1 Lecture 17

1.1 Triple Integrals

Example. Let T be the region bounded by the $xy-$plane and the paraboloid

$$g(x,y) = 4 - \left(x^2 + y^2\right)$$

Compute its volume, that is $\iiint_T dV$. The xy-plane is $z = 0$. For T we have

$$0 \leq z \leq 4 - \left(x^2 + y^2\right)$$

We want to determine the projection of T in the $xy-$plane. We intersect $z = g(x,y)$ with $z = 0$. This means $g(x,y) = 0$, and hence

$$x^2 + y^2 = 4$$

We take all points inside this circle, that is

$$D = \left\{(x,y) : x^2 + y^2\right\} \leq 4$$

Its volume is

$$\begin{aligned} \iiint_T dV &= \iint_D \left[\int_0^{g(x,y)} dz dA\right] \\ &= \iint_D \left(4 - x^2 - y^2\right) dA \end{aligned}$$

We compute this in polar coords, the disc D is described by

$$0 \leq r \leq z, \quad 0 \leq \theta \leq 2\pi$$

Using $r^2 = x^2 + y^2$, we get

$$\begin{aligned} \iint_D \left(4 - x^2 - y^2\right) dA &= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \left(4 - r^2\right) r d\theta dr \\ &= 2\pi \int_{r=0}^{2} \left(4r - r^3\right) dr \\ &= 2\pi \left[4\frac{r^2}{2} - \frac{r^4}{4}\right]_0^2 \\ &= 8\pi \end{aligned}$$

These are analogous formlae for the case

$$\begin{aligned} u_1(y,z) \leq x \leq u_2(y,z) \\ u_1(x,z) \leq x \leq u_2(x,z) \end{aligned}$$

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1.2 Change of Variables

The general coors (u, v, w) are defined by

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$$

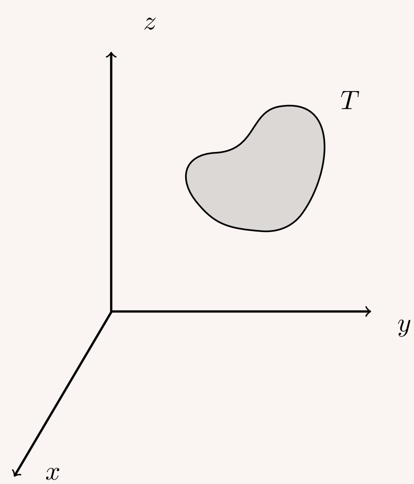
Definition 1. The Jacobian determinant is

$$J(u, v, w) = \left| \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_u & z_w \end{bmatrix} \right|$$

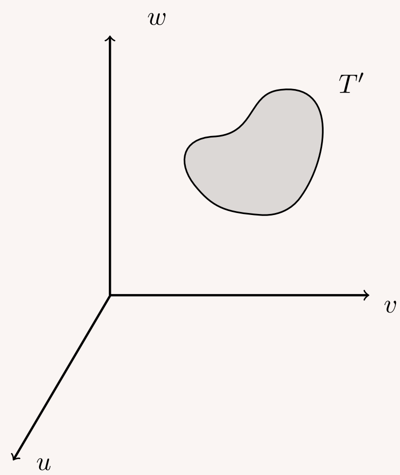
Here we have that $x_u = \frac{\partial x}{\partial u}$, and so on.

Theorem 2. Change of Variables

Let T be a region described in (x, y, z) coords.



Let T' be the corresponding region in (u, v, w).



Then we have

$$\int_T f dV = \int_{T'} f\left(x\left(u,v,w\right),y\left(u,v,w\right),z\left(u,v,w\right)\right)\cdot\left|J\left(u,v,w\right)\right| dudvdw$$

