} \cdot \vec{T} ds = F \int_C ds = FS$$

◇

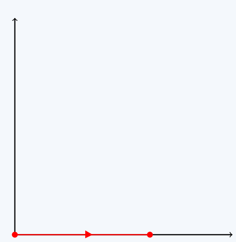
Now lets consider changes of orientation

Proposition. We have that

$$\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$$

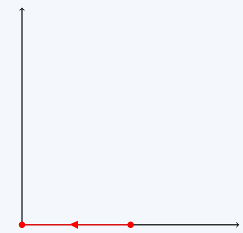
Example. Consider the segment

$$\vec{r}\left(t\right) = \left(t, 0\right), \quad 0 \leq t \leq 1$$



The opposite parametrization is

$$\vec{r}_{\text{opp}}\left(t\right) = \left(-t, 0\right), \quad -1 \leq t \leq 0$$



We have

$$\vec{r}'(t) = (1,0), \quad \vec{r}_{\text{opp}}(t) = (-1,0)$$

Consider $\vec{F}(x,y) = (x,0)$. For C we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (t,0) \cdot (1,0)\, dt = \int_0^1 t dt = \frac{1}{2}$$

For $-C$, we have

$$\int_{-C} \vec{F} \cdot d\vec{r} = \int_{-1}^0 (-t,0) \cdot (-1,0)\, dt = \int_{-1}^0 t dt = -\frac{1}{2}$$

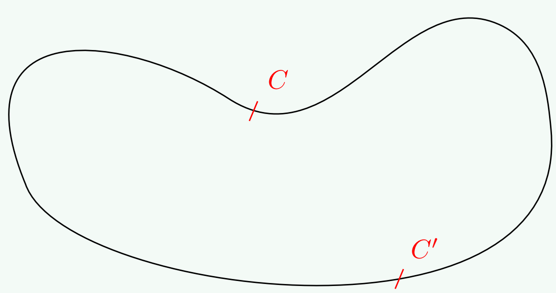
◇

14.2 Conservative Vector Fields

In general, $\int_C \vec{F} \cdot d\vec{r}$ depends on the curve C. However, sometimes it only depends on the endpoints.

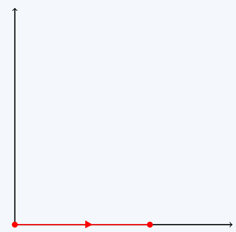
Definition 24. A vector field $\vec{F}$ is conservative if $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of C. That is, if C and C' have the same endpoints, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}$$



Example. Consider $\vec{F} = (1,1)$ and

$$C : x(t) = t, \quad y(t) = 0, \quad 0 \le t \le 1$$

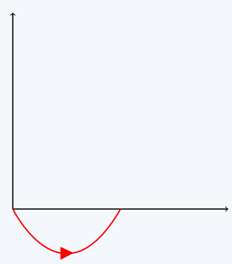


The endpoints are $(0,0)$ and $(1,0)$. We compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 dt = 1$$

Now, lets consider a different curve.

$$C' : x(t) = t, \quad y(t) = t(t-1), \quad 0 \le t \le 1$$



We have the same endpoints as C. The velocity is $\vec{r}'(t) = (1, 2t-1)$. Then

$$\begin{aligned}\int_{C'} \vec{F} \cdot d\vec{r} &= \int_0^1 (1,1) \cdot (1, 2t-1)\, dt \\ &= \int_0^1 2tdt - 2 \cdot \frac{1}{2} = 1\end{aligned}$$

At this stage, we cannot conclude $\vec{F}$ is consrvative (although it is). Note that

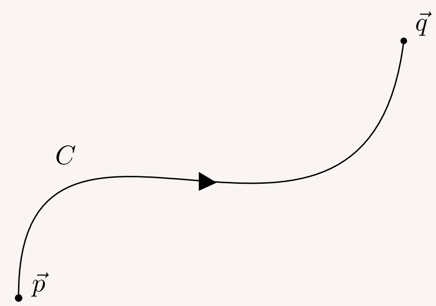
$$\vec{F} = \nabla f \quad \wedge \quad f(x,y) = x + y$$

We will prove that being a gradient field is the condition we want. $\diamond$

Theorem 25. The Gradient Theorem.

Suppose $\vec{F} = \nabla f$, consider a curve C starting at $\vec{p}$, and ending at $\vec{q}$, then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{q}) - f(\vec{p})$$



Proof. Pick a parametrization

$$\vec{r}(t), \quad t_0 \leq t \leq t_1$$

Note that: $\vec{r}(t_0) = \vec{p}$ and $r(t_1) = \vec{q}$. Using $\vec{F} = \nabla f$, we have

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \\ &= \int_{t_0}^{t_1} \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt\end{aligned}$$

From the chain rule, we get

$$\frac{df(\vec{r}(t))}{dt} = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$

Using this, we obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \frac{df(\vec{r}(t))}{dt} dt$$

Using the fundamental theorem of calculus, we get

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(\vec{r}(t_1)) - f(\vec{r}(t_0)) \\ &= f(\vec{q}) - f(\vec{p})\end{aligned}$$

□

If $\vec{F} = \nabla f$, then $\vec{F}$ is conservative, since $\int_C \vec{F} \cdot d\vec{r} = f(\vec{q}) - f(\vec{p})$ only depends on $\vec{p}$ and $\vec{q}$

Example. Consider the previous example with $\vec{F} = (1, 1)$. We saw $\vec{F} = \nabla f$ with $f(x, y) = x+y$.

For any curve starting at $\vec{p} = (0,0)$, and ending at $\vec{q} = (1,0)$, we have

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(\vec{q}) - f(\vec{p}) \\ &= f(1,0) - f(0,0) \\ &= 1 - 0 = 1\end{aligned}$$

Note that we could take

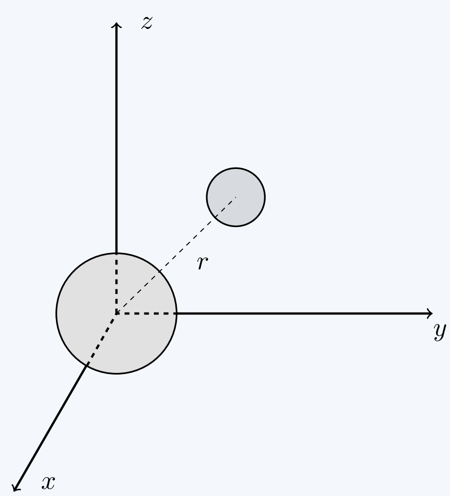
$$\tilde{f}(x,y) = x + y + c$$

With c being a constant. This gives the same result.

