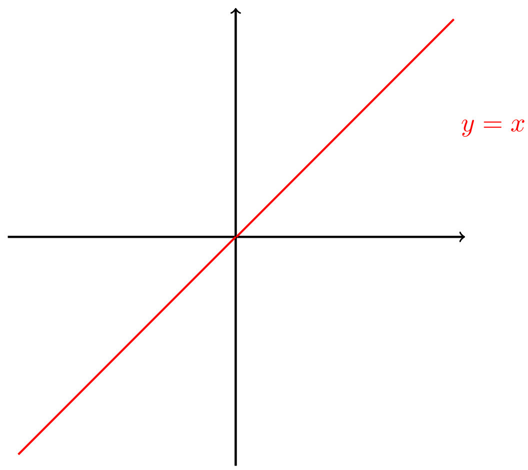
1 Lecture 7

1.1 Parametrized Curve

A curve is described as a sset of points in $\mathbb{R}^2$ and $\mathbb{R}^3$. For instance a line is described by

$$f = \left\{ (x, y) \in \mathbb{R}^2 : x = y \right\}$$



This is a static picture. But how do we give a dynamical picture? We'll use parametrized curves.

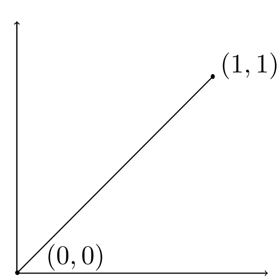
Definition 1. A parametrization of a curve c in $\mathbb{R}^2$, is given by

$$\vec{r}\,(t) = (x\,(t)\,, y\,(t))\,, \quad t_0 \leq t \leq t_1$$

Such that $\vec{r}\,(t) \in c$ for all time t.

- A parametrization describes motion. Think of t as the time.
- A parametrization is not unique.
- Various natural assumptions, such as continuity and differentiability.

Example. Consider the funtion $\vec{r}\,(t)$ with $0 \leq t \leq 1$.



$$\vec{r}\,(0) = (0,0)\,, \quad \vec{r}\,(1) = (1,1)$$

We have the portion of the line where $y = x$. Notce that here, $x\,(t) = t$, $y\,(t) = t$ and $y\,(t) = x\,(t) = t$ for all t.

Lets consider a different function,

$$\vec{r}\,(t) = (2t, 2t)\,, \quad 0 \leq t \leq \frac{1}{2}$$

When we parametrize, we get

$$\vec{r}\,(0) = (0,0)\,, \quad \vec{r}\left(\frac{1}{2}\right) = (1,1)$$

We are moving along the curve at twice the speed.

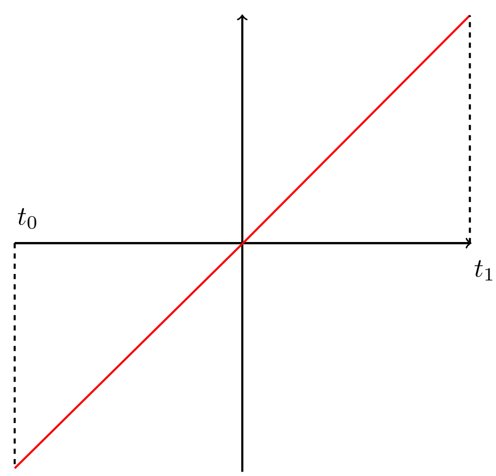
Example. Given $f(x)$, we consider

$$\vec{r}\,(t) = (t, f\,(t))\,, \quad t_0 \leq t \leq t_1$$

This describes a portion of the graph f, with

$$\text{Start: } (t_0, f\,(t_0))\,, \quad \text{End: } (t_1, f\,(t_1))$$

For instance, consider the line $y = mx + c$, we have



We have that

$$\vec{r}\,(t) = (t, mt + c)\,, \quad t_0 \leq t \leq t_1$$

Example. We want to describe a line with

$$\text{Start: } A = (x_0, y_0)\,, \quad \text{End: } B = (x_1, y_1)$$

Then we take the parametrization

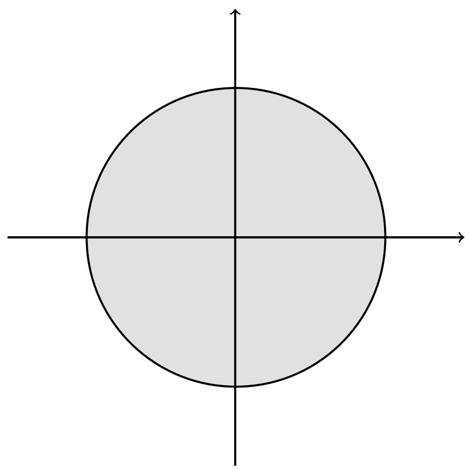
$$\vec{r}(t) = (1-t)A + tB, \quad 0 \le t \le 1$$

