

# **DAVE3700 - Matte 3000**

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June 22, 2022

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## 1 Introduction

## 2 Domains, Graphs and Level Sets

### 2.1 Domain of definition

A function may not be defined for all real numbers.

**Example.**  $f(x) = \frac{1}{x}$  is not defined for  $x = 0$

◇

**Definition 1.** The domain of a function  $f$  is the set of numbers for which it is defined. We write the domain of  $f$  as  $D_f$ .

For instance, for  $f(x) = \frac{1}{x}$  we have that

$$D_f = \{x \in \mathbb{R} : x \neq 0\}$$

This is the largest possible domain, we can also consider smaller domains. We have the interval from 1 to 2.

$$[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$$

**Example.** Find the largest domain of  $f(x, y) = \frac{1}{y-x}$ . The denominator should be non-zero, we get

$$D_f = \{(x, y) \in \mathbb{R}^2 : y - x \neq 0\}$$

◇

**Example.** Same exercise with  $f(x, y) = \sqrt{y - x^2}$ .

Argument:  $y - x^2 \geq 0$  (because square root). We will then have  $y \geq x^2$

◇

### 2.2 Graphs of functions

The plot of a function  $f$  describes its behaviour visually. Mathematically, a plot corresponds to the notion of a graph.

**Definition 2.** The graph of a function  $f(x, y)$  with domain  $D_f$  is the set of points  $(x, y, z)$  such that:

$$(x, y) \in D_f \text{ and } z = f(x, y)$$

We write  $G_f$  for the set

$$G_f = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_f \text{ and } z = f(x, y)\}$$

The graph of a function of two variables will, in general, be a surface.

**Example.** Lets consider  $f(x, y) = 1$ , with domain  $\mathbb{R}^2$ . The graph of  $f$  is:

$$G_f = \{(x, y, 1) : (x, y) \in \mathbb{R}^2\}$$

All points have  $z = 1$ , this is a plane. More generally, the graph of  $f(x, y) = ax + by + c$  is a plane with linear dependence on  $x$  and  $y$ .  $\diamond$

**Example.** Consider the graph of:

$$f(x, y) = x^2 + y^2, \quad D_F = \mathbb{R}^2$$

This surface is called a paraboloids.  $\diamond$

**Example.** A sphere of radius  $r$  is defined by

$$x^2 + y^2 + z^2 = r^2$$

All points  $x, y, z$  satisfy the equation.  $\diamond$

Is this the graph of a function? No!

There is no unique value of  $z$ , associated with  $(x, y)$  because:

$$z = \pm \sqrt{r^2 - x^2 - y^2}$$

Both satisfy the sphere equation. Lets consider the graph of

$$f(x, y) = \sqrt{r^2 - x^2 - y^2}$$

With the domain  $x^2 + y^2 \leq r^2$ . The graph is a half sphere.

### 2.3 Level Sets

Another way to visualize functions.

**Definition 3.** A level set of a function  $f(x, y)$  is constant. Essentially, this is a topographic map.

**Example.** Consider the function

$$f(x, y) = x^2 + y^2$$

The level sets for  $c > 0$  are circles.

$$f(x, y) = x^2 + y^2 = c = (\sqrt{c})^2$$

This is a circle with radius  $\sqrt{c}$ .

Now consider the case  $c < 0$ , then:

$$f(x, y) = x^2 + y^2 = c$$

which doesn't work, because the level sets are empty.

For  $c = 0$ , we only have the point  $(x, y) = (0, 0)$ . Generally, level sets of  $f(x, y)$  is a curve.

◇

### 3 Derivatives

#### 3.1 Partial Derivatives

In the case of one variable, we have

$$\frac{df}{dt} = \lim_{n \rightarrow 0} \frac{f(x+n) - f(x)}{n}$$

Similarly, for two or more variables, we have the following definition

**Definition 4.** The partial derivative of  $f(x, y)$  with respect to  $x$ :

$$\frac{\partial f}{\partial x} = \lim_{n \rightarrow 0} \frac{f(x+n, y) - f(x, y)}{n}$$

Also written as  $f_x$ , for  $\frac{\partial f}{\partial x}$ , we have  $f_y$

Note, the expression above is  $\frac{\partial f}{\partial x}(x, y)$ , which is the value at the point  $(x, y)$

#### 3.2 Higher order derivatives

Given  $\frac{\partial f}{\partial x}$ , we can take further derivatives. We have

$$\frac{\partial^2 f}{\partial^2 x} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$$

Also written as  $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ . In most cases,  $f_{xy}$  and  $f_{yx}$  coincide.

**Theorem 5.** Schwartz theorem: Suppose  $f_{xy}$  and  $f_{yx}$  exist, and are continuous, then

$$f_{xy} = f_{yx}$$

Similar definitions and results for the case of more variables:  $x_1, \dots, x_n$ , with  $n$  variables.

#### 3.3 Chain Rule

Suppose  $f(x) = g(h(x))$ , for instance

$$f(x) = (\cos x)^2 \text{ with } g(x) = x^2, h(x) = \cos x$$

then the chain rule is

$$\frac{df}{dt}(x_0) = \frac{dg}{dh}(h(x_0)) \cdot \frac{dh}{dt}(x_0)$$

Generalization to more variables.

**Theorem 6.** Chain rule: consider  $f(x, y)$   $x$  and  $y$  depending on a variable  $t$ . Then:

$$\frac{df}{dt}|_{t_0} = \frac{\partial f}{\partial x}(x(t_0), y(t_0)) \frac{dx}{dt}|_{t_0} + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \frac{dy}{dt}|_{t_0}$$

The "short form" of this result is

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

**Example.** Consider  $f(x, y) = xy$ , where

$$x(t) = \cos t, y(t) = \sin t$$

This cannot be computed directly with  $\frac{df}{dt}$ .

$$f(t) = f(x(t), y(t)) = f(\cos t, \sin t) = \cos t \cdot \sin t$$

We can compute

$$\frac{df}{dt} = (\cos t)' \sin t + \cos t (\sin t)' = -(\sin t)^2 + (\cos t)^2$$

Using the chain rule, we get

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

◇

### 3.4 The Gradient

Define an operation that takes a scalar function, and returns a vector function.

**Definition 7.** The gradient of  $f(x, y)$  at  $(x_0, y_0)$  is

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

## 4 Directional Derivatives and Critical Points

### 4.1 Directional Derivatives

We have seen the following

- $f_x$  = the rate of change along the  $x$ -direction.
- $f_y$  = the rate of change along the  $y$ -direction.

What about general directions?

**Definition 8.** Let  $\vec{u} = (a, b)$ , the directional derivative along  $\vec{u}$  at  $(x, y)$  is

$$D_{\vec{u}} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x + ha, y + hb) - f(x, y)}{h}$$

Note that

$$\vec{u} = (1, 0) \rightarrow D_{\vec{u}} = f_x$$

$$\vec{u} = (0, 1) \rightarrow D_{\vec{u}} = f_y$$

To compare directions, we take  $|\vec{t}| = 1$ . Here  $\vec{u}$  is the length of  $\vec{u}$ , that is

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

**Proposition 1.** We have the following result

$$D_{\vec{u}} \cdot f = \nabla f \cdot \vec{u}$$

**Proof.** Consider the following function

$$g(t) = f(x + ta, y + tb)$$

Its derivative at  $t = 0$  is

$$\begin{aligned} \frac{dg}{dt}(0) &= \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + ha, y + hb) - f(x, y)}{h} \\ &= D_{\vec{u}} \cdot f(x, y) \end{aligned}$$

On the other hand, using the chain rule, we get

$$\begin{aligned}\frac{dg}{dt}(0) &= \frac{\partial f}{\partial x} \frac{d(x+ta)}{dt} + \frac{\partial f}{\partial y} \frac{d(y+tb)}{dt} \\ &= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b = \nabla f \cdot \vec{u}\end{aligned}$$

□

We can now state another property of  $\nabla f$ . The direction where  $f$  changes the most.

**Proposition 2.**  $|D_{\vec{u}} \cdot f|$  is the largest when  $\vec{u}$  is parallel to  $\nabla f$ .

**Proof.** Recall that, given two vectors,  $\vec{v}$  and  $\vec{w}$ , we have that

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos \alpha$$

When is  $|\vec{v} \cdot \vec{w}|$  the largest? We have

$$|\vec{v} \cdot \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot |\cos \alpha|, \quad |\cos \alpha| \leq 1$$

It is the largest when the following condition is true

$$|\cos \alpha| = 1, \quad \alpha = 0 \vee \pi$$

Which means that the vectors are pointing in the same, or opposite direction. Applying this to  $D_{\vec{u}} \cdot f$ , we get

$$\begin{aligned}|D_{\vec{u}} \cdot f| &= |\nabla f \cdot \vec{u}| \\ &= |\nabla f| \cdot |\vec{u}| \cdot |\cos \alpha|\end{aligned}$$

For fixed values of  $|\vec{u}|$ , this is the largest when  $\alpha = 0 \vee \pi$ . That is  $\nabla f$  and  $\vec{u}$  are parallel. □

## 4.2 Critical Points

How do we find the maxima and minima of  $f(x)$ ? Lets take a look at  $f'(x_0) = 0$ .

**Definition 9.** We say that  $x_0, y_0$  is a critical point of  $f$  if:

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \quad \wedge \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

Similarly for

## 5 Lecutre 5

### 5.1 Hessian Matrix

**Example.** Consider again the function  $f(x, y) = x^2 - y^2$ .

$$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{yx} = 0$$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \det H = -4 < 0$$

Hence  $(x_0, y_0)$  is a saddle point.  $\diamond$

**Example.** Consider the function  $f(x, y) = x^2 + y^2$ , we have

$$(f_x, f_y) = (2x, 2y)$$

The only critical point is  $(x_0, y_0) = (0, 0)$ .