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \tilde{f}(1,0) - \tilde{f}(0,0) \\ &= 1 + c - c = 0\end{aligned}$$

◇

Example. Consider two objects A and B of mass M and m, as in the picutre



The force exerted by A on B is

$$\vec{F}(x,y,z) = -G\frac{Mm}{r^2}\hat{r}$$

Here we have $r = \sqrt{x^2 + y^2 + z^2}$ ant $\hat{r} = \frac{(x,y,z)}{r}$. It is a unit vector pointing at B, from A. $\vec{F}$, can be rewritten as

$$\vec{F}(x, y, z) = -GMm\frac{(x, y, z)\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}{}$$

Consider the function

$$V = \frac{GMm}{r} = GMm\frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

We compute

$$\frac{\partial V}{\partial x} - GMm\frac{1}{2} \cdot \frac{2x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Similarily for y and z, then

$$\nabla V = -GMm\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$\diamond$

15 Lecture 14

15.1 Conservative Feilds (cont.)

We will now explore other criteria for conservative fields.

Proposition. Suppose $\vec{F}$ is conservative, then $\nabla \times \vec{F} = 0$

Proof. Since $\vec{F} = \nabla f$, we have

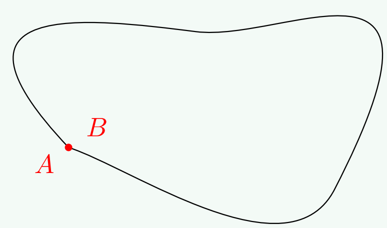
$$\nabla \times \vec{F} = \nabla \times \nabla f = 0$$

This is by an identity previously discussed. □

The converse for this is also true.

Note. It is easy to check if $\vec{F}$ is conservative by computing $\nabla \times \vec{F}$.

Definition 26. A curve is closed if its endpoints coincide.



Notation. The line integral of F along a closed curve is called the circulation. It is written as

$$\oint_C \vec{F} \cdot d\vec{r}$$

Proposition. If $\vec{F}$ is conservative, then

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

,

for any closed curve C.

Proof. By the gradient theorem, we have

$$\oint_C \vec{F} \cdot d\vec{r} = f(\vec{p}) - f(\vec{p}) = 0$$

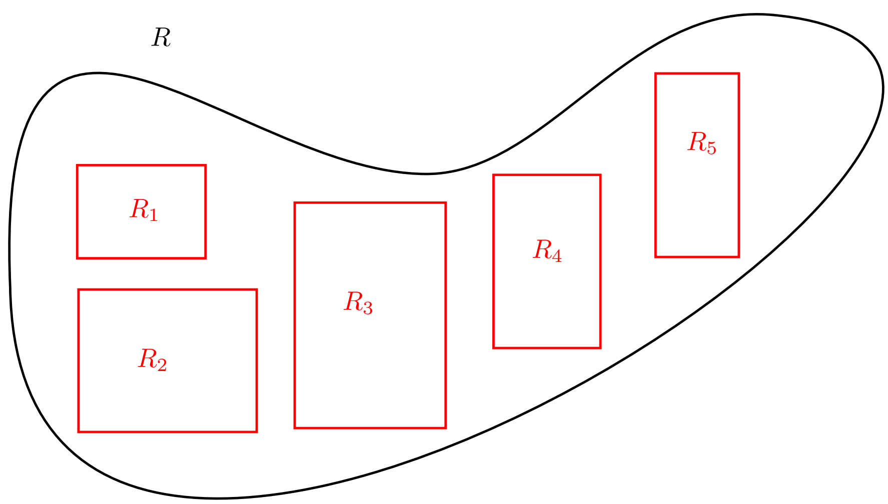
Since the endpoints coincide □

In summary, we have the following equivalent conditions

- $\vec{F}$ is conservative
- $\vec{F} = \nabla f$
- $\nabla \times \vec{F} = 0$
- $oint_C \vec{F} \cdot d\vec{r} = 0$, for any closed curve C.

15.2 Double Integrals

In two dimensions, we have the following method for computing integrals



We approximate a region R by rectangles R_i, with areas ΔA_i

Consider a function $f(x, y)$, pick a sample point (x_i, y_i) in each rectangle R_i. Then we consider the sum

$$\sum_i f(x_i^\star, y_i^\star) \Delta A_i$$

The limit ΔA_i, when it exists, gives the double integral.

Definition 27. The double integral of $f(x, y)$ over the region R is

$$\iint_R f dA = \lim_{\Delta A_i \to 0} \sum_{n=i} f\left(x_i^\star, y_i^\star\right) \Delta A_i$$

When $f = 1$, this gives the area of R, or the size of the region R. When $f > 0$, the integral is also the volume under f.

15.3 Some Properties

We still need concrete forumlas to compute $\iint_R f dA$. First, some general properties.

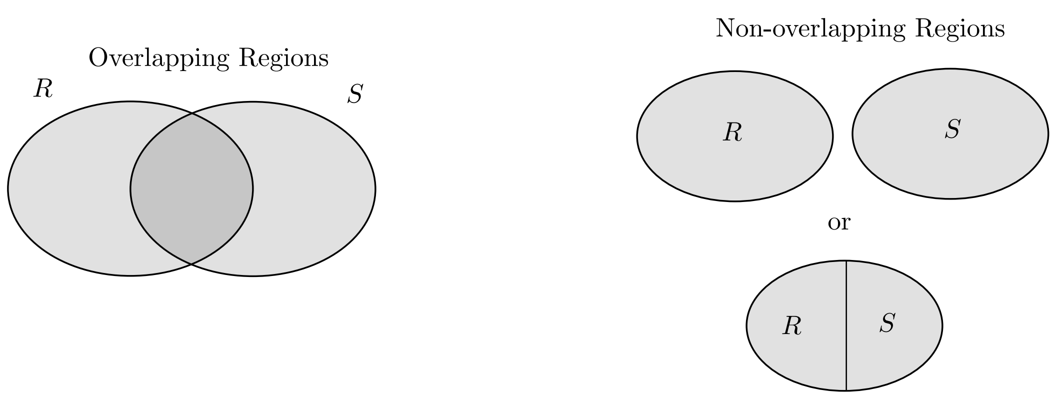
Proposition. Linearity.

