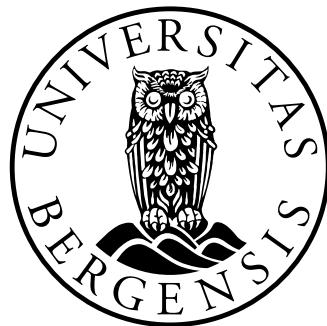


# **Performance of Distributed and Shared Memory Parallel Sparse Matrix Vector Multiplication**

**Kristian Sørdal**



Thesis for Master of Science Degree at the University  
of Bergen, Norway

2025

©Copyright Kristian Sørdal

The material in this publication is protected by copyright law.

Year: 2025

Title: Performance of Distributed and Shared Memory  
Parallel Sparse Matrix Vector Multiplication

Author: Kristian Sørdal

# Acknowledgements



*To err is human; but to really foul things up requires a computer.*

Paul R. Ehrlich



# Abstract

SPARSE MATRIX VECTOR MULTIPLICATION is an important kernel used in scientific computing. It is a problem that lends itself well to parallelization. The problem is bounded by, and scales with the memory bandwidth of the system. Therefore in order to efficiently perform *SpMV* on large distributed memory systems, it is important to reduce the communication between nodes, in order to extract as much as possible out of the memory bandwidth of the system.

This thesis aims to investigate the results of *SpMV* when run using different communication strategies.



# Contents

<b>Acknowledgements</b>	<b>iii</b>
<b>Abstract</b>	<b>vii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Research Question . . . . .	1
<b>2 Theory</b>	<b>3</b>
2.1 Sparse Matrix-Vector Multiplication . . . . .	3
2.2 Definitions . . . . .	4
2.3 Latency . . . . .	4
2.4 Parallel Architectures . . . . .	5
2.4.1 Shared Memory Architecture . . . . .	5
2.4.2 Distributed Memory Architecture . . . . .	5
2.4.3 Non-Uniform Memory Access . . . . .	6
<b>3 Background</b>	<b>7</b>
3.1 CSR Storage Format . . . . .	7
3.1.1 Computational Intensity . . . . .	8
3.2 Other storage formats . . . . .	8
3.2.1 COO Format . . . . .	8
3.2.2 CSC Format . . . . .	9
3.2.3 ELLPack Format . . . . .	10
3.3 Sequential SpMV . . . . .	10
3.4 Shared Memory SpMV . . . . .	11
3.4.1 Scheduling options . . . . .	12
3.4.2 Dynamic Scheduling . . . . .	12
3.4.3 First-Touch Policy . . . . .	13
3.4.4 Performance Implications of the First-Touch Policy	13
3.5 Distributed Memory SpMV . . . . .	13
3.5.1 Load Balancing and Graph Partitioning . . . . .	14

3.5.2	METIS . . . . .	15
3.5.3	Hypergraph Partitioning . . . . .	16
<b>4</b>	<b>Communication Strategies</b>	<b>19</b>
4.1	Exchange entire vector . . . . .	19
4.2	Exchange only separators . . . . .	20
4.2.1	Reordering . . . . .	21
4.3	Exchange only required separators . . . . .	23
4.4	Exchange only required separator values . . . . .	23
4.5	Memory Scalable Strategies . . . . .	24
4.5.1	IPUs and emerging architectures . . . . .	25
<b>5</b>	<b>Results</b>	<b>27</b>
5.1	Theoretical Maximum . . . . .	27
5.2	Matrices . . . . .	27
5.3	Single Node Performance - AMD EPYC 7601 . . . . .	28
5.3.1	Communication Volume . . . . .	29
5.3.2	Communication Time . . . . .	32
5.3.3	Communication Volume AMD EPYC 7302P . . .	37
5.3.4	Communication Time AMD EPYC 7302P . . .	40

# **Chapter 1**

## **Introduction**

### **1.1 Research Question**

This thesis aims to investigate the impact different communication strategies have on the performance of parallel SpMV. The implementation of SpMV uses a shared memory layout on each socket of a node, and distributed memory for communication across nodes.



# Chapter 2

## Theory

### 2.1 Sparse Matrix-Vector Multiplication

Sparse Matrix-Vector Multiplication (SpMV) is a fundamental operation encountered in many areas of scientific computing. It is especially prominent in solving large systems of linear equations and in large-scale simulations. The matrices involved are typically both very large and very sparse.

A matrix can in theory be considered sparse if it is worthwhile to treat zero values separately. In theory, this translates to a matrix being less than full, i.e. less than  $\mathcal{O}(n^2)$  nonzeros for a  $n \times n$  matrix. However, in the context of sparse linear algebra, sparse means that there is a constant number of nonzeros per row, i.e.  $\mathcal{O}(n)$  nonzeros per row. The matrices used in scientific computing, such as matrices based on meshes, or graphs such as social networks all have this property. Optimizing the performance of SpMV, particularly through parallel computing techniques, is crucial for enhancing the efficiency of many scientific applications.

However, SpMV is notoriously difficult to optimize, both in sequential and parallel implementations. One major reason is its inherently low computational intensity.

## 2.2 Definitions

Term	Definition
Node	A node, or a compute node, is a computeunit within a larger parallel computing system.
Dual/single socket	A dual or single socket node refers to the amount of processors on a node. Dual socketed nodes have two processors, that are connected through an interconnect.

## 2.3 Latency

The execution time of computational operations can vary significantly, and it is important to have some understanding of the latency times associated with typical operations. Referencing these latency numbers can be valuable when interpreting benchmark results and the performance of different programs.

Operation	Time [ns]
L1 cache reference	1
Branch misprediction	3
L2 cache reference	4
Mutex lock/unlock	17
Main memory reference	100
Compress 1K bytes with Zippy	2000
Send 2kB over 10 Gbps network	1600
Send 1K bytes over 1 Gbps network	10 000
Read 4K bytes randomly from SSD*	20 000
Round trip within same datacenter	500 000
Read 1MB sequentially from memory	1 000 000
Disk seek	10 000 000
Read 1MB sequentially from disk	10 000 000
TCP packet round trip between continents	150 000 000

Figure 2.1: Latency numbers for common operation, adapted from [8]

## 2.4 Parallel Architectures

There are two main architectures used in the parallel computing industry: Shared Memory Architecture and Distributed Memory Architecture. The following sections gives an overview of the key difference between the two.

### 2.4.1 Shared Memory Architecture

On a system with shared memory architecture, every processing unit (PU) have access to the same memory, treat it as a global address space. On such systems, the biggest challenge is that of *cache coherency*, where in order to prevent race conditions, every read of the cache must reflect the latest write (adapted from [7]).

### 2.4.2 Distributed Memory Architecture

On systems with distributed memory architecture, every processor have their own local memory, not accessible by other processors. When a process needs to access memory from another process, explicit communication of the data stored at that memory address needs to occur, and happens through whichever network the processors are connected with (adapted from [7]). Figure 2.2 shows the difference between shared and distributed memory architectures.

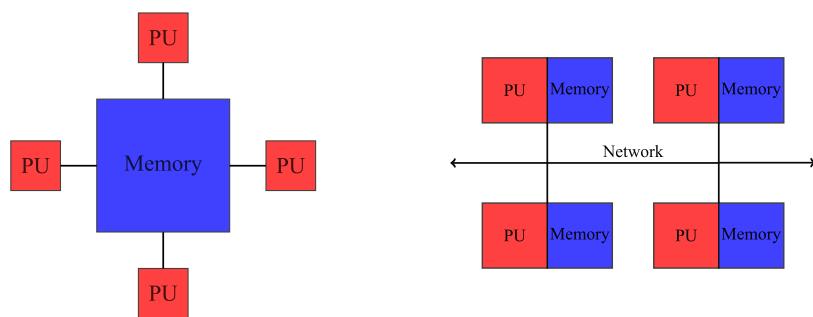


Figure 2.2: Shared and Distributed Memory Architecture (adapted from [6])

### 2.4.3 Non-Uniform Memory Access

Non-Uniform Memory Access (NUMA) refers to a multiprocessor system architecture in which memory access latencies depend on the location of the memory relative to a given processor. Unlike traditional symmetric multiprocessing (SMP) systems, where all processors share equal access times to a centralized memory pool, NUMA architectures consist of multiple processor sockets or nodes, each directly connected to its own local memory. These nodes are interconnected by a high-speed communication network, typically an interconnect, facilitating access to remote memory residing on other nodes.

# Chapter 3

## Background

### 3.1 CSR Storage Format

CSR (Compressed Sparse Row) is the most widely used storage format for sparse matrices. As its name suggests, it compresses the amount of memory used to store a matrix without loss of information. It does so by utilizing three vectors  $A_p, A_j, A_x$ . Figure 3.1 shows an example of a matrix stored in CSR format, adapted from [3].

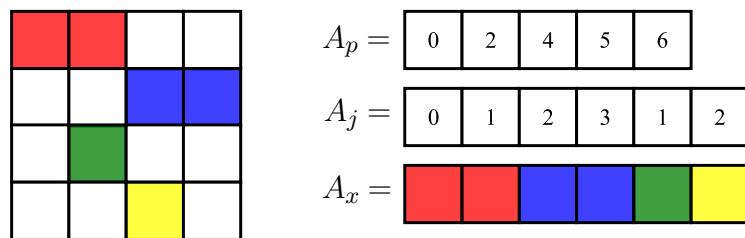


Figure 3.1: Matrix represented in the CSR format.

The first vector,  $A_p$  stores the indices of the first non-zero in the vectors  $A_p$  and  $A_x$ . For a given entry  $A_p[i]$ ,  $A_p[i]$  is the index of the first non-zero in the  $i^{\text{th}}$  row.  $A_j[j]$  and  $A_x[j]$  denotes the column index and value of the  $j^{\text{th}}$  non-zero, respectively.

Throughout the remainder of this thesis, we operate under the assumption that all matrices are represented in CSR format, unless explicitly noted otherwise.

### 3.1.1 Computational Intensity

The *computational intensity* of an operation describes the relation between the number of floating-point operations (FLOPS) and the number of memory accesses required. It is formally defined as:

$$\text{Computational intensity} = \frac{\text{FLOPS}}{\text{Memory accesses}} \quad (3.1)$$

Operations with low computational intensity, such as SpMV, are often *memory bound* rather than *compute bound*. This means that increasing the computational power of a system (e.g., faster processors) does not necessarily lead to proportional speedups in SpMV performance, as memory bandwidth remains the limiting factor.

## 3.2 Other storage formats

There exists many other lesser used storage formats for matrices

### 3.2.1 COO Format

The Coordinate List (COO) format stores the matrix as a set of triples on the form  $i, j, x$ , where  $i$  is the row index,  $j$  is the column index, and  $x$  is the value stored at  $(i, j)$  in the matrix. In this format all 0 entries are ignored.

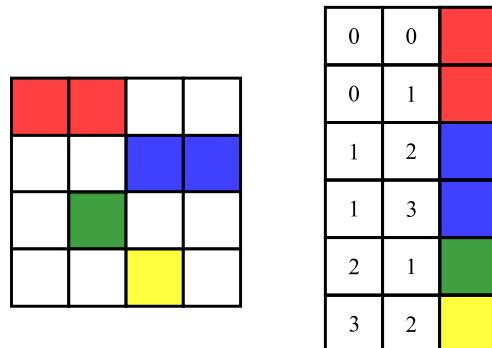


Figure 3.2: Example Matrix represented in the COO format.

### 3.2.2 CSC Format

The Compressed Sparse Column (CSC) format is similar to the CSR format, but instead of compressing the rows, we compress the columns. In this matrix format, we have irregular memory writes, but the reads are more regular. This can however be a problem

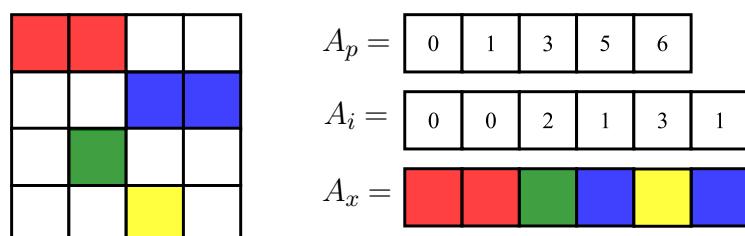


Figure 3.3: Matrix represented in the CSC format.