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \det H = 4 > 0$$

Since  $f_{xx} = 2 > 0$ , we conclude that  $(0, 0)$  is a local minima. In this case, it is actually a global minimum, because  $f(x, y) = x^2 + y^2 \geq 0$ .  $\diamond$

### 5.2 Global extremal values

A function can have many maxima and minimas. Usually, we are interested in the largest and smallest values.

**Definition 10.** Let  $f(x, y)$  be with domain  $D_f$ . Then we have

- $(x_0, y_0)$  is a global maxima if  $f(x_0, y_0) \geq f(x, y)$  for all  $(x, y) \in D_f$ .
- $(x_0, y_0)$  is a global minima if  $f(x_0, y_0) \leq f(x, y)$  for all  $(x, y) \in D_f$ .

Trivial example: for  $f(x, y) = 1$ , all points are global maxima and minima.  
Note that global maxima and minima need not be critical points.

**Example.** We have  $f(x) = x$  with  $D_f = [-1, 1]$ .

- Global maxima at  $x = 1, f(1) = 1$ .
- Global minima at  $x = -1, f(-1) = -1$

We have no critical points because  $f'(x) = 1 \neq 0$ . Also note that maxima and minima depend on the chosen domain.

If we take  $D_f = [-2, 3]$ , then

Max:  $x = 3$ , Min:  $x = -2$

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**Theorem 11.** Let  $f$  be continuous with domain  $D_f$ . Suppose  $D_f$  is closed and bounded, then there is at least one global maxima, and one global minima.

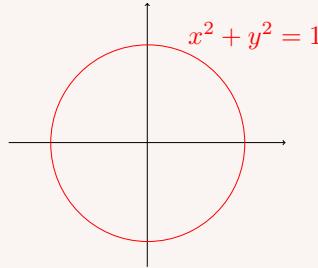


Figure 1: The circle is the boundary.

Some terminology:

$$\text{open} = \{x^2 + y^2 < 1\}$$

$$\text{closed} = \{x^2 + y^2 \leq 1\}$$

The method for finding maxima and minima is as follows:

1. Find critical points of  $f$  in  $D_f$ , and characterize them.
2. Study the points that are on the boundary.
3. Compare them.

**Example.** Consider  $f(x, y) = x^2 + y^2$  with domain

$$D = \{(x, y) \in \mathbb{R} : x^2 + y^2 \leq 1\}$$

The domain in this case is a disc. The red circle is the boundary.

We compute  $f_x = 2x$ ,  $f_y = 2y$ . The only critical point is the origin at  $(x_0, y_0) = (0, 0)$ . This is a global minimum since  $f(0, 0)$  and  $f(x, y) \geq 0$ .

Now, lets consider the boundary

$$C = \{x^2 + y^2 = 1\}$$

For any point  $(x_0, y_0)$  on the circle  $C$ , we have

$$f(x_0, y_0) = x_0^2 + y_0^2 = 1$$

We claim that this point is a global maximum. For any  $(x, y)$  in domain  $D_f$ , we have

$$f(x, y) = x^2 + y^2 \leq 1$$

The value  $f(x, y) = 1$  is obtain only at the boundary  $C$ . Any point on the circle is a global maxima.  $\diamond$

### 5.3 Constrained optimization

In this section, we will discuss how to find maxima and minima of  $f(x, y)$  with constraint  $g(x, y) = 0$ . Think of  $g = 0$  ad a budget, or a geometrical constraint.

**Example.** We want to minimize  $f(x, y) = x^2 + y^2$  with the constraint  $g(x, y) = xy - 1 = 0$ .

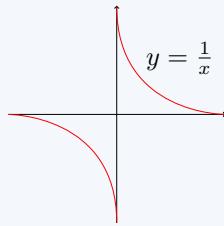


Figure 2:  $y = \frac{1}{x}$

We have the following strategy

- Solve  $g = 0$  for one variable. For instance  $y = \frac{1}{x}$ .
- Consider:  $h(x) = f\left(x, \frac{1}{x}\right) = x^2 + x^{-2}$

Now we can study this function of one variable with no constraints. We can proceed as usual.

$$\frac{dh}{dt} = 2x - 2x^{-3} = 0$$

This is equivalent to  $x^4 = 1$ . The real solutions are  $x = \pm 1$ . Since  $y = \frac{1}{x}$ , we get the critical points:

$$(x, y) = (1, 1), \quad (x, y) = (-1, -1)$$

$\diamond$

## 6 Lecture 6

### 6.1 Substitution Method

We want to maximize / minimize  $f(x, y)$  with constraint  $g(x, y) = 0$ . We can solve  $g(x, y) = 0$  for one variable  $y = y(x)$ .

**Example.** Consider  $f(x, y) = x^2 + y^2$  and  $g(x, y) = xy - 1 = 0$ . In this case, we have  $f = \frac{1}{x}$  from  $g = 0$ , we get

$$h(x) = f(x, x^{-1}) = x^2 + x^{-2}$$

We have found the minima at  $(1, 1)$  and  $(-1, -1)$ .  $\diamond$

This method isn't always feasible, so let's look at some alternatives.

### 6.2 Lagrange's Method

**Example.** Let's look at the level curves, which are circles. We have that  $f(x, y) = x^2 + y^2 = c$ .

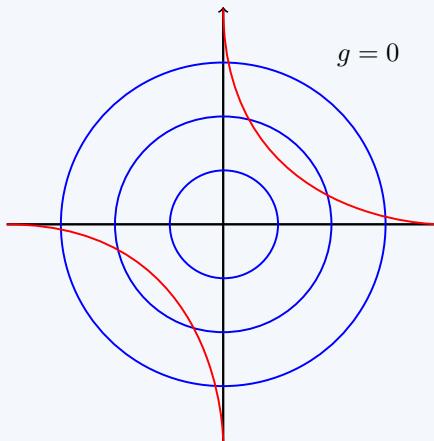
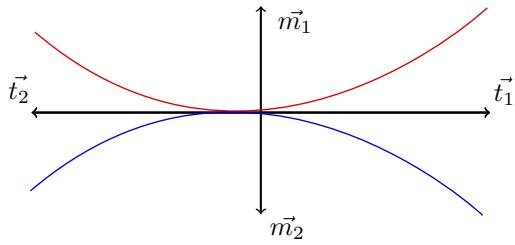


Figure 3:  $f(x, y)$

Smaller circles correspond to smaller values of  $f(x, y)$ , but we must also satisfy  $g(x, y) = 0$ . In the best case,  $f(x, y) = c$  is just touching  $g(x, y) = 0$ . If this is worked out geometrically, we get  $(1, 1)$  and  $(-1, -1)$ .  $\diamond$

This idea is used in Lagrange's method. We want  $f(x, y) = c$  to be parallel to  $g(x, y) = 0$ . More precisely: Their vectors should be parallel.



Equivalently, their normal vectors are also parallel. Recall that a normal vector to  $g = 0$  is given by  $\nabla f$ . Similarly,  $\nabla f$  is normal to  $f = c$ .

### 6.3 Method for Langrange

Suppose we want to find a local maxima and minima of  $f(x, y)$  with constraint  $g(x, y) = 0$ . We proceed as follows

1. Find all possible solutions to the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0$$

2. Plug in all solutions from 1. into  $f(x, y)$  and identify the largest and smallest.

- The number  $\lambda$  is called the *Lagrange Multiplier*.
- Easy extension to  $n$  variables.
- Can also be generalized to multiple constraints,  $g_1, \dots, g_n$

**Example.** We have the following functions

$$f(x, y) = x^2 + y^2, \quad g(x, y) = xy - 1 = 0$$

We compute the gradients

$$\nabla f(x, y) = (2x, 2y), \quad \nabla g(x, y) = (x, y)$$

The Lagrange equations are

$$2x = \lambda y, \quad 2y = \lambda x, \quad xy - 1 = 0$$

Observe that  $(x, y, \lambda) = (0, 0, 0)$  is not a solution. For  $x \neq 0$ , we get  $y = \frac{1}{x}$ , from the third equation.

$$\begin{aligned}
 2x = \lambda y &\Rightarrow 2x = \lambda \frac{1}{x} \Rightarrow \lambda = 2x^2 \\
 2y = \lambda x &\Rightarrow \frac{2}{x} = 2xs^2 \cdot x \Rightarrow x^4 = 1 \\
 x^4 = 1 &\Rightarrow x = \pm 1, \quad y = \frac{1}{x}
 \end{aligned}$$

$$(x, y) = (1, 1) \quad \wedge \quad (x, y) = (-1, -1)$$

Are these points the minima? We have

$$f(1, 1) = f(-1, -1) = 2$$

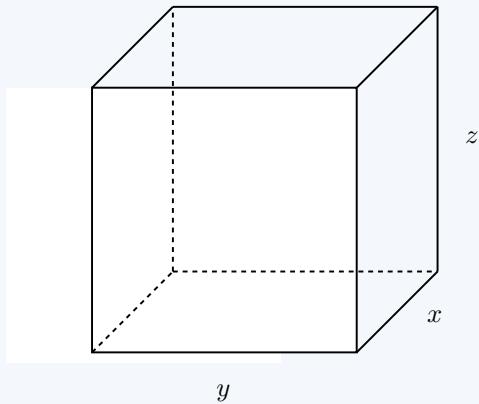
Let's compare this to some other points, such that  $g(x, y) = 0$ . For instance,  $(x, y) = (2, \frac{1}{2})$

$$g\left(2, \frac{1}{2}\right) = 2 \cdot \frac{1}{2} - 1 = 0$$

$$f\left(2, \frac{1}{2}\right) = 2^2 + \left(\frac{1}{2}\right)^2 > 2$$

This tells us that  $(x, y) = (\pm 1, \pm 1)$  are local minima.  $\diamond$

**Example.** Consider a box of surface area  $24 \text{ cm}^2$ .



Determine the dimensions  $(x, y, z)$  such that the volume is max.  
We have the surface area  $2xy + 2xz + 2yz$ . Our constraint is

$$g(x, y, z) = 2xy + 2xz + 2yz - 24 = 0$$

The goal is to maximize  $f(x, y, z) = 0$ , with the constraint  $g(x, y, z) = 0$ .  
The equation  $\nabla f = \lambda \nabla g$  gives

$$yz = 2\lambda(y + 2), \quad xz = 2\lambda(x + 2), \quad xy = 2\lambda(x + y)$$

Observe that  $\lambda \neq 0$ , since  $x, y, z > 0$ . To solve the equations, we can multiply by  $x, y$  and  $z$ , respectively, then we get

$$x(y + z) = y(x + z) = x(y + x)$$

Consider  $x(y + z) = y(x + z)$ .

