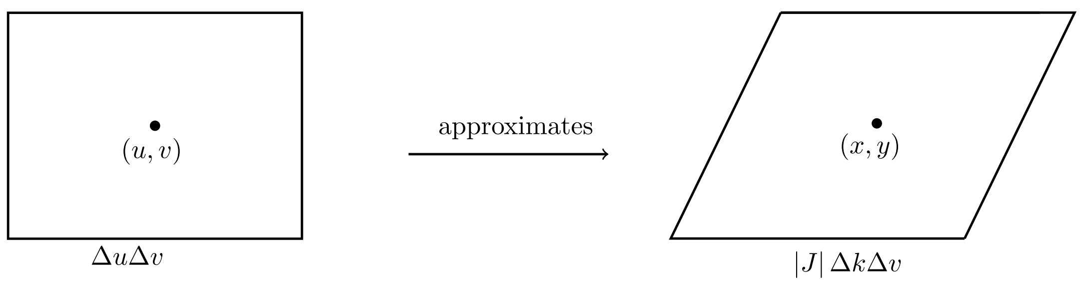
1 Lecture 16

1.1 Change of variables (cont.)

We have seen that the *Jacobian Determinant* J, and

$$\iint_R f dA = \iint_S f\left(x\left(u,v\right), y\left(u,v\right)\right)\left|J\left(u,v\right)\right| dudv$$

The Jacobian appears since

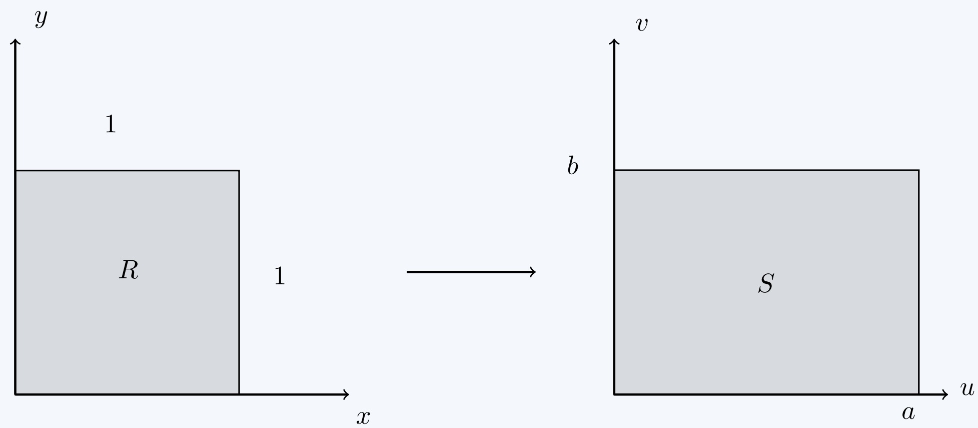


Example. Consider (u,v) defined by

$$u = ax, \quad v = by, a, b > 0$$

Now consider the square

$$R = [1,0] \times [0,1]$$



It becomes the rectangle $S = [0, a] \times [0, b]$. Let us also write

$$x = \frac{u}{a}, \quad y = \frac{v}{b}$$

Then we compute the Jacobian.

$$J = \left| \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \right| = \frac{1}{ab}$$

We have $\iint_R 1dA = 1$, using the theorem, we compute

$$\begin{aligned} \iint_R 1dA &= \iint_S |J(u, v)| \, dudv \\ &= \int_{u=0}^{a} \int_{v=0}^{b} \frac{1}{ab} dudv \\ &= 1 \end{aligned}$$

