Einführung in Visual Computing

186.822

Geometric Transformations

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Transformations in the Rendering Pipeline

- object capture/creation
- **scene objects in object space**
- modeling
- viewing
- projection

vertex stage („vertex shader“)

- **transformed vertices in clip space**
- clipping + homogenization
- **scene in normalized device coordinates**
- viewport transformation
- rasterization
- shading

pixel stage („fragment shader“)

- **raster image in pixel coordinates**

Geometric Transformations in the Rendering Pipeline

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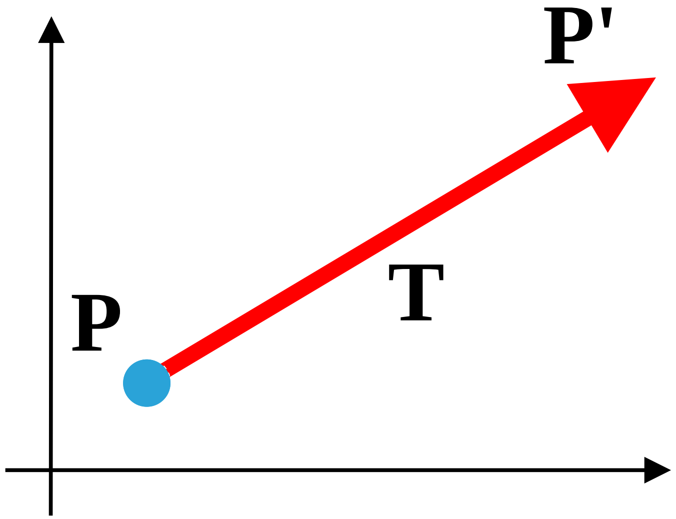
Basic Transformations: Translation

translating a point from position P to P' with translation vector T



$$x' = x + t_x \qquad y' = y + t_y$$

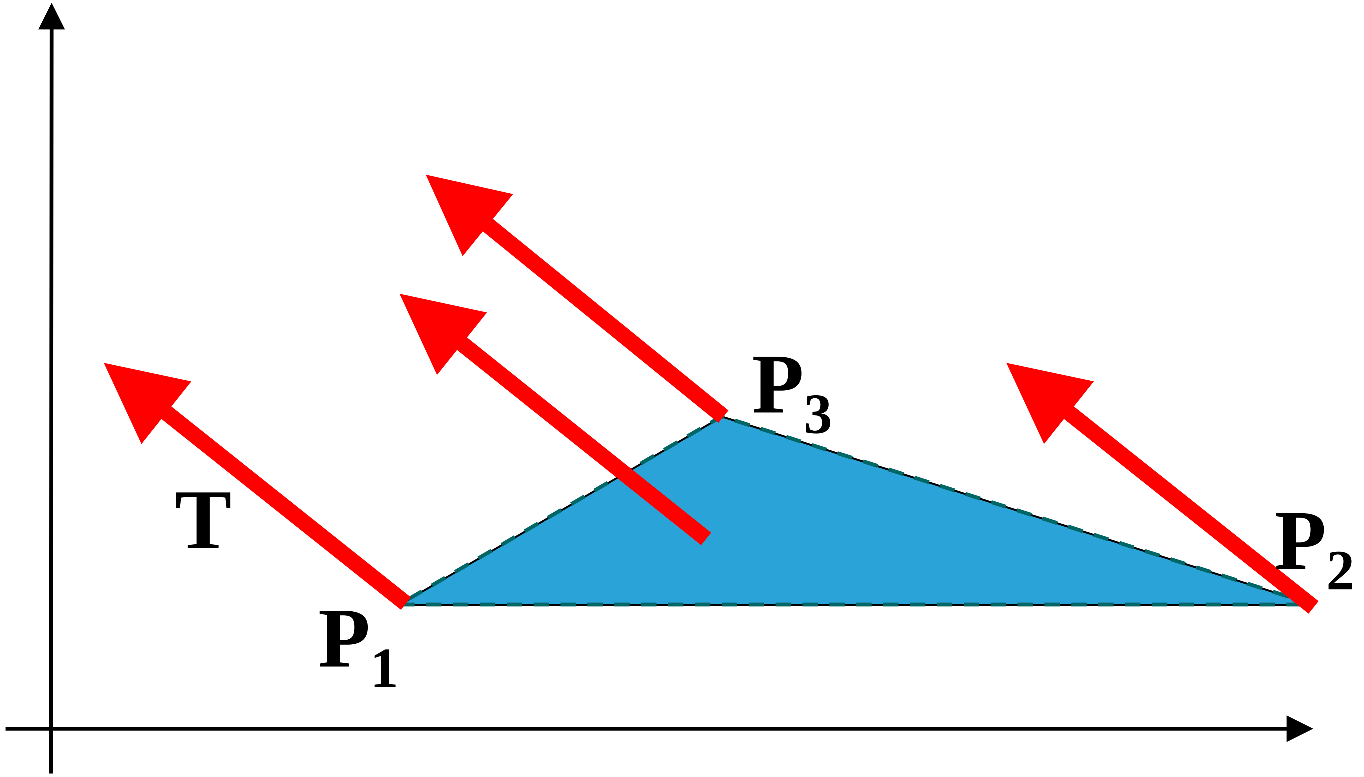
$$\boxed{\mathbf{P}' = \mathbf{P} + \mathbf{T}}$$

notation: $$\mathrm{P} = \begin{pmatrix} x \\ y \end{pmatrix},\ \mathrm{P}' = \begin{pmatrix} x' \\ y' \end{pmatrix},\ \mathrm{T} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