We can express this as

$$\int_T f dV = \int_{T'} f dV'$$

Where $dv = dxdydz$ and $dV' = \left|J\left(u,v,w\right)\right| dudvdw$

1.3 Cylindrical Coordinates

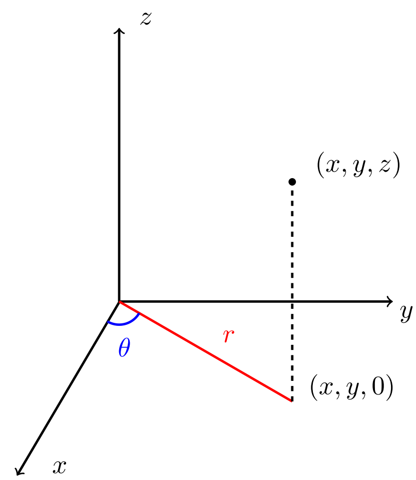
Definition 3. The cylindrical coordinates (r,θ,x) are

$$x = r\cos\theta, \quad y = r\sin\theta, z = z$$

Their range are respectively

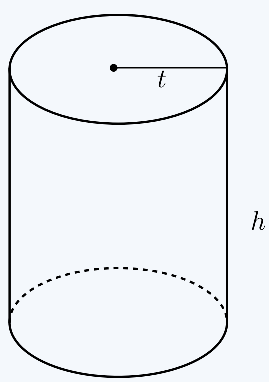
$$0 \leq r \leq \infty, \quad 0 \leq \theta \leq 2\pi, \quad -\infty \leq z \leq \infty$$

The geometrical interpretation is shown in the figure below.



z is the same as in the cartesian case, (r, θ) are the same as in the polar case. How does this relate to cylinders?

Example. We consider a solid cylinder with radius t and height h.



This cylinder is described by

$$T = \left\{(x, y, z) : x^2 + y^2 \leq t^2, \quad 0 \leq z \leq h\right\}$$

Passing to cylindrical coords, the disc $x^2 + y^2 \leq t^2$ becomes

$$0 \leq r \leq t^2, \quad 0 \leq \theta \leq 2\pi$$

Then we get the region

$$T' = \{(r, \theta, z) : 0 \leq r \leq t, \quad 0 \leq \theta \leq t, \quad 0 \leq z \leq h\}$$

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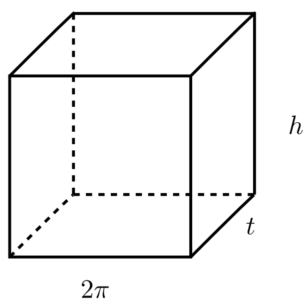
Note. In cylindrical coords, the cylinder T becomes the box T'



Proposition. For the cylindrical coords, we have

$$dV = r d\theta dr dz$$

Proof. The Jacobian is

$$J(r, \theta, z) = \left| \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

The determinant is

$$J(r, \theta, z) - \left| \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \right| = r$$

□

Example. Consider again the cylinder T as before, then we compute

$$\begin{aligned}\iiint_T dV &= \int_{r=0}^{t}\int_{\theta=0}^{2\pi}\int_{z=0}^{h} r dz d\theta dr \\ &= 2\pi h \int_{r=0}^{t} r dr \\ &= 2\pi h \cdot \frac{1}{2}t^2 \\ &= \pi t^2 h\end{aligned}$$

This returns the familiar formula. ◇

1.4 Spherical Coordinates

Definition 4. The spherical coords (ρ, φ, θ) are

$$x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho \cos\varphi$$

Their range is

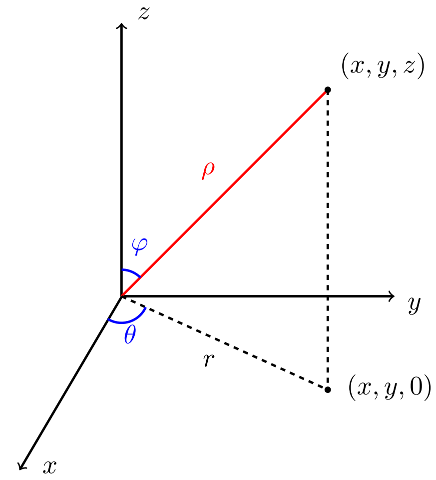
$$0 \leq \rho \leq \infty, \quad 0 \leq \varphi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

> **Note.** Only φ goes to π.

Observe. We have the following relations

$$\rho = \sqrt{x^2 + y^2 + z^2}, \tan\theta = \frac{y}{x}$$

The geometrical meaning is as follows


z
(x, y, z)
ρ
φ
y
θ
r
$(x, y, 0)$
x