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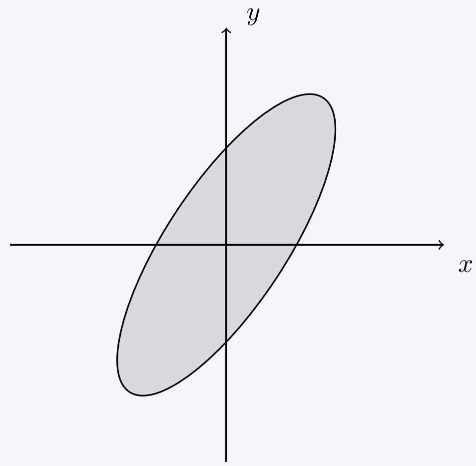
Example. Consider the elliptical region

$$R : x^2 - xy + y^2 \leq 2, \quad f(x, y) = x^2 - xy + y^2$$



We want to compute $\iint_R f dA$. Consider (u, v) defined by

$$x = \sqrt{2} - \sqrt{\frac{2}{3}}u, \quad y = \sqrt{2} + \sqrt{\frac{2}{3}}v$$

$$x^2 - xy + y^2 = 2u^2 + 2v^2$$

In (u, v) coordinates, we get

$$\begin{aligned} S &= \left\{(u, v) : 2u^2 + 2v^2 \leq 2\right\} \\ &= \left\{(u, v) : u^2 + v^2 \leq 1\right\} \end{aligned}$$

This equals a circle. By the theorem, we have that

$$\iint_R f dA = \iint_S f\left(x\left(u, v\right), y\left(u, v\right)\right) \left|J\left(u, v\right)\right| dudv$$

For f, we have

$$f\left(x, y\right) = x^2 - xy + y^2 = 2\left(u^2 + v^2\right)$$

The Jacobian is

$$J\left(u, v\right) = \left|\begin{bmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{bmatrix}\right| = \frac{2}{\sqrt{3}} - \left(-\frac{2}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}$$

$$\iint_R f dA = \iint_S 2\left(u^2 + v^2\right) \frac{4}{\sqrt{3}} dudv$$

Since S is a disc, we introduce

$$u = r\cos\theta, \quad v = r\sin\theta$$

The disc is then $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. We obtain

$$\begin{aligned}
\iint_R f dA &= \frac{8}{\sqrt{3} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r^2 \cdot r d\theta dr} \\
&= \frac{16\pi}{\sqrt{3}} \\
&= \int_{r=0}^{1} r^3 dr \\
&= \frac{4\pi}{\sqrt{3}}
\end{aligned}$$

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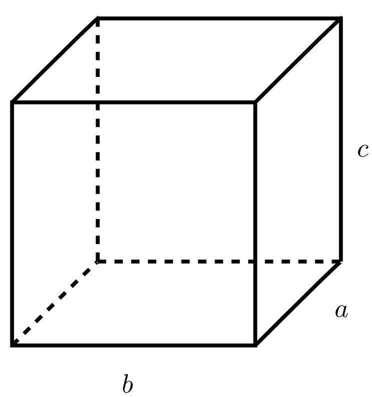
Note. The aboslute value in $|J(u,v)|$ is important!

1.2 Triple Integrals

When doing a triple integral, we proceed as with double integrals. The analogue of the rectangle are boxes. A box is a region of the following form

$$B = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$$



$a = x_1 - x_0$

$b = y_1 - y_0$

$c = z_1 - z_0$

Its volume is

$$\mathrm{vol}(B) = (x_1 - x_0)(y_1 - y_0)(z_1 - z_0)$$

We approximate a region with boxes B_i of volumes Δv_i. Pick a sample point $(x_i^\star, y_i^\star, z_i^\star)$ in each B_i. Given $f(x,y,z)$ we consider

$$\sum_i f\left(x_i^\star, y_i^\star, z_i^\star\right)\Delta_i$$

in the unit of the integral.

Definition 1. The triple integral of f over t is

$$\iiint_T f dV = \lim_{\Delta v_i \to 0} \sum_i f\left(x_i^\star, y_i^\star, z_i^\star\right)\Delta_i$$

When $f = 1$ we get the volume of T. In the case of $f > 0$, it is less intuitive. How do we interpret $\int_T f dV$? We want to think of f as a local density.