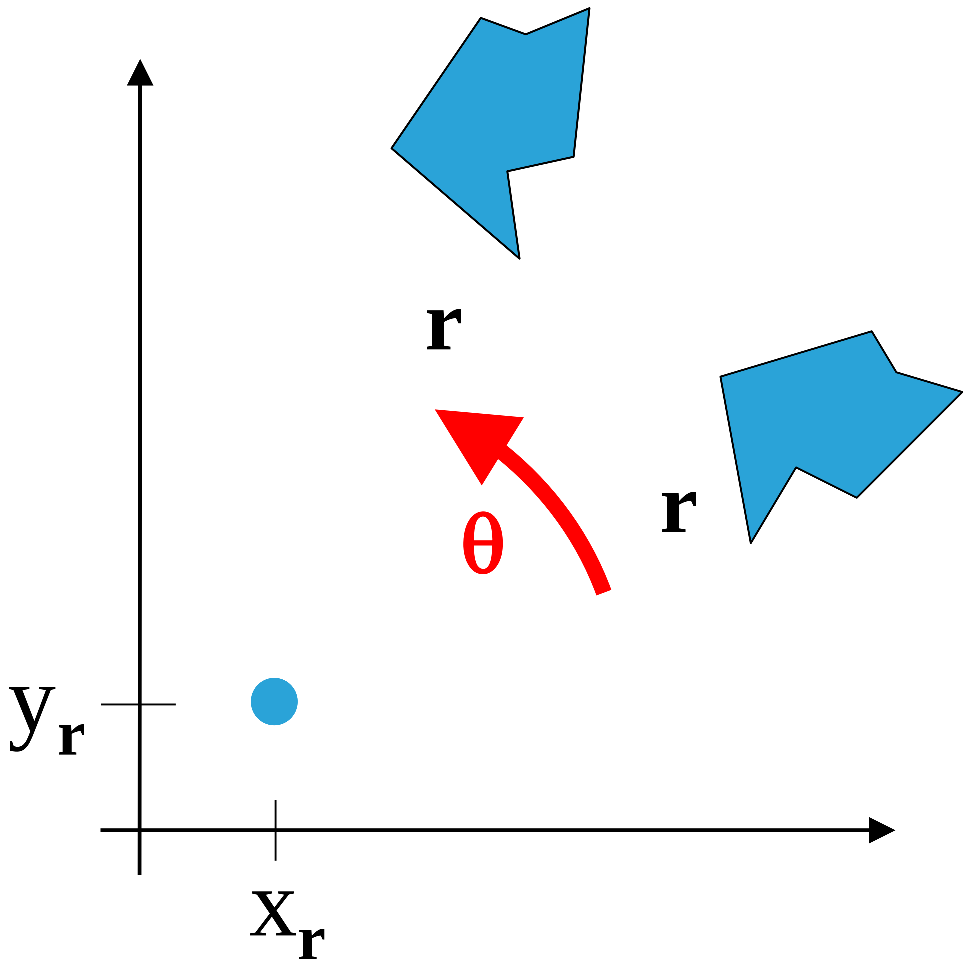
Basic Transformations: Translation

rigid body transformation

object transformed by transforming boundary points



example:
rotation of an object by an angle θ around the pivot point (x_r, y_r)



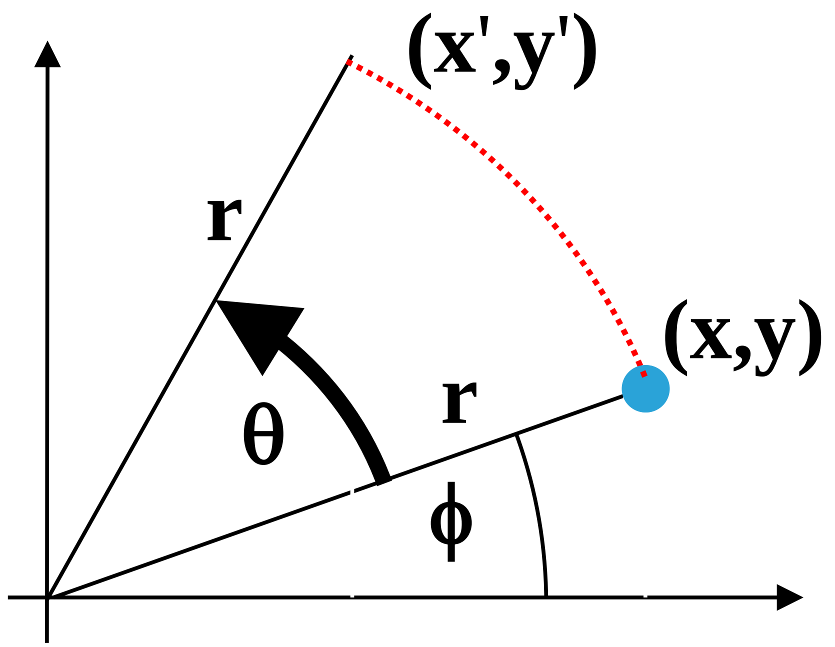
Basic Transformations: Rotation

positive angle $\Rightarrow$ ccw rotation

$$x = r \cdot \cos\phi \qquad y = r \cdot \sin\phi$$

$$\begin{aligned} x' &= r \cdot \cos(\phi + \theta) \\ &= \underline{r \cdot \cos\phi} \cdot \cos\theta - \underline{r \cdot \sin\phi} \cdot \sin\theta \\ &= \quad \mathbf{x} \quad \cdot \cos\theta - \quad \mathbf{y} \quad \cdot \sin\theta \end{aligned}$$

$$\begin{aligned} y' &= r \cdot \sin(\phi + \theta) \\ &= r \cdot \cos\phi \cdot \sin\theta + r \cdot \sin\phi \cdot \cos\theta \end{aligned}$$



$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

formulation with a transformation matrix:

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

$$P' = R \cdot P \quad \text{with} \quad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$R \cdot P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

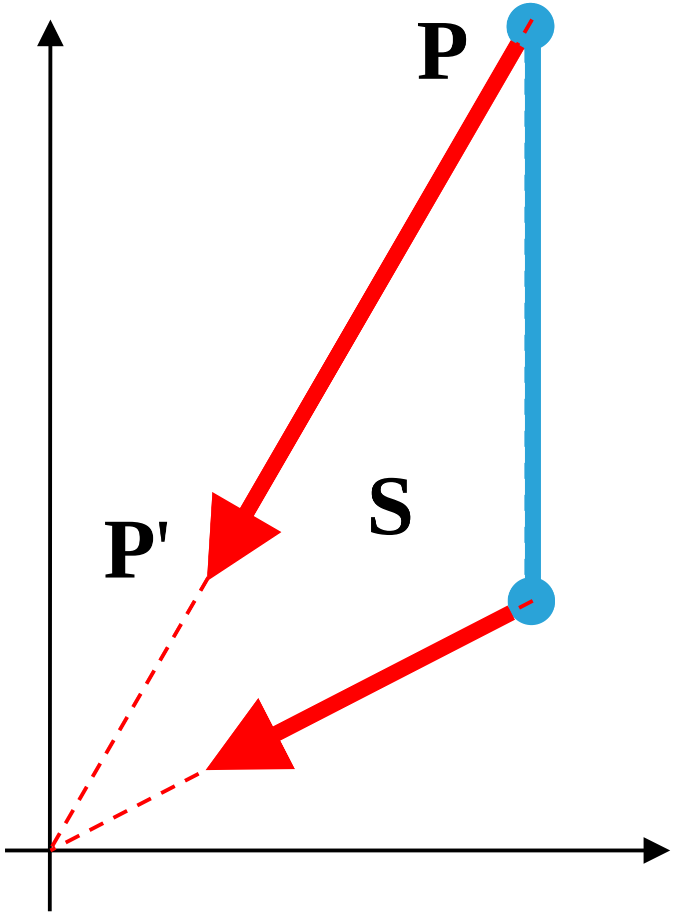
Basic Transformations: Scaling

$$x' = x \cdot s_x, \qquad y' = y \cdot s_y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

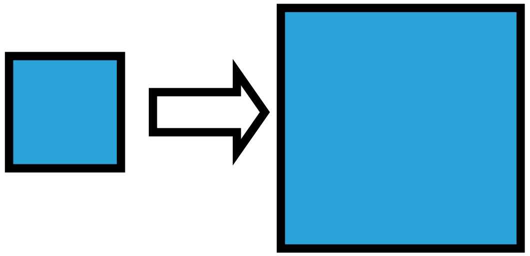
$$\boxed{P' = S \cdot P}$$

example: a line scaled using $s_x = s_y = 0.33$ is reduced in size and moved closer to the coordinate origin

