Contents

1 Domains, Graphs and Level Sets

1.1 Domain of definition

A function may not be defined for all real numbers.
Example. $f(x) = \frac{1}{x}$ is not defined for $x = 0$

Definition 1. The domain of a function f is the set of numbers for which it is defined. We write the domain of f as D_f.

For instance, for $f(x) = \frac{1}{x}$ we have that

$$D_f = \{x \in \mathbb{R} : x \neq\}$$

This is the lasrgerst possible domain, we can also consider smaller domains. We have the interval from 1 to 2.

$$[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$$

Example. Find the largest domain of $f(x, y) = \frac{1}{y-x}$. The denominator should be non-zero, we get

$$D_f = \left\{(x, y) \in \mathbb{R}^2 : y - x \neq 0\right\}$$

Example. Same excersise with $f(x, y) = \sqrt{y - x^2}$.
Argument: $y - x^2 \geq 0$ (because square root). We will then have $y \geq x^2$

1.2 Graphs of functions

The plot of a function f descrives its behaviour visually. Mathematically, a plot corresponds to the notion of a graph.

Definition 2. The graph of a function $f(x, y)$ with domain D_f is the set of points (x, y, z) such that:

$$(x, y) \in D_f \text{ and } z = f(x, y)$$

We write G_f for the set

$$G_f = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_f \text{ and } z = f\left(x, y\right)\right\}$$

The graph of a function of two variables will, in general, be a surface.

Example. Lets consider $f(x, y) = 1$, with domain $\mathbb{R}^2$. The graph of f is:

$$G_f = \left\{(x, y, 1) : (x, y)\, \mathbb{R}^2\right\}$$

All points have $z = 1$, this is a plane. More generally, the graph of$f(x, y) = ax + by + c$ is a plane with linear dependence on x and y.

Example. Consider the graph of:

$$f(x, y) = x^2 + y^2, \quad D_F = \mathbb{R}^2$$

This surface is called a parabloids.

Example. A sphere of radius r is defined by

$$x^2 + y^2 + z^2 = r^2$$

All points x, y, z satisfy the equation.
Is this the graph of a function? No!
There is no unique value of z, associated with (x, y) because:

$$z = \pm\sqrt{r^2 - x^2 - y^2}$$

Both satisfy the sphere equation. Lets consider the graph of

$$f(x, y) = \sqrt{r^2 - x^2 - y^2}$$

With the domain $x^2 + y^2 \leq r^2$. The grap is a half sphere.

1.3 Level Sets

Another way to visualize functions.

Definition 3. A level set of a function $f(x, y)$ is constant. Essentialy, this is a topographic map.

Example. Consider the function

$$f(x, y) = x^2 + y^2$$

The level sets for $c > 0$ are circles.

$$f(x, y) = x^2 + y^2 = c = \left(\sqrt{c}\right)^2$$

This is a circle with radius $\sqrt{c}$.
Now consider the case $c < 0$, then:

$$f(x, y) = x^2 + y^2 = c$$

which doesn't work, because the level sets are empty.
For $c = 0$, we only have the point $(x, y) = (0, 0)$. Generally, level sets of $f(x, y)$ is a curve.

2 Lecture 3

2.1 Partial Derivatives

In the case of one variable, we have

$$\frac{df}{dt} = \lim_{n \to 0} \frac{f(x + n) - f(x)}{n}$$

Similarily, for two of more variables, we have the following definition

Definition 4. The partial derivative of $f(x, y)$ with respect to x:

$$\frac{\partial f}{\partial x} = \lim_{n \to 0} \frac{f(x + n, y) - f(x, y)}{n}$$

Also written as f_x, for $\frac{\partial f}{\partial y}$, we have f_y

Note, the expression above is $\frac{\partial f}{\partial x}(x, y)$, which is the value at the point (x, y)

2.2 Higher order derivatives

Given $\frac{\partial f}{\partial x}$, we can take further derivatives. We have

$$\frac{\partial^2 f}{\partial^2 x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Also written as $f_{xx}, f_{yy}, f_{xy}, f_{yx}$. In most cases, f_{xy} and f_{yx} coincide.

