
1 Lecture 8

1.1 Determining Motion

Given acceleration $\vec{a}(t)$, can we find $\vec{v}(t)$ and $\vec{s}(t)$? Yes, with some initial conditions given, we can. This is done by integration, consider

$$\vec{v}(t) = \frac{d\vec{s}}{dt}$$

This is a differential equation for $\vec{s}(t)$. To solve it, we integrate both sides in t , from t_1 , to t_2 . We get

$$\int_{t_0}^{t_1} \vec{v}(t) dt = \int_{t_0}^{t_1} \frac{d\vec{s}}{dt} dt$$

The fundamental theorem of calculus gives

$$\vec{s}(t_1) - \vec{s}(t_0) = \int_{t_0}^{t_1} \vec{v}(t) dt$$

We can determine $\vec{s}(t)$ for any t if we know $\vec{v}(t)$ and the initial condition $\vec{s}(t_0)$.

Example. Consider an object with acceleration

$$\vec{a}(t) = (1, t) = \vec{i} - j\vec{j}$$

We have the following initial conditions

$$\vec{s}(0) = (2, 0) = 2\vec{i} \quad \wedge \quad \vec{v}(0) = 0$$

We want to determine $\vec{s}(t)$. First, to determine $\vec{v}(t)$, we compute

$$\begin{aligned} \int_0^t \vec{a}(t) dt &= \vec{i} \int_0^t 1 dt + \vec{j} \int_0^t t dt \\ &= t\vec{i} + \frac{1}{2}t^2\vec{j} \end{aligned}$$

Here $t_0 = 0$, since $\vec{v}(0) = 0$, then

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t) dt = t\vec{i} + \frac{1}{2}t^2\vec{j}$$

To determine \vec{s} , we compute

$$\begin{aligned} \int_0^t \vec{v}(t) dt &= \vec{i} \int_0^t t dt + \vec{j} \int_0^t \frac{1}{2}t^2 dt \\ &= \frac{1}{2}t^2\vec{i} + \frac{1}{6}t^3\vec{j} \end{aligned}$$

Since $\vec{s}(0) = (2, 0) = 2\vec{i}$, we get

$$\vec{s}(t) = \vec{s}(0) + \int_0^t \vec{v}(t) dt = \left(\frac{t^2}{2} + 2\right)\vec{i} + \frac{1}{6}t^3\vec{j}$$

1.2 Arc Length

A formula that we can concretely use to compute the length of a curve (using a parametrization). Consider a curve c , with parametrization

$$\vec{s}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Definition 1. The arc length of c is given by

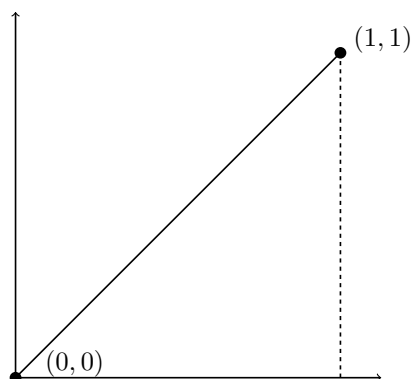
$$S = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Idea: summing segments of length



To compute S , we need a parametrization of c . Does S depend on this choice? No!

Example. Consider the line segment below



From elementary geometry, its length is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Consider the parametrization

$$\vec{s}(t) = (t, t), \quad 0 \leq t \leq 1$$

We have $(x'(t), y'(t)) = (1, 1)$. Then

$$S = \int_0^1 \sqrt{1^2 + 1^2} dt = \sqrt{2} \int_0^1 1 dt = \sqrt{2}$$

Instead we choose

$$\vec{s}(t) = (2t, 2t), \quad 0 \leq t \leq \frac{1}{2}$$

We have $(x'(t), y'(t)) = (2, 2)$. Then

$$S = \int_0^{\frac{1}{2}} \sqrt{2^2 + 2^2} dt = \sqrt{8} \int_0^{\frac{1}{2}} 1 dt = \sqrt{8} \cdot \frac{1}{2} = \sqrt{2}$$

Question: What is the distance crossed up to time t ?

Definition 2. The arc length parameter is

$$S(t) = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The difference is that we integrate up to t , not t_1 . Special case when $S(t_1) = S$. There is an important relation between $S(t)$ and $\vec{v}(t)$.

Proposition 1. We have

$$|\vec{v}(t)| = \frac{dS}{dt}$$

Proof. The fundamental theorem of calculus states that if

$$F(x) = \int_a^x f(x) dt \rightarrow F'(x) = f(x)$$

Applying this to $S(t)$, then

$$S'(t) = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

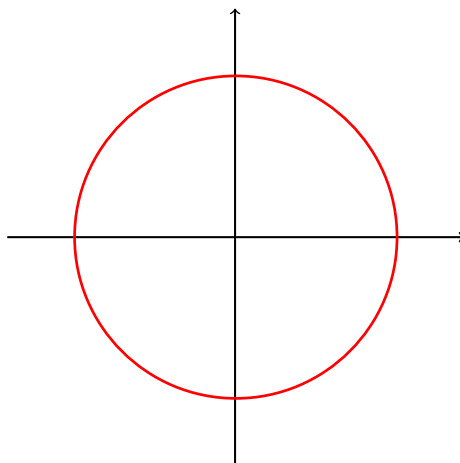
On the other hand, we have that

$$\vec{v}(t) = (x'(t), y'(t)) \quad \wedge \quad |\vec{v}(t)| = (t) = \sqrt{x'(t)^2 + y'(t)^2}$$

The two expressions coincide. □

Example. Consider a circle of radius R , with

$$\vec{S}(t) = (R \cos t, R \sin t), \quad 0 \leq t \leq 2\pi$$



We want to compute $S(t)$, we have

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{R^2 (\sin t)^2 + R^2 (\cos t)^2} = R$$

We want to check that $\frac{dS}{dt} = |\vec{v}(t)|$. We have

$$\vec{v}(t) = \frac{d\vec{S}}{dt} = (-R \sin t, R \cos t)$$

Its length is equal to $|\vec{v}(t)| = R$. Since $S(t) = R(t)$, we see that

$$\frac{dS}{dt} = |\vec{v}(t)|$$