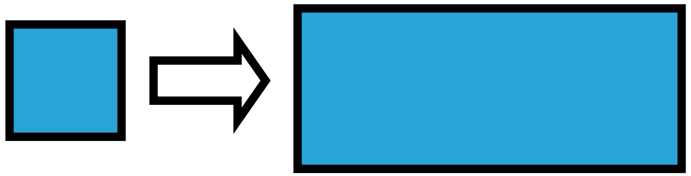


Basic Transformations: Scaling

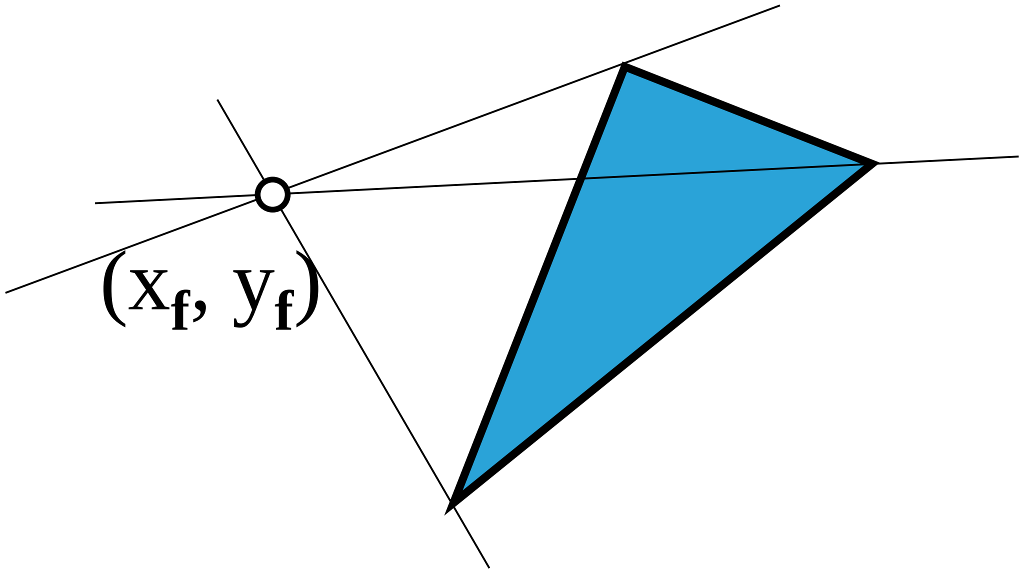
- uniform scaling: $S_{\mathbf{x}} = S_{\mathbf{y}}$



- differential scaling: $S_{\mathbf{x}} \neq S_{\mathbf{y}}$



- fixed point:



Transformation Matrices

- scaling

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

- rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

- x-mirroring

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

- translation

$$(x'\ y') = (x + t_x,\ y + t_y) \quad \ldots ?$$

Homogeneous Coordinates (1)

instead of $\begin{pmatrix} x \\ y \end{pmatrix}$ use $\begin{pmatrix} x_h \\ y_h \\ h \end{pmatrix}$ with $x = x_h/h, \quad y = y_h/h$

very often $h=1$, i.e. $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

in this way all transformations can be formulated in matrix form

Homogeneous Coordinates (2)

notation:

- translation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$P' = T(t_x, t_y) \cdot P$$

- rotation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$P' = R(\theta) \cdot P$$

- scaling

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$P' = S(s_x, s_y) \cdot P$$

Inverse Matrices

- translation $$T^{-1}(t_x, t_y) = T(-t_x, -t_y)$$
- rotation $$R^{-1}(\theta) = R(-\theta)$$
- scaling $$S^{-1}(s_x, s_y) = S(1/s_x, 1/s_y)$$

Composite Transformations (1)

n transformations are applied after each other on a point P, these transformations are represented by matrices
$M_1, M_2, \ldots, M_n$.

$$P' = M_1 \cdot P$$

$$P'' = M_2 \cdot P'$$

$$\ldots$$

$$P^{(n)} = M_n \cdot P^{(n-1)}$$

shorter: $P^{(n)} = (M_n \cdot \ldots (M_2 \cdot (M_1 \cdot P)) \ldots)$

Composite Transformations (2)

$$\boxed{P^{(n)} = (M_n \cdot \ldots (M_2 \cdot (M_1 \cdot P)\,)\ \ldots\ )}$$

matrix multiplications are **associative**:

$$(\mathbf{M_1} \cdot \mathbf{M_2}) \cdot \mathbf{M_3} = \mathbf{M_1} \cdot (\mathbf{M_2} \cdot \mathbf{M_3})$$

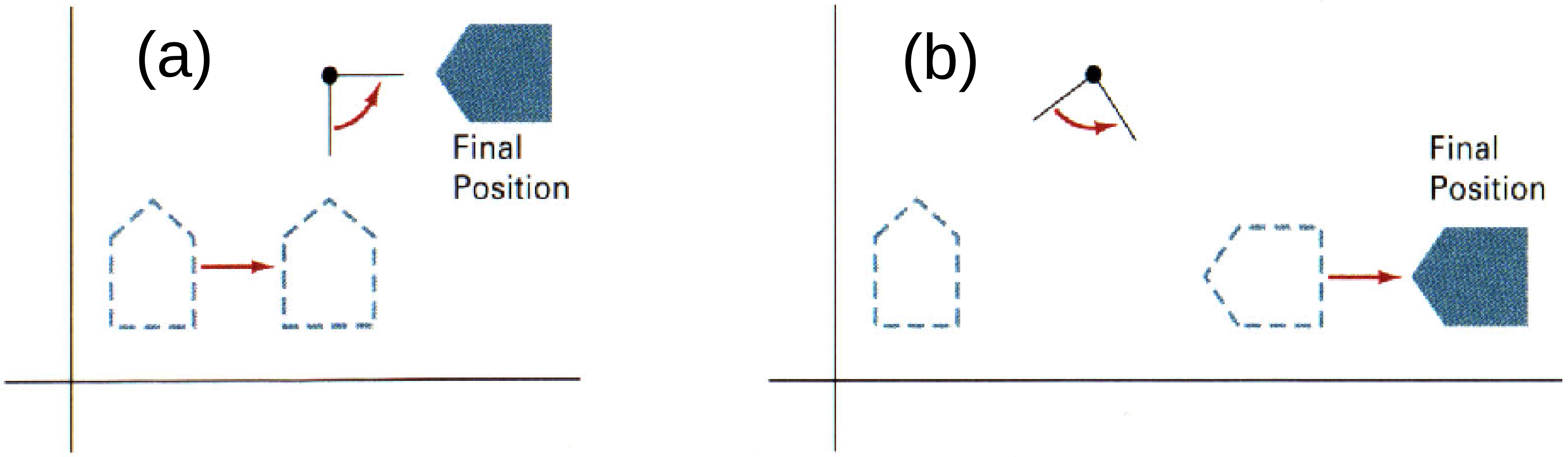
(but not commutative: $M_1 \cdot M_2 \neq M_2 \cdot M_1$)

Transformations are not commutative!

Reversing the order in which a sequence of transformations is performed may affect the transformed position of an object!

→ in (a), an object is first translated, then rotated.

→ in (b), the object is rotated first, then translated.



Composite Transformations (2)

$$\boxed{P^{(n)} = (M_n \cdot \ldots (M_2 \cdot (M_1 \cdot P)\,)\ldots)}$$

matrix multiplications are **associative:**

$$(\mathbf{M}_1 \cdot \mathbf{M}_2) \cdot \mathbf{M}_3 = \mathbf{M}_1 \cdot (\mathbf{M}_2 \cdot \mathbf{M}_3)$$

(but not commutative: $M_1 \cdot M_2 \neq M_2 \cdot M_1$)

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therefore the total transformation can also be written as: $P^{(n)} = (\, \mathbf{M_n \cdot \ldots \cdot M_2 \cdot M_1} \,) \cdot P$

constant for whole images, objects, etc.!!!