Theorem 1. Scwartz theorem: Suppose f_{xy} and f_{yx} exist, and are continous, then

$$f_{xy} = f_{yx}$$

Similar definitions and results for the case of more variables: $x_1, \ldots, x_n$, with n variables.

2.3 Chain Rule

Suppose $f(x) = g(h(x))$, for instance

$$f(x) = (\cos x)^2 \text{ with } g(x) = x^2, h(x) = \cos x$$

then the chain rule is

$$\frac{df}{dt}(x_0) = \frac{dg}{dh}(h(x_0)) \cdot \frac{dh}{dt}(x_0)$$

Generalization to more variables.

Theorem 2. Chain rule: consider $f(x, y)$ x and y depending on a variable t. Then:

$$\frac{df}{dt} t_0 = \frac{\partial f}{\partial x}(x(t_0), y(t_0)) \frac{dx}{dt} t_0 + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \frac{dy}{dt} t_0$$

The "short form" of this result is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example. Consiter $f(x, y) = xy$, where

$$x(t) = \cos t, y(t) = \sin t$$

This cannot be computed directly with $\frac{df}{dt}$.

$$f(t) = f(x(t), y(t)) = f(\cos t, \sin t) = \cos t \cdot \sin t$$

We can compute

$$\frac{df}{dt} = (\cos t)' \sin t + \cos t (\sin t)' = -(\sin t)^2 + (\cos t)^2$$

Usin the chain rule, we get

$$\frac{df}{dt} j = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

2.4 The Gradient

Define an operation that takes a scalar function, and returns a vector function.

Definition 5. The gradient of $f(x,y)$ at (x_0,y_0) is

$$\nabla f(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0))$$

3 Lecture 4

3.1 Directional Derivatives

We have seen the following

- $f_x =$ the rate of change along the $x-$direction.
- $f_y =$ the rate of change along the $y-$direction.

What about general directions?

Definition 6. Let $\vec{u} = (a,b)$, the directional derivative along $\vec{u}$ at (x,y) is

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h}$$

Note that

$$\vec{u} = (1,0) \to D_{\vec{u}} = f_x$$
$$\vec{u} = (1,0) \to D_{\vec{u}} = f_y$$

To compare directions, we take $\left|\vec{t}\right| = 1$. Here $\vec{u}$ is the length of $\vec{u}$, that is

$$|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}}$$

Proposition 1. We have the following result

$$D_{\vec{u}} \cdot f = \nabla f \cdot \vec{u}$$

Proof. Consider the following function

$$g(t) = f(x+ta, y+tb)$$

Its derivative at $t=0$ is

$$\begin{aligned}\frac{dg}{dt}(0) &= \lim_{h\to 0}\frac{g(h)-g(0)}{h}\\ &= \lim_{h\to 0}\frac{f(x+ha,y+hb)-f(x,y)}{h}\\ &= D_{\vec{u}}\cdot f(x,y)\end{aligned}$$

On the other hand, using the chain rule, we get

$$
\begin{aligned}
\frac{dg}{dt}(0) &= \frac{\partial f}{\partial x}\frac{d(x+ta)}{dt} + \frac{\partial f}{\partial y}\frac{d(y+tb)}{dt} \\
&= \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b = \nabla f \cdot \vec{u}
\end{aligned}
$$

□

We can now state another property of ∇f. The direction where f changes the most.

Proposition 2. $|D_{\vec{u}} \cdot f|$ is the largest when $\vec{u}$ is paralell to ∇f.