Example. Consider a 3D-object described by a 3D-region T, with $\rho(x, y, z)$ its mass density. Then its mass is given by

$$m = \iiint_T \rho dV$$

$\diamond$

Proposition. Linearity

If a and b are constants, then

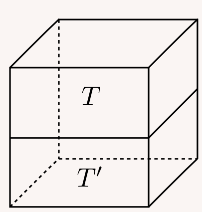
$$\iiint_T (af + bg)\, dV = a \iiint_T f dV + b \iiint_T g dV$$

Proposition. Partitions

Let T and T' be non-overlapping regions, then

$$\iiint_{T \cup T'} f dV = \iiint_T f dV + \iiint_{T'} f dV$$

v



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1.3 Integration Over Boxes

Boxes are usually the easiest regions to consider when doing a triple integral, as mentioned before.

Proposition. Let $T = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$, then

$$\iiint_T f dV = \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z)\, dxdydz$$

The order can be exchanged.

Example. As expected, $f = 1$ gives the volume.

$$\begin{aligned}\iiint_T f dV &= \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z)\, dxdydz \\ &= (x_1 - x_0) \int_{z_0}^{z_1} \int_{y_0}^{y_1} 1 dydz \\ &= (x_1 - x_0)(y_1 - y_0)(z_1 - z_0)\end{aligned}$$

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Example. Consider $T = [0, 1] \times [0, 1] \times [-1, 1]$ and $f(x, y, z) = 2$. Compute $\int_T f dV = \int_T 2dV$. We have

$$\begin{aligned}\iiint_T f dV &= \int_{z=-1}^{1} \int_{y=0}^{1} \int_{x=0}^{1} zdxdydz \\ &= \int_{z=-1}^{1} \int_{y=0}^{1} zdydz \\ &= \left[\frac{1}{2} z^2\right]_{-1}^{1} \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0\end{aligned}$$

Interpretation: the contributions of f cancel over $-1 \leq z \leq 1$, notice that

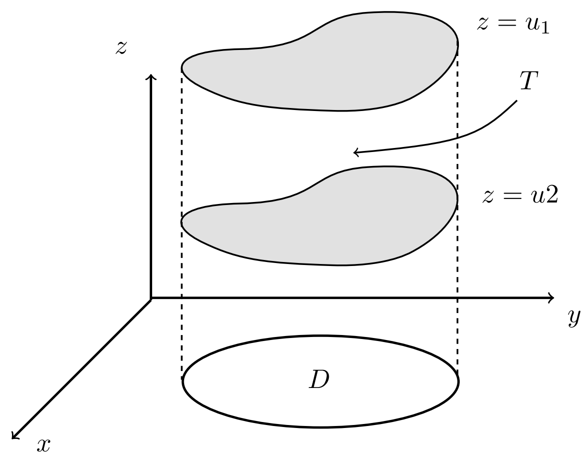
$$f(x, y, z) = -f(x, y, z)$$

and $[-1, 1]$ is unchanged under $z \to -2$. ◇

1.4 General Regions

This is an analoguq of x and y-simple for double integrals. Suppose that we have

$$T = \{(x, y, z) : (x, y) \in D, \quad u_1(x, y) \leq z \leq u_2(x, y)\}$$



We can interpret D as the projection of T in the $xy-$plane. It is also the domain og u_1 and u_2.

Proposition. Suppose T is a region as above, then

$$\iiint_T f dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

Example. Suppose T is of the form

$$T = \{(x, y, z) : (x, y) \in D, \quad 0 \leq x \leq f(x, y)\}$$

Then we have

$$\begin{aligned} \iiint_T 1 dV &= \iint_D \left[\int_0^{f(x,y)} dx 1 dz \right] dA \\ &= \iint_D f(x, y)\, dA \end{aligned}$$

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We get the double integral of f over D, or the volume under the surface $f(x, y)$. More generally, consider

$$u_1(x, y) \leq z \leq u_2(x, y)$$

Then we have

$$\iiint_T 1 dV = \iint_D (u_2 - u_1)\, dA$$

Which can be read as "volume under u_2" - "volume under u_1"