
1 Line Integrals, Parametrization and Vector Fields

1.1 Identities between operations

We have seen three operations defined by ∇ .

Gradient: ∇f

Divergence: $\nabla \cdot \vec{F}$

Curl: $\nabla \times \vec{F}$

There are many of them, we look at only one.

Proposition 1. For any scalar field f we have

$$\nabla \times (\nabla f) = 0$$

Proof. We have $\nabla f = (f_x, f_y, f_z)$. Then

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \vec{i}(f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{cy})$$

But partial derivatives can be exchanged, $f_{xy} = f_{yx}$. Then we find that

$$\nabla \times (\nabla f) = 0$$

□

We are going to use this when we discuss conservative fields.

1.2 Line Integrals

Consider the curve C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

Consider the points $\vec{r}(t)$ and $\vec{r}(t + \Delta t)$ where Δt is small.

$$\Delta x = x(t + \Delta t) - x(t), \quad \Delta y = y(t + \Delta t) - y(t)$$

The distance between the two points is

$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

When Δt is small, we can use the following approximation

$$\Delta x \approx \frac{dx}{dt} \Delta t, \quad \Delta y \approx \frac{dy}{dt} \Delta t$$

Then, for the distance, we get

$$S = \sqrt{\left(\frac{dx}{dt} \Delta t\right)^2 + \left(\frac{dy}{dt} \Delta t\right)^2} = \Delta t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Definition 1. The line integral of $f(x, y)$ along a curve C is defined by

$$\int_C f dS = \int_{t_0}^{t_1} f(x(y), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Some observations:

- Note that f is restricted to $\vec{r}(t) = (x(t), y(t))$.
- When $f = 1$, we recover the arc length.

Example. Consider C defined by

$$x(t), y(t) = 0, \quad 0 \leq t \leq 1$$

First, we compute

$$\sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{1^2 + 0^2} = 1$$

Now, consider $f(x, y) = x^2 + y$. Compute $\int_C f dS$. Restricting f to C gives

$$f(x(t), y(t)) = x(t)^2 + y(t) = t^2 + 0 = t^2$$

Then we obtain

$$\int_C f dS = \int_0^1 t^2 \cdot 1 dt = \frac{1}{3}$$

- Line integrals can be used to compute the mass of an object (1-dimensional)
- The curve C describes the object
- The function $\int_C f dS$ is the mass.

1.3 Parametrization and orientation

Next results as for the arc length.

Proposition 2. The integral $\int_C f dS$ does not depend on the parametrization of C .

We will consider a special case.

Example. Consider C with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\text{Start: } A = (x(t_0), y(t_0)), \quad \text{End: } B = (x(t_1), y(t_1))$$

We want to go from B to A , which can be done in the following manner

$$\vec{r}_{\text{opp}} = (x(-t), y(-t)), \quad -t_1 \leq t \leq -t_0$$

Note that

$$\vec{r}_{\text{opp}}(-t_1) = (x(t_1), y(t_1)) = B$$

$$\vec{r}_{\text{opp}}(-t_0) = (x(t_0), y(t_0)) = A$$

Example. Consider C = segment from $(0,0)$, to $(1,0)$, take $f(x,y) = x$. Show that $\int_C f dS = \frac{1}{2}$ using $\vec{r}(t)$ and $\vec{r}_{\text{opp}}(t)$.

If C parametrized by $\vec{r}(t)$, we will use $-C$ when considering $\vec{r}_{\text{opp}}(t)$. We have that

$$\int_C f dS = \int_{-C} f dS$$

The situation will be different for vector fields.

1.4 Case of Vector Fields

Consider the curve with

$$\vec{r}(t) = (x(t), y(t)), \quad t_0 \leq t \leq t_1$$

$$\vec{r}'(t) = (x'(t), y'(t)) \quad \text{Velocity Vector}$$

Definition 2. The line integral of \vec{F} along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

$$\text{Here } \vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$$

- In physics, we have that \vec{F} = Force and $\int_C \vec{F} d\vec{r}$ = Work done by \vec{F} along C .
- The elementary case $W = FS$, where W is work, F is Force and S is the displacement.

More explicitly, we consider

$$\vec{F}(x, y) = (P(x, y), Q(x, y))$$

Then we have that

$$\vec{F}(\vec{r}(t) \cdot \vec{r}'(t)) = P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)$$

The line integral is then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} P(x(t), y(t)) \cdot x'(t) dt + \int_{t_0}^{t_1} Q(x(t), y(t)) \cdot y'(t) dt$$

We will write the expression as

$$\int_C \vec{F} d\vec{r} = \int_C P dx + \int_C Q dy$$

Example. Consider a curve C with

$$\vec{r}(t) = (t, t^2), \quad 0 \leq t \leq 1$$

We have $x(t) = t$ and $y(t) = t^2$, its derivative is

$$\vec{r}'(t) = (x'(t), y'(t)) = (1, 2t)$$

Now consider the vector field

$$\vec{F}(x, y) = (x + y, x)$$

That is $P = x + y$ and $Q = x$. When restricted to C , we get

$$\vec{F}(\vec{r}(t)) = (x(t) + y(t), y(t)) = (t + t^2, t)$$

We want to compute

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = (t + t^2, t) \cdot (1, 2t) = (3t^2 + t)$$

We then obtain the following

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + t) dt = \frac{3}{2}$$