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1 Domains, Graphs and Level Sets

1.1 Domain of definition

A function may not be defined for all real numbers.

Example. $f(x) = \frac{1}{x}$ is not defined for $x = 0$

Definition 1. The domain of a function f is the set of numbers for which it is defined. We write the domain of f as D_f .

For instance, for $f(x) = \frac{1}{x}$ we have that

$$D_f = \{x \in \mathbb{R} : x \neq 0\}$$

This is the largest possible domain, we can also consider smaller domains. We have the interval from 1 to 2.

$$[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$$

Example. Find the largest domain of $f(x, y) = \frac{1}{y-x}$. The denominator should be non-zero, we get

$$D_f = \{(x, y) \in \mathbb{R}^2 : y - x \neq 0\}$$

Example. Same exercise with $f(x, y) = \sqrt{y - x^2}$.

Argument: $y - x^2 \geq 0$ (because square root). We will then have $y \geq x^2$

1.2 Graphs of functions

The plot of a function f describes its behaviour visually. Mathematically, a plot corresponds to the notion of a graph.

Definition 2. The graph of a function $f(x, y)$ with domain D_f is the set of points (x, y, z) such that:

$$(x, y) \in D_f \text{ and } z = f(x, y)$$

We write G_f for the set

$$G_f = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_f \text{ and } z = f(x, y)\}$$

The graph of a function of two variables will, in general, be a surface.

Example. Let's consider $f(x, y) = 1$, with domain \mathbb{R}^2 . The graph of f is:

$$G_f = \{(x, y, 1) : (x, y) \in \mathbb{R}^2\}$$

All points have $z = 1$, this is a plane. More generally, the graph of $f(x, y) = ax + by + c$ is a plane with linear dependence on x and y .

Example. Consider the graph of:

$$f(x, y) = x^2 + y^2, \quad D_F = \mathbb{R}^2$$

This surface is called a paraboloids.

Example. A sphere of radius r is defined by

$$x^2 + y^2 + z^2 = r^2$$

All points x, y, z satisfy the equation.

Is this the graph of a function? No!

There is no unique value of z , associated with (x, y) because:

$$z = \pm \sqrt{r^2 - x^2 - y^2}$$

Both satisfy the sphere equation. Lets consider the graph of

$$f(x, y) = \sqrt{r^2 - x^2 - y^2}$$

With the domain $x^2 + y^2 \leq r^2$. The graph is a half sphere.

1.3 Level Sets

Another way to visualize functions.

Definition 3. A level set of a function $f(x, y)$ is constant. Essentially, this is a topographic map.

Example. Consider the function

$$f(x, y) = x^2 + y^2$$

The level sets for $c > 0$ are circles.

$$f(x, y) = x^2 + y^2 = c = (\sqrt{c})^2$$

This is a circle with radius \sqrt{c} .

Now consider the case $c < 0$, then:

$$f(x, y) = x^2 + y^2 = c$$

which doesn't work, because the level sets are empty.

For $c = 0$, we only have the point $(x, y) = (0, 0)$. Generally, level sets of $f(x, y)$ is a curve.

2 Lecture 3

2.1 Partial Derivatives

In the case of one variable, we have

$$\frac{df}{dt} = \lim_{n \rightarrow 0} \frac{f(x + n) - f(x)}{n}$$

Similarly, for two or more variables, we have the following definition

Definition 4. The partial derivative of $f(x, y)$ with respect to x :

$$\frac{\partial f}{\partial x} = \lim_{n \rightarrow 0} \frac{f(x + n, y) - f(x, y)}{n}$$

Also written as f_x , for $\frac{\partial f}{\partial x}$, we have f_y

Note, the expression above is $\frac{\partial f}{\partial x}(x, y)$, which is the value at the point (x, y)

2.2 Higher order derivatives

Given $\frac{\partial f}{\partial x}$, we can take further derivatives. We have

$$\frac{\partial^2 f}{\partial^2 x} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

Also written as $f_{xx}, f_{yy}, f_{xy}, f_{yx}$. In most cases, f_{xy} and f_{yx} coincide.

Theorem 1. Schwartz theorem: Suppose f_{xy} and f_{yx} exist, and are continuous, then

$$f_{xy} = f_{yx}$$

Similar definitions and results for the case of more variables: x_1, \dots, x_n , with n variables.

2.3 Chain Rule

Suppose $f(x) = g(h(x))$, for instance

$$f(x) = (\cos x)^2 \text{ with } g(x) = x^2, h(x) = \cos x$$

then the chain rule is

$$\frac{df}{dt}(x_0) = \frac{dg}{dh}(h(x_0)) \cdot \frac{dh}{dt}(x_0)$$

Generalization to more variables.