Proof. Recall that, given two vectors, $\vec{v}$ and $\vec{w}$, we have that

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos\alpha$$

When is $|\vec{v} \cdot \vec{w}|$ the largest? We have

$$|\vec{v} \cdot \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot |\cos\alpha|, \quad |\cos\alpha| \leq 1$$

It is the largest when the following condition is true

$$|\cos\alpha| = 1, \quad \alpha = 0 \vee \pi$$

Which means that the vectores are pointing in the same, or opposite direction. Applying this to $D_{\vec{u}} \cdot f$, we get

$$
\begin{aligned}
|D_{\vec{u}} \cdot f| &= |\nabla f \cdot \vec{u}| \\
&= |\nabla f| \cdot |\vec{u}| \cdot |\cos\alpha|
\end{aligned}
$$

For fixed values of $|\vec{u}|$, this is the largest when $\alpha = 0 \vee \pi$. That is ∇f and $\vec{u}$ are paralell. □

3.2 Critical Points

How do we find the maxima and minima of $f(x)$? Lets take a look at $f'(x_0) = 0$.

Definition 7. We say that x_0, y_0 is a critical point of f if:

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \quad \wedge \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

Similarily for

4 Lecutre 5

4.1 Hessian Matrix

Example. Consider again the function $f(x, y) = x^2 - y^2$.

$$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{yx} = 0$$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \det H = -4 < 0$$

Hence (x_0, y_0) is a saddle point.

Example. Consider the function $f(x,y) = x^2 + y^2$, we have

$$(f_x, f_y) = (2x, 2y)$$

The only critical point is $(x_0, y_0) = (0,0)$.

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \det H = 4 > 0$$

Since $f_{xx} = 2 > 0$, we conclude that $(0,0)$ is a local minima. In this case, it is actually a global minimum, because $f(x,y) = x^2 + y^2 \geq 0$.

4.2 Global extremal values

A function can have many maxima and minimas. Usually, we are intrested in the largest and smallest values.

Definition 8. Let $f(x,y)$ be with domain D_f. Then we have

- (x_0, y_0) is a global maxima if $f(x_0, y_0) \geq f(x,y)$ for all $(x,y) \in D_f$.
- (x_0, y_0) is a global minima if $f(x_0, y_0) \geq f(x,y)$ for all $(x,y) \in D_f$.

Trivial example: for $f(x,y) = 1$, all points are global maxima and minima.

Note that global maxima and minima need not be critical points.

Example. We have $f(x) = x$ with $D_f = [-1, 1]$.

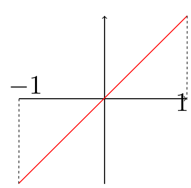


Figure 1: $f(x)$

- Global maxima at $x = 1, f(1) = 1$.
- Global minima at $x = -1, f(-1) = -1$

We have no critical points because $f'(x) = 1 \neq 0$. Also not that maxima and minima depend on the chosen domain.

If we take $D_f = [-2, 3]$, then

$$\text{Max: } x = 3, \quad \text{Min: } x = -2$$

Theorem 3. Let f be continous with domain D_f. Suppose D_f is closed and bounded, then there is at least one global maxima, and one global minima.

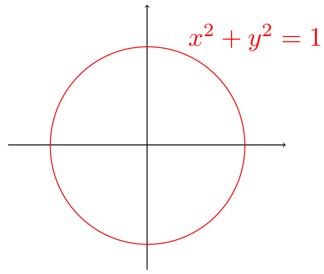


Figure 2: The circle is the boundary.

Some terminology:

$$\text{open } = \left\{x^2 + y^2 < 1\right\}$$
$$\text{closed } = \left\{x^2 + y^2 \leq 1\right\}$$

The method for finding maxima and minima is as follows:

1. Find critical points of f in D_f, and characterize them.
2. Study the points that are on the boundary.
3. Compare them.

Example. Consider $f(x, y) = x^2 + y^2$ with domain

$$D = \left\{(x, y) \in \mathbb{R} : x^2 + y^2 \leq 1\right\}$$

The domain in this case is a disc. The red circle is the boundary.

We compute $f_x = 2x, f_y = 2y$. The only critical point is the origin at $(x_0, y_0) = (0, 0)$. This is a global minium since $f(0, 0)$ and $f(x, y) \geq 0$.