$$x(y + z) = y(x + z) \Rightarrow (x - y)z = 0$$

$$z \neq 0, \quad x = y$$

It's the same for the other equations, so  $x = y = z$ . We also need to use  $g = 0$ . Setting  $x = y = z$ , we get

$$\begin{aligned} g(x, x, x) &= 2x^2 + 2x^2 + 2x^2 - 24 = 0 \\ x^2 &= 4 \\ x &= 2 \end{aligned}$$

Lagrange's method gives:

$$(x, y, z) = (2, 2, 2) \Rightarrow V = 8$$

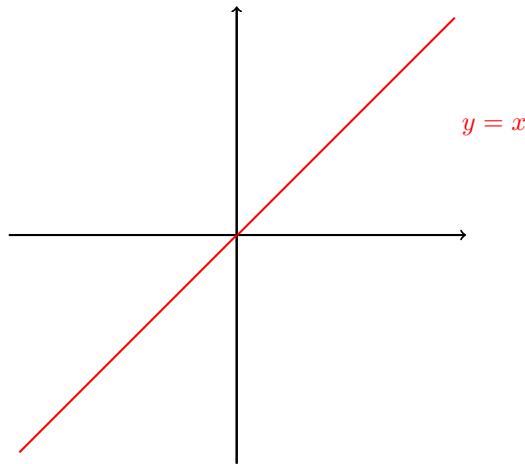
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## 7 Lecture 7

### 7.1 Parametrized Curve

A curve is described as a set of points in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . For instance a line is described by

$$f = \{(x, y) \in \mathbb{R}^2 : x = y\}$$



This is a static picture. But how do we give a dynamical picture? We'll use parametrized curves.

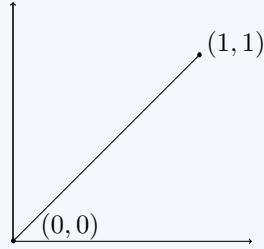
**Definition 12.** A parametrization of a curve  $c$  in  $\mathbb{R}^2$ , is given by

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Such that  $\vec{r}(t) \in c$  for all time  $t$ .

- A parametrization describes motion. Think of  $t$  as the time.
- A parametrization is not unique.
- Various natural assumptions, such as continuity and differentiability.

**Example.** Consider the function  $\vec{r}(t)$  with  $0 \leq t \leq 1$ .



$$\vec{r}(0) = (0, 0), \quad \vec{r}(1) = (1, 1)$$

We have the portion of the line where  $y = x$ . Notice that here,  $x(t) = t$ ,  $y(t) = t$  and  $y(t) = x(t) = t$  for all  $t$ .

Lets consider a different function,

$$\vec{r}(t) = (2t, 2t), \quad 0 \leq t \leq \frac{1}{2}$$

When we parametrize, we get

$$\vec{r}(0) = (0, 0), \quad \vec{r}\left(\frac{1}{2}\right) = (1, 1)$$

We are moving along the curve at twice the speed. ◊

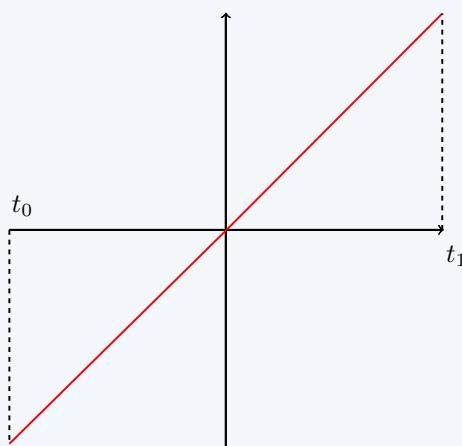
**Example.** Given  $f(x)$ , we consider

$$\vec{r}(t) = (t, f(t)), \quad t_0 \leq t \leq t_1$$

This describes a portion of the graph  $f$ , with

$$\text{Start: } (t_0, f(t_0)), \quad \text{End: } (t_1, f(t_1))$$

For instance, consider the line  $y = mx + c$ , we have



We have that

$$\vec{r}(t) = (t, mt + c), \quad t_0 \leq t \leq t_1$$

◇

**Example.** We want to describe a line with

$$\text{Start: } A = (x_0, y_0), \quad \text{End: } B = (x_1, y_1)$$

Then we take the parametrization

$$\vec{r}(t) = (1-t)A + tB, \quad 0 \leq t \leq 1$$

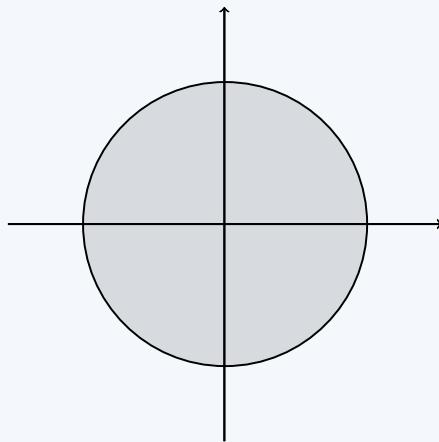
More explicitly, we have

$$\vec{r}(t) = ((1-t)x_0 + tx, (1-t)y_0 + ty)$$

Note that  $\vec{r}(0) = A$  and  $\vec{r}(1) = B$ .

◇

**Example.** Consider  $\vec{r}(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$ . What curve does this describe?



It describes a circle.

$$x(t)^2 + y(t)^2 = (\cos(t)^2 + \sin(t)^2) = 1$$

We start at  $(1, 0)$  and move counter-clockwise. ◊

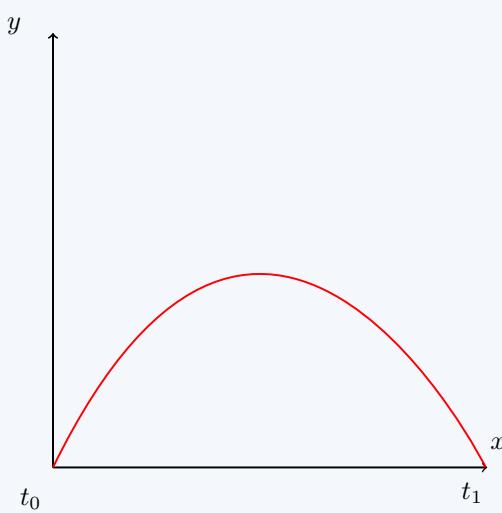
**Example.** Here is an example from physics. Consider

$$x(t) = v_x t, \quad y(t) = v_y t - \frac{1}{2} g t^2, \quad 0 \leq t \leq \frac{2v_y}{g}$$

This describes the motion of an object with initial velocity  $\vec{v} = (v_x, v_y)$ , under gravity. We write  $t_0 = 0$  and  $t_1 = \frac{2v_y}{g}$ . Note that

$$\vec{r}(t_0) = (0, 0), \quad \vec{r}(t_1) = \left( \frac{2v_x v_y}{g}, 0 \right)$$

The object falls back to the ground at time  $t_1$ .



Well known fact: This motion is parabolic, we will rederive this.  
From  $x(t) = v_x t$ , we get  $t = \frac{x(t)}{v_x}$ . Then

$$y(t) = v_y t - \frac{1}{2} g t^2 \Rightarrow \frac{v_x}{v_y} x(t) - \frac{1}{2} \frac{g}{v_x^2} x(t)^2$$

This is the expression of a parabola

$$y = ax^2 + bx + c, \quad a \neq 0$$

It can also be written as

$$y(t) = -\frac{1}{2} \frac{g}{v_x^2} \left( x(t) - \frac{v_y}{g} \right)^2 + \frac{1}{2} \frac{v_y}{g} \frac{v_y}{v_x}$$

◇

## 7.2 Kinematics

Kinematics describes position, velocity and acceleration of an object.

**Definition 13.** The position vector is  $\vec{r}(t)$ . The velocity vector is  $\vec{v}(t) \frac{d\vec{r}}{dt}$ . The acceleration vector is  $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2}$ .

If we write  $\vec{r}(t) = (x(t), y(t))$ , then

$$\vec{v}(t) = \left( \frac{dx}{dt}, \frac{dy}{dt} \right) = (x'(t), y(t))$$

Similarly

$$\vec{a}(t) = \left( \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right) = (x''(t), y''(t))$$

**Example.** Consider again the gravity example. Here we have

$$\vec{r}(t) = \left( v_x t, v_y t - \frac{1}{2} g t^2 \right)$$

The velocity is

$$\vec{v}(t) = (v_x, v_y - gt)$$

Note that  $v(0) = (v_x, v_y)$  is the initial velocity of the object. For acceleration, we get

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (0, -g)$$

◊

## 8 Lecture 8

### 8.1 Determining Motion

Given acceleration  $\vec{a}(t)$ , can we find  $\vec{v}(t)$  and  $\vec{s}(t)$ ? Yes, with some initial conditions given, we can. This is done by integration, consider

$$\vec{v}(t) = \frac{d\vec{s}}{dt}$$

This is a differential equation for  $\vec{s}(t)$ . To solve it, we integrate both sides in  $t$ , from  $t_1$ , to  $t_2$ . We get

$$\int_{t_0}^{t_1} \vec{v}(t) dt = \int_{t_0}^{t_1} \frac{d\vec{s}}{dt} dt$$

The fundamental theorem of calculus gives

$$\vec{s}(t_1) - \vec{s}(t_0) = \int_{t_0}^{t_1} \vec{v}(t) dt$$

We can determine  $\vec{s}(t)$  for any  $t$  if we know  $\vec{v}(t)$  and the initial condition  $\vec{s}(t_0)$ .