More explicitly, we have

$$\vec{r}(t) = ((1-t)x_0 + tx, (1-t)y_0 + ty)$$

Note that $\vec{r}(0) = A$ and $\vec{r}(1) = B$.

Example. Consider $\vec{r}(t) = (\cos t, \sin t), 0 \le t \le 2\pi$. What curve does this describe?



It describes a circle.

$$x(t)^2 + y(t)^2 = \left(\cos(t)^2 + \sin(t)^2\right) = 1$$

We start at $(1,0)$ and move counter-clockwise.

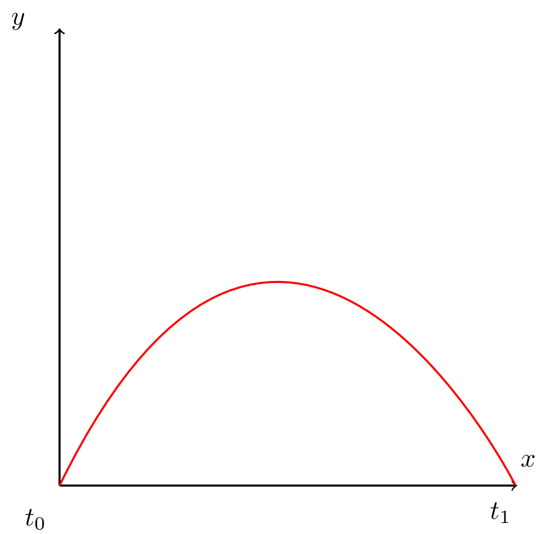
Example. Here is an example from physics. Consider

$$x(t) = v_x t, \quad y(t) = v_y t - \frac{1}{2}gt^2, \quad 0 \le t \le \frac{2v_y}{g}$$

This describes the motion of an object with initial velocity $\vec{v} = (v_x, v_y)$, under gravity. We write $t_0 = 0$ and $t_1 = \frac{2v_y}{g}$. Note that

$$\vec{r}(t_0) = (0,0), \quad \vec{r}(t_1) = \left(\frac{2v_x v_y}{g}, 0\right)$$

The object falls back to the ground at time t_1.



Well known fact: This motion is parabolic, we will rederive this.

From $x(t) = v_x t$, we get $t = \frac{x(t)}{v_x}$. Then

$$y(t) = v_y t - \frac{1}{2} g t^2 \Rightarrow \frac{v_x}{v_y} x(t) - \frac{1}{2} \frac{g}{v_x^2} x(t)^2$$

This is the expression of a parabola

$$y = ax^2 + bx + c, \quad a \neq 0$$

It can also be written as

$$y(t) = -\frac{1}{2} \frac{g}{v_x^2} \left(x(t) - \frac{v_y}{g} \right)^2 + \frac{1}{2} \frac{v_y}{g} \frac{v_y}{v_x}$$

1.2 Kinematics

Kinematics describes position, velocity and acceleration of an object.

Definition 2. The position vector is $\vec{r}(t)$. The velocity vector is $\vec{v}(t)\,\frac{d\vec{r}}{dt}$. The acceleration vector is $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2}$.

If we write $\vec{r}(t) = (x(t), y(t))$, then

$$\vec{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (x'(t), y(t))$$

Similarly

$$\vec{a}(t) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right) = (x''(t), y''(t))$$

Example. Consider again the gravity example. Here we have

$$\vec{r}\,(t) = \left(v_x t, v_y t - \frac{1}{2} g t^2\right)$$

The velocity is

$$\vec{v}\,(t) = (v_x, v_y - gt)$$

Note that $v\,(0) = (v_x, v_y)$ is the initial velocity of the object. For acceleration, we get

$$\vec{a}\,(t) = \frac{d\vec{v}}{dt} = (0, -g)$$