Composite Transformations (3)

simple composite transformations

- composite translations

$$T(t_{x2},t_{y2})\cdot T(t_{x1},t_{y1}) = T(t_{x1}+t_{x2},t_{y1}+t_{y2})$$

- composite rotations

$$R(\theta_2)\cdot R(\theta_1) = R(\theta_1+\theta_2)$$

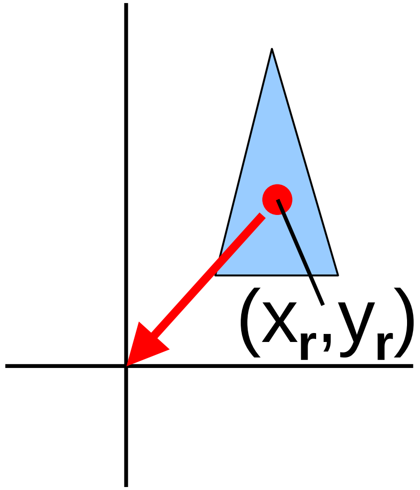
- composite scaling

$$S(s_{x2},s_{y2})\cdot S(s_{x1},s_{y1}) = S(s_{x1}\cdot s_{x2},s_{y1}\cdot s_{y2})$$

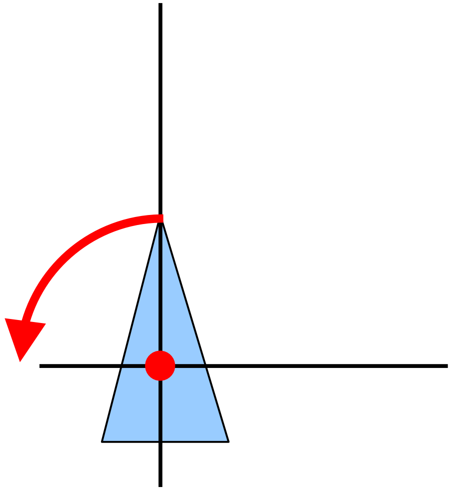
Composite Transformations (4)

general pivot-point rotation

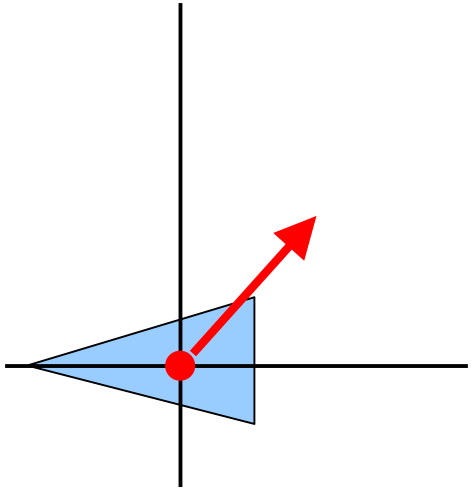
$$T(x_r, y_r) \cdot R(\theta) \cdot T(-x_r, -y_r) = R(x_r, y_r, \theta)$$



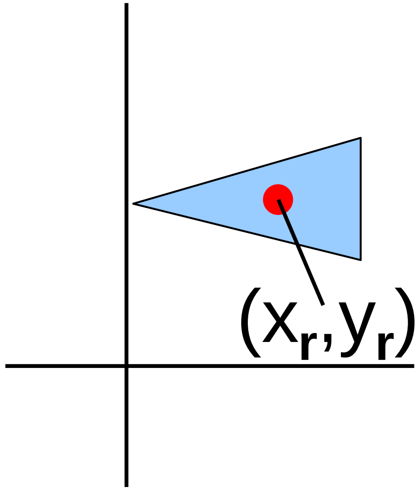
original position and pivot point



translation of object so that pivot point is at origin



rotation about origin

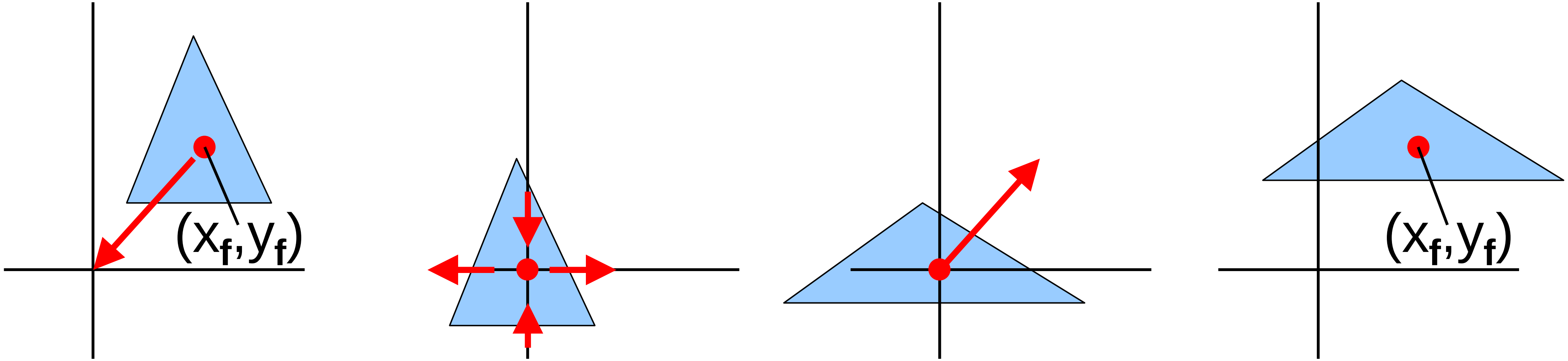


translation so that the pivot point is returned

Composite Transformations (5)

general fixed-point scaling

$$T(x_f, y_f) \cdot S(s_x, s_y) \cdot T(-x_f, -y_f) = S(x_f, y_f, s_x, s_y)$$



original position and fixed point

translate object so that fixed point is at origin

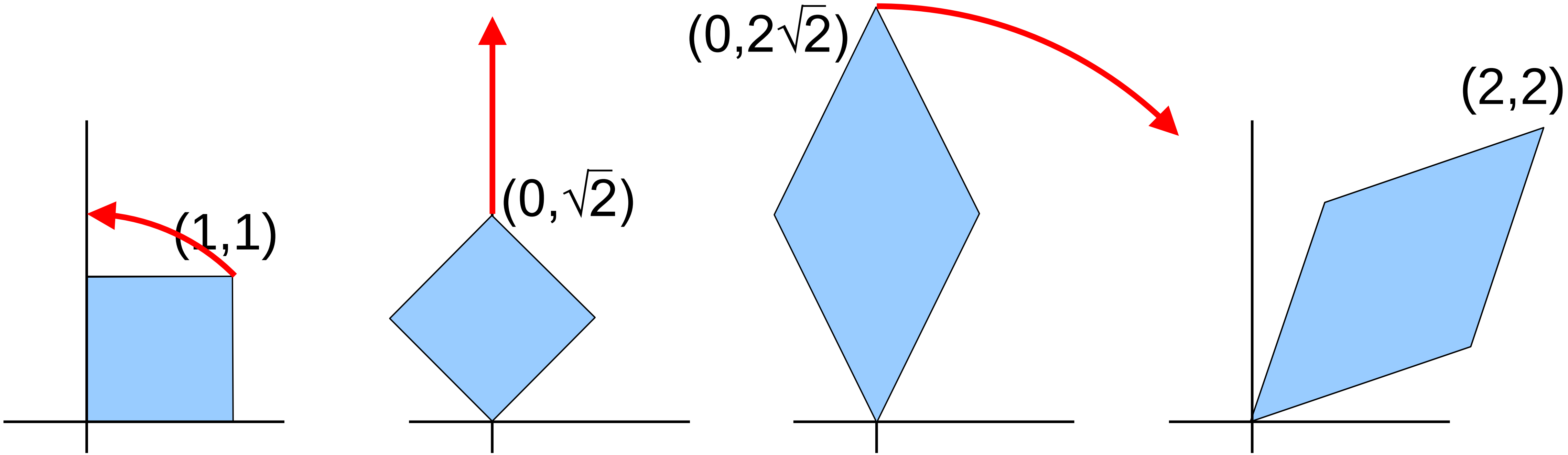
scale object with respect to origin

translate so that the fixed point is returned

Composite Transformations (6)

general scaling direction

$$R^{-1}(\theta)\cdot S(s_1,s_2)\cdot R(\theta)$$



original position

after 45° rotation

after (1,2) scaling

after - 45 ° rotation back

Example

translate by (3,4), then rotate by 45° and then scale up by factor 2 in x-direction

