Contents

1 Domains, Graphs and Level Sets

1.1 Domain of definition

A function may not be defined for all real numbers.
Example. $f(x) = \frac{1}{x}$ is not defined for $x = 0$

Definition 1. The domain of a function f is the set of numbers for which it is defined. We write the domain of f as D_f.

For instance, for $f(x) = \frac{1}{x}$ we have that

$$D_f = \{x \in \mathbb{R} : x \neq\}$$

This is the lasrgerst possible domain, we can also consider smaller domains. We have the interval from 1 to 2.

$$[1,2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$$

Example. Find the largest domain of $f(x,y) = \frac{1}{y-x}$. The denominator should be non-zero, we get

$$D_f = \{(x,y) \in \mathbb{R}^2 : y - x \neq 0\}$$

Example. Same excersise with $f(x,y) = \sqrt{y - x^2}$.
    Argument: $y - x^2 \geq 0$ (because square root). We will then have $y \geq x^2$

1.2 Graphs of functions

The plot of a function f descrives its behaviour visually. Mathematically, a plot corresponds to the notion of a graph.

Definition 2. The graph of a function $f(x,y)$ with domain D_f is the set of points (x,y,z) such that:

$$(x,y) \in D_f \text{ and } z = f(x,y)$$

We write G_f for the set

$$G_f = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D_f \text{ and } z = f(x,y)\}$$

The graph of a function of two variables will, in general, be a surface.

Example. Lets consider $f(x,y) = 1$, with domain $\mathbb{R}^2$. The graph of f is:

$$G_f = \{(x,y,1) : (x,y) \, \mathbb{R}^2\}$$

All points have $z = 1$, this is a plane. More generally, the graph of $f(x,y) = ax + by + c$ is a plane with linear dependence on x and y.

Example. Consider the graph of:

$$f(x,y) = x^2 + y^2, \quad D_F = \mathbb{R}^2$$

This surface is called a parabloids.

Example. A sphere of radius r is defined by

$$x^2 + y^2 + z^2 = r^2$$

All points x, y, z satisfy the equation.
Is this the graph of a function? No!
There is no unique value of z, associated with (x, y) because:

$$z = \pm\sqrt{r^2 - x^2 - y^2}$$

Both satisfy the sphere equation. Lets consider the graph of

$$f(x, y) = \sqrt{r^2 - x^2 - y^2}$$

With the domain $x^2 + y^2 \leq r^2$. The grap is a half sphere.

1.3   Level Sets

Another way to visualize functions.

> **Definition 3.** A level set of a function $f(x, y)$ is constant. Essentialy, this is a topographic map.

Example. Consider the function

$$f(x, y) = x^2 + y^2$$

The level sets for $c > 0$ are circles.

$$f(x, y) = x^2 + y^2 = c = \left(\sqrt{c}\right)^2$$

This is a circle with radius $\sqrt{c}$.
Now consider the case $c < 0$, then:

$$f(x, y) = x^2 + y^2 = c$$

which doesn't work, because the level sets are empty.
For $c = 0$, we only have the point $(x, y) = (0, 0)$. Generally, level sets of $f(x, y)$ is a curve.

2   Lecture 3

2.1   Partial Derivatives

In the case of one variable, we have

$$\frac{df}{dt} = \lim_{n \to 0} \frac{f(x + n) - f(x)}{n}$$

Similarily, for two of more variables, we have the following definition

> **Definition 4.** The partial derivative of $f(x, y)$ with respect to x:
>
> $$\frac{\partial f}{\partial x} = \lim_{n \to 0} \frac{f(x + n, y) - f(x, y)}{n}$$
>
> Also written as f_x, for $\frac{\partial f}{\partial y}$, we have f_y

Note, the expression above is $\frac{\partial f}{\partial x}(x, y)$, which is the value at the point (x, y)

2.2 Higher order derivatives

Given $\frac{\partial f}{\partial x}$, we can take further derivatives. We have

$$\frac{\partial^2 f}{\partial^2 x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Also written as $f_{xx}, f_{yy}, f_{xy}, f_{yx}$. In most cases, f_{xy} and f_{yx} coincide.

Theorem 1. Scwartz theorem: Suppose f_{xy} and f_{yx} exist, and are continous, then

$$f_{xy} = f_{yx}$$

Similar definitions and results for the case of more variables: $x_1, \ldots, x_n$, with n variables.

2.3 Chain Rule

Suppose $f(x) = g(h(x))$, for instance

$$f(x) = (\cos x)^2 \text{ with } g(x) = x^2, h(x) = \cos x$$

then the chain rule is

$$\frac{df}{dt}(x_0) = \frac{dg}{dh}(h(x_0)) \cdot \frac{dh}{dt}(x_0)$$

Generalization to more variables.

