1 Lecture 6

1.1 Substitution Method

We want to maximize / minimize $f(x, y)$ with constraint $g(x, y) = 0$. We can solve $g(x, y) = 0$ for one variable $y = y(x)$.

Example. Consider $f(x, y) = x^2 + y^2$ and $g(x, y) = xy - 1 = 0$. In this case, we have $f = \frac{1}{x}$ from $g = 0$, we get

$$h(x) = f\left(x, x^{-1}\right) = x^2 + x^{-2}$$

We have found the minima at $(1, 1)$ and $(-1, -1)$.
This method isn't always feasible, so lets look at some alternatives.

1.2 Lagrange's Method

Example. Lets look at the level curves, which are circles. We have the that $f(x, y) = x^2 + y^2 = c$.

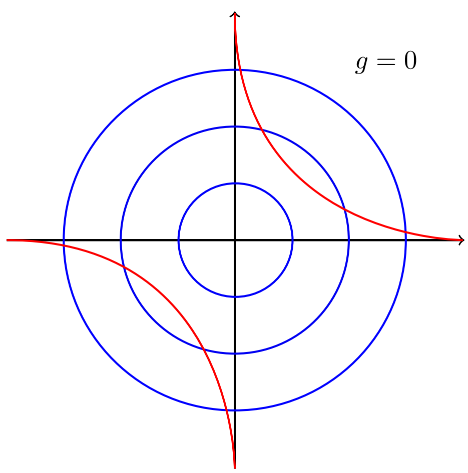
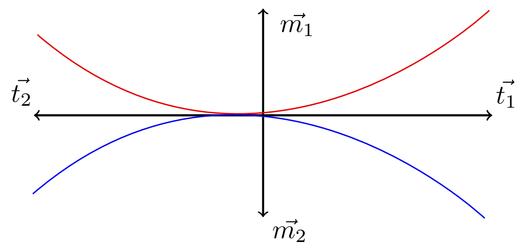


Figure 1: $f(x, y)$

Smaller circles correspont to smaller values of $f(x, y)$, but we must also satisfy $g(x, y) = 0$. In the best case, $f(x, y) = c$ is just touching $g(x, y) = 0$. If this is worked out geometrically, we get $(1, 1)$ and $(-1, -1)$.

This idea is used in Lagranges method. We want $f(x, y) = c$ to be parallell to $g(x, y) = 0$. More precisely: Their vectors should be parallell.



Equivalently, their normal vectors are also paralell. Recall that a normal vector to $g = 0$ is given by ∇f. Similarly, ∇f is normal to $f = c$.

1.3 Method for Langrange

Suppose we want to find a local maxima and minima of $f(x, y)$ with constraint $g(x, y) = 0$. We proceed as follows

1. Find all possible solutions to the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0$$

2. Plug in all solutions from 1. into $f(x, y)$ and identify the largest and smallest.

- The number λ is called the *Lagrange Multiplier*.
- Easy extension to n variables.
- Can also be generalized to multiple constraints, $g_1, \ldots, g_n$

Example. We have the following functions

$$f(x, y) = x^2 + y^2, \quad g(x, y) = xy - 1 = 0$$

We compute the gradients

$$\nabla f(x, y) = (2x, 2y), \quad \nabla g(x, y) = (x, y)$$

The Lagrange equiations are

$$2x = \lambda y, \quad 2y = \lambda x, \quad xy - 1 = 0$$

Observe that $(x, y, \lambda) = (0, 0, 0)$ is not a solution. For $x \neq 0$, we get $y = \frac{1}{x}$, from the third equation.

$$
\begin{aligned}
&2x = \lambda y \Rightarrow 2x = \lambda \frac{1}{x} \Rightarrow \lambda = 2x^2 \\
&2y = \lambda x \Rightarrow \frac{2}{x} = 2xs^2 \cdot x \Rightarrow x^4 = 1 \\
&x^4 = 1 \Rightarrow x \pm 1, \quad y = \frac{1}{x}
\end{aligned}
$$

$$(x, y) = (1, 1) \quad \wedge \quad (x, y) = (-1, -1)$$

Are theses points the minima? We have

$$f(1, 1) = f(-1, -1) = 2$$

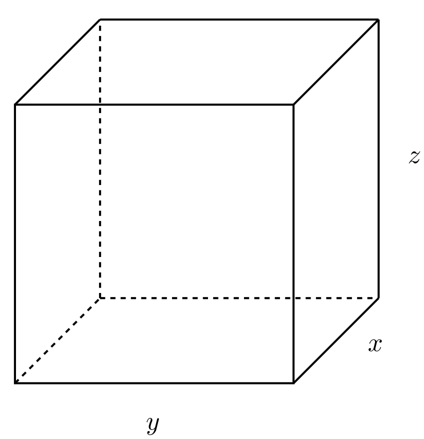
Lets Compare this to some other points, such that $g(x, y) = 0$. For instance, $(x, y) = \left(2, \frac{1}{2}\right)$

$$g\left(2, \frac{1}{2}\right) = 2 \cdot \frac{1}{2} - 1 = 0$$

$$f\left(2, \frac{1}{2}\right) = 2^2 + \left(\frac{1}{2}\right)^2 > 2$$

This tells us that $(x, y) = (\pm 1, \pm 1)$ are local minima.

Example. Consider a box of surface area 24 cm^2.



Determine the dimensions (x, y, z) such that the volume is max.

We have the surface area $2xy + 2xz + 2yz$. Our constraint is

$$g(x, y, z) = 2xy + 2xz + 2yz - 24 = 0$$

The goal is to maximize $f(x, y, z) = 0$, with the consstraint $g(x, y, z) = 0$.

The equation $\nabla f = \lambda \nabla g$ gives

$$yz = 2\lambda\left(y+2\right), \quad xz = 2\lambda\left(x+2\right), \quad xy = 2\lambda\left(x+y\right)$$

Obsereve that $\lambda \neq$, since $x, y, z > 0$. To solve the equations, we can multiply by x, y and z, respectively, then we get

$$x\left(y+z\right) = y\left(x+z\right) = x\left(y+x\right)$$

Consider $x\left(y+z\right) = y\left(x+z\right)$.

$$x\left(y+z\right) = y\left(x+z\right) \Rightarrow \left(x-y\right)z = 0$$

$$z \neq 0, \quad x = y$$

Its the same for the other equations, so $x = y = z$. We also need to use $g = 0$. Setting $x = y = z$j, we get

$$\begin{aligned} g\left(x,x,x\right) = 2x^2 + 2x^2 + 2x^2 - 24 = 0 \\ x^2 = 4 \\ x = 2 \end{aligned}$$

Lagranges method gives:

$$\left(x,y,z\right) = \left(2,2,2\right) \Rightarrow V = 8$$