### 3.2.3 ELLPack Format

For an  $M \times N$  matrix with a maximum of  $K$  non-zeros per row, the ELLPack format stores the non-zeros in an  $M \times K$  matrix `data`, and an  $M \times K$  matrix `indices`. The `data` matrix store the values of the non-zeros, and `indices` store the column index of every element. Rows that have fewer than  $K$  non-zeros are padded with zeros. Adapted from [9].

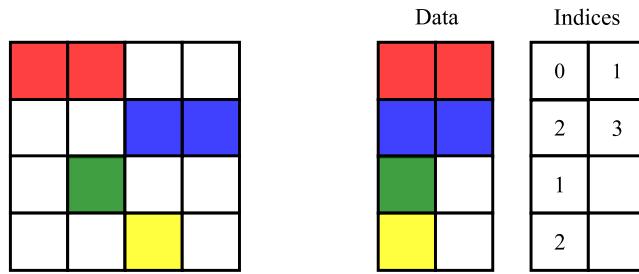


Figure 3.4: Matrix Represented in the ELLPACK format.

## 3.3 Sequential SpMV

A sequential implementation of SpMV on a matrix stored in the CSR format can be implemented in the following manner:

---

**Algorithm 1:** Sequential CSR-based SpMV

---

**Input :**  $A_p, A_j, A_x, x$

**Output :**  $y$

```

for  $i \leftarrow 0$  to  $n$  do
    sum  $\leftarrow 0$ 
    for  $j \leftarrow A_p[i]$  to  $A_p[i+1]$  do
        sum = sum +  $A_x[j] \cdot x[A_j[j]]$ 
     $y[i] \leftarrow$  sum

```

---

For SpMV on well structured matrices such as those similar to the matrix illustrated in Figure 3.5, each non-zero element typically incurs a data movement cost of approximately 12 bytes and results in 2 floating-point operations (FLOPs). This estimation may appear inconsistent with the data access pattern described in algorithm 3.3, where two double-precision values and one integer are accessed per non-zero, corresponding to a total of 20 bytes. The apparent discrepancy arises from the caching behavior of the input vector  $x$ : after the initial access, elements of  $x$  are frequently reused and thus remain in cache, reducing the effective memory traffic associated with subsequent accesses.

In contrast, for highly unstructured matrices, the absence of data locality can significantly degrade cache reuse. In the worst case scenario, each non-zero may trigger the loading of an entire 64 byte cache line, with only a small fraction of it being used. Consequently, the effective data movement can increase to as much as 76 bytes per 2 FLOPs.

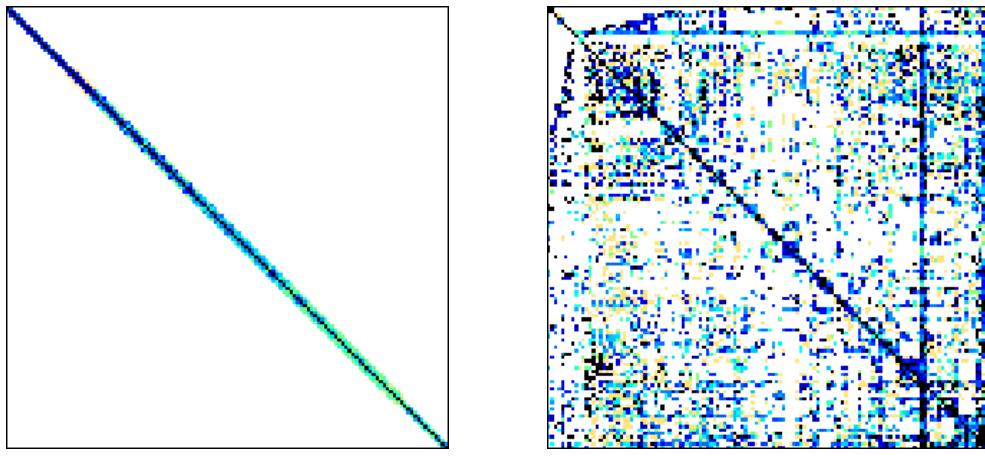


Figure 3.5: Well structured (a) and poorly structured (b) matrices.

## 3.4 Shared Memory SpMV

SpMV can be parallelized using the OpenMP directive `#pragma omp parallel for`. By default, this tells OpenMP to use `static` scheduling when parallelizing the outer iteration loop. When static scheduling is used, the span of the iteration that each thread will execute is precomputed, and stays static, as the naming suggests. There are other scheduling options, such as `dynamic` and `guided`, which will be discussed in later sections.

An implementation of shared memory SpMV is outlined below.

---

**Algorithm 2:** Shared Memory CSR-based SpMV

---

**Input :**  $A_p, A_j, A_x, x$

**Output :**  $y$

```
#pragma omp parallel for
for i ← 0 to n do
    sum ← 0
    for j ← Ap[i] to Ap[i + 1] do
        sum = sum + Ax[j] · x[Aj[j]]
    y[i] ← sum
```

---

### 3.4.1 Scheduling options

As seen in algorithm 2, the outer loop is parallelized, which translates to dividing the rows of the matrix evenly among the threads. This work fine for well structured matrices, but for matrices with dense rows, such as the matrix shown in Figure 3.6, we obtain large imbalances in the computational load for each thread, which impacts performance.

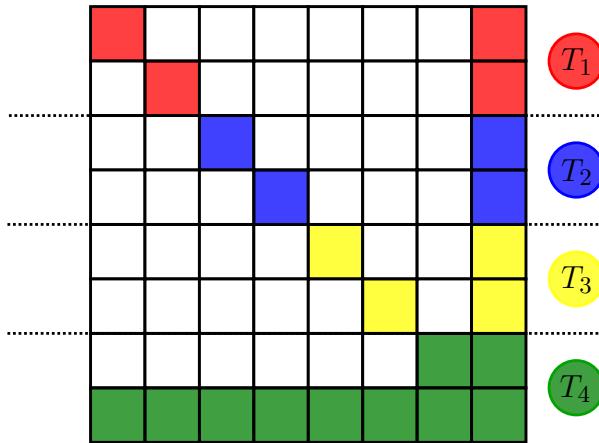


Figure 3.6: Row distribution among threads under static scheduling.

### 3.4.2 Dynamic Scheduling

Dynamic scheduling in OpenMP involves precomputing the range of iterations without assigning specific iteration subsets to individual threads

in advance. Under static scheduling, if thread  $A$  receives sparse rows and thread  $B$  receives dense rows, thread  $A$  will become idle prematurely while thread  $B$  remains computationally engaged. In contrast, dynamic scheduling allocates iterations at runtime to threads as they become available, thus potentially balancing the computational load more effectively, particularly when iteration workloads vary significantly.

At first glance, dynamic scheduling appears to resolve the workload imbalance inherent in static scheduling, even though it comes with more overhead. While this assumption holds true on smaller systems, such as personal laptops equipped with fewer physical cores and a single processor socket, it does not readily apply to larger, dual-socket systems utilized in this thesis. Here, the first-touch memory policy becomes particularly significant.

### 3.4.3 First-Touch Policy

The first-touch policy stipulates that the initial thread accessing a memory page allocates this page to its local memory domain. Consequently, if thread first accesses a memory page, it is stored locally to thread . Subsequent accesses by thread to this page result in non-local memory access, compelling thread to retrieve the data from either remote or main memory. Such accesses incur performance penalties due to significantly higher latency compared to local memory access.

### 3.4.4 Performance Implications of the First-Touch Policy

Table 2.1 illustrates that accesses to main memory exhibit substantially higher latency compared to local cache (e.g., L1 cache) accesses. In dual-socket configurations, accessing memory residing on the remote socket involves inter-socket communication through an interlink, further exacerbating latency. Consequently, dynamic (or guided) scheduling alone does not mitigate the performance degradation arising from poorly structured matrices, as it inadvertently exacerbates memory locality issues inherent to the first-touch policy on NUMA architectures.

## 3.5 Distributed Memory SpMV

In distributed memory systems, the computational workload is distributed across multiple nodes, each typically comprising a dual-socket architecture

with local memory. Unlike shared memory systems, data access across nodes is non-trivial and requires explicit communication, most commonly implemented using the Message Passing Interface (MPI).

For distributed sparse matrix-vector multiplication (SpMV), the computation proceeds similarly to the shared memory case (algorithm 2). The sparse matrix is divided across nodes, and each node computes its local segment of the output vector  $y$ . At the end of each iteration, partial results are assembled to produce the global vector, necessitating inter-node communication.

While the thread scheduling and memory locality issues discussed in 3.4.1 remain relevant, distributed memory systems introduce the additional challenge of communication overhead. As shown in Table 2.1, inter-node communication is significantly more expensive than memory accesses. Consequently, minimizing the total communication volume per SpMV iteration is critical for performance.

### 3.5.1 Load Balancing and Graph Partitioning

Even with a well-balanced distribution, excessive communication between nodes can become a performance bottleneck if data dependencies span many partitions. An effective partitioning strategy must therefore strike a careful balance between distributing the computational load evenly and reducing the amount of communication required between ranks. This trade-off is naturally captured by the graph partitioning problem.

The graph partitioning problem involves dividing an undirected graph  $G(V, E)$ , where  $V$  is the set of vertices and  $E$  is the set of edges into  $K$  disjoint subsets (or partitions) such that each subset contains a comparable amount of vertices, and the number of edges connecting different subsets is minimized. Each vertex  $v_i \in V$  may be assigned a weight  $w_i$ , and each edge  $e_{ij} \in E$  may carry a cost  $c_{ij}$ . The sum of weights  $W_k$  of each partition  $P_k$  must not exceed a specified imbalance threshold  $\epsilon$  relative to the average weight of each partition. An edge can be classified as either an internal edge (those that connect nodes in the same partition), or an external edge (those that connect vertices in different partitions), and the cutsize is defined as the sum of the costs of all external edges. The goal is to minimize the cutsize, while preserving the balance between the size of each partition. This problem is known to be NP-Hard, even for obtaining bipartitions on unweighted graphs (adapted from [2]).

Algorithms that aim to solve the graph partitioning problem rely

on heuristics that provide good approximations in practice. Optimizing for one objective at the expense of the other is trivial but undesirable: assigning all vertices to a single partition minimizes communication to zero but results in extreme load imbalance, whereas naïvely assigning vertices to maintain balance without regard to connectivity may lead to high communication costs.

### 3.5.2 METIS

The graph partitioner used for the experiments in this thesis is called METIS. METIS is a software package for partitioning large irregular graphs, and has other uses such as partitioning meshes and computing fill-reducing orderings of sparse matrices. METIS provides algorithms that are based on multilevel graph partitioning. These algorithms partition the graph through a coarsening and refinement phase. In the coarsening phase, the size of the graph is reduced by way of collapsing the vertices and edges, and then partition the smaller graph. In the refinement phase, the graph is gradually expanded to its original size, and the portion of the graph that is close to the boundary is the primary focus, ensuring the quality of the partition is kept (adapted from [5]). algorithm 3 shows how to partition and reorder the vertices of a graph according to the generated partition, using METIS'  $K$ -way partition algorithm.

The way METIS partitions graph does not necessarily optimize for minimizing the communication volume, but rather obtaining partitions that are roughly equal in size. It is possible that there are partitions that have a larger imbalance between the size of partitions, but much smaller communication volumes. In order to obtain partitions that optimize for minimizing communication, it is necessary to look to hypergraph partitioners.