Theorem 2. Chain rule: consider $f(x, y)$ x and y depending on a variable t. Then:

$$\frac{df}{dt}t_0 = \frac{\partial f}{\partial x}(x(t_0), y(t_0))\frac{dx}{dt}t_0 + \frac{\partial f}{\partial y}(x(t_0), y(t_0))\frac{dy}{dt}t_0$$

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3 Line Integrals, Parametrization and Vector Fields

3.1 Identities between operations

We have seen three operations defined by ∇.

$$\text{Gradient: } \nabla f \qquad \text{Divergence: } \nabla \cdot \vec{F} \qquad \text{Curl: } \nabla \times \vec{F}$$

There are many of them, we look at only one.

Proposition 1. For any scalar field f we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$. Then

$$\nabla \times \vec{F} = \left|\left|\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix}\right|\right| = \vec{i}(f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{cy})$$

But partial derivatives can be exhanged, $f_{xy} = f_{yx}$. Then we find that

$$\nabla \times (\nabla f) = 0$$

$\square$

We are going to use this when we discuss conservative fields.

3.2   Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Consider the points $\vec{r}(t)$ and $\vec{r}(t + \Delta t)$ where Δt is small.

$$\Delta x = x(t + \Delta t) - x(t), \quad \Delta y = y(t + \Delta t) - y(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt} \Delta t, \quad \Delta y \approx \frac{dy}{dt} \Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt}\Delta t\right)^2 + \left(\frac{dy}{dt}\Delta t\right)^2} = \Delta t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Definition 5. The line integral of $f(x, y)$ along a curve C is defined by

$$\int_C f dS = \int_{t_0}^{t_1} f(x(y), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Some observations:

- Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$.

- When $f = 1$, we recover the arc length.

Example. Consider C definet by

$$x(t), y(t) = 0, \quad 0 \leq t \leq 1$$

First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} = 1$$

Now, consider $f(x, y) = x^2 + y$. Compute $\int_C f dS$. Restricting f to C gives

$$f(x(t), y(t)) = x(t)^2 + y(t) = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f dS = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

- Line integrals can be used to compute the mass of an objects (1 - dimensional)

- The curve C describes the object

- The function $\int_C f dS$ is the mass.

3.3   Parametrization and orientation

Next results as for the arc legnth.

Proposition 2. The integral $\int_C f dS$ does not depend on the parametrization of C.

We will consider a special case.
Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\text{Start: } A = (x(t_0), y(t_0)), \quad \text{End: } B = (x(t0), y(t_1))$$

We want to go from B to A, which can be done in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\vec{r}_{\text{opp}}(-t_1) = (x(t_1), y(t_1)) = B$$

$$\vec{r}_{\text{opp}}(-t_0) = (x(t_0), y(t_0)) = A$$

Example. Consider C = segment from (0,0), to (1,0), take $f(x,y) = x$. Show that $\int_C f dS = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$.

If C parametrized by $\vec{r}(t)$, we will use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have that

$$\int_C f dS = \int_{-C} f dS$$

The situation will be different for vector fields.

3.4   Case of Vector Fields

Consider the curve with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\vec{r'}(t) = (x'(t), y'(t)) \quad \textit{Velocity Vector}$$

Definition 6. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r'}(t) \, dt$$

Here $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$

- In physics, we have that $\vec{F} =$ Force and $\int_C \vec{F} d\vec{r} =$ Work done by $\vec{F}$ along C.

- The elementary case $W = FS$, where W is work, F is Force and S is the displacement.

More explicitly, we consider

$$\vec{F}(x, y) = (P(x, y), Q(x, y))$$

Then we have that

$$\vec{F}\left(\vec{r}(t) \cdot \vec{r'}(t)\right) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t) \, dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t) \, dt$$

We will write the expression as

$$\int_C \vec{F} d\vec{r} = \int_C P \, dx + \int_C Q \, dy$$

Example. Consider a curve C with

$$\vec{r}(t) = \left(t, t^2\right), \quad 0 \le t \le 1$$

We have $x(t) = t$ and $y(t) = t^2$, its derivative is

$$\vec{r'}(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is $P = x + y$ and $Q = x$. When restricted to C, we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = \left(t + t^2, t\right)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r'}(t) = \left(t + t^2, t\right) \cdot (1, 2t) = \left(3t^2 + t\right)$$

We then obation the following

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(3t^2 + t\right) dt = \frac{3}{2}$$