Now, lets consider the boundary

$$C = \left\{x^2 + y^2 = 1\right\}$$

For any point (x_0, y_0) on the circle C, we have

$$f(x_0, y_0) = x_0^2 + y_0^2 = 1$$

We claim that this point is a global maximum. For any (x, y) in domain D_f, we have

$$f(x, y) = x^2 + y^2 \leq 1$$

The value $f(x, y) = 1$ is obtain only at the boundary C. Any point on the circle is a global maxima.

4.3 Constrained optimization

In this section, we will discuss how to find maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. Think of $g = 0$ ad a budget, or a geometrical constraint.
Example. We want to minimize $f(x, y) = x^2 + y^2$ with the constraint $g(x, y) = xy - 1 = 0$.

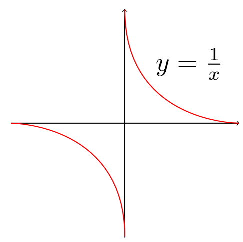


Figure 3: $y = \frac{1}{x}$

We have the following strategy

- Solve $g = 0$ for one variable. For instance $y = \frac{1}{x}$.
- Consider: $h(x) = f\left(x, \frac{1}{x}\right) = x^2 + x^{-2}$

Now we can study this function of one variable with no constraints. We can proceed as usual.

$$\frac{dh}{dt} = 2x - 2x^{-3} = 0$$

This is equivalent to $x^4 = 1$. The real solutions are $x = \pm 1$. Since $y = \frac{1}{x}$, we get the critical points:

$$(x, y) = (1, 1), \quad (x, y) = (-1, -1)$$

5 Lecture 6

5.1 Substitution Method

We want to maximize / minimize $f(x, y)$ with constraint $g(x, y) = 0$. We can solve $g(x, y) = 0$ for one variable $y = y(x)$.

Example. Consider $f(x, y) = x^2 + y^2$ and $g(x, y) = xy - 1 = 0$. In this case, we have $f = \frac{1}{x}$ from $g = 0$, we get

$$h(x) = f\left(x, x^{-1}\right) = x^2 + x^{-2}$$

We have found the minima at $(1, 1)$ and $(-1, -1)$.
This method isn't always feasible, so lets look at some alternatives.

5.2 Lagrange's Method

Example. Lets look at the level curves, which are circles. We have the that $f(x,y) = x^2 + y^2 = c$.

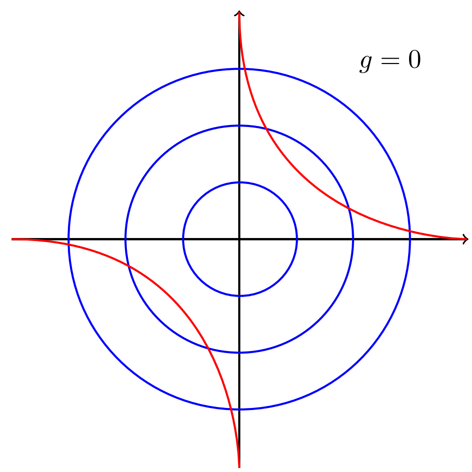
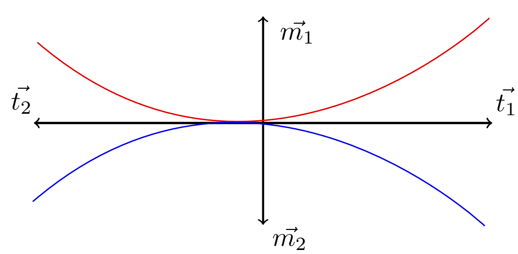


Figure 4: $f(x,y)$

Smaller circles correspont to smaller values of $f(x,y)$, but we must also satisfy $g(x,y) = 0$. In the best case, $f(x,y) = c$ is just touching $g(x,y) = 0$. If this is worked out geometrically, we get $(1,1)$ and $(-1,-1)$.

This idea is used in Lagranges method. We want $f(x,y) = c$ to be parallell to $g(x,y) = 0$. More precisesly: Their vectors should be parallell.



Equivalently, their normal vectors are also paralell. Recall that a normal vector to $g = 0$ is given by ∇f. Similarly, ∇f is normal to $f = c$.