Theorem 2. Chain rule: consider $f(x, y)$ x and y depending on a variable t . Then:

$$\frac{df}{dt}|_{t_0} = \frac{\partial f}{\partial x}(x(t_0), y(t_0)) \frac{dx}{dt}|_{t_0} + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \frac{dy}{dt}|_{t_0}$$

The "short form" of this result is

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Example. Consider $f(x, y) = xy$, where

$$x(t) = \cos t, y(t) = \sin t$$

This cannot be computed directly with $\frac{df}{dt}$.

$$f(t) = f(x(t), y(t)) = f(\cos t, \sin t) = \cos t \cdot \sin t$$

We can compute

$$\frac{df}{dt} = (\cos t)' \sin t + \cos t (\sin t)' = -(\sin t)^2 + (\cos t)^2$$

Using the chain rule, we get

$$\frac{df}{dt}|_j = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

2.4 The Gradient

Define an operation that takes a scalar function, and returns a vector function.

Definition 5. The gradient of $f(x, y)$ at (x_0, y_0) is

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

3 Line Integrals, Parametrization and Vector Fields

3.1 Identities between operations

We have seen three operations defined by ∇ .

Gradient: ∇f

Divergence: $\nabla \cdot \vec{F}$

Curl: $\nabla \times \vec{F}$

There are many of them, we look at only one.

Proposition 1. For any scalar field f we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$. Then

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix} \right| = \vec{i}(f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{cy})$$

But partial derivatives can be exchanged, $f_{xy} = f_{yx}$. Then we find that

$$\nabla \times (\nabla f) = 0$$

□

We are going to use this when we discuss conservative fields.

3.2 Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Consider the points $\vec{r}(t)$ and $\vec{r}(t + \Delta t)$ where Δt is small.

$$\Delta x = x(t + \Delta t) - x(t), \quad \Delta y = y(t + \Delta t) - y(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt} \Delta t, \quad \Delta y \approx \frac{dy}{dt} \Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt} \Delta t\right)^2 + \left(\frac{dy}{dt} \Delta t\right)^2} = \Delta t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Definition 6. The line integral of $f(x, y)$ along a curve C is defined by

$$\int_C f dS = \int_{t_0}^{t_1} f(x(y), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Some observations:

- Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$.
- When $f = 1$, we recover the arc length.

Example. Consider C defined by

$$x(t), y(t) = 0, \quad 0 \leq t \leq 1$$

First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} = 1$$

Now, consider $f(x, y) = x^2 + y$. Compute $\int_C f dS$. Restricting f to C gives

$$f(x(t), y(t)) = x(t)^2 + y(t) = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f dS = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

- Line integrals can be used to compute the mass of an object (1-dimensional)
- The curve C describes the object
- The function $\int_C f dS$ is the mass.

3.3 Parametrization and orientation

Next results as for the arc length.

Proposition 2. The integral $\int_C f dS$ does not depend on the parametrization of C .

We will consider a special case.

Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\text{Start: } A = (x(t_0), y(t_0)), \quad \text{End: } B = (x(t_1), y(t_1))$$

We want to go from B to A , which can be done in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\vec{r}_{\text{opp}}(-t_1) = (x(t_1), y(t_1)) = B$$

$$\vec{r}_{\text{opp}}(-t_0) = (x(t_0), y(t_0)) = A$$

Example. Consider C = segment from $(0,0)$, to $(1,0)$, take $f(x,y) = x$. Show that $\int_C f dS = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$.

If C parametrized by $\vec{r}(t)$, we will use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have that

$$\int_C f dS = \int_{-C} f dS$$

The situation will be different for vector fields.

3.4 Case of Vector Fields

Consider the curve with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\vec{r}'(t) = (x'(t), y'(t)) \quad \text{Velocity Vector}$$

Definition 7. The line integral of \vec{F} along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

$$\text{Here } \vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$$

- In physics, we have that \vec{F} = Force and $\int_C \vec{F} d\vec{r}$ = Work done by \vec{F} along C .
- The elementary case $W = FS$, where W is work, F is Force and S is the displacement.

More explicitly, we consider

$$\vec{F}(x, y) = (P(x, y), Q(x, y))$$

Then we have that

$$\vec{F}(\vec{r}(t) \cdot \vec{r}'(t)) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t) dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t) dt$$

We will write the expression as

$$\int_C \vec{F} d\vec{r} = \int_C P dx + \int_C Q dy$$

Example. Consider a curve C with

$$\vec{r}(t) = (t, t^2), \quad 0 \leq t \leq 1$$

We have $x(t) = t$ and $y(t) = t^2$, its derivative is

$$\vec{r}'(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is $P = x + y$ and $Q = x$. When restricted to C , we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = (t + t^2, t)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = (t + t^2, t) \cdot (1, 2t) = (3t^2 + t)$$

We then obtain the following

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + t) dt = \frac{3}{2}$$