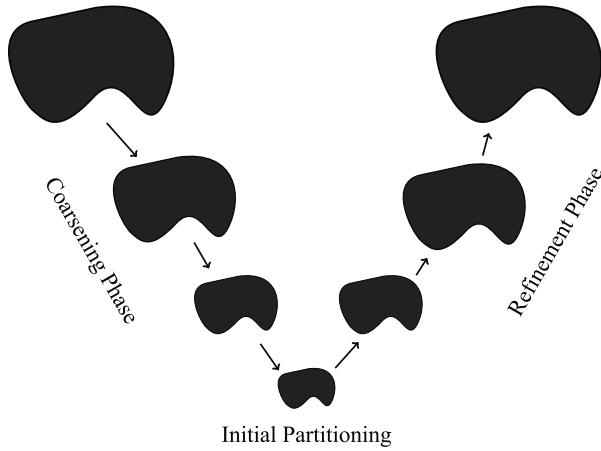


Figure 3.7: Illustration of multilevel graph partitioning (adapted from [5]).

### 3.5.3 Hypergraph Partitioning

A *hypergraph*  $H = (V, N)$  is defined as a set of vertices  $V$  and a set of nets (hyperedges)  $N$  among those vertices. Every net  $n_j \in N$  is a subset of vertices, i.e.,  $n_j \subseteq V$ . The vertices in a net  $n_j$  are called its *pins* and denoted as  $\text{pins}[n_j]$ . The size of a net is equal to the number of its pins, i.e.,  $s_j = |\text{pins}[n_j]|$ . The set of nets connected to a vertex  $v_i$  is denoted as  $\text{nets}[v_i]$ . The degree of a vertex is equal to the number of nets it is connected to, i.e.,  $d_i = |\text{nets}[v_i]|$ .

[2]

Hypergraph partitioners try to solve the same problems, and employ similar algorithms as those graph partitioners use, however they use these algorithms on hypergraphs, and not regular graphs. Çatalyürek and Ayakanat showed in [2] that using the hypergraph partitioner they developed, they could get up to 63% less communication volume, with an average reduction in communication volume of 30% - 38%, when compared to METIS.

As the improvement of the communication strategies are the main focus of the contents of this thesis, the specific partitioner tool that is used is not the biggest concern, and the programs implemented have

opted for the usage of METIS, which as mentioned is a regular graph partitioner. It would be possible to refactor the code in such a way that a hypergraph partitioner such as PaToH can be used.

---

**Algorithm 3:** Partitioning and reordering a matrix.

---

**Input :**  $g, n_p, p$

**Output :** Partitioned and reordered  $g$

```

if  $n_p = 1$  then
     $p[0] \leftarrow 0$ 
     $p[1] \leftarrow g.n_r$ 
    return  $g$ 

partitionVector  $\leftarrow$  METIS_PartGraphKway(arguments
    specifying constraints for partition)
newId  $\leftarrow [0] \cdot g.n_r$ 
oldId  $\leftarrow [0] \cdot g.n_r$ 
id  $\leftarrow 0$ 
p[0]  $\leftarrow 0$ 
for  $r \in \{0, \dots, r\}$  do
    for  $i \in \{0, \dots, g.n_r\}$  do
        if  $partitionVector[i] = r$  then
            oldId[id]  $\leftarrow i$ 
            newId[i]  $\leftarrow id$ 
            id  $\leftarrow id + 1$ 
    partitionVector[r + 1]  $\leftarrow id$ 
newRowPtr  $\leftarrow [0] \cdot g.n_r + 1$ 
newColIdx  $\leftarrow [0] \cdot g.n_c$ 
newValues  $\leftarrow [0] \cdot g.n_c$ 
for  $i \in \{0, \dots, g.n_r - 1\}$  do
    d  $\leftarrow g.rowPtr[oldId[i] + 1] - g.rowPtr[oldId[i]]$ 
    newRowPtr[i + 1]  $\leftarrow newRowPtr[i] + d$ 
    for  $j \in \{0, \dots, d - 1\}$  do
        newColIdx[newRowPtr[i] + j]  $\leftarrow g.colIdx[g.rowPtr[oldId[i]]$ 
         $+ j]$ 
        newValues[newRowPtr[i] + j]  $\leftarrow g.values[g.rowPtr[oldId[i]]$ 
         $+ j]$ 
    for  $j \in \{newRowPtr[i], \dots, newRowPtr[i + 1] - 1\}$  do
        newColIdx[j]  $\leftarrow newId[newColIdx[j]]$ 
g.rowPtr  $\leftarrow newRowPtr$ 
g.colIdx  $\leftarrow newColIdx$ 
g.values  $\leftarrow newValues$ 
return g

```

---

# Chapter 4

## Communication Strategies

In parallel implementations of Sparse Matrix-Vector Multiplication (SpMV), effective communication management is critical due to its significant influence on overall performance. Communication often emerges as a bottleneck in distributed-memory systems because the speed at which data moves between nodes is significantly lower than within-node memory access speeds. Consequently, reducing communication volume and optimizing communication patterns can yield substantial performance improvements.

This chapter evaluates a series of progressively optimized communication strategies employed in distributed-memory parallel SpMV. Starting from the simplest method, exchanging the entire result vector between all nodes, the strategies become increasingly selective and efficient, focusing specifically on exchanging only the essential data elements required by each node. These approaches leverage knowledge of the matrix structure, partitioning methods, and computational dependencies to minimize communication overhead.

### 4.1 Exchange entire vector

The most straightforward approach is to have each rank send all of its computed values of  $y$  to every other rank. This ensures that all processes possess a complete and updated copy of the output vector before the next iteration. This strategy can be implemented using MPI's collective communication operation `MPI_Allgatherv`, which accommodates variable message sizes from each rank. Figure 4.2 illustrates the state of the  $y$  vector before and after communication using this strategy.

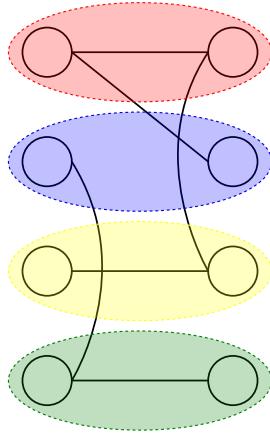


Figure 4.1: examplegraph

---

**Algorithm 4:** 1a - Exchange entire vector

---

```

for each iteration do
    spmv(g,x,y)
    MPI_Allgatherv(local_y, sendcount, MPI_DOUBLE, y,
                   recvcounts, displs, MPI_DOUBLE, MPI_COMM_WORLD)
    swap pointers of x and y

```

---

## 4.2 Exchange only separators

An improvement to the previous strategy can be achieved by recognizing that only separator values, those required by multiple processes, must be communicated. Non-separator values are used exclusively by the process that computed them and therefore do not need to be communicated.

To facilitate this strategy, separator values are reordered such that they appear at the beginning of each process's local segment of  $y$ . Once this structure is established, communication is performed using `MPI_Allgatherv`, transmitting only the subset of  $y$  that contains separator values. The number of separators on each process must be known beforehand, which can be computed by counting the number of elements that have neighbours belonging to a different partition.

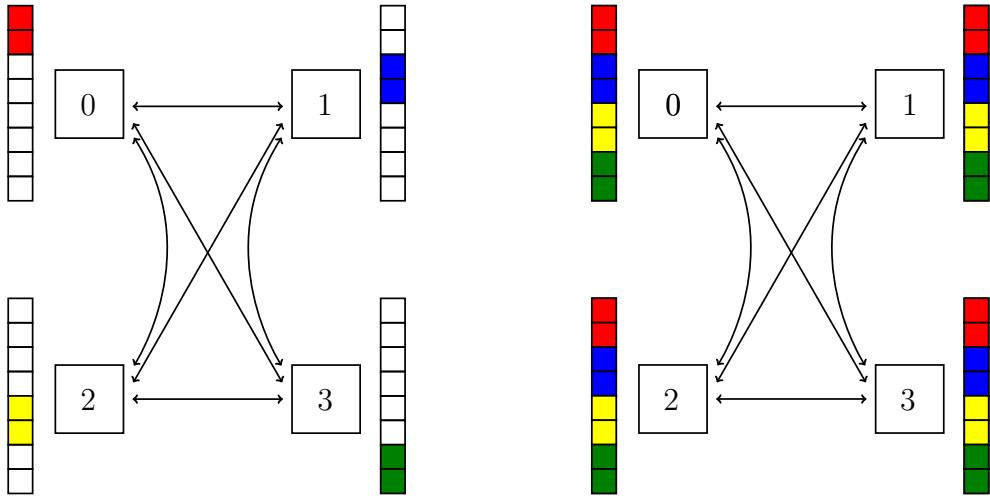


Figure 4.2: Contents of each ranks  $x$  vector before and after communication under the *Exchange Entire Vector* communication pattern.

#### 4.2.1 Reordering

After partitioning the matrix into different parts, we obtain a partition vector  $p$ , where the  $p[i]$  stores the index of the partition the  $i^{\text{th}}$  entry in  $A_p$ . It is necessary to reorder the entries in  $A_p$  in accordance with the partition vector, such that all entries belonging to the same partition are stored in sequence. The algorithm below gives an outline of how this can be achieved. Here  $n_p$  is the number of partitions,  $n_r$  is the size of  $A_p$ , and  $n_c$  is the size of  $A_j$  and  $A_x$ .

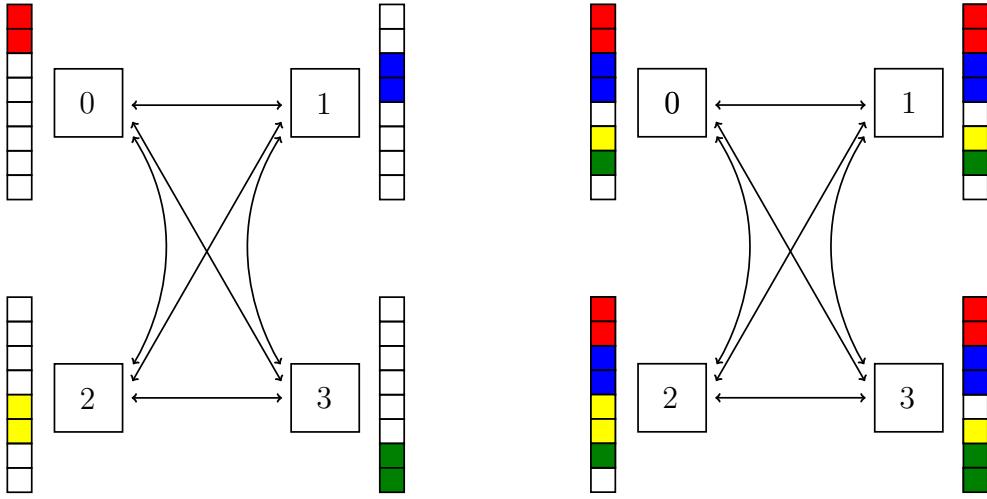


Figure 4.3: Contents of each ranks  $x$  vector before and after communication under the *Exchange Separators* communication pattern.

---

**Algorithm 5:** Reordering of Separators

---

```

Input :  $n_p, n_r, n_c, p, A_p, A_j, A_x$ 
Output : Reordered  $A_p, A_j, A_x$ 

newId  $\leftarrow [0] \cdot n_r$ 
oldId  $\leftarrow [0] \cdot n_r$ 
id  $\leftarrow 0$ 
 $p_0 \leftarrow 0$ 

for  $r \in \{0, \dots, n_p - 1\}$  do
  for  $i \in \{0, \dots, n_r - 1\}$  do
    if  $p[i] = r$  then
      oldId[id]  $\leftarrow i$ 
      newId[i]  $\leftarrow id$ 
      id  $\leftarrow id + 1$ 
     $p[r + 1] \leftarrow id$ 

  newV  $\leftarrow [0] \cdot (n_r + 1)$ 
  newE  $\leftarrow [0] \cdot n_c$ 
  newA  $\leftarrow [0] \cdot n_c$ 

  for  $i \in \{0, \dots, n_r - 1\}$  do
    degree  $\leftarrow A_p[\text{oldId}[i] + 1] - A_p[\text{oldId}[i]]$ 
    newV[i + 1]  $\leftarrow newV[i] + degree$ 

  for  $i \in \{0, \dots, n_r - 1\}$  do
    degree  $\leftarrow A_p[\text{oldId}[i] + 1] - A_p[\text{oldId}[i]]$ 
    for  $j \in \{0, \dots, degree - 1\}$  do
      newE[newV[i] + j]  $\leftarrow A_j[A_p[\text{oldId}[i]] + j]$ 
      newA[newV[i] + j]  $\leftarrow A_x[A_p[\text{oldId}[i]] + j]$ 
    for  $j \in \{newV[i], \dots, newV[i + 1] - 1\}$  do
      newE[j]  $\leftarrow newId[newE[j]]$ 

```

### 4.3 Exchange only required separators

Further reduction to the communication volume can be achieved by observing that not all separator values are required by every process. As the number of partitions increases, the set of dependencies between partitions tends towards sparsity. As a consequence of this, certain sets of separators may only need to be communicated to a given subset of processes. Using this strategy, each process only communicates its set of separator values to the processes that require them.