Let a and b be two constants, then

$$\int_R (af + bg)\, dA = a \int_R f dA + b \int_R g dA$$

Proof. This follows the linearity of limits. □

The next property is related to portions of the region of integrations.



Proposition. Partitions

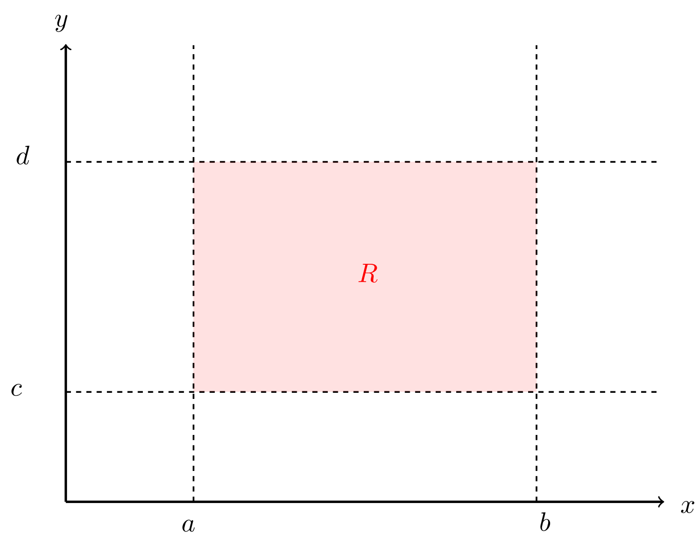
Let R and S be non-overlapping regions. Then we have

$$\int_{R \cup B} f dA = \int_R f dA + \int_S f dA$$

Idea. The total area is the sum of the areas.

15.4 Integrations Over Rectangles

Integrations over a rectangle is the easiest case of a double integral.



General rectangle:

$$R = (a, b) \times (c, d)$$

Proposition. Let $R = (a, b) \times (c, d)$, then

$$\iint_R f dA = \int_a^b \left(\int_c^d f(x, y) \, dy \right) dx = \int_c^d \left(\int_a^b f(x, y) \, dx \right) dy$$

We reduce to the case of two ordinary integrals.

Example. By simple geometry, the area of a rectangle is $(b-a) \cdot (d-c)$. The double integral gives

$$\iint_R 1 dA = \int_a^b \left(\int_c^d 1 dy \right) dx = \int_a^b (d-c) \, dx = (b-a) \cdot (d-c)$$

◇

Example. Compute $\iint_R f dA$ with

$$f(x,y) = xy, \quad R = [0,1] \times [0,2]$$

We compute

$$\begin{aligned}\iint_R xy dA &= \int_{y=0}^{2} \left(\int_{x=0}^{1} xy dx \right) dy \\ &= \frac{1}{2} \int_{y=0}^{2} y dy \\ &= \frac{1}{2} \cdot \frac{2^2}{2} \\ &= 1\end{aligned}$$

We can also use

$$\begin{aligned}\iint_R xy dA &= \int_{x=0}^{1} \left(\int_{y=0}^{2} xy dy \right) dx \\ &= \frac{2^2}{2} \int_{x=0}^{1} x dx \\ &= 2 \cdot \frac{1}{2} \\ &= 1\end{aligned}$$

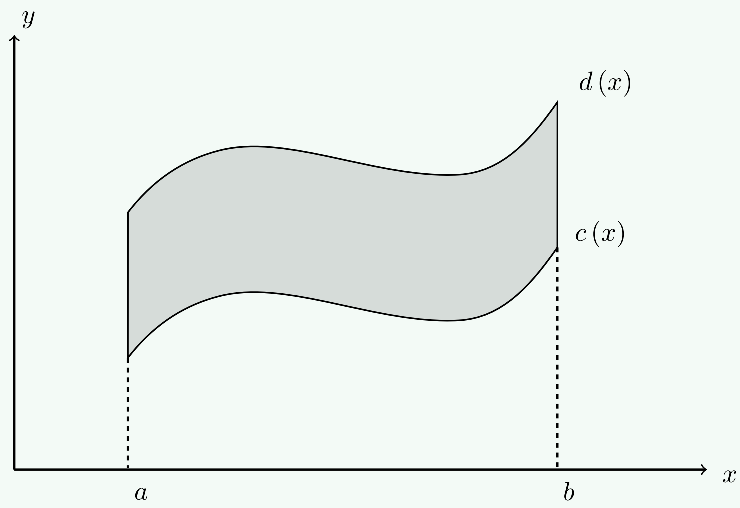
◇

15.5 General Regions

In this section we will consider simple regions.

Definition 28. A region D is called *y-simple* if it is of the following form.

$$D = \{(x,y) : a \leq x \leq b, \quad c(x) \leq y \leq d(x)\}$$



This is the region below $d(x)$, and above $c(x)$. When $c(x) = \mathrm{c}$ and $d(x) = d$, we get a rectangle.

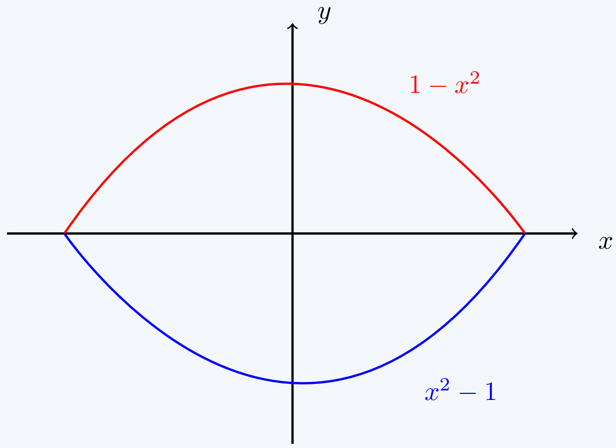
Proposition. Suppose D is a y-simple region. Then:

$$\iint_D f dA = \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y)\, dy \right) dx$$

Note. We first integrate in y, and then in x.