5.3 Method for Langrange

Suppose we want to find a local maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. We proceed as follows

1. Find all possible solutions to the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0$$

2. Plug in all solutions from 1. into $f(x, y)$ and identify the largest and smallest.

   - The number λ is called the *Lagrange Multiplier*.
   - Easy extension to n variables.
   - Can also be generalized to multiple constraints, $g_1, \ldots, g_n$

Example. We have the following functions

$$f(x, y) = x^2 + y^2, \quad g(x, y) = xy - 1 = 0$$

We compute the gradients

$$\nabla f(x, y) = (2x, 2y), \quad \nabla g(x, y) = (x, y)$$

The Lagrange equiations are

$$2x = \lambda y, \quad 2y = \lambda x, \quad xy - 1 = 0$$

Observe that $(x, y, \lambda) = (0, 0, 0)$ is not a solution. For $x \neq 0$, we get $y = \frac{1}{x}$, from the third equation.

$$\begin{aligned}
&2x = \lambda y \Rightarrow 2x = \lambda \frac{1}{x} \Rightarrow \lambda = 2x^2 \\
&2y = \lambda x \Rightarrow \frac{2}{x} = 2xs^2 \cdot x \Rightarrow x^4 = 1 \\
&x^4 = 1 \Rightarrow x \pm 1, \quad y = \frac{1}{x}
\end{aligned}$$

$$(x, y) = (1, 1) \quad \wedge \quad (x, y) = (-1, -1)$$

Are theses points the minima? We have

$$f(1, 1) = f(-1, -1) = 2$$

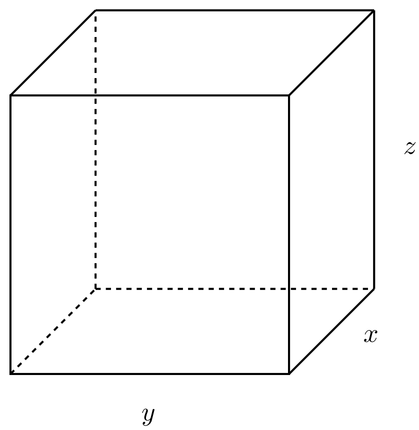
Lets Compare this to some other points, such that $g(x, y) = 0$. For instance, $(x, y) = \left(2, \frac{1}{2}\right)$

$$g\left(2, \frac{1}{2}\right) = 2 \cdot \frac{1}{2} - 1 = 0$$

$$f\left(2, \frac{1}{2}\right) = 2^2 + \left(\frac{1}{2}\right)^2 > 2$$

This tells us that $(x, y) = (\pm 1, \pm 1)$ are local minima.

Example. Consider a box of surface area 24 cm^2.



Determine the dimensions (x, y, z) such that the volume is max.

We have the surface area $2xy + 2xz + 2yz$. Our constraint is

$$g(x, y, z) = 2xy + 2xz + 2yz - 24 = 0$$

The goal is to maximize $f(x, y, z) = 0$, with the consstraint $g(x, y, z) = 0$.

The equation $\nabla f = \lambda \nabla g$ gives

$$yz = 2\lambda (y + 2), \quad xz = 2\lambda (x + 2), \quad xy = 2\lambda (x + y)$$

Obsereve that $\lambda \neq$, since $x, y, z > 0$. To solve the equations, we can multiply by x, y and z, respectively, then we get

$$x(y + z) = y(x + z) = x(y + x)$$

Consider $x(y + z) = y(x + z)$.

$$x(y + z) = y(x + z) \Rightarrow (x - y)z = 0$$

$$z \neq 0, \quad x = y$$

Its the same for the other equations, so $x = y = z$. We also need to use $g = 0$. Setting $x = y = z$j, we get

$$\begin{aligned} g(x, x, x) = 2x^2 + 2x^2 + 2x^2 - 24 = 0 \\ x^2 = 4 \\ x = 2 \end{aligned}$$

Lagranges method gives:

$$(x, y, z) = (2, 2, 2) \Rightarrow V = 8$$

6 Line Integrals, Parametrization and Vector Fields

6.1 Identities between operations

We have seen three operations defined by ∇.