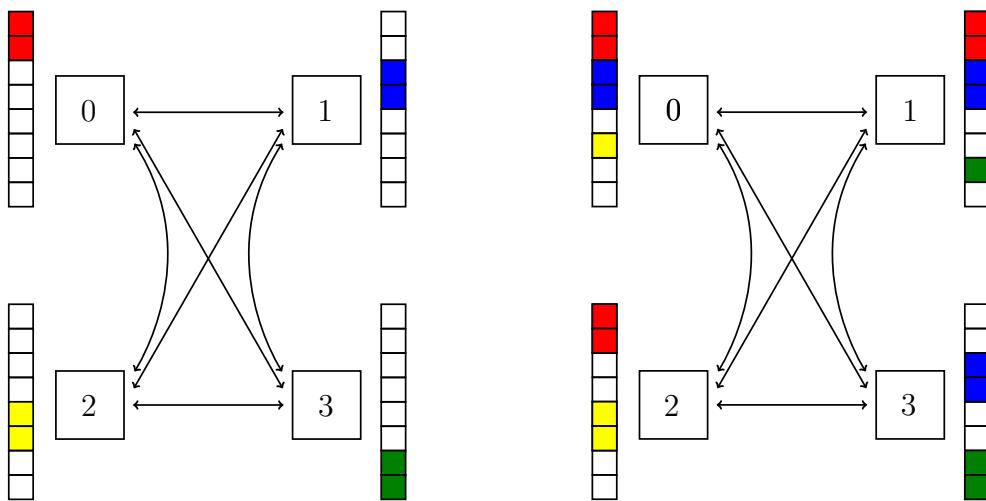


Figure 4.4: Contents of each rank  $x$  vector before and after communication under the *Exchange Required Separators* communication pattern.

### 4.4 Exchange only required separator values

The final strategy aims to minimize communication overhead by transmitting only the exact subset of separator values that are both computed by and required for inter-process computation. If a specific separator value computed by one process is needed by exactly one other process, then only that single recipient receives the value.

This approach eliminates all unnecessary data transfers but introduces additional complexity in managing communication schedules. Dependencies must be mapped at a fine-grained level, and communication patterns

must be explicitly tailored to the structure of the matrix and its partitioning.

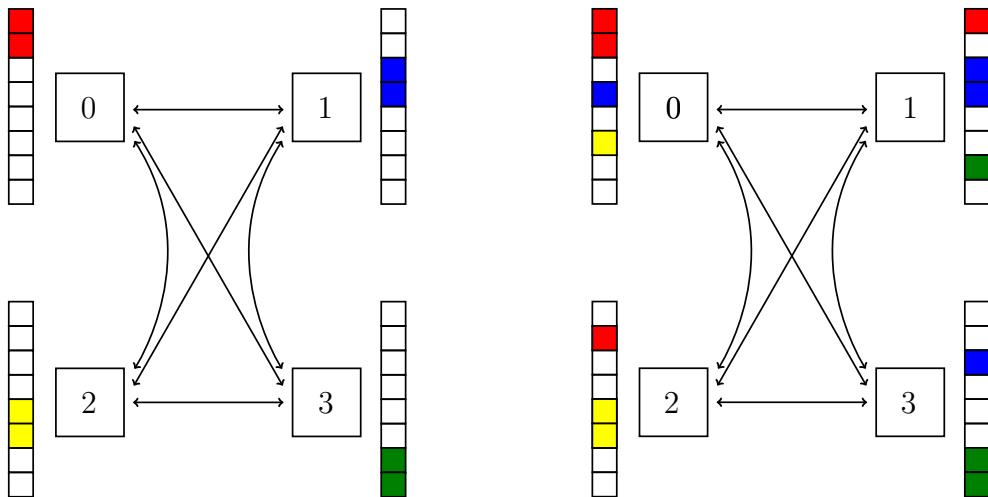
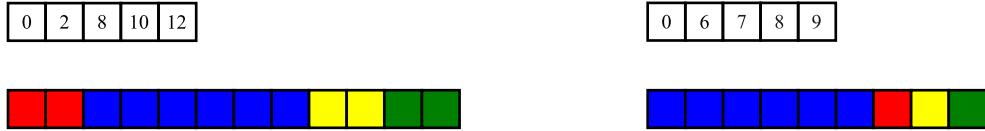


Figure 4.5: Contents of each ranks  $x$  vector before and after communication under the *Exchange Required Elements* communication pattern.

## 4.5 Memory Scalable Strategies

The communication strategies discussed so far all have a common problem that prevents them from scaling to large matrices. These strategies all store the entire vector  $x$ , and will run into performance issues when  $x$  is so large that it doesn't fit into memory. Usually, this is not a problem when SpMV is ran on CPUs, as they have large amounts of memory. Even on GPUs this problem might not be encountered, as modern GPUs have sufficient memory for large matrices.

Instead of storing the entire vector, each rank only stores its local part of the vector. In addition, it is necessary to allocate enough space for the separators elements that are needed from the other ranks. In order to achieve this,  $x$  is renumbered such that every ranks part of the vector is 0-indexed. For the local part of the  $x$  vector this is done by simply subtracting the position of the first local entry assigned to that rank from each entry in the local vector. It is slightly more complicated to renumber the separator elements, and the technicalities of how to do this will not be discussed here.

Figure 4.6: Global indexing vs. Local indexing of  $x$ .

#### 4.5.1 IPUs and emerging architectures

In the ever evolving world of computer chip manufacturing, there has in the past years emerged a new trend in the field of architecture design. This comes in the form of packaging thousands of small processor cores into a single device, where each core has its own local memory, and no device-level shared memory [1]. An example of such a processor is the relatively new Cerebras Wafer-Scale Engine (WSE-3), which is the largest chip ever built. With a spec sheet of 4 trillion transistors, 900 000 cores, memory bandwidth of 21PB/s, and 44 GB of on-wafer memory (see [4]). Such architectures necessitates a careful distribution of relevant data across each processor, and is where memory scalable communication strategies are beneficial.



# Chapter 5

## Results

### 5.1 Theoretical Maximum

As has been established, the maximum performance scales with the available memory bandwidth of the system, and not with the amount of processes used. For SpMV, a total of 6 bytes are read per FLOP, which means that the theoretical maximum is given by 5.1. This is under the assumption that all resources on the system is dedicated to the SpMV operation, which is not the case in reality. At most, somewhere in the neighbourhood of 80% of the systems resources can be expected to be utilized.

$$\text{Maximum Theoretical Performance} = \frac{\text{Memory bandwidth}}{6} \quad (5.1)$$

### 5.2 Matrices

For the results presented in the following sections, the matrices shown in Figure 5.1 were selected as they are in general well-structure and symmetrical.

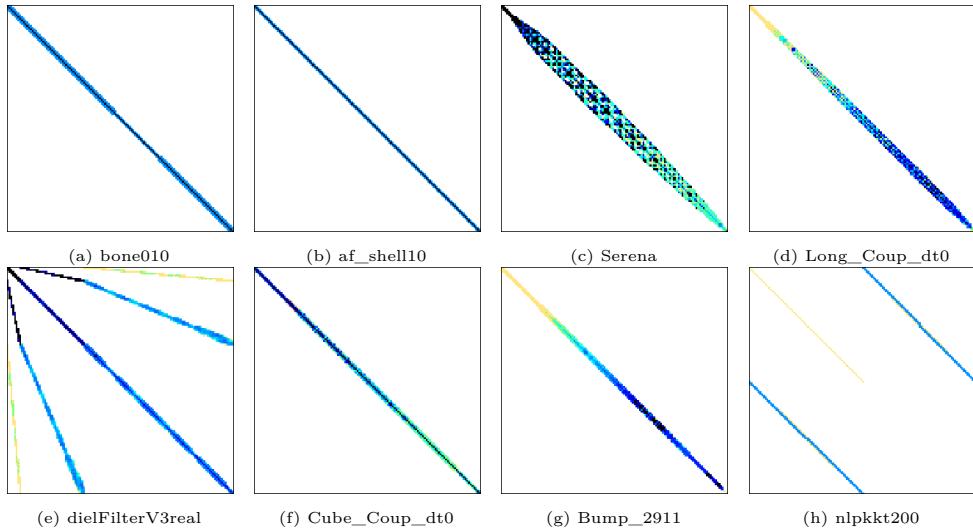


Figure 5.1: Matrices used to generate results.

Name	Purpose
bone010	Trabecular Bone Micro-Finite Element Model
af_shell10	Sheet metal forming matrix
Serena	Gas reservoir simulation for $CO_2$ sequestration
Long_Coup_dt0	3D coupled consolidation problem (geological formation)
dielFilterV3real	High-order vector finite element method in EM
Cube_Coup_dt0	3D coupled consolidation problem (3D cube)
Bump_2911	3D geomechanical reservoir simulation
nlpkkt200	Symmetric indefinite KKT matrix

All programs have been compiled with `-march=native` and `-O3` compilation flags.

### 5.3 Single Node Performance - AMD EPYC 7601

The following results illustrate the communication volume as a percentage of the total number of elements in the  $x$  vector for the given matrix per iteration of SpMV. Each plot shows three communication strategies, these being the *Exchange Separators*, *Exchange required separators* and *Exchange required elements* strategies, respectively. Results for the *Exchange*

*entire vector* communication strategy have been left out, as the communication volume remains constant at 100% for all entries. Furthermore, the plots show three entries per communication strategy, in order to illustrate the communication volume of the rank that communicates the maximum of all ranks, the minimum of all ranks, and the average across all ranks.