1. $M_1 = T(3,4) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

2. $M_2 = R(45°) = \begin{pmatrix} \cos 45° & -\sin 45° & 0 \\ \sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $M_3 = S(2,1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\boxed{M = M_3 \cdot M_2 \cdot M_1}$

Example

translate by (3,4), then rotate by 45° and then scale up by factor 2 in x-direction

$$M = M_3 \cdot M_2 \cdot M_1 =$$

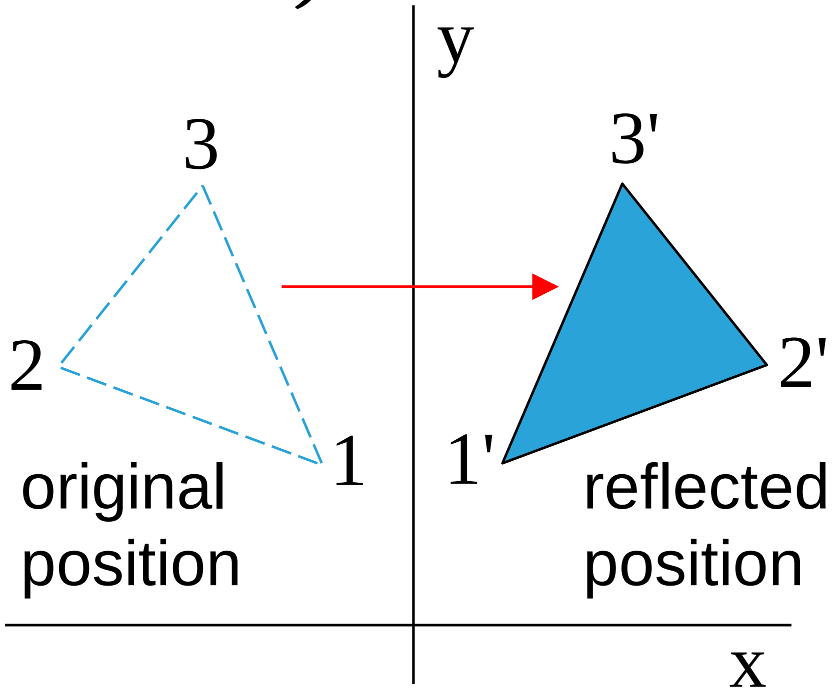
$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 45° & -\sin 45° & 0 \\ \sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 45° & -\sin 45° & 3\cos 45° - 4\sin 45° \\ \sin 45° & \cos 45° & 3\sin 45° + 4\cos 45° \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \underline{\underline{\begin{pmatrix} 2\cos 45° & -2\sin 45° & 6\cos 45° - 8\sin 45° \\ \sin 45° & \cos 45° & 3\sin 45° + 4\cos 45° \\ 0 & 0 & 1 \end{pmatrix}}}$$

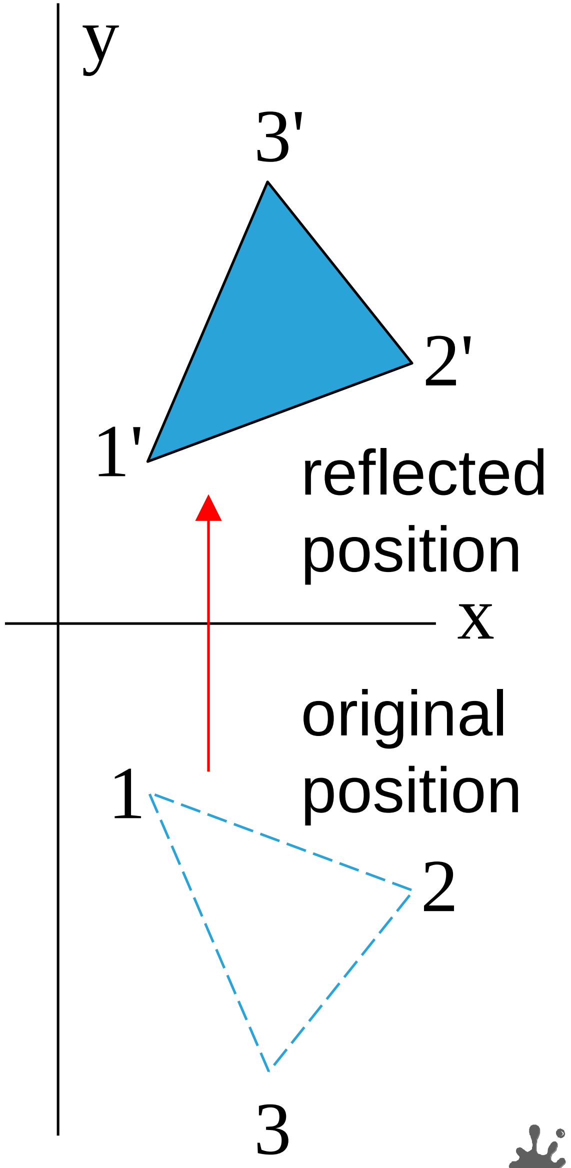
about **y**-axis:

$$Rf_{\mathbf{y}} = \begin{pmatrix} \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$



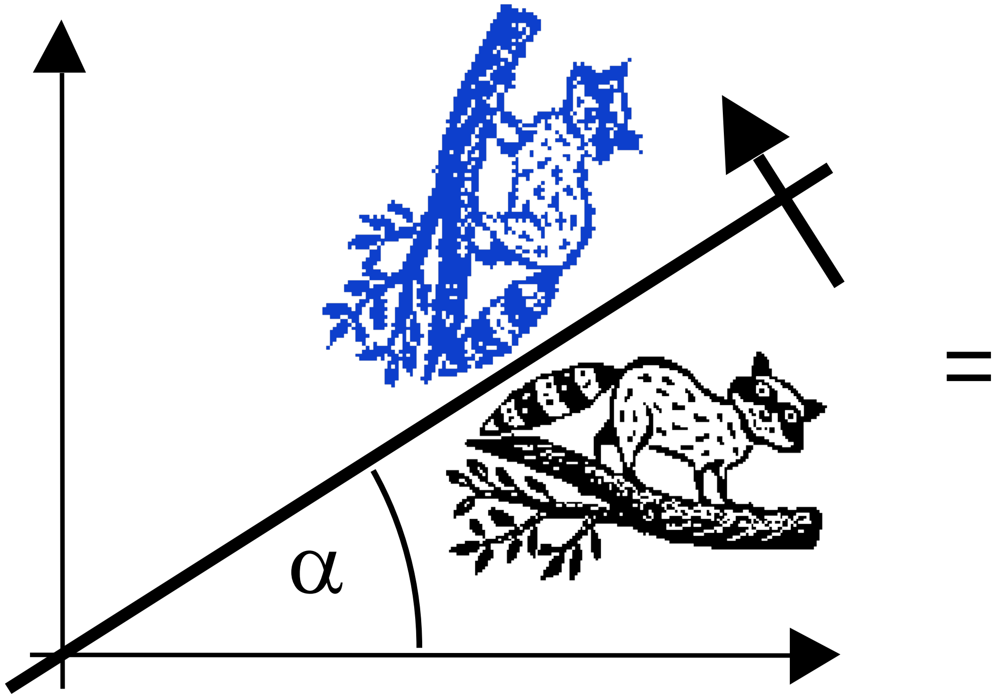
about **x**-axis:

$$Rf_{\mathbf{x}} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$



Example

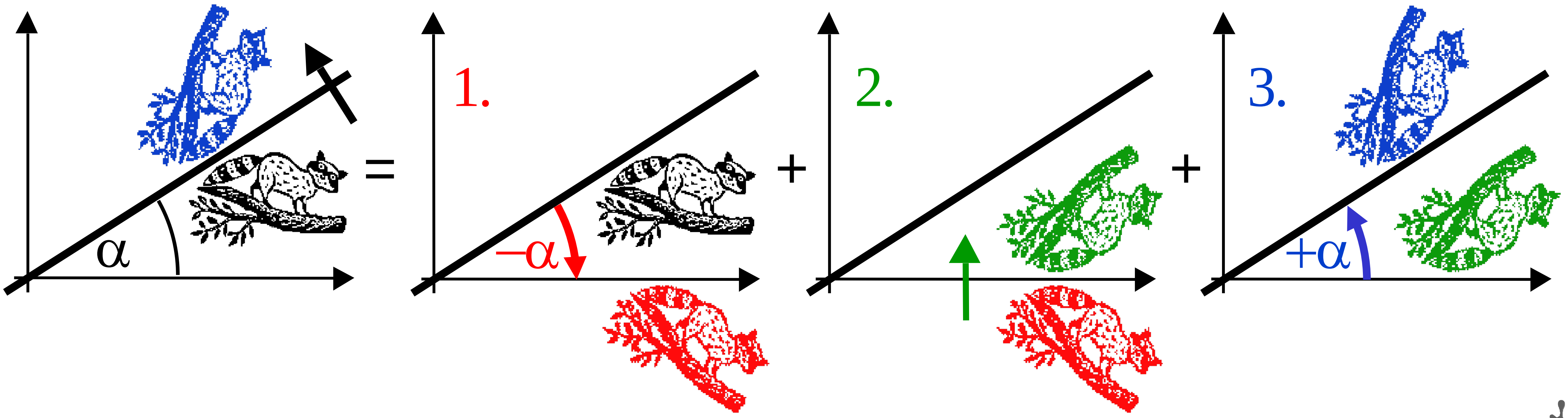
reflection about the axis with angle α



Example

reflection about the axis with angle α

1. rotation by –α
2. mirroring about x-axis
3. rotation by +α



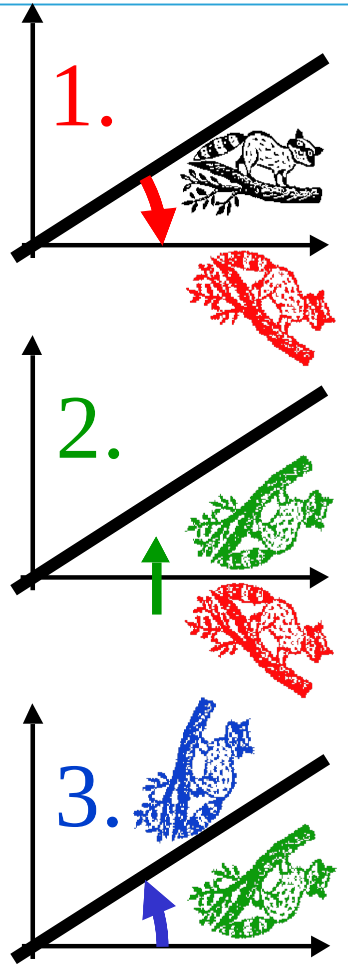
Example

reflection about the axis with angle α

1. $M_1 = R(-\alpha) = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $M_2 = S(1,-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $M_3 = R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$



$$\boxed{P' = M_3 \cdot (M_2 \cdot (M_1 \cdot P)) = (M_3 \cdot M_2 \cdot M_1) \cdot P}$$

Example

reflection about the axis with angle α

$$\mathbf{M_3 \cdot M_2 \cdot M_1} =$$

$$= \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \cdot \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

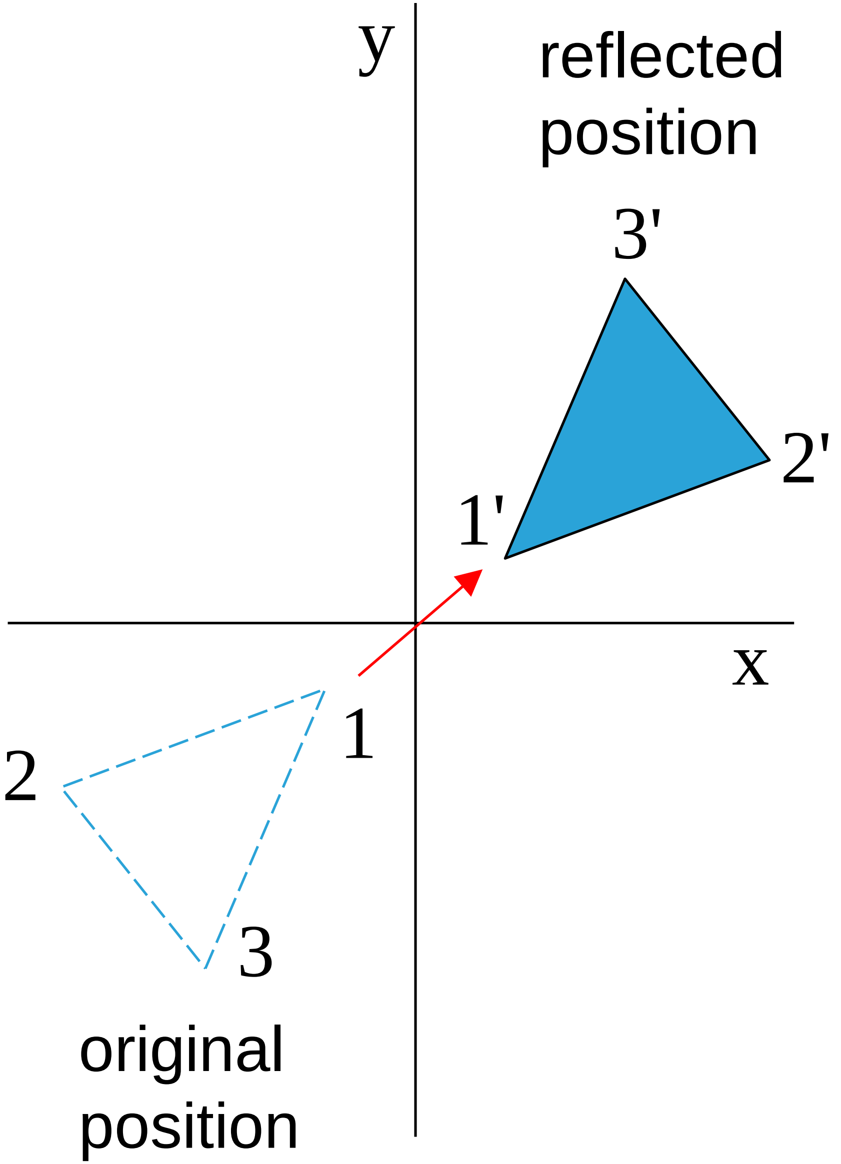
$$= \underbrace{\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}} =$$