**Example.** Consider an object with acceleration

$$\vec{a}(t) = (1, t) = \vec{i} - j\vec{j}$$

We have the following initial conditions

$$\vec{s}(0) = (2, 0) = 2\vec{i} \quad \wedge \quad \vec{v}(0) = 0$$

We want to determine  $\vec{s}(t)$ . First, to determine  $\vec{v}(t)$ , we compute

$$\begin{aligned} \int_0^t \vec{a}(t) dt &= \vec{i} \int_0^t 1 dt + \vec{j} \int_0^t t dt \\ &= t\vec{i} + \frac{1}{2}t^2\vec{j} \end{aligned}$$

Here  $t_0 = 0$ , since  $\vec{v}(0) = 0$ , then

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t) dt = t\vec{i} + \frac{1}{2}t^2\vec{j}$$

To determine  $\vec{s}$ , we compute

$$\begin{aligned} \int_0^t \vec{v}(t) dt &= \vec{i} \int_0^t t dt + \vec{j} \int_0^t \frac{1}{2}t^2 dt \\ &= \frac{1}{2}t^2\vec{i} + \frac{1}{6}t^3\vec{j} \end{aligned}$$

Since  $\vec{s}(0) = (2, 0) = 2\vec{i}$ , we get

$$\vec{s}(t) = \vec{s}(0) + \int_0^t \vec{v}(t) dt = \left( \frac{t^2}{2} + 2 \right) \vec{i} + \frac{1}{6} t^3 \vec{j}$$

◊

## 8.2 Arc Length

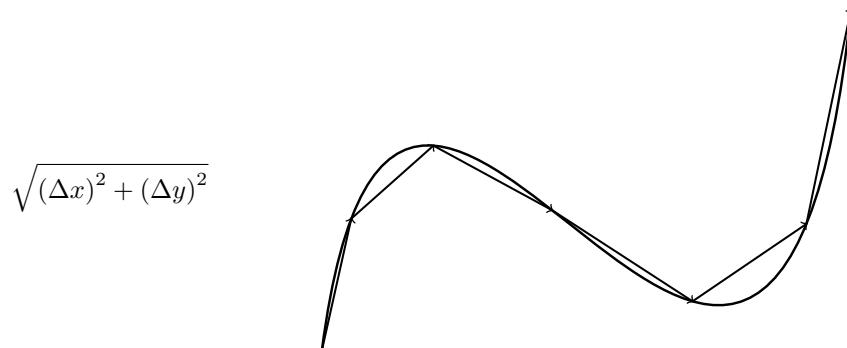
A formula that we can concretely use to compute the length of a curve (using a parametrization). Consider a curve  $c$ , with parametrization

$$\vec{s}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

**Definition 14.** The arc length of  $c$  is given by

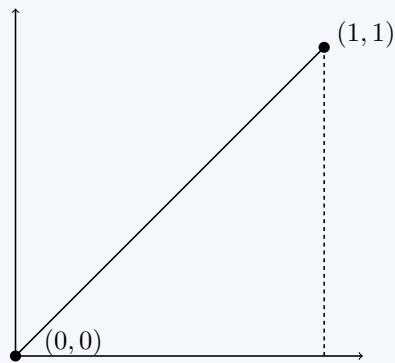
$$S = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Idea: summing segments of length



To compute  $S$ , we need a parametrization of  $c$ . Does  $S$  depend on this choice? No!

**Example.** Consider the line segment below



From elementary geometry, its length is  $\sqrt{1^2 + 1^2} = \sqrt{2}$ . Consider the parametrization

$$\vec{s}(t) (t, t), \quad 0 \leq t \leq 1$$

We have  $(x'(t), y'(t)) = (1, 1)$ . Then

$$S = \int_0^1 \sqrt{1^2 + 1^2} dt = \sqrt{2} \int_0^1 1 dt = \sqrt{2}$$

Instead we choose

$$\vec{s}(t) = (2t, 2t), \quad 0 \leq t \leq \frac{1}{2}$$

We have  $(x'(t), y'(t)) = (2, 2)$ . Then

$$S = \int_0^{\frac{1}{2}} \sqrt{2^2 + 2^2} dt = \sqrt{8} \int_0^{\frac{1}{2}} 1 dt = \sqrt{8} \cdot \frac{1}{2} = \sqrt{2}$$

◊

**Question:** What is the distance crossed up to time  $t$ ?

**Definition 15.** The arc length parameter is

$$S(t) = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The difference is that we integrate up to  $t$ , not  $t_1$ . Special case when  $S(t_1) = S$ . There is an important relation between  $S(t)$  and  $\vec{v}(t)$ .

**Proposition 3.** We have

$$|\vec{v}(t)| = \frac{dS}{dt}$$

**Proof.** The fundamental theorem of calculus states that if

$$F(x) = \int_a^x f(t) dt \rightarrow F'(x) = f(x)$$

Applying this to  $S(t)$ , then

$$S'(t) = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

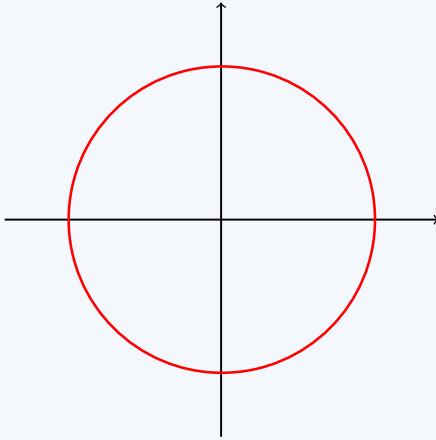
On the other hand, we have that

$$\vec{v}(t) = (x'(t), y'(t)) \quad \wedge \quad |\vec{v}(t)| = (t) = \sqrt{x'(t)^2 + y'(t)^2}$$

The two expressions coincide. □

**Example.** Consider a circle of radius R, with

$$\vec{S}(t) = (R \cos t, R \sin t), \quad 0 \leq t \leq 2\pi$$



We want to compute  $S(t)$ , we have

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{R^2 (\sin t)^2 + R^2 (\cos t)^2} = R$$

We want to check that  $\frac{dS}{dt} = |\vec{v}(t)|$ . We have

$$\vec{v}(t) = \frac{d\vec{S}}{dt} = (-R \sin t, R \cos t)$$

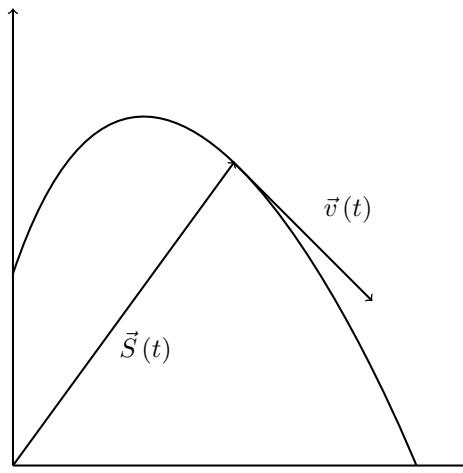
Its length is equal to  $|\vec{v}(t)| = R$ . Since  $S(t) = R(t)$ , we see that

$$\frac{dS}{dt} = |\vec{v}(t)|$$

◇

### 8.3 Tangent Vectors

Geometrically,, the velocity  $\vec{v}(t)$  is tangent to a curve. It is useful to define a tangent vector of length 1.



**Definition 16.** The unit tangent vector is

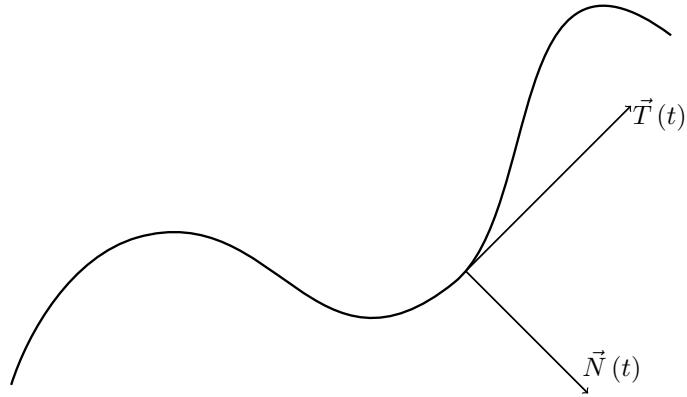
$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

Note that  $\vec{T}$  has length 1 since

$$\vec{T}(t) \cdot \vec{T}(t) = \frac{\vec{v}(t) \cdot \vec{v}(t)}{|\vec{v}(t)|^2} = 1$$

#### 8.4 Normal Vectors

Normal vectors are normal to the curve, or in other words, they are orthogonal. Recall that for implicit curves  $f(x, y) = 0$ , a normal vector is given by  $\nabla f$ .



Lets now consider parametrised curves. We have

$$S(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

**Definition 17.** A unit normal vector to the curve is defined by

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left| \frac{d\vec{T}}{dt} \right|}$$

We need to check that  $\vec{N}$  is orthogonal to  $\vec{T}$ , that is:  $\vec{N}(t) \cdot \vec{T}(t) = 0$  for all  $t$ .