Example. Consider

$$D = \left\{ (x, y) : -1 \leq x \leq 1, \quad x^2 - 1 \leq y \leq 1 - x^2 \right\}$$



◇

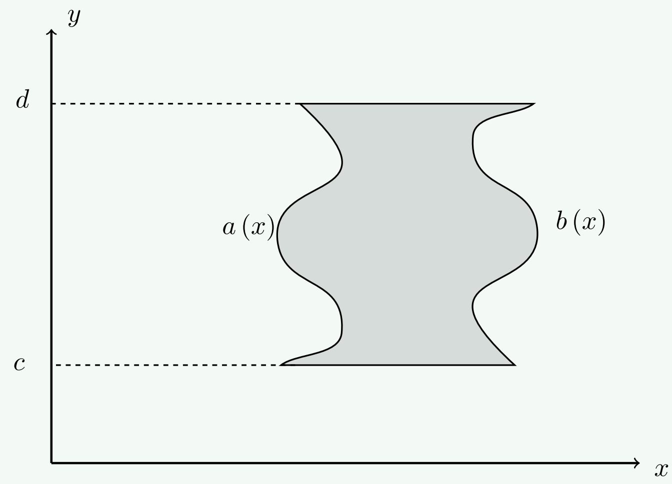
D is y-simple with $c(x) = x^2 - 1$ and $d(x) = x^2 - 1$. We compute the area of D, we have

$$\begin{aligned}\iint_D dA &= \int_{-1}^{1} \left(\int_{x^2-1}^{1-x^2} 1 dy \right) dx \\ &= \int_{-1}^{1} \left(\left(1 - x^2\right) - \left(x^2 - 1\right) \right) dx \\ &= 2 \int_{-1}^{1} \left(1 - x^2\right) dx \\ &= 2 \left(x - \frac{1}{3} x^3 \right)_{-1}^{1} \\ &= \frac{8}{3}\end{aligned}$$

We can exchange the role of x and y.

Definition 29. A region D is *x-simple* if it is of the form

$$D = \{(x, y) : a(y) \leq x \leq b(y), \quad c \leq y \leq d\}$$



Proposition. If D is x-simple, then

$$\iint_D f dA = \int_c^d \left(\int_{a(y)}^{b(y)} f\left(x, y\right) dx \right) dy$$

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16.1 General Regions (cont.)

As we saw in the previous lecture, we have the following regions

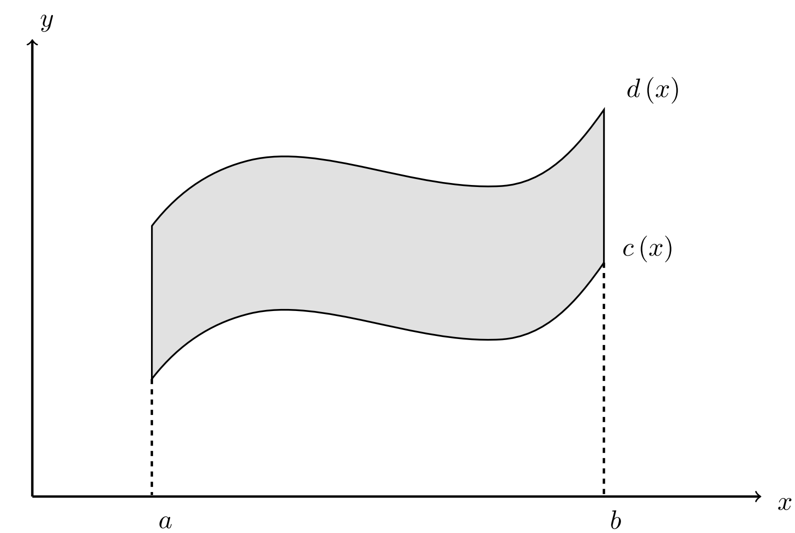


Figure 4: y-simple

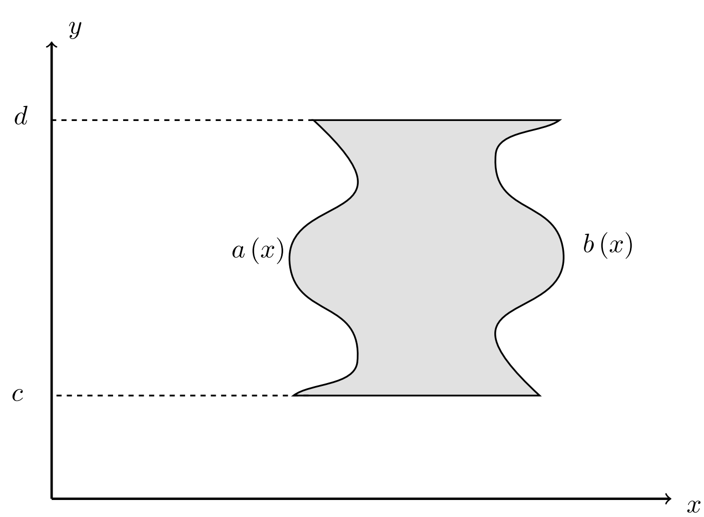
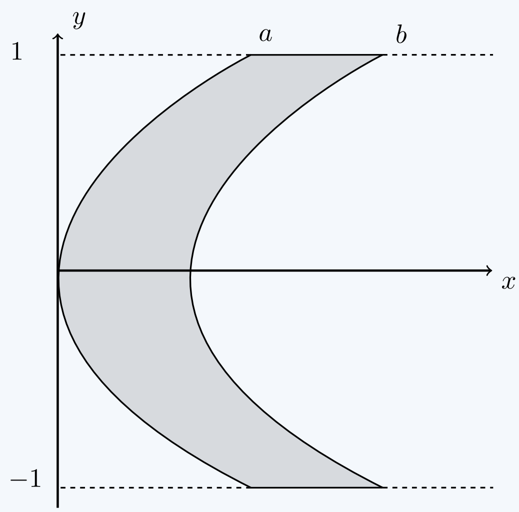