$$= \begin{pmatrix} \cos^2\alpha - \sin^2\alpha & 2\sin\alpha\cos\alpha & 0 \\ 2\sin\alpha\cos\alpha & \sin^2\alpha - \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Other Transformations: Reflection about a Point

reflection about origin

$$Rf_O\,(=R(180°)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Reflection with Respect to a General Line

reflection with respect to the line $\mathbf{y=mx+b}$

$$T(0,b) \cdot R(\theta) \cdot S(1,-1) \cdot R(-\theta) \cdot T(0,-b)$$

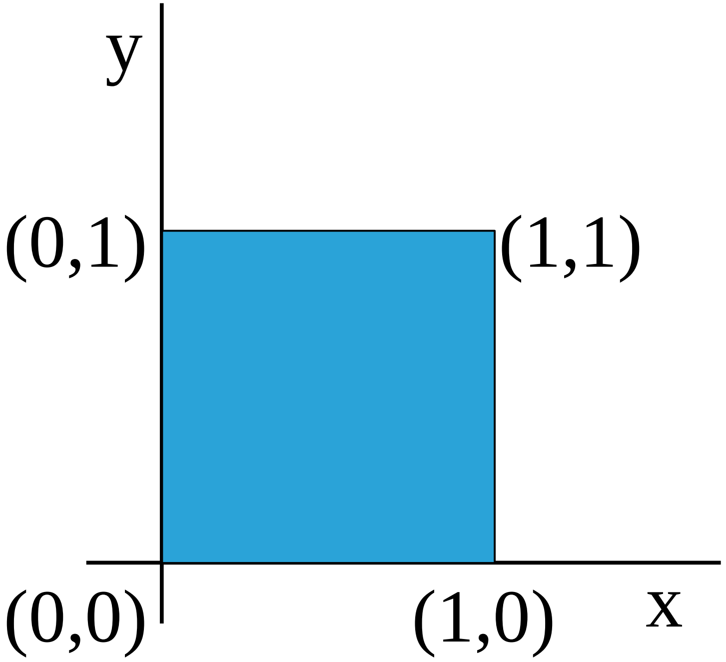
$$m = \tan(\theta)$$

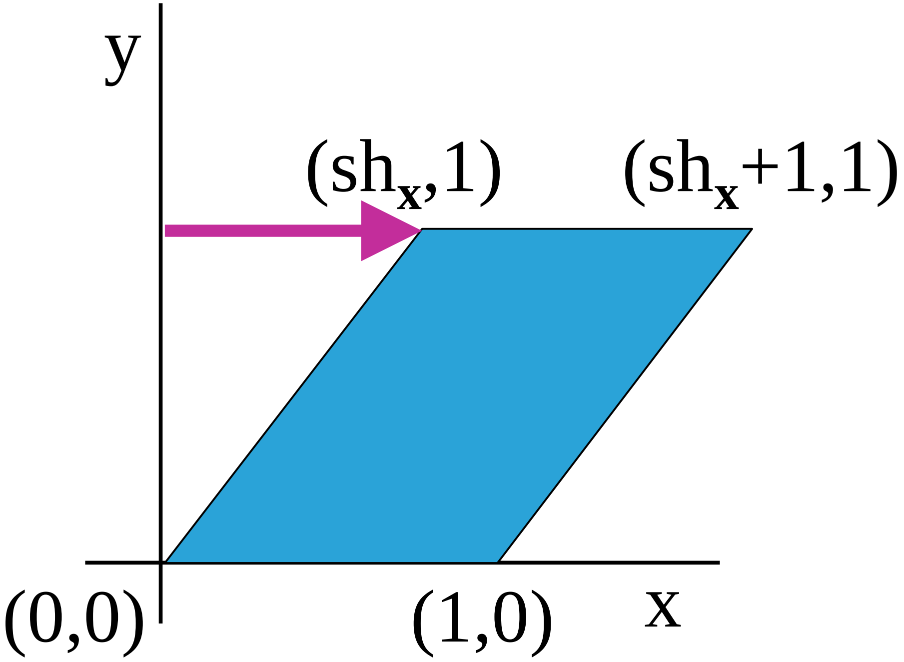
Other Transformations: Shear (1)