### 5.3.1 Communication Volume

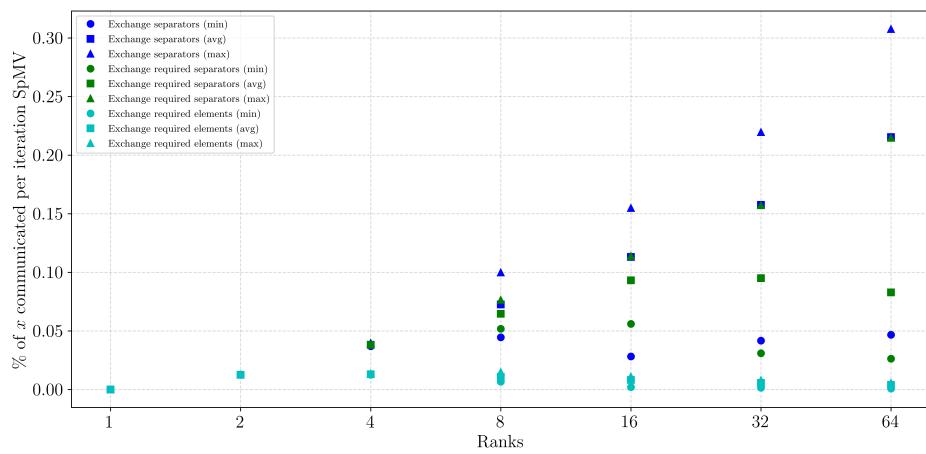


Figure 5.2: Serena

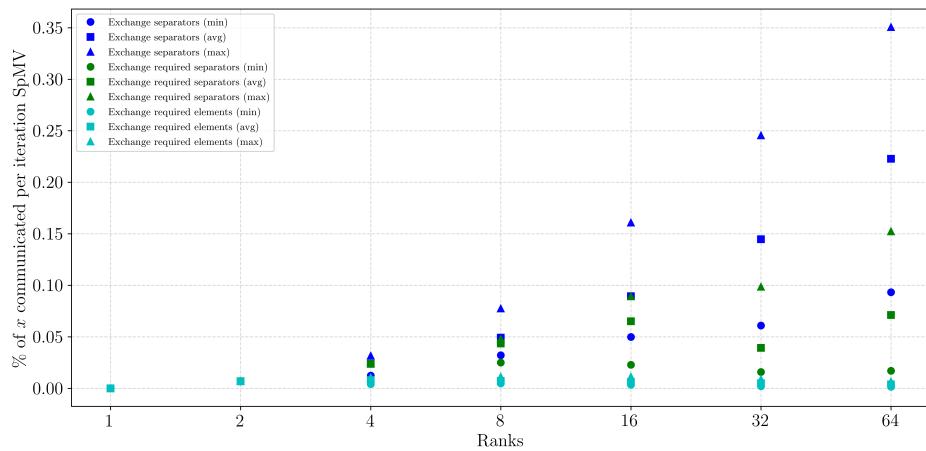


Figure 5.3: bone010

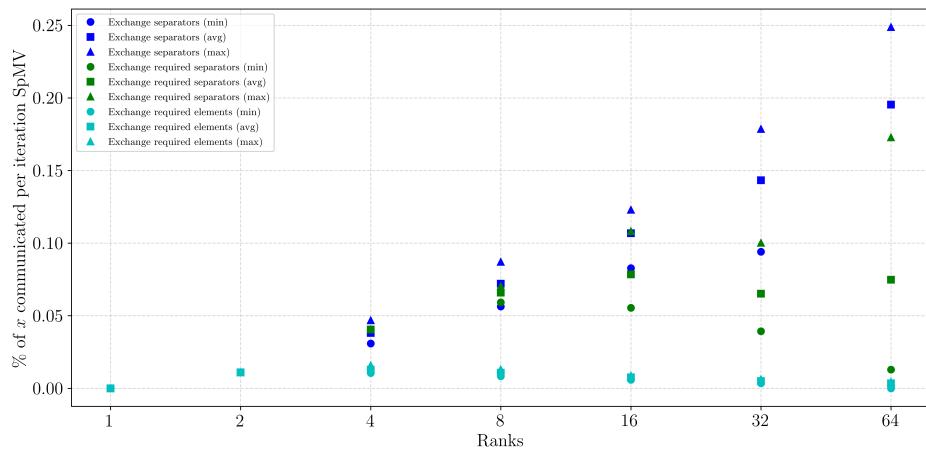


Figure 5.4: Bump\_2911

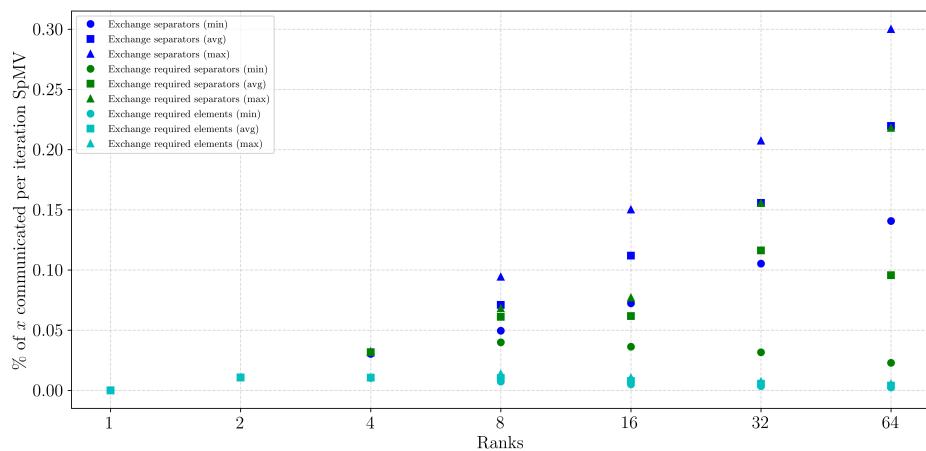


Figure 5.5: Cube\_Coup\_dt0

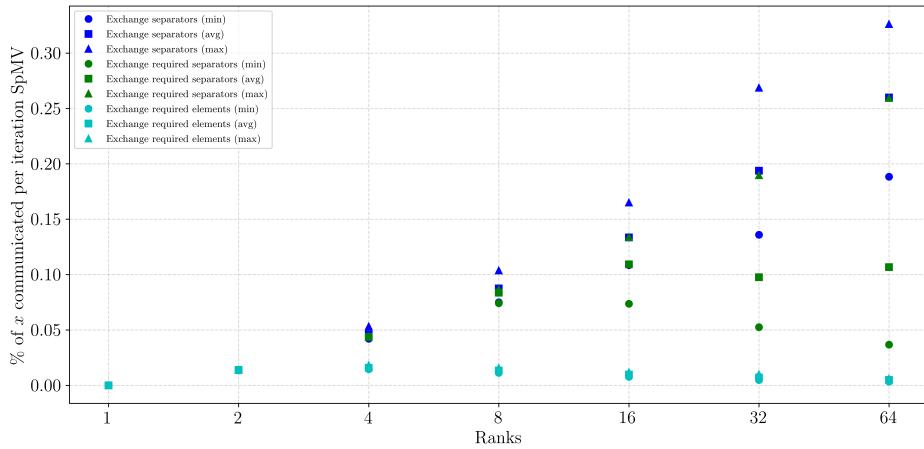


Figure 5.6: Long\_Coup\_dt0

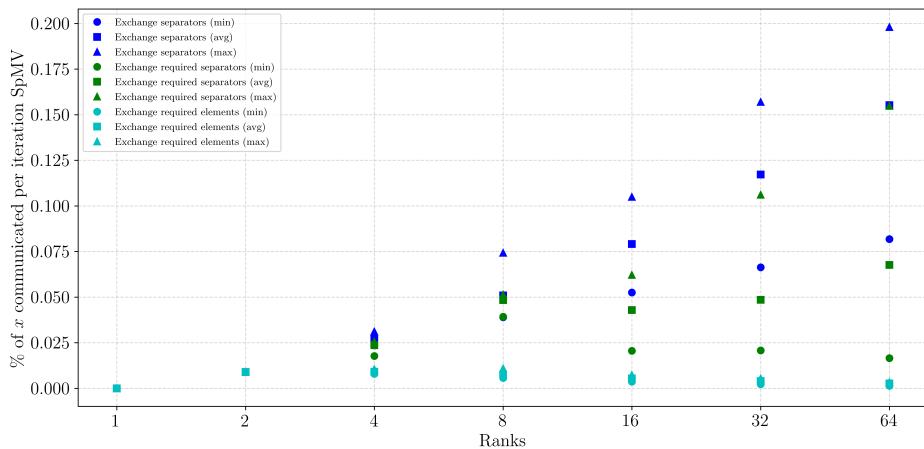


Figure 5.7: nlpkkt200

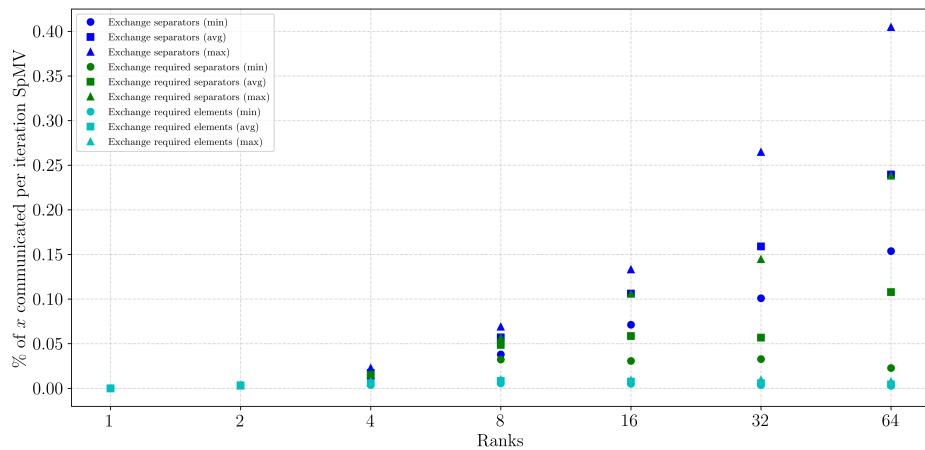


Figure 5.8: dielFilterV3real

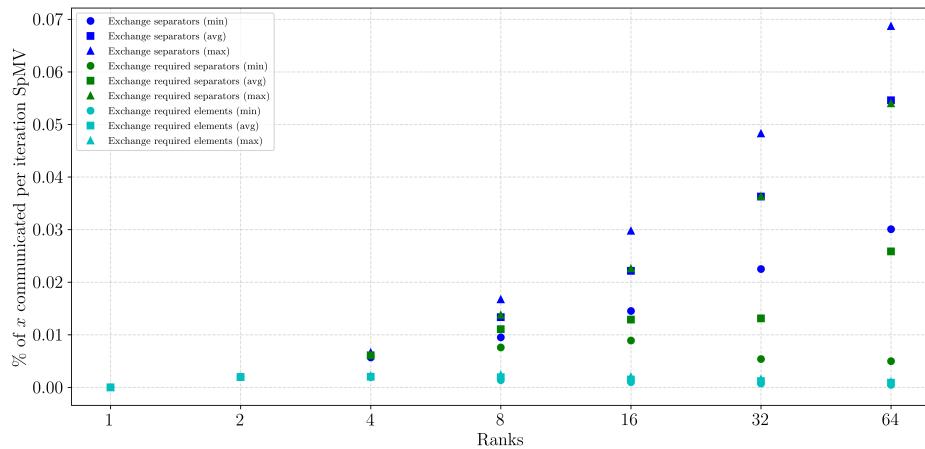


Figure 5.9: af\_shell10

### 5.3.2 Communication Time

The following plots illustrate the time spent communicating per iteration of SpMV.