Figure 5: x-simple

Example. Consider the region with $-1 \leq x \leq 1$ and bounder by

$$a(y) = y^2, \quad b(y) = y^2 + \frac{1}{2}$$

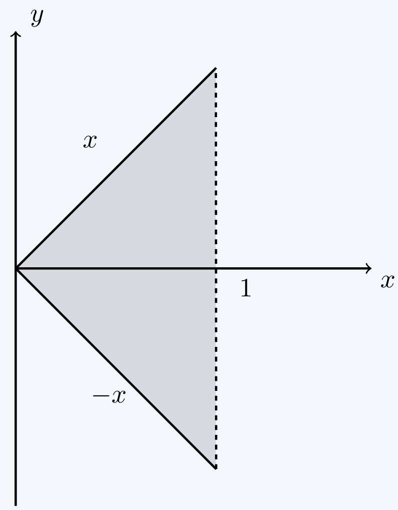


This is an x-simple region. $\diamond$

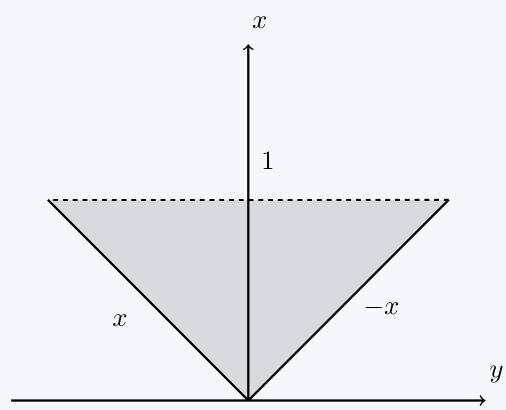
More general regions can be partitioned into x-simple and y-simple regions. A region can be described in many different ways.

Example. Consider the region

$$D = \{(x, y) : 0 \leq x \leq 1, \quad -x \leq y \leq x\}$$



This is a y-simple region. We can also describe this triangle using x-simple regions.



We have two regions:

$$\begin{aligned} D_1 &= \{(x, y) : y \leq x \leq 1, \quad 0 \leq y \leq 1\} \\ D_2 &= \{(x, y) : -y \leq x \leq 1, \quad -1 \leq y \leq 0\} \end{aligned}$$

We have that $D = D_1 \cup D_2$, and

$$\iint_D f dA = \iint_{D_1} f dA + \iint_{D_2} f dA$$

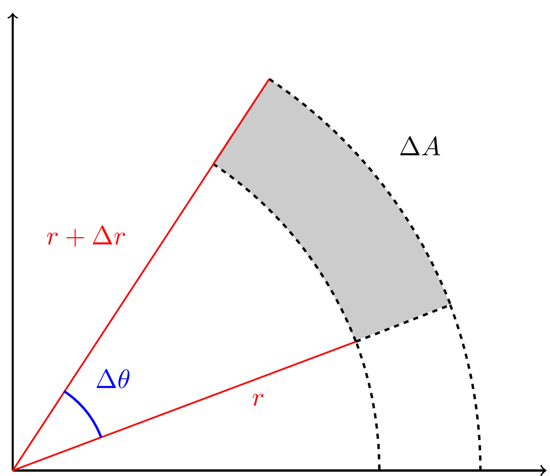
◇

16.2 Integrations in polar coordinates

We want to compute $\iint_D f dA$ using the polar coordinates (r, θ), recall that

$$x = r\cos\theta, \quad y = r\sin\theta$$

In cartesian coordinates, we use the "infinitesimal area $dA = dxdy$". We will now consider small polar regions.



Recall that area is equal to $\frac{1}{2}\theta r^2$.

$$\begin{aligned}\Delta A &= \text{"Large region"} - \text{"Small Region"} \\ &= \frac{1}{2}\Delta\theta\,(r + \Delta r)^2 - \frac{1}{2}\Delta\theta r^2 \\ &= r\Delta\theta r + \frac{1}{2}\Delta\theta\,(\Delta r)^2\end{aligned}$$

Neglecting th $(\Delta r)^2$ term, we get

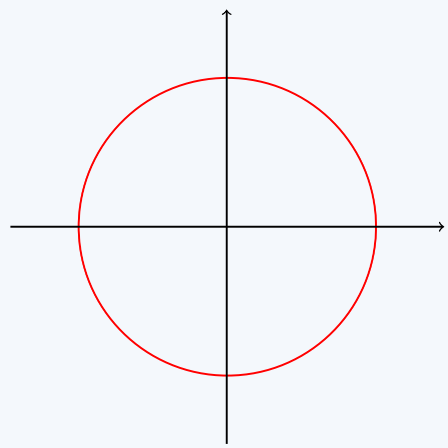
$$\Delta A \approx r\Delta\theta\Delta r$$

Proposition. Suppose R is described in polar coordinates by $a \leq r \leq b$ and $\alpha \leq \theta \leq \beta$, then

$$\iint_R f dA = \int_{r=a}^{b} \left[\int_{\theta=\alpha}^{\beta} f\left(r\cos\theta, r\sin\theta\right) r d\theta \right] dr$$

Here, $dA = rd\theta dr$ is the infinitesimal area. Also, $f\left(r\cos\theta, r\sin\theta\right)$ is simply $f\left(x,y\right)$ in polar coordinates.

Example. Consider a disc of radius t. Its area is πt^2



It is described by $R = \{(1,\theta) : 0 \leq r \leq t, 0 \leq \theta \leq 2\pi\}$. We compute

$$\iint_R dA = \int_0^t \left[\int_0^{2\pi} r d\theta \right] dr$$

$$\begin{aligned} \iint_R dA &= \int_0^t \left[\int_0^{2\pi} r d\theta \right] dr \\ &= 2\pi \int_0^t r dr \\ &= 2\pi \left[\frac{r^2}{2} \right]_0^t \\ &= \pi t^2 \end{aligned}$$

$\diamond$

Example. We have R as before, we want to compute $\iint_R f dA$ where

$$f(x, y) = x^2 + y^2$$

Using polar coordinates, we get

$$f(r\cos\theta, r\sin\theta) = r^2(\cos\theta) + r^2 + (\sin\theta)^2 = r^2$$

Our double integral is

$$\begin{aligned}
\iint_R f dA &= \int_0^t \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, r d\theta dr \\
&= \int_0^t \int_0^{2\pi} r^3 d\theta dr \\
&= 2\pi \int_0^t r^3 dr \\
&= 2\pi \frac{1}{4} t^4 \\
&= \frac{\pi}{2} t^4
\end{aligned}$$

◇

16.3 Change in variables

We want to consider general coordinates (u, v) defined by

$$x = x(u, v), \quad y = y(u, v)$$

How to integrate with (u, v)?