**Proposition 4.** We have that

$$\vec{N}(t) \cdot \vec{T}(t) = 0$$

**Proof.** Since  $\vec{T}$  is a unit vector, we have that, for all  $t$

$$\vec{T}(t) \cdot \vec{T}(t) = 1$$

Take the time derivative, the  $(\vec{T} \cdot \vec{T}) = 0$ . We also have

$$\frac{d}{dt} (\vec{T} \cdot \vec{T}) = \frac{d\vec{T}}{dt} \cdot \vec{T} + \vec{T} \frac{d\vec{T}}{dt} = 2 \frac{d\vec{T}}{dt} \cdot \vec{T}$$

Since  $(\vec{T} \cdot \vec{T}) = 0$ , we get  $\frac{d\vec{T}}{dt} \cdot \vec{T} = 0$ . Dividing by  $\left| \frac{d\vec{T}}{dt} \right|$ , we get  $\vec{N}(t) \cdot \vec{T}(t) = 0$

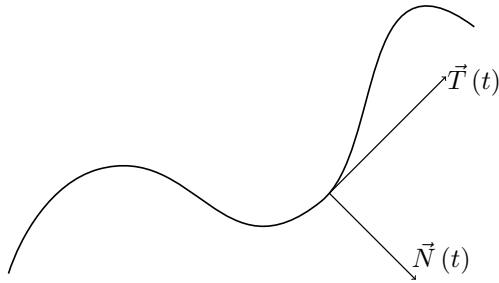
□

## 9 Lecture 9

### 9.1 Normal Vectors - Continued

Yesterday we saw that

$$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left| \frac{d\vec{T}}{dt} \right|}$$



**Example.** Consider the circle

$$\vec{r}(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$$

The velocity is

$$\vec{v}(t) = \vec{r}' = (-\sin t, \cos t)$$

We have  $|\vec{v}(t)| = 1$ , since

$$\vec{v}(t) \cdot \vec{v}(t) = (\sin^2 t + \cos^2 t) = 1$$

We find that  $\vec{T}(t) = \vec{v}(t)$ . To find  $\vec{N}$ , we need first

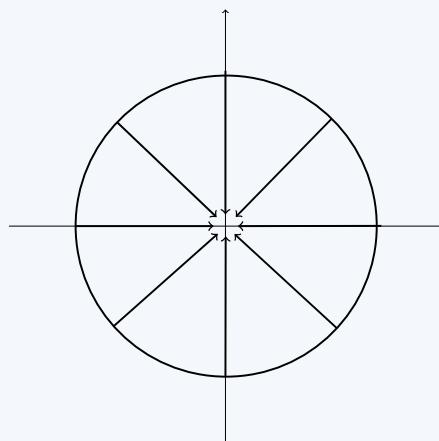
$$\frac{d\vec{T}}{dt} = \frac{d\vec{v}}{dt} = (-\cos t, -\sin t)$$

We check that  $\left| \frac{d\vec{T}}{dt} \right| = 1$ , then

$$\vec{N}(t) = (-\cos t, -\sin t) = -\vec{r}(t)$$

We compute more explicitly:

$$\vec{N}(t) \cdot \vec{T}(t) = (-\cos t, -\sin t) \cdot (-\sin t, \cos t) = \cos t \cdot \sin t - \sin t \cdot \cos t = 0$$



◊

We revisit the implicit case  $f(x, y) = 0$ .

**Proposition 5.** Let  $C$  be defined by  $f(x, y) = 0$ . A unit normal to  $C$  is given by  $\vec{n} = \frac{\nabla f}{|\nabla f|}$

**Proof.** Suppose  $C$  is parametrized by

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

By definition of  $C$ , we have

$$f(\vec{r}(t)) = f(x(t), y(t)), \quad t_0 \leq t \leq t_1$$

We have that  $\frac{d}{dt} f(\vec{r}(t)) = 0$ , but using the chain rule, we get

$$\frac{df(\vec{r}(t))}{dt} = \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

Hence, we get

$$\nabla f(\vec{r}(t)) \cdot \vec{v}(t) = 0$$

Therefore  $\nabla f(\vec{r}(t))$  is normal to the curve  $C$ , or orthogonal to the tangent  $\vec{v}(t)$

□

**Example.** The circle of radius 1 can be described implicitly by  $f(x, y) = 0$  with

$$f(x, y) = x^2 + y^2 - 1$$

The gradient is  $\nabla f(x, y) = (2x, 2y)$ , not of length 1, since

$$|\nabla f(x, y)| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2}$$

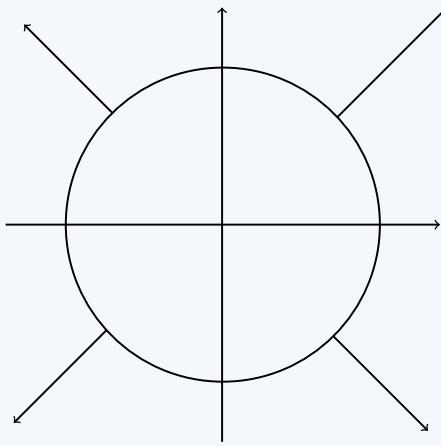
We are only interested in  $(x, y)$  such that  $f(x, y) = 0$ , that is  $x^2 + y^2 = 1$ , then

$$|\nabla f| = 2$$

Hence we get

$$\vec{n} \frac{\nabla f}{|\nabla f|} = \frac{(2x, 2y)}{2} = (x, y)$$

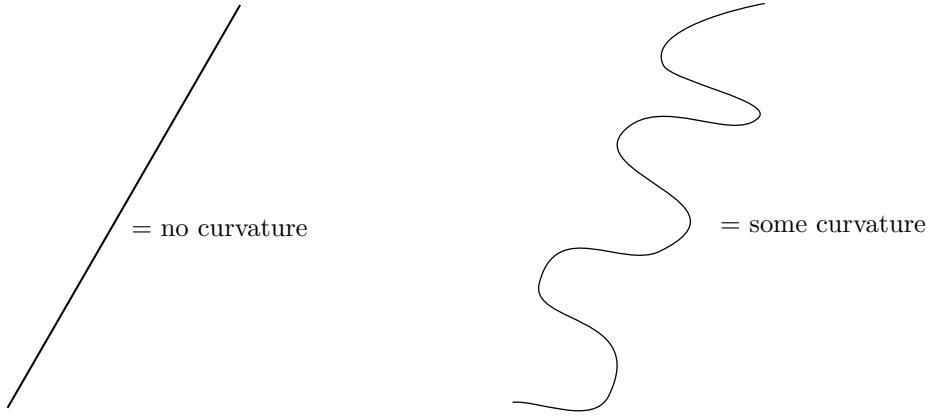
The normal vectors point outside of the circle, note:  $\vec{n} = -\vec{N}$ , compare with the parameter.



◊

## 9.2 Curvature

We want to compute how much a curve "curves".



How do we quantify this?

**Definition 18.** Let  $C$  be a parametrized curve, and  $\vec{T}$  its unit tangent vector. The curvature is then defined by

$$K = \left| \frac{d\vec{T}}{dS} \right|$$

Where  $S$  is the arc length parameter of  $C$ . We also define the radius of curvature as

$$\rho = \frac{1}{K}$$

Note: We consider  $S$ , not  $t$ . Reason:  $K$  does not depend on parametrization.

**Example.** Consider the following

$$\vec{r}(t) = (0, 0), \quad 0 \leq t \leq 1$$

We expect  $K$  to be zero. We can easily compute

$$S(t) = \int_0^t \sqrt{1^2 - 0^2} dt = t$$

Here we have  $S(t) = t$ . Now we compute  $\vec{T}$ . We have  $\vec{v}(t) = (1, 0)$ , and  $|\vec{v}(t)| = 1$ . Then  $\vec{T}(t) = \vec{v}(t) = (1, 0)$ . Furthermore, we get that

$$\frac{d\vec{T}}{dS} = 0 \quad \wedge \quad K = 0$$

◇

**Example.** Consider a circle of radius  $R$  with

$$\vec{r}(t) = (R \cos t, R \sin t), \quad 0 \leq t \leq 2\pi$$

The velocity is

$$\vec{v}(t) = (-R \sin t, R \cos t)$$

We get that  $|\vec{v}(t)| = R$ , then

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = (-\sin t, \cos t)$$

To compute  $K$ , we need to express  $\vec{T}$  in terms of  $\vec{S}(t)$ . We saw that

$$S(t) = R \cdot t \Rightarrow t = \frac{S}{R}$$

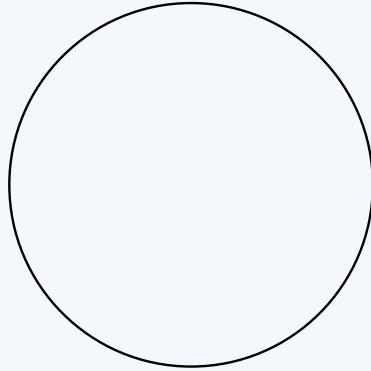
Now we compute

$$\frac{d\vec{T}}{dS} = \left( -\frac{1}{R} \cos \frac{S}{R}, -\frac{1}{R} \sin \frac{S}{R} \right)$$

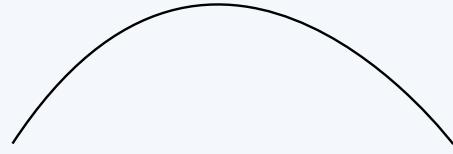
Finally, we have

$$K = \left| \frac{d\vec{T}}{dS} \right| = \sqrt{\frac{1}{R^2} \left( \cos \frac{S}{R} \right)^2 + \frac{1}{R^2} \left( \sin \frac{S}{R} \right)^2} = \frac{1}{R}$$

We have non-zero, constant curvature, we also have that  $\rho = R$ . Note that  $K \rightarrow 0$ , as  $r \rightarrow \infty$



Large Curvature



Small Curvature

◇

**Proposition 6.** We have that

$$K = \frac{1}{|\vec{v}|} = \left| \frac{d\vec{T}}{dt} \right|$$

**Example.** Consider again

$$\vec{r}(t) = (R \cos t, R \sin t)$$

We have seen that

$$|\vec{v}(t)|, \vec{T}(t) = (-\sin t, \cos t)$$

To get  $K$ , we compute

$$\frac{d\vec{T}}{dt} = (-\cos t, -\sin t)$$

◇

## 10 Lecture 10

### 10.1 Curves in polar form

Polar coordinates are an alternative description to cartesian coordinates  $(x, y)$ .

