1 Lecture 10

1.1 Curves in polar form

Polar coordinates are an alternative description to cartesian coordinates (x, y).

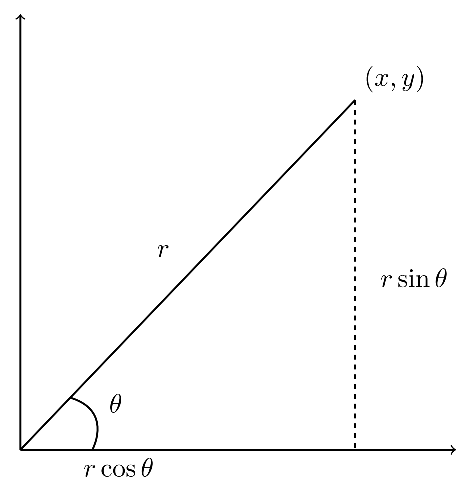
Definition 1. The polar coordinates r, θ are defined by

$$x = r\cos\theta, \quad y = r\sin\theta$$

Their range is respectively

$$0 \leq r \leq \infty, \quad 0 \leq \theta \leq 2\pi$$

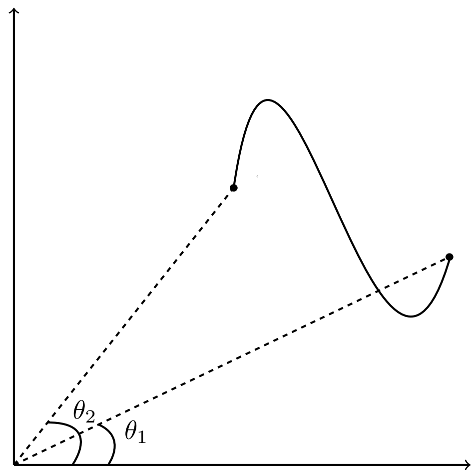
Geometrical meaning is that r is the distance from the origin, and θ is the angle.



From (x, y) to (r, θ), we can use

$$r^2 = x^2 + y^2$$

We can describe curves using (r, θ). The idea is to give r as a function of θ. The curve will be "traced" as we vary θ. It is an analogue of $y = f(x)$.



Example. Consider the curve

$$r(\theta) = 1, \quad 0 \leq \theta \leq 2\pi$$

What curve is it? All points have distance 1 from origin ($r = 1$)

Using $r^2 = x^2 + y^2$, we find that $x^2 + y^2 = 1$. We have a circle of radius 1. ◇

Example. The next curve is called the cardioid, it is defined by

$$r(\theta) = 1 - \cos\theta, \quad 0 \leq \theta \leq 2\pi$$

It is described in cartesian coordinates by

$$\left(x^2 + y^2 + x\right)^2 = x^2 + y^2$$

Polar coordinates work best in the prescence of spherical symmetry. The length of C can be computed using polar coordinates. ◇

Proposition. Let C be given in polar form by

$$r = r(\theta), \quad \alpha \leq \theta \leq \beta$$

Then its arc length can be computed by

$$S = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 d\theta}$$

Proof. Parametrize C by

$$\vec{r}(\theta) = (x(\theta), y(\theta)), \quad \alpha \leq \theta \leq \beta$$

where we set

$$x(\theta) = r(\theta)\cos\theta, \quad y(\theta) = r(\theta)\sin\theta$$

The derivates are, with $r' = \frac{dr}{d\theta}$

$$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = r'\sin\theta + r\cos\theta$$

After some computation we get

$$(x')^2 + (y')^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

Therefore we get

$$\begin{aligned} S &= \int_\alpha^\beta \sqrt{(x')^2 + (y')^2} d\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \end{aligned}$$

□

Example. We have a circle given by

$$r(\theta) = R, \quad 0 \leq \theta \leq 2\pi$$

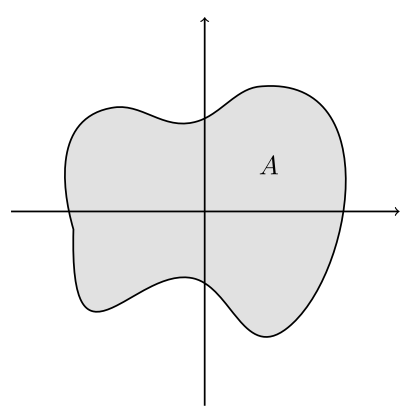
We have $\frac{dr}{d\theta} = 0$, then

$$S = \int_0^{2\pi} \sqrt{R^2 + 0^2} d\theta = R\int_0^{2\pi} d\theta = 2\pi R$$

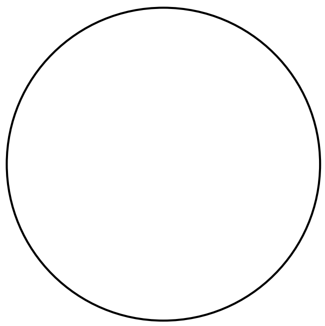
This gives us the circumference of the circle. ⋄

1.2 Areas In Polar Form

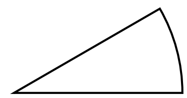
We want to compute the area inside a closed curve in polar form.