Definition 30. The *jacobian determinant* is defined by

$$J(u, v) = \left| \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right| = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

Example. Consider the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

We want to compute $J(r, \theta)$, we have

$$J(r, \theta) = \left| \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right| = \left| \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \right|$$

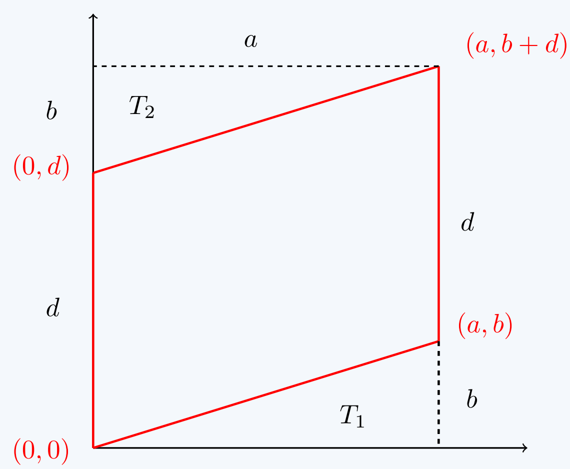
We obtain

$$\begin{aligned}
J(r, \theta) &= \cos\theta \cdot r\cos\theta - \sin\theta \cdot (-r\sin\theta) \\
&= r\left((\cos\theta)^2 + (\sin\theta)^2\right) \\
&= r
\end{aligned}$$

This correspons to r in $dA = rd\theta dr$. ◇

The determinant should be intepreted as an area (up to certain signs).

Example. Given $\vec{v} = (a, b)$ and $\vec{u} = (0, d)$, consider $(0,0)$, $\vec{v} = (a, b)$, $\vec{w} = (0, d)$, $\vec{v} + \vec{w} = (a, b + d)$. These four points define a paralellogram.



◇

we claim that

$$a = \left| \det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right| = |ad|$$

we consider $a, b, d > 0$, then

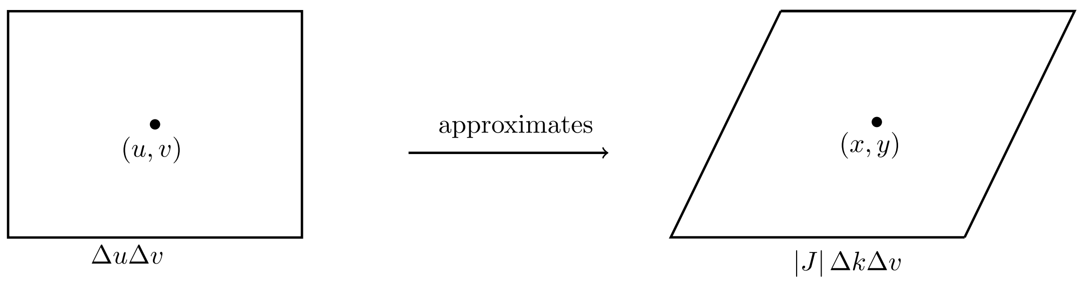
$$a = a\left(b + d\right) - \frac{1}{2}ab = \frac{1}{2}ab = ad$$

this is the same as the determinant. more generally, with

$$\left(0, 0\right), \vec{v} = \left(a, b\right), \quad \vec{w} = \left(c, d\right), \quad \vec{v} + \vec{w} = \left(a + c, b + d\right)$$

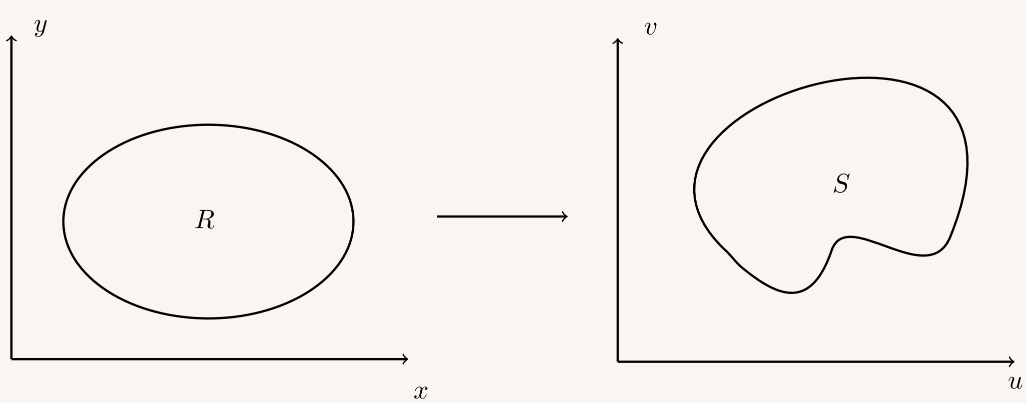
$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ab - bc|$$

the idea of change of variables can be described by the following figure.



Theorem 31. Change of variables.

Let R be a region in cartesian coordinates



Let S be the corresponding region in (u, v)-coordinates, then

$$\iint_R f dA = \iint_S f\left(x\left(u, v\right), y\left(u, v\right)\right) \left|J\left(u, v\right)\right| dudv$$

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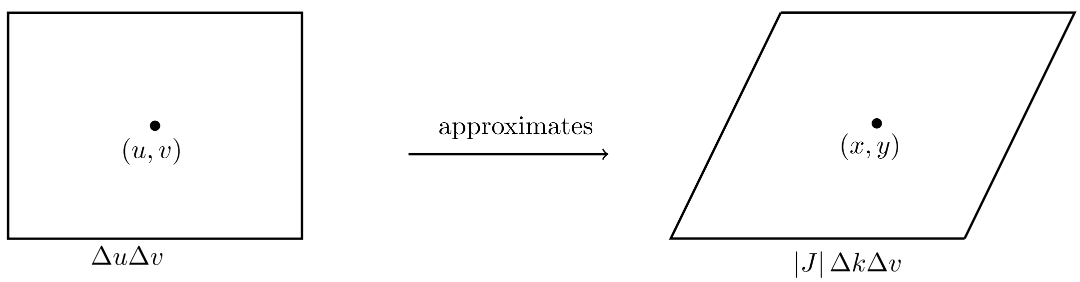
17.1 Change of variables (cont.)