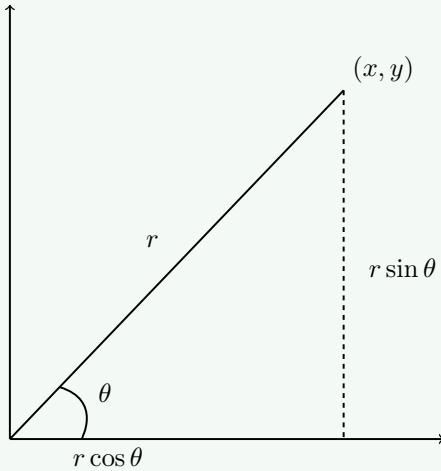
**Definition 19.** The polar coordinates  $r, \theta$  are defined by

$$x = r \cos \theta, \quad y = r \sin \theta$$

Their range is respectively

$$0 \leq r \leq \infty, \quad 0 \leq \theta \leq 2\pi$$

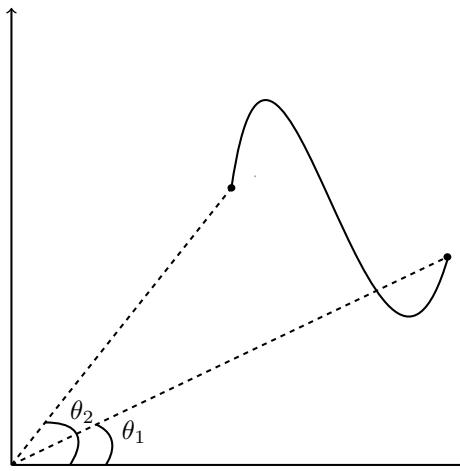
Geometrical meaning is that  $r$  is the distance from the origin, and  $\theta$  is the angle.



From  $(x, y)$  to  $(r, \theta)$ , we can use

$$r^2 = x^2 + y^2$$

We can describe curves using  $(r, \theta)$ . The idea is to give  $r$  as a function of  $\theta$ . The curve will be "traced" as we vary  $\theta$ . It is an analogue of  $y = f(x)$ .



**Example.** Consider the curve

$$r(\theta) = 1, \quad 0 \leq \theta \leq 2\pi$$

What curve is it? All points have distance 1 from origin ( $r = 1$ )

Using  $r^2 = x^2 + y^2$ , we find that  $x^2 + y^2 = 1$ . We have a circle of radius 1.  $\diamond$

**Example.** The next curve is called the cardioid, it is defined by

$$r(\theta) = 1 - \cos \theta, \quad 0 \leq \theta \leq 2\pi$$

It is described in cartesian coordinates by

$$(x^2 + y^2 + x)^2 = x^2 + y^2$$

Polar coordinates work best in the presence of spherical symmetry. The length of  $C$  can be computed using polar coordinates.  $\diamond$

**Proposition 7.** Let  $C$  be given in polar form by

$$r = r(\theta), \quad \alpha \leq \theta \leq \beta$$

Then its arc length can be computed by

$$S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 d\theta}$$

**Proof.** Parametrize  $C$  by

$$\vec{r}(\theta) = (x(\theta), y(\theta)), \quad \alpha \leq \theta \leq \beta$$

where we set

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta$$

The derivates are, with  $r' = \frac{dr}{d\theta}$

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta$$

After some computation we get

$$(x')^2 + (y')^2 = r^2 + \left( \frac{dr}{d\theta} \right)^2$$

Therefore we get

$$\begin{aligned} S &= \int_{\alpha}^{\beta} \sqrt{(x')^2 + (y')^2} d\theta \\ &= \int_{\alpha}^{\beta} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} d\theta \end{aligned}$$

□

**Example.** We have a circle given by

$$r(\theta) = R, \quad 0 \leq \theta \leq 2\pi$$

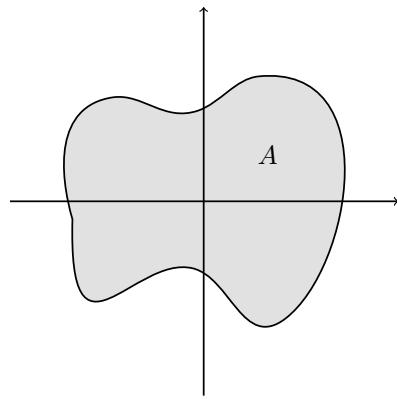
We have  $\frac{dr}{d\theta} = 0$ , then

$$S = \int_0^{2\pi} \sqrt{R^2 + 0^2} d\theta = R \int_0^{2\pi} d\theta = 2\pi R$$

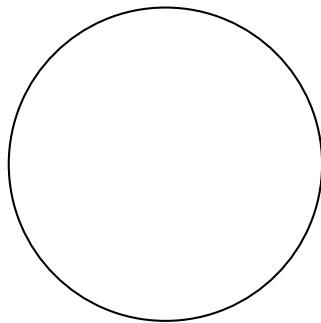
This gives us the circumference of the circle. ◇

## 10.2 Areas In Polar Form

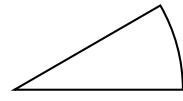
We want to compute the area inside a closed curve in polar form.



Basic observation:



Has area  $\pi r^2$



Has area  $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}r^2\theta$

Add small regions with angle  $\Delta\theta$  and area  $\frac{1}{2}r^2\Delta\theta$ .

**Proposition 8.** Consider a closed curve described by

$$r = r(\theta), \quad \alpha \leq \theta \leq \beta$$

Then its area is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r(\theta)^2 d\theta$$

**Example.** We have a circle of radius  $R$

$$r(\theta) = R, \quad 0 \leq \theta \leq 2\pi$$

we get

$$A = \frac{1}{2} \int_0^{2\pi} R^2 d\theta = \frac{1}{2} R^2 \int_0^{2\pi} d\theta = \pi R^2$$

◊

**Example.** Consider the cardioid

$$r(\theta) = 1 - \cos \theta, \quad 0 \leq \theta \leq 2\pi$$

The area is given by:

$$\begin{aligned} A &= \int_0^{2\pi} (1 - \cos \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (1 - 2\cos \theta + (\cos \theta)^2) d\theta \end{aligned}$$

To compute this, we use

$$\int \cos \theta d\theta = \sin \theta + C, \quad \int (\cos \theta)^2 d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin \theta \cos \theta + C$$

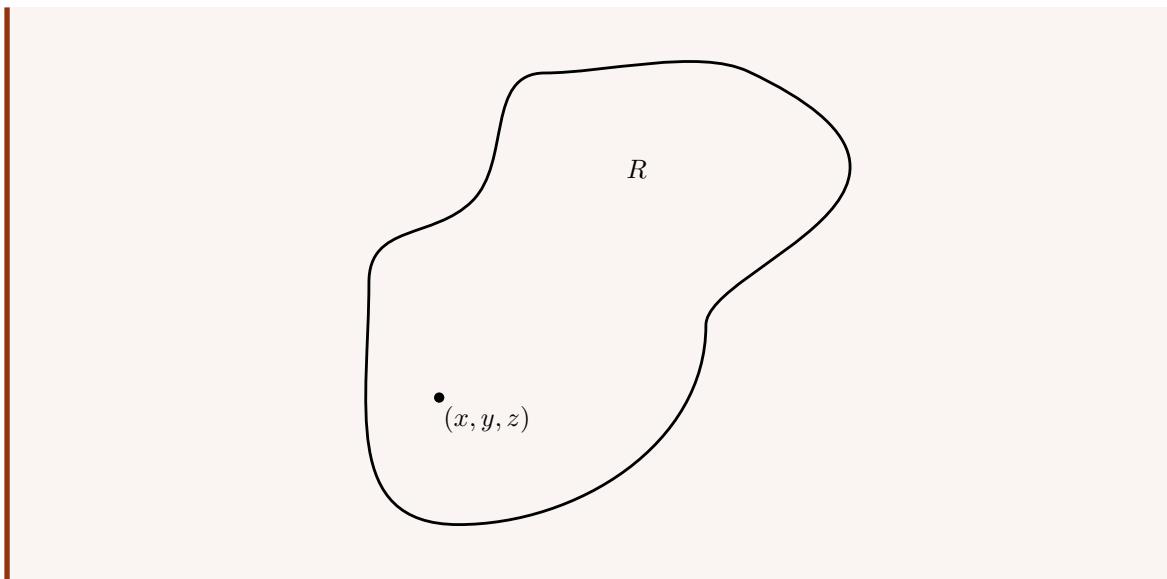
Finally, we obtain

$$A = \frac{1}{2} \cdot 2\pi + 0 + \frac{1}{2} \cdot \frac{1}{2} 2\pi = \frac{3}{2}\pi$$

◊

## 11 Scalar and Vector Fields

**Idea 1.** A field describes a property of a region  $R$



Mathematically described by

$$f : \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad m \text{ inputs, } n \text{ outputs}$$

For scalar fields the output is a scalar. For vector fields the output is a vector.

**Example.** The temperature is a scalar field.

$$T : (x, y, z) \rightarrow T(x, y, z)$$

The wind velocity is a vector field

$$\vec{W} : (x, y, z) \rightarrow \vec{W}(x, y, z)$$

◇

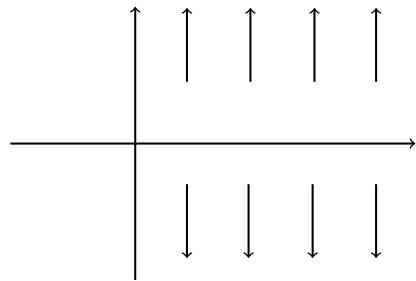
**Notation.** The notation for vector fields:

$$\begin{aligned} \vec{F}(x, y, z) &= (P(x, y, z), Q(x, y, z), R(x, y, z)) \\ &= P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k} \end{aligned}$$

$P, Q, R$  are component functions.