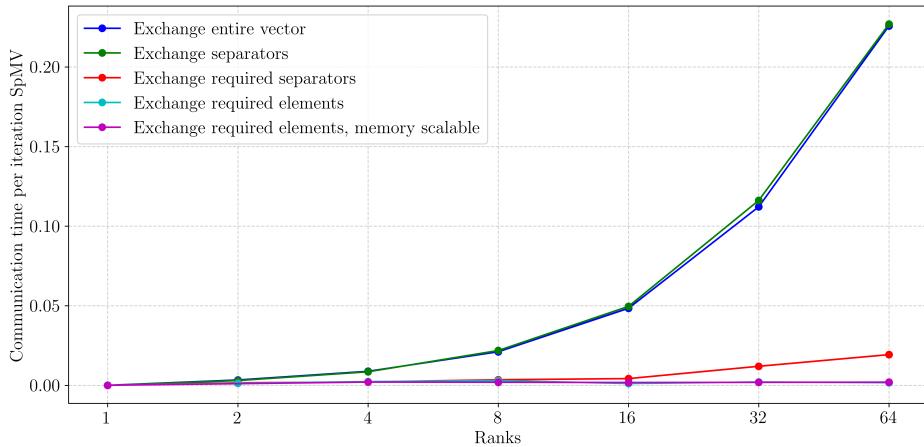


Figure 5.10: Serena

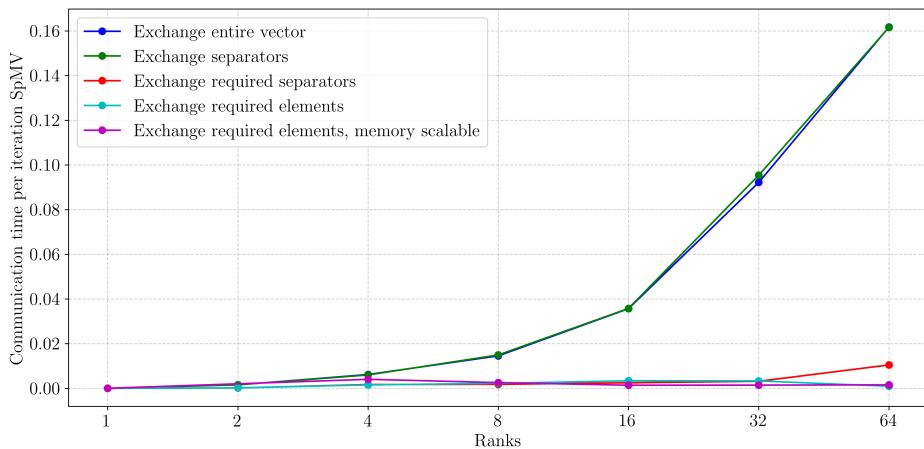


Figure 5.11: bone010

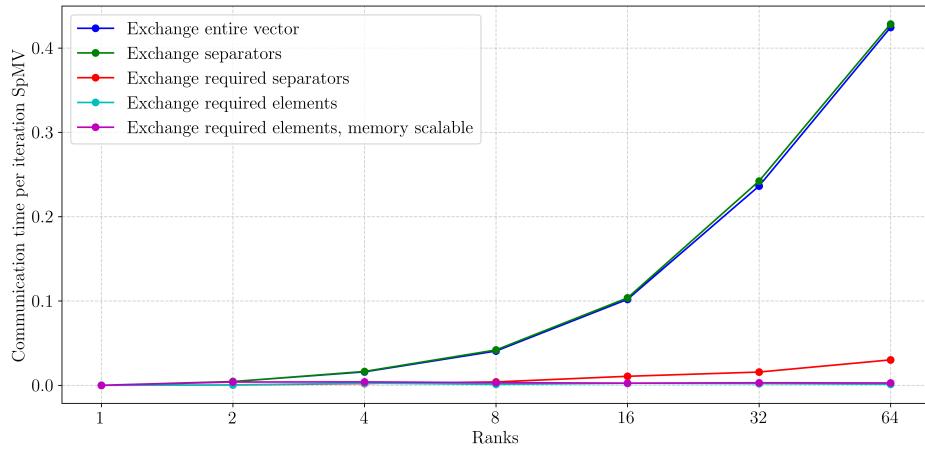


Figure 5.12: Bump\_2911

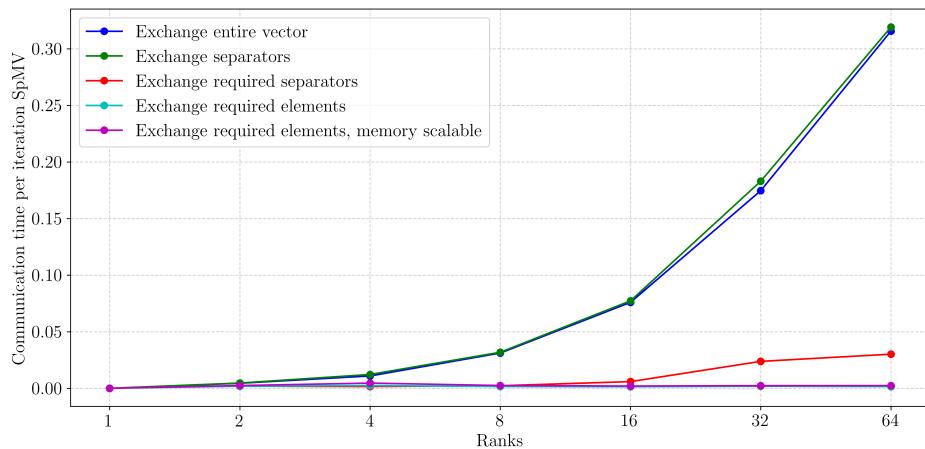


Figure 5.13: Cube\_Coup\_dt0

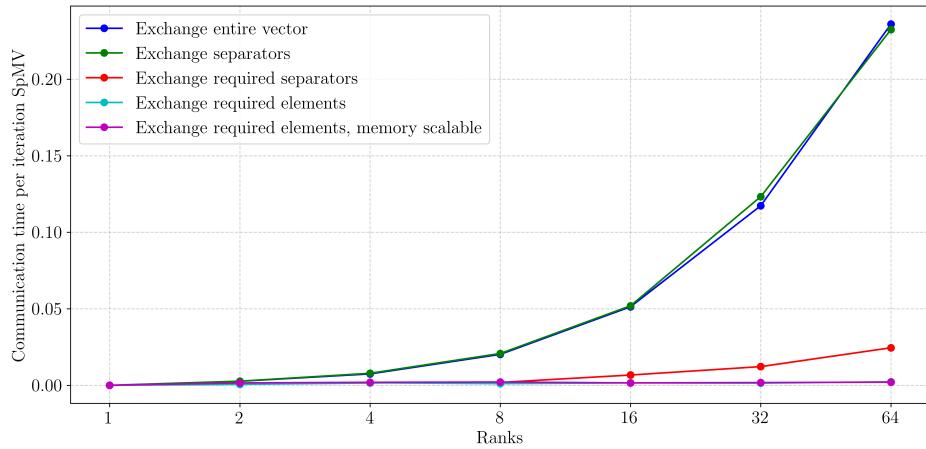


Figure 5.14: Long\_Coup\_dt0

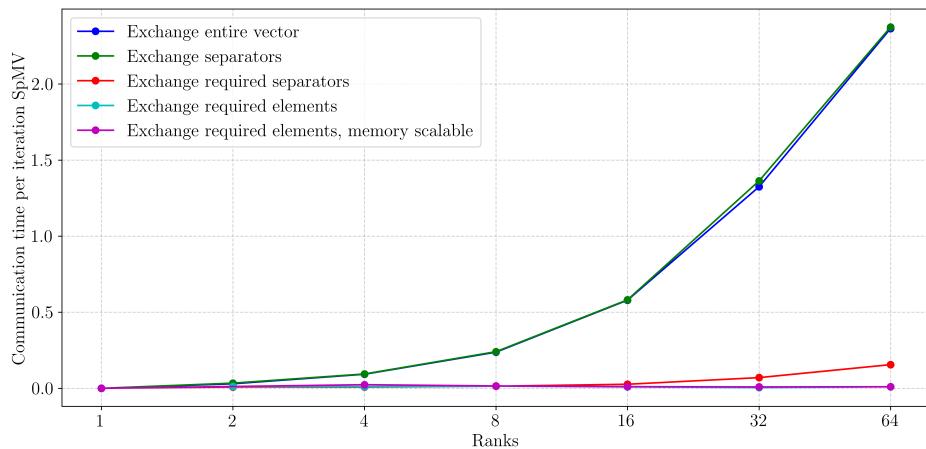


Figure 5.15: nlpkkt200

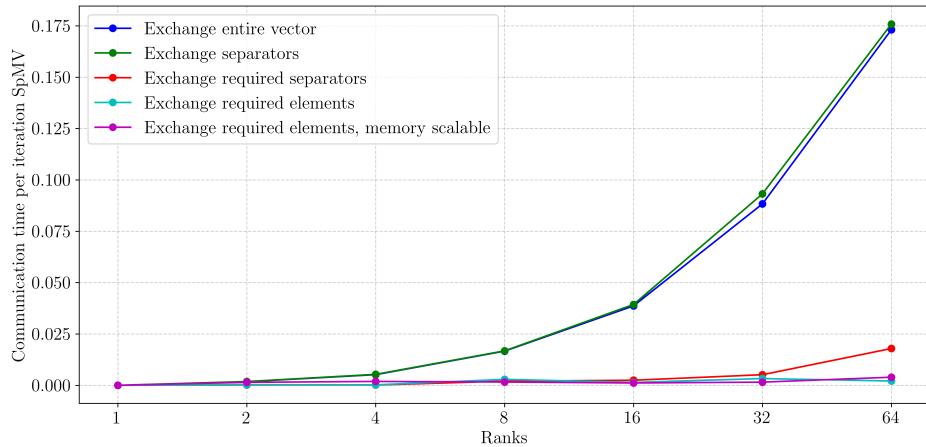


Figure 5.16: dielFilterV3real

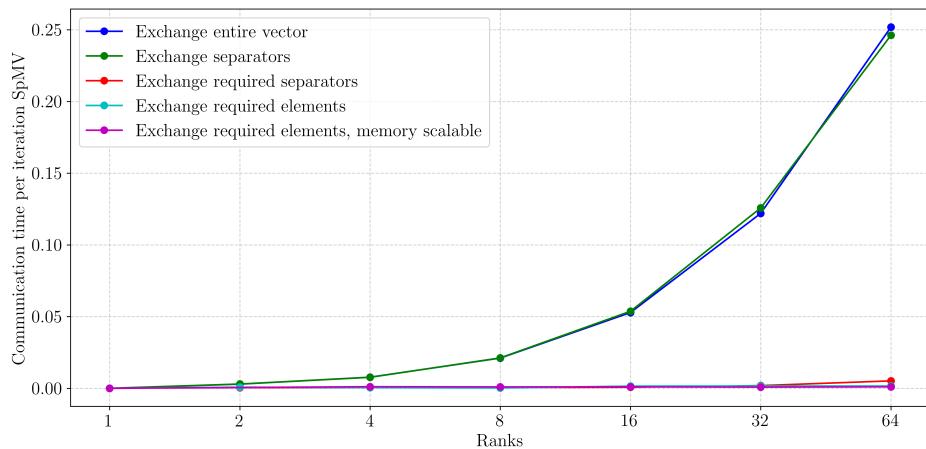


Figure 5.17: af\_shell10

### 5.3.3 Communication Volume AMD EPYC 7302P

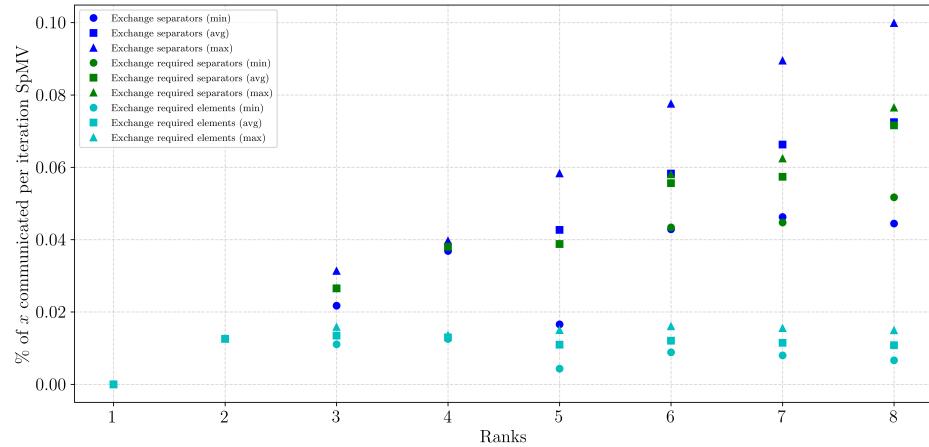