We have seen that the *Jacobian Determinant* J, and

$$\iint_R f dA = \iint_S f\left(x\left(u,v\right), y\left(u,v\right)\right)\left|J\left(u,v\right)\right| dudv$$

The Jacobian appears since

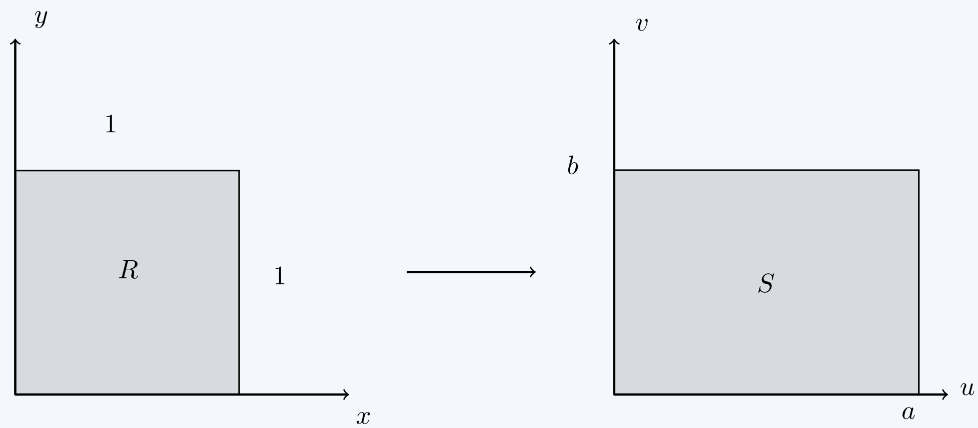


Example. Consider (u,v) defined by

$$u = ax, \quad v = by, a, b > 0$$

Now consider the square

$$R = [1,0] \times [0,1]$$



It becomes the rectangle $S = [0, a] \times [0, b]$. Let us also write

$$x = \frac{u}{a}, \quad y = \frac{v}{b}$$

Then we compute the Jacobian.

$$J = \left| \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \right| = \frac{1}{ab}$$

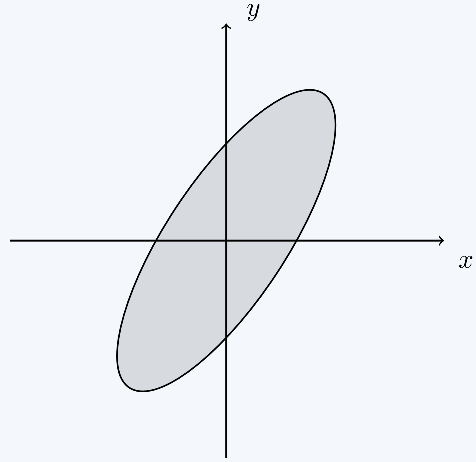
We have $\iint_R 1 dA = 1$, using the theorem, we compute

$$\begin{aligned} \iint_R 1 dA &= \iint_S |J(u, v)| \, dudv \\ &= \int_{u=0}^{a} \int_{v=0}^{b} \frac{1}{ab} dudv \\ &= 1 \end{aligned}$$

◇

Example. Consider the elliptical region

$$R : x^2 - xy + y^2 \leq 2, \quad f(x, y) = x^2 - xy + y^2$$



We want to compute $\iint_R f dA$. Consider (u,v) defined by

$$x = \sqrt{2} - \sqrt{\frac{2}{3}}u, \quad y = \sqrt{2} + \sqrt{\frac{2}{3}}v$$

$$x^2 - xy + y^2 = 2u^2 + 2v^2$$

In (u,v) coordinates, we get

$$\begin{aligned} S &= \left\{(u,v) : 2u^2 + 2v^2 \leq 2\right\} \\ &= \left\{(u,v) : u^2 + v^2 \leq 1\right\} \end{aligned}$$

This equals a circle. By the theorem, we have that

$$\iint_R f dA = \iint_S f\left(x\left(u,v\right), y\left(u,v\right)\right) \left|J\left(u,v\right)\right| dudv$$

For f, we have

$$f\left(x,y\right) = x^2 - xy + y^2 = 2\left(u^2 + v^2\right)$$

The Jacobian is

$$J\left(u,v\right) = \left|\begin{bmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{bmatrix}\right| = \frac{2}{\sqrt{3}} - \left(-\frac{2}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}$$

$$\iint_R f dA = \iint_S 2\left(u^2 + v^2\right) \frac{4}{\sqrt{3}} dudv$$

Since S is a disc, we introduce

$$u = r\cos\theta, \quad v = r\sin\theta$$

The disc is then $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. We obtain

$$\begin{aligned}\iint_R f dA &= \frac{8}{\sqrt{3}\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2 \cdot r d\theta dr} \\ &= \frac{16\pi}{\sqrt{3}} \\ &= \int_{r=0}^{1} r^3 dr \\ &= \frac{4\pi}{\sqrt{3}}\end{aligned}$$

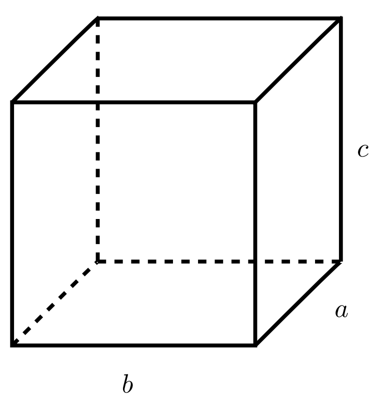
$\diamond$

Note. The aboslute value in $|J(u,v)|$ is important!

17.2 Triple Integrals

When doing a triple integral, we proceed as with double integrals. The analogue of the rectangle are boxes. A box is a region of the following form

$$B = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$$



$$a = x_1 - x_0$$

$$b = y_1 - y_0$$

$$c = z_1 - z_0$$

Its volume is

$$\mathrm{vol}(B) = (x_1 - x_0)(y_1 - y_0)(z_1 - z_0)$$

We approximate a region with boxes B_i of volumes Δv_i. Pick a sample point $(x_i^\star, y_i^\star, z_i^\star)$ in each B_i. Given $f(x,y,z)$ we consider

$$\sum_i f\left(x_i^\star, y_i^\star, z_i^\star\right)\Delta_i$$

in the unit of the integral.

Definition 32. The triple integral of f over t is

$$\iiint_T f dV = \lim_{\Delta v_i \to 0} \sum_i f\left(x_i^\star, y_i^\star, z_i^\star\right)\Delta_i$$

When $f = 1$ we get the volume of T. In the case of $f > 0$, it is less intuitive. How do we interpret $\int_T f dV$? We want to think of f as a local density.