In 2D we visualize vector fields by vector plots. For instance, take

$$\vec{F}(x, y) = (0, y)$$



A vector field  $\vec{F}$  and a scalar field  $f$  can be related as follows

**Definition 20.** If  $\vec{F} = \nabla f$ , we say that  $\vec{F}$  is a gradient field, and  $f$  is a potential.

## 11.1 Gradient, Divergence and Curl

These are operations defined in terms of the formal vector

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$$

**Definition 21.** The gradient of a scalar field  $f$  is

$$\text{grad } f = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

The output is a vector.

The divergence of a vector field  $\vec{F}(P, Q, R)$  is

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The output is a scalar field.

The curl of a vector field  $\vec{F}$  is

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \right|$$

These operations can all be obtained from  $\nabla$ .

Operation	Input	Output	Symbol
Gradient	Scalar	Vector	$\nabla f$
Divergence	Vector	Scalar	$\nabla \cdot f$
Curl	Vector	Vector	$\nabla \times f$

## 12 Lecture 11

### 12.1 Operations on fields

**Example.** Consider  $\vec{F}(x, y, z) = (x^2, y^2, z^2)$ , that is

$$P = x^2, \quad Q = y^2, \quad R = z^2$$

Its divergence is

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2x + 2y + 2z$$

We can check that  $\vec{F}$  is a gradient field. We have that  $\vec{F} = \nabla f$ , with the potential

$$f(x, y, z) = \frac{1}{3} (x^2 + y^2 + z^2)$$

◊

$\nabla \times \vec{F}$  is computed as a determinant. We can use the cofactor, or the Laplace expansion.

$$\begin{aligned} \nabla \times \vec{F} &= \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \right| = \begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{bmatrix} \vec{i} - \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{bmatrix} \vec{j} + \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{bmatrix} \vec{k} \\ &= \vec{i} (\partial_y R - \partial_z Q) - \vec{j} (\partial_x R - \partial_z P) + \vec{k} (\partial_x Q - \partial_y P) \end{aligned}$$

This is a concrete formula for  $\nabla \times \vec{F}$ .

**Example.** Consider  $\vec{F} = xy\vec{i} + (x+z)\vec{j} + yz\vec{k}$ . We compute

$$\nabla \times \vec{F} = \vec{i} (\partial_y R - \partial_z Q) - \vec{j} (\partial_x R - \partial_z P) + \vec{k} (\partial_x Q - \partial_y P)$$

where

$$P = xy, \quad Q = x + z, \quad R = yz$$

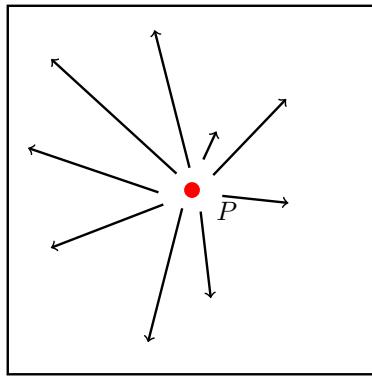
$$\nabla \times \vec{F} = (2 - 1) \vec{i} + (1 - x) \vec{k}$$

◊

**Note.** Gradient and divergence can be defined in any dimension. The curl is only defined in up to three dimensions.

### 12.2 Interpretation of Divergence

Think of  $\vec{F}$  as the velocity of a fluid.  $\nabla \cdot \vec{F}$  at a point  $P$  is the amount of fluid entering / leaving a small region around  $P$ .



**Example.** Consider  $\vec{F}_1(x, y) = (x, y)$ , then

$$\nabla \cdot \vec{F}_1 = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + 2$$

This corresponds to fluid leaving the region. Similarly for  $\vec{F}_2 = (-x, -y)$ , then

$$\nabla \cdot \vec{F}_2 = \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -2$$

This corresponds to fluid entering the region. Finally consider  $\vec{F}_3 = (0, 1)$ , then

$$\nabla \cdot \vec{F}_3 = 0$$

This is an equilibrium situation. ◊

### 12.3 Interpretation of Curl

$\nabla \times \vec{F}$  measures the "rotation" of  $\vec{F}$ .

**Example.** Consider  $\vec{F}(x, y, z) = (x^2, 0, 0)$ . We expect no rotation, we compute

$$\nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 0 & 0 \end{bmatrix} \right| = 0$$



◊

**Example.** Consider  $\vec{F}(x, y, z) = (-\omega y, \omega x, 0)$ , where  $\omega$  is a non-zero constant. We compute

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -\omega y & \omega x \end{bmatrix} \vec{k} = 2\omega \vec{k}$$

$\nabla \times \vec{F} \neq 0$  gives non-zero rotation. Also  $\nabla \times \vec{F} = \text{const}$  gives that there is some rotation everywhere.

For a physical interpretation of this, we can write

$$\vec{v} = \vec{F}, \quad \vec{\omega} = (0, 0, \omega), \quad \vec{r} = (x, y, z)$$

Then we can check that  $\vec{v} = \vec{\omega} \cdot \vec{r}$ . This is the velocity corresponding to the angular velocity  $\vec{\omega}$

◊

## 12.4 Scalar Field from Gradient

Suppose that we know  $\nabla f$ . Can we recover  $f$ ? Yes, up to the initial conditions. The strategy is the following, first we write

$$\vec{F} = \nabla f = (P, Q, R)$$

By the definition of the gradient field, we have

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

Comparing, we get

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q, \quad , \quad \frac{\partial f}{\partial z} = R$$

**Example.** Suppose we are given the gradient field

$$\vec{F} = \nabla f = \left( 2x + 2y, \frac{1}{2}x^2 + 3y \right)$$

We want to find  $f$ , we must have

$$\frac{\partial f}{\partial x} = 2x + 2y, \quad \frac{\partial f}{\partial y} = \frac{1}{2}x^2 + 3y$$

Integrating the first equation in  $x$  gives us

$$\begin{aligned} f(x, y) &= \int (2x + xy) dx \\ &= x^2 + \frac{1}{2}x^2y + g(y) \end{aligned}$$

Where  $g(y)$  is the integration constant. It can depend on  $y$ .  
Now we compute  $\frac{\partial f}{\partial y}$ . We get

$$\frac{\partial f}{\partial y} = \frac{1}{2}x^2 + \frac{dg}{dy}$$

But we also have that  $\frac{\partial f}{\partial y} = \frac{1}{2}x^2 + 3y$ .  
Comparing them, we get

$$\frac{1}{2}x^2 + \frac{dg}{dy} = \frac{1}{2}x^2 + 3y$$

Then  $\frac{dg}{dy} = 3y$ . Integrating in  $y$  we get

$$g(y) = \int 3y dy = \frac{3}{2}y^3 + C$$

Here,  $C$  is a constant. Finally, inserting it, we get

$$\begin{aligned} f(x, y) &= x^2 + \frac{1}{2}x^2y + g(k) \\ &= x^2 + \frac{1}{2}x^2y + \frac{3}{2}y^3 + C \end{aligned}$$

The constant  $C$  is usually not important. It can be fixed by an initial condition. For instance  $f(0, 0)$  implies that  $C = 0$

◊

## 13 Lecture 12

### 13.1 Identities Between Operations

We have seen three operations defined by  $\nabla$ .

Gradient:  $\nabla f$ , Divergence:  $\nabla \cdot \vec{F}$ , Curl:  $\nabla \times \vec{F}$

There are many identities, we'll now look at one.

**Proposition 9.** For any scalar field  $f$ , we have

$$\nabla \times (\nabla f) = 0$$

**Proof.** We have  $\nabla f = (f_x, f_y, f_z)$ , then

$$\nabla \times (\nabla f) = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix} \right| = (f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{xy})$$

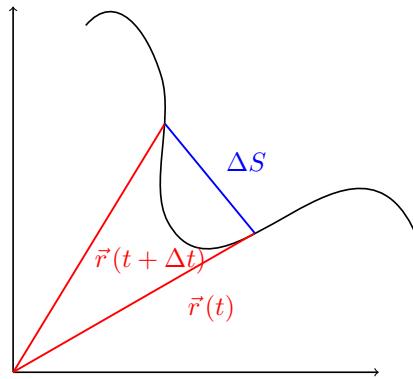
But partial derivatives can be exchanged. Then we find that  $\nabla \times (\nabla f) = 0$  □

We are going to use this when we discuss conservative fields.

### 13.2 Line Integrals

Consider the curve  $C$  with

$$\vec{r}(t) = (x(t), y(t)), t_0 \leq t \leq t_1$$



$$\Delta x = x(t + \Delta t) - x(t), \quad \Delta y = y(t + \Delta t) - y(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x^2) + (\Delta y)^2}$$

When  $\Delta t$  is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt} \Delta t, \quad \Delta y \approx \frac{dy}{dt} \Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt} \Delta t^2\right) + \left(\frac{dy}{dt} \Delta t^2\right)} = \Delta t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Letting  $\Delta t \rightarrow 0$  leads to the following

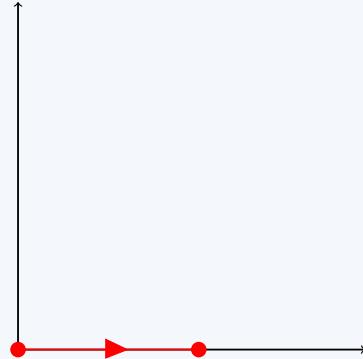
**Definition 22.** The line integral of  $f(x, y)$  along a curve  $C$  is defined by

$$\int_C f ds = \int_{t_0}^{t_1} f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

**Observe.** Note that  $f$  is restricted to  $\vec{r}(t) = (x(t), y(t))$ . When  $f = 1$ , we recover the arc length.

**Example.** Consider  $C$  defined by

$$x(t) = t, \quad y(t) = 0, \quad 0 \leq t \leq 1$$



First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} dt = 1 dt$$

Now, consider  $f(x, y) = x^2 + y$ . We want to compute  $\int_C f ds$ . Restricting  $f$  to  $C$  gives

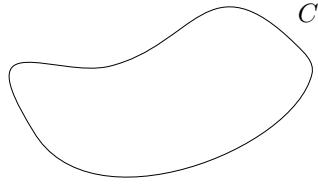
$$f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f ds = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

◊

**Note.** Line integrals can be used to compute the mass of a 1-dimensional object. The curve  $C$  describes the object, and the function  $\int_C f ds$  is the mass.