Gradient: ∇f Divergence: $\nabla \cdot \vec{F}$ Curl: $\nabla \times \vec{F}$

There are many of them, we look at only one.

Proposition 3. For any scalar field f we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$. Then

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix} \right| = \vec{i}\,(f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{cy})$$

But partial derivatives can be exhanged, $f_{xy} = f_{yx}$. Then we find that

$$\nabla \times (\nabla f) = 0$$

$\square$

We are going to use this when we discuss conservative fields.

6.2 Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Consider the points $\vec{r}(t)$ and $\vec{r}(t + \Delta t)$ where Δt is small.

$$\Delta x = x\,(t + \Delta t) - x\,(t)\,, \quad \Delta y = y\,(t + \Delta t) - y\,(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt}\Delta t, \quad \Delta y \approx \frac{dy}{dt}\Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt}\Delta t\right)^2 + \left(\frac{dy}{dt}\Delta t\right)^2} = \Delta t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Definition 9. The line integral of $f(x,y)$ along a curve C is defined by

$$\int_C f dS = \int_{t_0}^{t_1} f(x(y), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Some observations:

- Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$.
- When $f = 1$, we recover the arc length.

Example. Consider C definet by

$$x(t), y(t) = 0, \quad 0 \leq t \leq 1$$

First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} = 1$$

Now, consider $f(x,y) = x^2 + y$. Compute $\int_C f dS$. Restricting f to C gives

$$f(x(t), y(t)) = x(t)^2 + y(t) = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f dS = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

- Line integrals can be used to compute the mass of an objects (1 - dimensional)
- The curve C describes the object
- The function $\int_C f dS$ is the mass.

6.3 Parametrization and orientation

Next results as for the arc legnth.

Proposition 4. The integral $\int_C f dS$ does not depend on the parametrization of C.

We will consider a special case.

Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\text{Start: A} = (x(t_0), y(t_0)), \quad \text{End: B} = (x(t0), y(t_1))$$

We want to go from B to A, which can be done in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\vec{r}_{\text{opp}}(-t_1) = (x(t_1), y(t_1)) = \text{B}$$

$$\vec{r}_{\text{opp}}(-t_0) = (x(t_0), y(t_0)) = \text{A}$$

Example. Consider C = segement from (0,0), to (1,0), take $f(x,y)=x$. Show that $\int_C f dS = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$.

If C parametrized by $\vec{r}(t)$, we will use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have that

$$\int_C f dS = \int_{-C} f dS$$

The situation will be different for vector fields.

6.4 Case of Vector Fields

Consider the curve with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\vec{r'}(t) = (x'(t), y'(t)) \quad \textit{Velocity Vector}$$

Definition 10. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r'}(t)\, dt$$

Here $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$

- In physics, we have that $\vec{F}$ = Force and $\int_C \vec{F} d\vec{r}$ = Work done by $\vec{F}$ along C.
- The elementary case $W = FS$, where W is work, F is Force and S is the displacement.

More explicitly, we consider

$$\vec{F}(x,y) = (P(x,y), Q(x,y))$$

Then we have that

$$\vec{F}\left(\vec{r}(t) \cdot \vec{r'}(t)\right) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t)\, dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t)\, dt$$

We will write the expression as

$$\int_C \vec{F} d\vec{r} = \int_C P\ dx + \int_C Q\ dy$$

Example. Consider a curve C with

$$\vec{r}(t) = \left(t, t^2\right), \quad 0 \leq t \leq 1$$

We have $x(t) = t$ and $y(t) = t^2$, its derivative is

$$\vec{r'}(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is $P = x + y$ and $Q = x$. When restricted to C, we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = \left(t + t^2, t\right)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r'}(t) = \left(t + t^2, t\right) \cdot (1, 2t) = \left(3t^2 + t\right)$$

We then obation the following

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(3t^2 + t\right) dt = \frac{3}{2}$$