Example. Consider a 3D-object described by a 3D-region T, with $\rho\,(x, y, z)$ its mass density. Then its mass is given by

$$m = \iiint_T \rho dV$$

◇

Proposition. Linearity

If a and b are constants, then

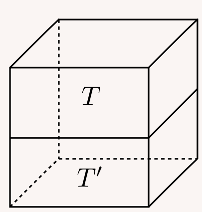
$$\iiint_T (af + bg)\, dV = a\iiint_T f dV + b\iiint_T g dV$$

Proposition. Partitions

Let T and T' be non-overlapping regions, then

$$\iiint_{T\cup T'} f dV = \iiint_T f dV + \iiint_{T'} f dV$$

v



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17.3 Integration Over Boxes

Boxes are usually the easiest regions to consider when doing a triple integral, as mentioned before.

Proposition. Let $T = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$, then

$$\iiint_T f dV = \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z)\, dx dy dz$$

The order can be exchanged.

Example. As expected, $f = 1$ gives the volume.

$$\begin{aligned}\iiint_T f dV &= \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z)\, dx dy dz \\ &= (x_1 - x_0) \int_{z_0}^{z_1} \int_{y_0}^{y_1} 1 dy dz \\ &= (x_1 - x_0)(y_1 - y_0)(z_1 - z_0)\end{aligned}$$

◇

Example. Consider $T = [0, 1] \times [0, 1] \times [-1, 1]$ and $f(x, y, z) = 2$. Compute $\int_T f dV = \int_T 2 dV$. We have

$$\iiint_T f dV = \int_{z=-1}^{1} \int_{y=0}^{1} \int_{x=0}^{1} z dx dy dz$$

◇

18 Line Integrals, Parametrization and Vector Fields

18.1 Identities between operations

We have seen three operations defined by ∇.

Gradient: ∇f Divergence: $\nabla \cdot \vec{F}$ Curl: $\nabla \times \vec{F}$

There are many of them, we look at only one.

Proposition. For any scalar field f we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$. Then

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix} \right| = \vec{i}\,(f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{cy})$$

But partial derivatives can be exhanged, $f_{xy} = f_{yx}$. Then we find that

$$\nabla \times (\nabla f) = 0$$

$\square$

We are going to use this when we discuss conservative fields.

18.2 Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Consider the points $\vec{r}(t)$ and $\vec{r}(t + \Delta t)$ where Δt is small.

$$\Delta x = x\,(t + \Delta t) - x\,(t)\,, \quad \Delta y = y\,(t + \Delta t) - y\,(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt}\Delta t, \quad \Delta y \approx \frac{dy}{dt}\Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt}\Delta t\right)^2 + \left(\frac{dy}{dt}\Delta t\right)^2} = \Delta t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Definition 33. The line integral of $f(x,y)$ along a curve C is defined by

$$\int_C f dS = \int_{t_0}^{t_1} f(x(y), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Some observations:

- Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$.
- When $f = 1$, we recover the arc length.

Example. Consider C definet by

$$x(t), y(t) = 0, \quad 0 \leq t \leq 1$$

First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} = 1$$

◇

Now, consider $f(x,y) = x^2 + y$. Compute $\int_C f dS$. Restricting f to C gives

$$f(x(t), y(t)) = x(t)^2 + y(t) = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f dS = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

- Line integrals can be used to compute the mass of an objects (1 - dimensional)
- The curve C describes the object
- The function $\int_C f dS$ is the mass.

18.3 Parametrization and orientation

Next results as for the arc legnth.

Proposition. The integral $\int_C f dS$ does not depend on the parametrization of C.

We will consider a special case.

Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\text{Start: A} = (x(t_0), y(t_0)), \quad \text{End: B} = (x(t0), y(t_1))$$

We want to go from B to A, which can be done in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \le t \le -t_0$$

Note that

$$\vec{r}_{\text{opp}}(-t_1) = (x(t_1), y(t_1)) = \text{B}$$

$$\vec{r}_{\text{opp}}(-t_0) = (x(t_0), y(t_0)) = \text{A}$$

◇

Example. Consider C = segement from (0,0), to (1,0), take $f(x,y) = x$. Show that $\int_C f dS = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$. ◇

If C parametrized by $\vec{r}(t)$, we will use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have that

$$\int_C f dS = \int_{-C} f dS$$

The situation will be different for vector fields.

18.4 Case of Vector Fields

Consider the curve with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \le t \le t_1$$

$$\vec{r'}(t) = (x'(t), y'(t)) \quad \textit{Velocity Vector}$$

Definition 34. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r'}(t)\, dt$$

Here $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$

- In physics, we have that $\vec{F}$ = Force and $\int_C \vec{F} d\vec{r}$ = Work done by $\vec{F}$ along C.
- The elementary case $W = FS$, where W is work, F is Force and S is the displacement.

More explicitly, we consider

$$\vec{F}(x,y) = (P(x,y), Q(x,y))$$

Then we have that

$$\vec{F}\left(\vec{r}(t) \cdot \vec{r'}(t)\right) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P\left(x\left(t\right), y\left(t\right)\right) \cdot x'\left(t\right) dt + \int_{t_0}^{t_1} Q\left(x\left(t\right), y\left(t\right)\right) \cdot y'\left(t\right) dt$$

We will write the expression as

$$\int_C \vec{F} d\vec{r} = \int_C P\ dx + \int_C Q\ dy$$

Example. Consider a curve C with

$$\vec{r}\left(t\right) = \left(t, t^2\right), \quad 0 \leq t \leq 1$$

We have $x\left(t\right) = t$ and $y\left(t\right) = t^2$, its derivative is

$$\vec{r'}\left(t\right) = \left(x'\left(t\right), y'\left(t\right)\right) = \left(1, 2t\right)$$

Now consider the vector field

$$\vec{F}\left(x, y\right) = \left(x + y, x\right)$$

That is $P = x + y$ and $Q = x$. When restricted to C, we get

$$\vec{F}\left(\vec{r}\left(t\right)\right) = \left(x\left(t\right) + y\left(t\right), y\left(t\right)\right) = \left(t + t^2, t\right)$$

We want to compute

$$\vec{F}\left(\vec{r}\left(t\right)\right) \cdot \vec{r'}\left(t\right) = \left(t + t^2, t\right) \cdot \left(1, 2t\right) = \left(3t^2 + t\right)$$

We then obation the following

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(3t^2 + t\right) dt = \frac{3}{2}$$

◇