### 13.3 Parametrization and Orientation

The next result is as for the arc length.

**Proposition 10.** The integral  $\int_C f ds$  does not depend on the parametrization of  $C$ .

We will consider a special case

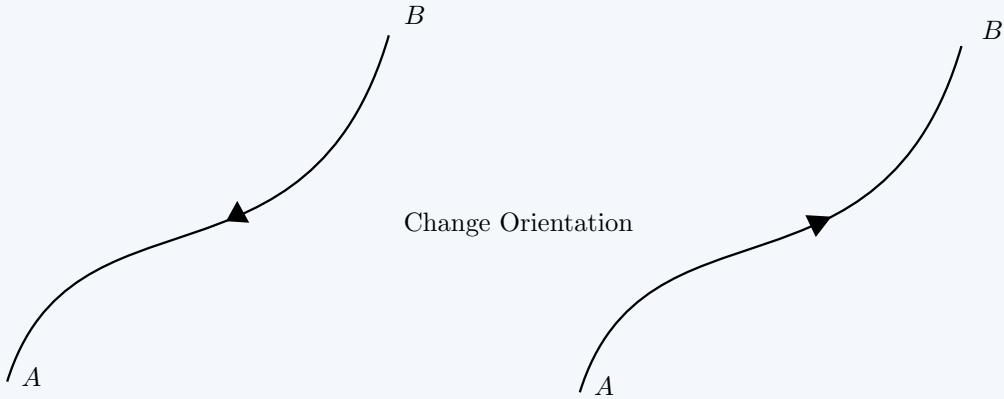
**Example.** Consider  $C$  with

$$\vec{r}(t) = (x(t), y(t)), t_0 \leq t \leq t_1$$

We have

$$\text{Start: } A = (x(t_0), y(t_0)), \quad \text{End: } B = (x(t_1), y(t_1))$$

We want to go from  $B$  to  $A$ .



We can do this in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\begin{aligned}\vec{r}_{\text{opp}}(-t_1) &= (x(t_1), y(t_1)) = B \\ \vec{r}_{\text{opp}}(-t_0) &= (x(t_0), y(t_0)) = A\end{aligned}$$

◇

**Example.** Consider the segment  $C$  from  $(0, 0)$  to  $(1, 0)$ . Take  $f(x, y) = x$ . Show that  $\int_C f ds = \frac{1}{2}$  using  $\vec{r}(t)$  and  $\vec{r}_{\text{opp}}(t)$ .

If  $C$  parametrized by  $\vec{r}(t)$ , we use  $-C$  when considering  $\vec{r}_{\text{opp}}(t)$ . We have

$$\int_C f ds = \int_{-C} f ds$$

The situation will be different for vector fields. ◇

### 13.4 Case of Vector Fields

Consider the curve with

$$\begin{aligned}\vec{r}(t) &= (x(t), y(t)), \quad t_0 \leq t \leq t_1 \\ \vec{r}'(t) &= (x'(t), y'(t)) \quad (\text{Velocity vector})\end{aligned}$$

**Definition 23.** The line integral of  $\vec{F}$  along  $C$  is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}' dt$$

Here  $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$

In physics, we have that  $\vec{F}$  is the force, and  $\int_C \vec{F} d\vec{r}$  is the work done by  $\vec{F}$  along  $C$ . The elementary case is given by  $W = FS$ , or work = force  $\cdot$  displacement. More explicitly, we consider

$$\vec{F}(x, y) = (P(x, y), Q(x, y))$$

Then we have

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

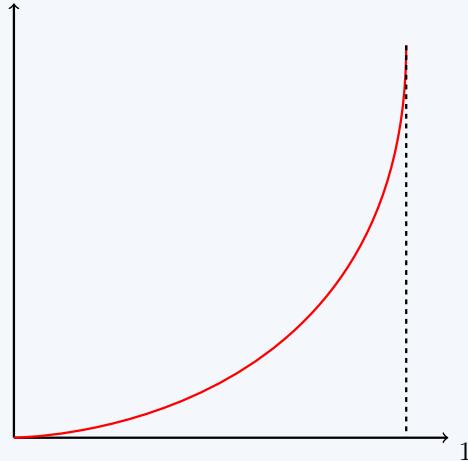
$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t) dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t) dt$$

We will write the expression as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + \int_C Q dy$$

**Example.** Consider the curve  $C$  with

$$\vec{r}(t) = (t, t^2), \quad 0 \leq t \leq 1$$



We have  $x(t) = t$  and  $y(t) = t^2$ , its derivative is

$$\vec{r}(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is  $P = x + y$  and  $Q = x$ , when this is restricted to  $C$ , we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = (t + t^2, t)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}' = (t + t^2, t) \cdot (1, 2t) = (3t^2 + t)$$

We finally obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + t) dt = \frac{3}{2}$$

◇

## 14 Line Integrals, Parametrization and Vector Fields

### 14.1 Identities between operations

We have seen three operations defined by  $\nabla$ .

Gradient:  $\nabla f$

Divergence:  $\nabla \cdot \vec{F}$

Curl:  $\nabla \times \vec{F}$

There are many of them, we look at only one.

**Proposition 11.** For any scalar field  $f$  we have

$$\nabla \times (\nabla f) = 0$$

**Proof.** We have  $\nabla f = (f_x, f_y, f_z)$ . Then

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \vec{i}(f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{xy})$$

But partial derivatives can be exchanged,  $f_{xy} = f_{yx}$ . Then we find that

$$\nabla \times (\nabla f) = 0$$

□

We are going to use this when we discuss conservative fields.

### 14.2 Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Consider the points  $\vec{r}(t)$  and  $\vec{r}(t + \Delta t)$  where  $\Delta t$  is small.

$$\Delta x = x(t + \Delta t) - x(t), \quad \Delta y = y(t + \Delta t) - y(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

When  $\Delta t$  is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt} \Delta t, \quad \Delta y \approx \frac{dy}{dt} \Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt} \Delta t\right)^2 + \left(\frac{dy}{dt} \Delta t\right)^2} = \Delta t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

**Definition 24.** The line integral of  $f(x, y)$  along a curve  $C$  is defined by

$$\int_C f dS = \int_{t_0}^{t_1} f(x(y), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Some observations:

- Note that  $f$  is restricted to  $\vec{r}(t) = (x(t), y(t))$ .
- When  $f = 1$ , we recover the arc length.

**Example.** Consider  $C$  defined by

$$x(t), y(t) = 0, \quad 0 \leq t \leq 1$$

First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} = 1$$

◊

Now, consider  $f(x, y) = x^2 + y$ . Compute  $\int_C f dS$ . Restricting  $f$  to  $C$  gives

$$f(x(t), y(t)) = x(t)^2 + y(t) = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f dS = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

- Line integrals can be used to compute the mass of an object (1-dimensional)
- The curve  $C$  describes the object
- The function  $\int_C f dS$  is the mass.

### 14.3 Parametrization and orientation

Next results as for the arc length.

**Proposition 12.** The integral  $\int_C f dS$  does not depend on the parametrization of  $C$ .

We will consider a special case.

**Example.** Consider  $C$  with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\text{Start: } A = (x(t_0), y(t_0)), \quad \text{End: } B = (x(t_1), y(t_1))$$

We want to go from  $B$  to  $A$ , which can be done in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\vec{r}_{\text{opp}}(-t_1) = (x(t_1), y(t_1)) = B$$

$$\vec{r}_{\text{opp}}(-t_0) = (x(t_0), y(t_0)) = A$$

◊

**Example.** Consider  $C$  = segment from  $(0,0)$ , to  $(1,0)$ , take  $f(x, y) = x$ . Show that  $\int_C f dS = \frac{1}{2}$  using  $\vec{r}(t)$  and  $\vec{r}_{\text{opp}}(t)$ .

◊

If  $C$  parametrized by  $\vec{r}(t)$ , we will use  $-C$  when considering  $\vec{r}_{\text{opp}}(t)$ . We have that

$$\int_C f dS = \int_{-C} f dS$$

The situation will be different for vector fields.

#### 14.4 Case of Vector Fields

Consider the curve with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\vec{r}'(t) = (x'(t), y'(t)) \quad \text{Velocity Vector}$$

**Definition 25.** The line integral of  $\vec{F}$  along  $C$  is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

$$\text{Here } \vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$$

- In physics, we have that  $\vec{F}$  = Force and  $\int_C \vec{F} d\vec{r}$  = Work done by  $\vec{F}$  along  $C$ .

- The elementary case  $W = FS$ , where  $W$  is work,  $F$  is Force and  $S$  is the displacement.

More explicitly, we consider

$$\vec{F}(x, y) = (P(x, y), Q(x, y))$$

Then we have that

$$\vec{F}(\vec{r}(t) \cdot \vec{r}'(t)) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t) dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t) dt$$

We will write the expression as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + \int_C Q dy$$

**Example.** Consider a curve  $C$  with

$$\vec{r}(t) = (t, t^2), \quad 0 \leq t \leq 1$$

We have  $x(t) = t$  and  $y(t) = t^2$ , its derivative is

$$\vec{r}'(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is  $P = x + y$  and  $Q = x$ . When restricted to  $C$ , we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = (t + t^2, t)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = (t + t^2, t) \cdot (1, 2t) = (3t^2 + t)$$

We then obtain the following

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + t) dt = \frac{3}{2}$$

◇