Figure 5.18: Serena

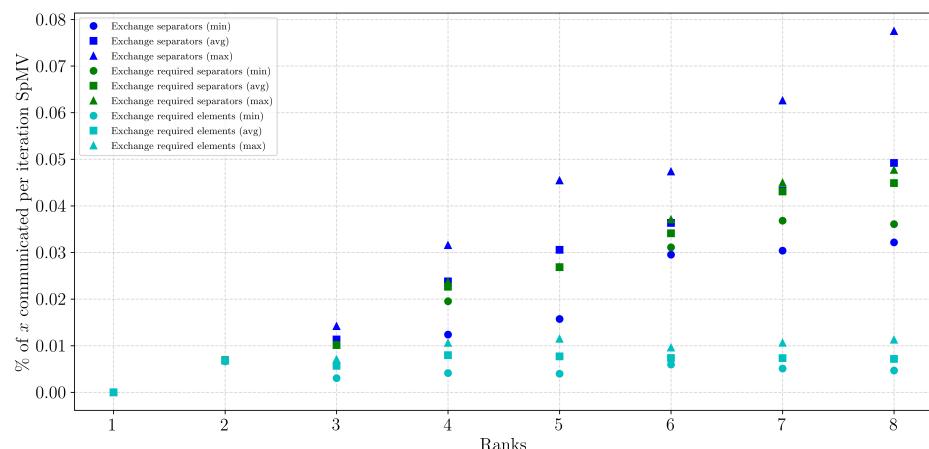


Figure 5.19: bone010

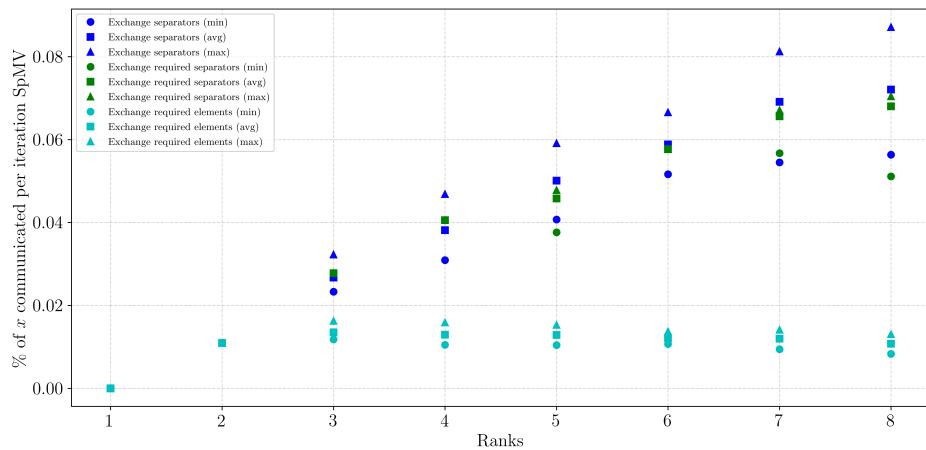


Figure 5.20: Bump\_2911

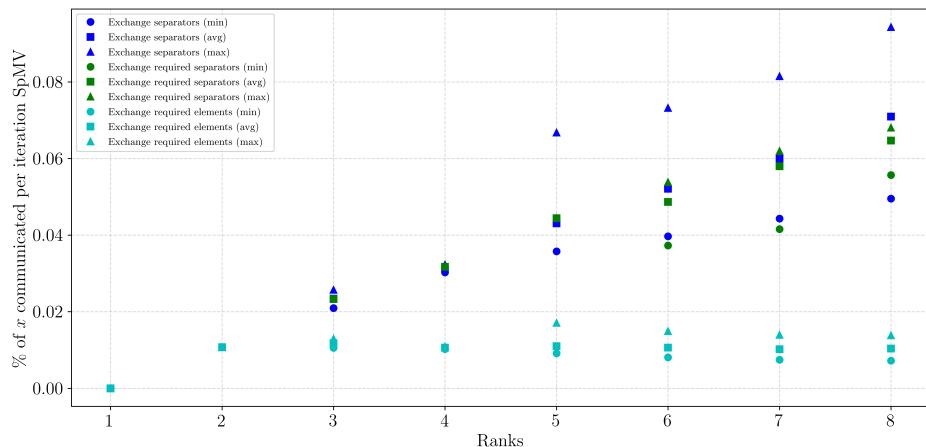


Figure 5.21: Cube\_Coup\_dt0

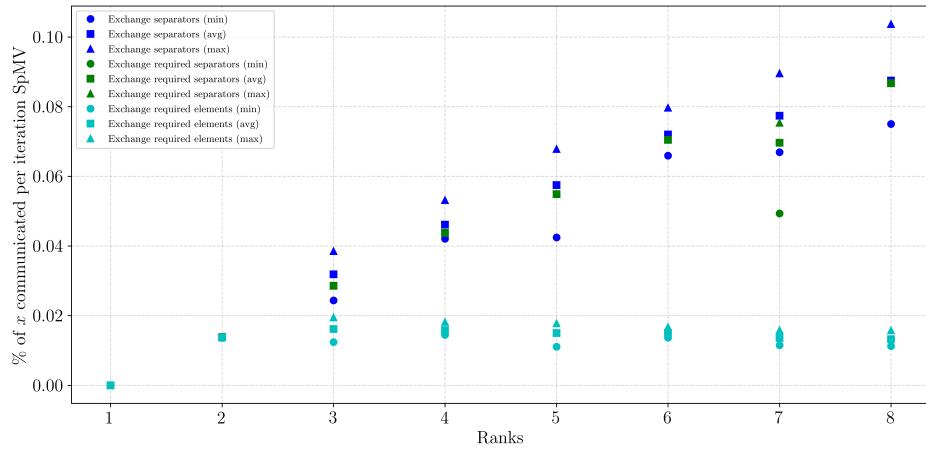


Figure 5.22: Long\_Coup\_dt0

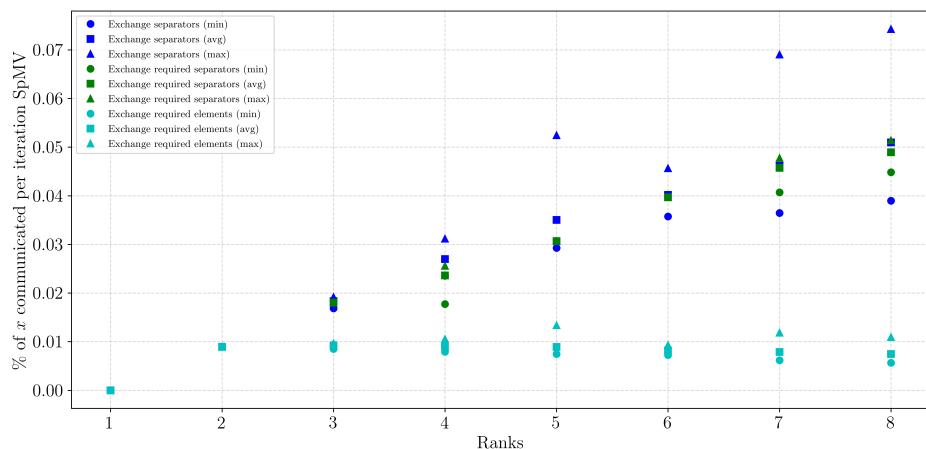


Figure 5.23: nlpkkt200

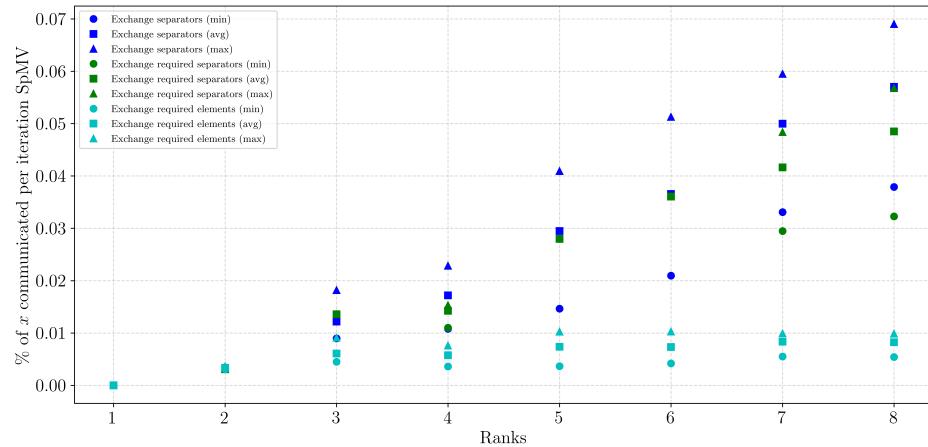


Figure 5.24: dielFilterV3real

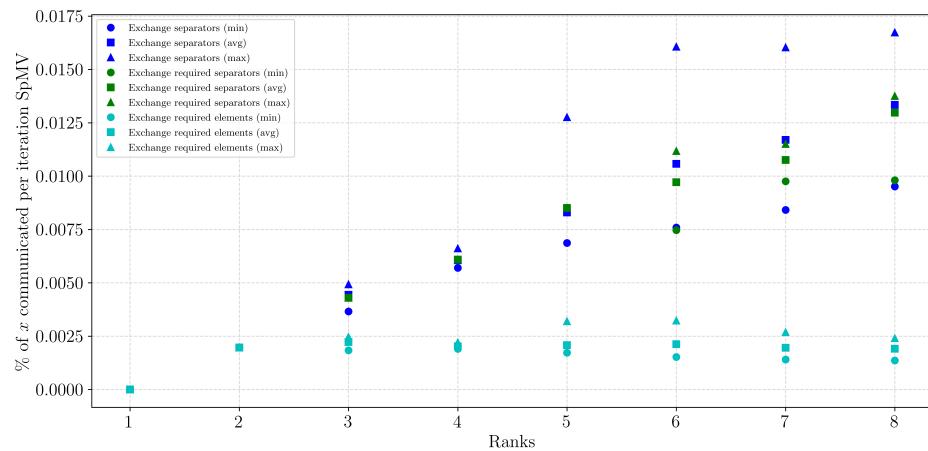


Figure 5.25: af\_shell10

### 5.3.4 Communication Time AMD EPYC 7302P

The following plots illustrate the time spent communicating per iteration of SpMV.