Basic observation:



Has area πr^2



Has area $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$

Add small regions with angle $\Delta\theta$ and area $\frac{1}{2} r^2 \Delta\theta$.

Proposition. Consider a closed curve described by

$$r = r(\theta), \quad \alpha \leq \theta \leq \beta$$

Then its area is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r(\theta)^2 \, d\theta$$

Example. We have a circle of radius R

$$r(\theta) = R, \quad 0 \leq \theta \leq 2\pi$$

we get

$$A = \frac{1}{2} \int_0^{2\pi} R^2 d\theta = \frac{1}{2} R^2 \int_0^{2\pi} d\theta = \pi R^2$$

◇

Example. Consider the cardioid

$$r(\theta) = 1 - \cos\theta, \quad 0 \leq \theta \leq 2\pi$$

The area is given by:

$$\begin{aligned} A &= \int_0^{2\pi} (1 - \cos\theta)^2 \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left(1 - 2\cos\theta + (\cos\theta)^2\right) d\theta \end{aligned}$$

To compute this, we use

$$\int \cos\theta d\theta = \sin\theta + C, \quad \int (\cos\theta)^2 \, d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

Finally, we obtain

$$A = \frac{1}{2}\cdot 2\pi + 0 + \frac{1}{2}\cdot\frac{1}{2}2\pi = \frac{3}{2}\pi$$

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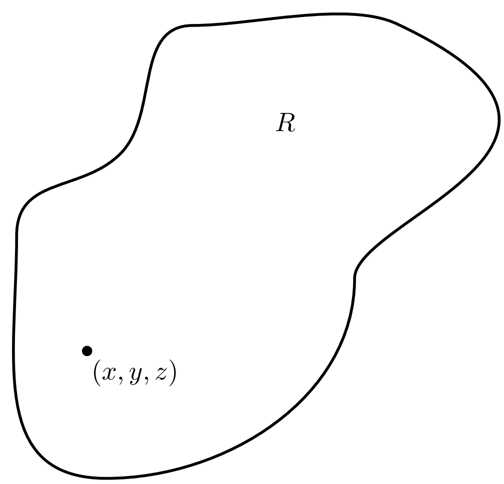
2 Scalar and Vector Fields

Idea. A field describes a property of a region R



Mathematically described by

$$f : \mathbb{R}^m \to \mathbb{R}^n, \quad m \text{ inputs}, n \text{ outputs}$$

For scalar fields the output is a scalar. For vector fields the output is a vector.

Example. The temperature is a scalar field.

$$T : (x, y, z) \to T(x, y, z)$$

The wind velocity is a vector field

$$\vec{W} : (x, y, z) \to \vec{W}(x, y, z)$$

◇

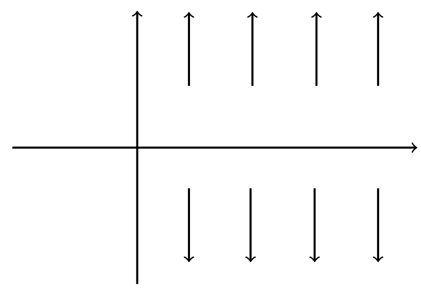
Notation. The notation for vector fields:

$$\begin{aligned} \vec{F}(x, y, z) &= (P(x, y, z), Q(x, y, z), R(x, y, z)) \\ &= P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k} \end{aligned}$$

P, Q, R are componente functions.

In 2D we visualize vector fields by vector plots. For instance, take

$$\vec{F}(x, y) = (0, y)$$



A vector field $\vec{F}$ and a scalar field f can be related as follows

Definition 2. If $\vec{F} = \Delta f$, we say that $\vec{F}$ is a gradient field, and f is a potential.

2.1 Gradient, Divergence and Curl

These are operation defined in terms of the formal vector

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Definition 3. The gradient of a scalar field f is

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

The output is a vector.

The divergence of a vector field $\vec{F}\,(P, Q, R)$ is

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The output is a scalar field.

The curl of a vector field $\vec{F}$ is

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{bmatrix} \right|$$

These operations can all be obtained from ∇.

Operation	Input	Output	Symbol
Gradient	Scalar	Vector	∇f
Divergence	Vector	Scalar	$\nabla \cdot f$
Curl	Vector	Vector	$\nabla \times f$