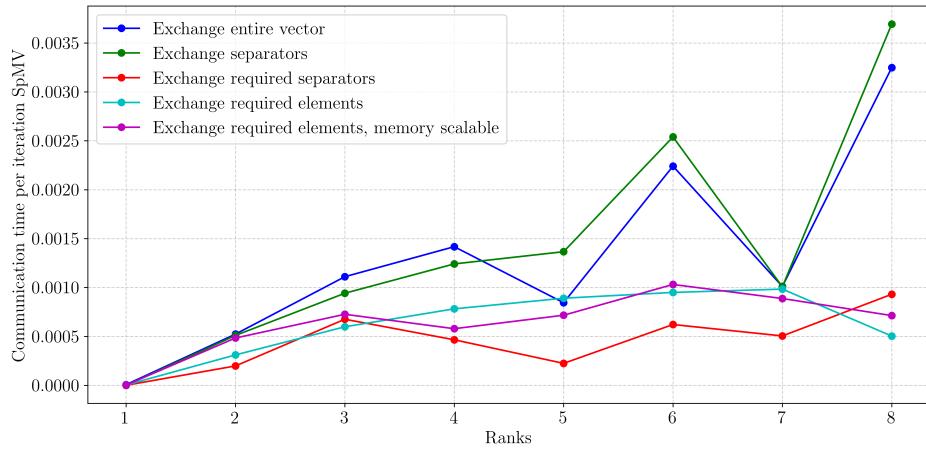


Figure 5.26: Serena

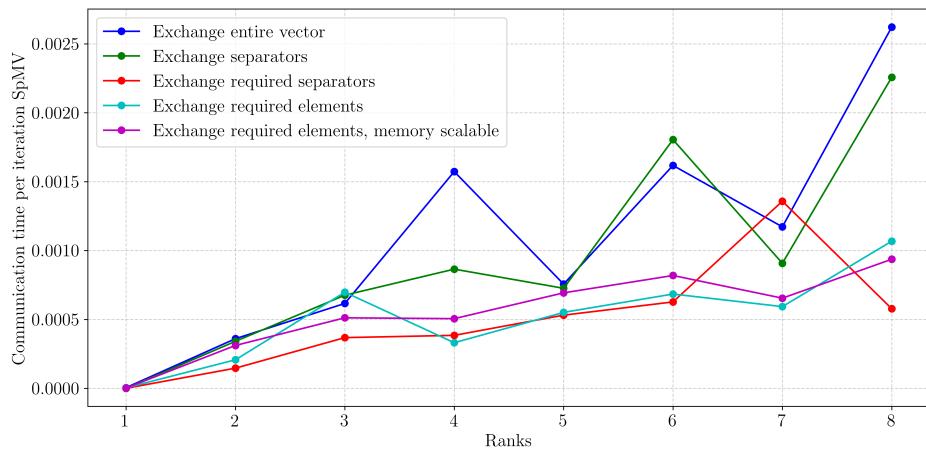


Figure 5.27: bone010

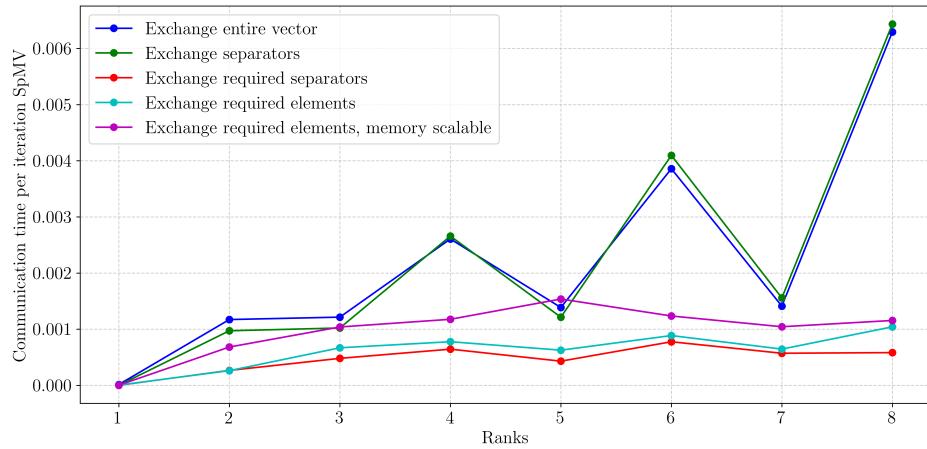


Figure 5.28: Bump\_2911

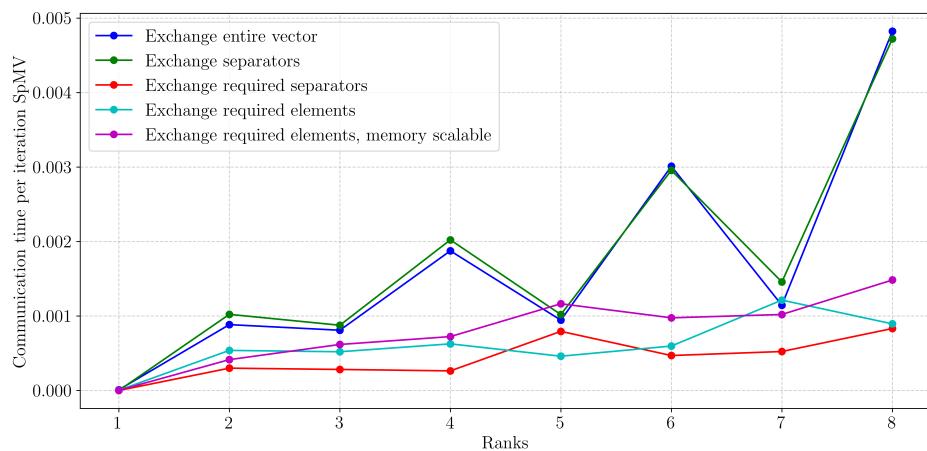


Figure 5.29: Cube\_Coup\_dt0

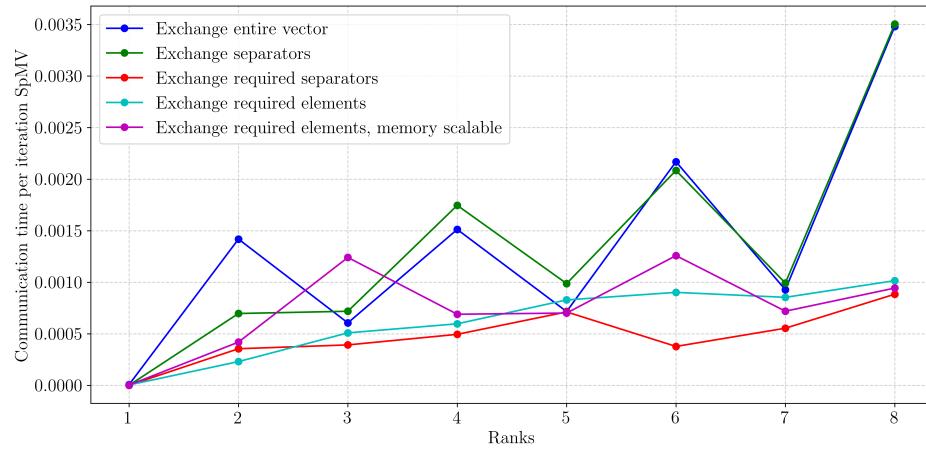


Figure 5.30: Long\_Coup\_dt0

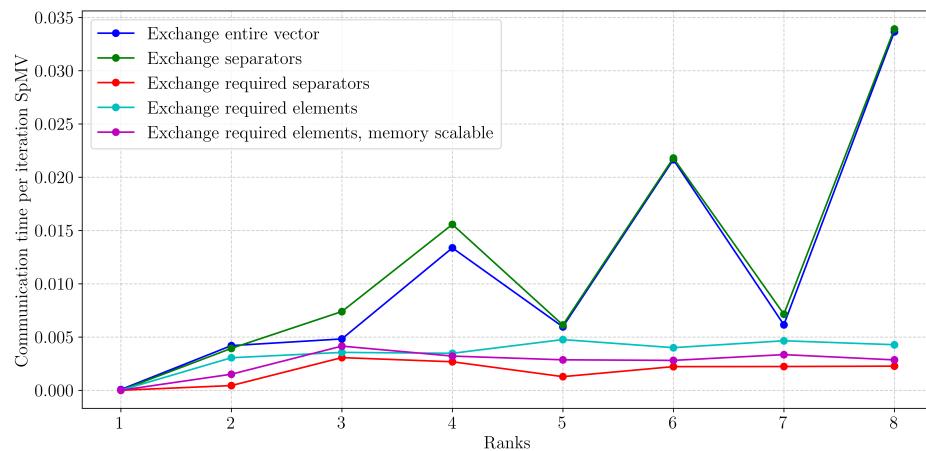


Figure 5.31: nlpkkt200

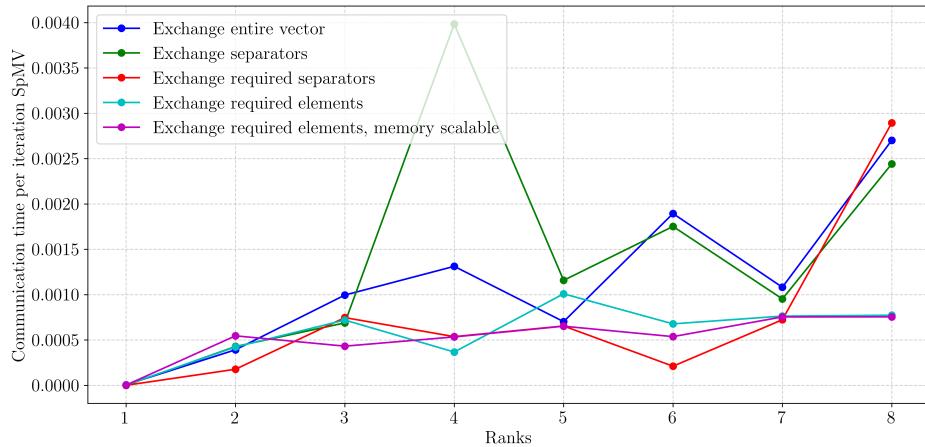


Figure 5.32: dielFilterV3real

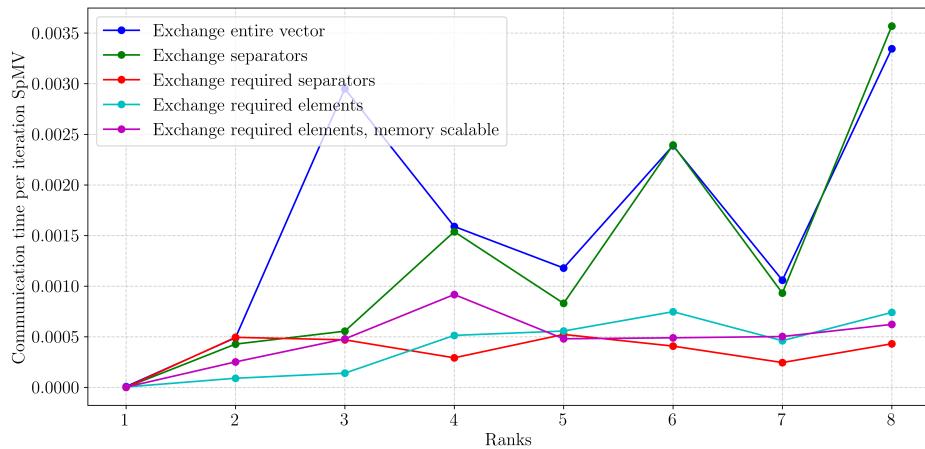


Figure 5.33: af\_shell10

Table 5.1: Hardware used in our experiments. STREAM Triad was run with the `-march=native` and `-O3` compilation flags.

	Xeon-A	FPGAQ	Naples	Rome	Milan	TX2	Hi1620
CPUs	Intel Xeon Gold 6130	AMD Epyc 7413	AMD Epyc 7601	AMD Epyc 7302P	AMD Epyc 7763	Cavium TX2 CN9980	HiSilicon Kunpeng 920-6426
Instr. set	x86-64	x86-64	x86-64	x86-64	x86-64	ARMv8.1	ARMv8.2
Microarch.	Skylake	Zen 3	Zen	Zen 2	Zen 3	Vulcan	TaiShan v110
Sockets	2	2	2	1	2	2	2
Cores	2 × 16	2 × 24	2 × 32	1 × 16	2 × 64	2 × 32	2 × 64
Freq. [GHz]	1.9–3.6	2.6–3.6	2.7–3.2	1.5–3.3	2.5–3.5	2.0–2.5	2.6
L1I/core [KiB]	32	64	64	32	32	32	64
L1D/core [KiB]	32	32	32	32	32	32	64
L2/core [KiB]	1024	512	512	512	512	256	512
L3/socket [MiB]	22	128	64	16	256	32	64
Mem. channels	2 × 6	2 × 8	2 × 8	1 × 8	2 × 8	2 × 8	2 × 8
Bandwidth [GB/s]	256	256(to be added)	342	204.8	409.6	342	342
Triad [GB/s]	147.1	137.4(to be added)	169.7	90.9	256.5	236.4	260.4



# Bibliography

- [1] Luk Burchard, Kristian Gregorius Hustad, Johannes Langguth, and Xing Cai. Enabling unstructured-mesh computation on massively tiled ai processors: An example of accelerating in silico cardiac simulation. *Frontiers in Physics*, Volume 11 - 2023, 2023.
- [2] U.V. Catalyurek and C. Aykanat. Hypergraph-partitioning-based decomposition for parallel sparse-matrix vector multiplication. *IEEE Transactions on Parallel and Distributed Systems*, 10(7):673–693, 1999.
- [3] Gautam Gupta, Sivasankaran Rajamanickam, and Erik G. Boman. GAMGI: Communication-reducing algebraic multigrid for gpus. In *Proceedings of the 28th ACM SIGPLAN Annual Symposium on Principles and Practice of Parallel Programming (PPoPP)*, pages 61–75. ACM, 2024.
- [4] Cerebras Systems Inc. Wse-3 datasheet. <https://cerebras.net/product/system/>, 2024. DS03 v3 821.
- [5] George Karypis and Vipin Kumar. METIS: A Software Package for Partitioning Unstructured Graphs, Partitioning Meshes, and Computing Fill-Reducing Orderings of Sparse Matrices. Technical report, University of Minnesota, Department of Computer Science / Army HPC Research Center, 1997. Version 3.0.3.
- [6] Lawrence Livermore National Laboratory. Introduction to parallel computing tutorial. <http://hpc.llnl.gov/documentation/tutorials/introduction-parallel-computing-tutorial>, 2024. Accessed: 2025-05-08.
- [7] Nakul Manchanda and Karan Anand. Non-uniform memory access (numa). *New York University*, 4, 2010.

- [8] Peter Norvig. Latency numbers every programmer should know. <https://norvig.com/21-days.html#Latency>, 2021. Accessed: 2025-04-29.
- [9] Cong Zheng, Shuo Gu, Tong-Xiang Gu, Bing Yang, and Xing-Ping Liu. Biell: A bisection ellpack-based storage format for optimizing spmv on gpus. *Journal of Parallel and Distributed Computing*, 74(7):2639–2647, 2014. Special Issue on Perspectives on Parallel and Distributed Processing.