x-direction shear

- along x-axis
- reference line y=0

$$\begin{pmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

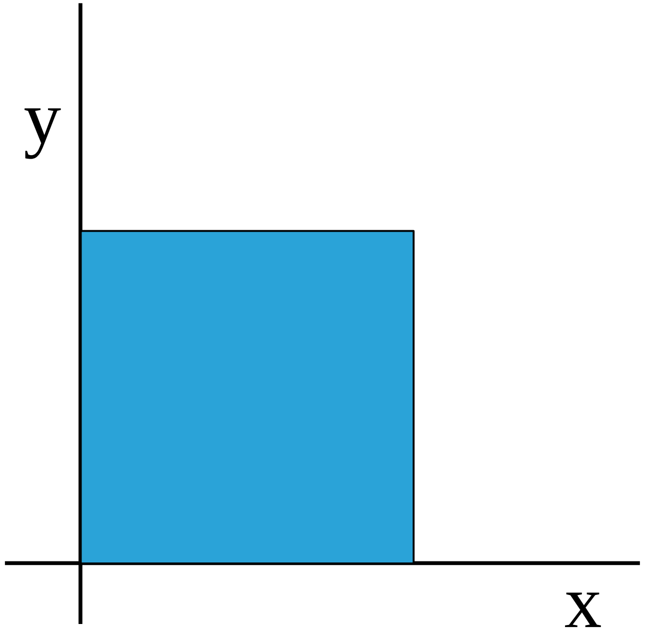


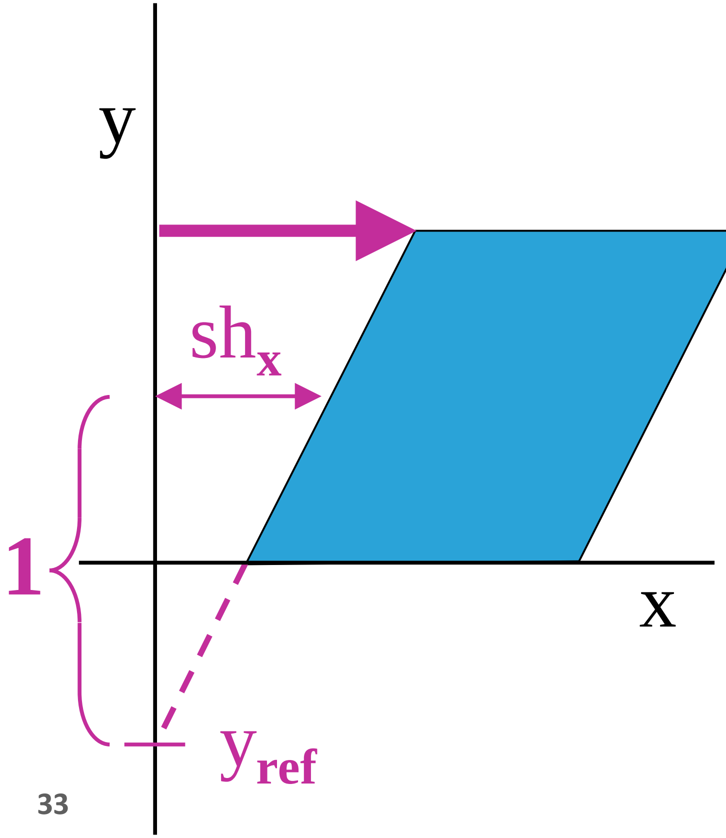


general x-direction shear

- along x-axis
- reference line $y = y_{ref}$

$$\begin{pmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

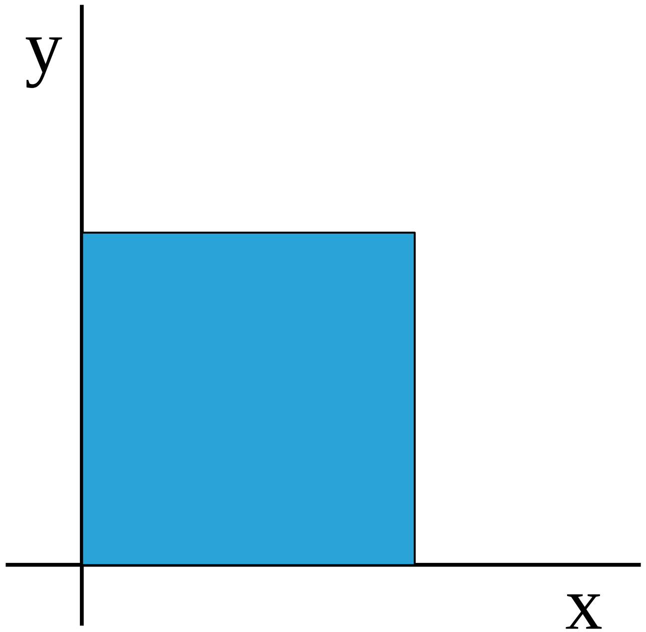


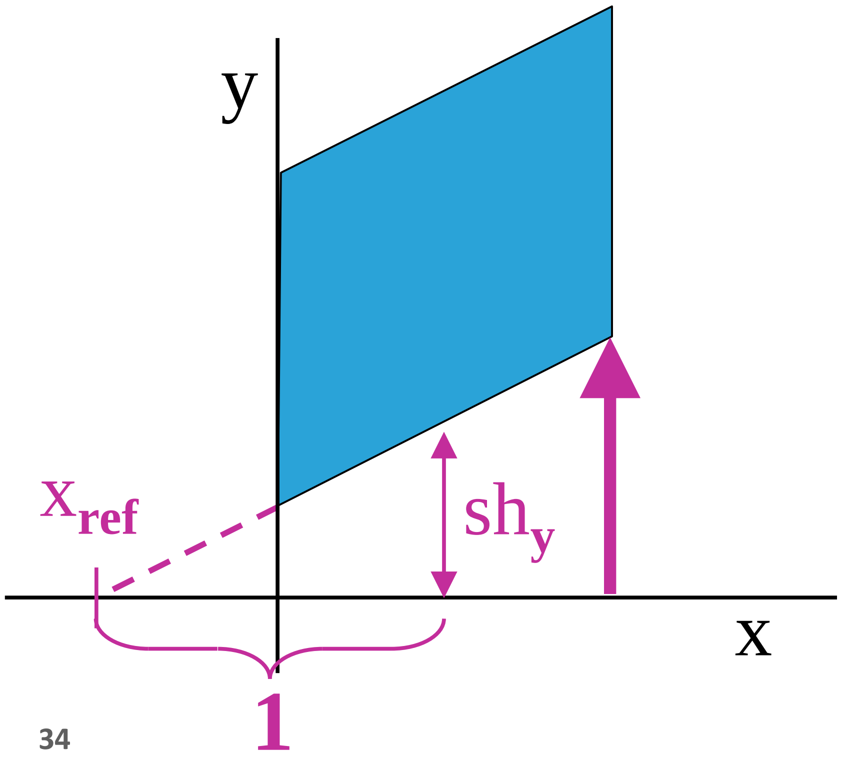


general y-direction shear

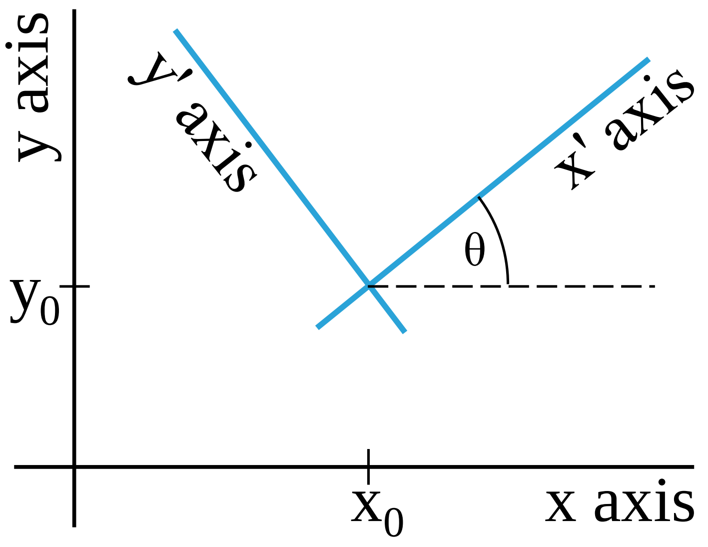
- along y-axis
- reference line $x=x_{ref}$

$$\begin{pmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{pmatrix}$$



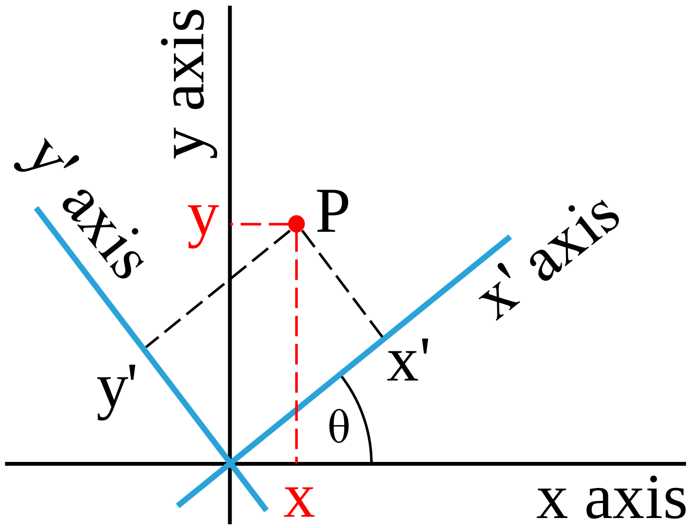


Transformation between Coordinate Systems



$$M_{xy,x'y'} = R(-\theta)\cdot T(-x_0, -y_0)$$

a Cartesian $x'y'$ system positioned at (x_0, y_0) with orientation θ in an xy Cartesian system



position of the reference frames after translating the origin of the $x'y'$ system to the coordinate origin of the xy system

Affine Transformations

$$x' = a_{xx}x + a_{xy}y + b_x$$

$$y' = a_{yx}x + a_{yy}y + b_y$$

- collinear $\Rightarrow$ points on a line stay on a line
- parallel lines $\Rightarrow$ parallel lines
- ratios of distances along a line are preserved
- finite points $\Rightarrow$ finite points
- any affine transformation is a combination of translation, rotation, scaling, (reflection, shear)
- translation, rotation, reflection only:
  - angle, length preserving

3D Transformations

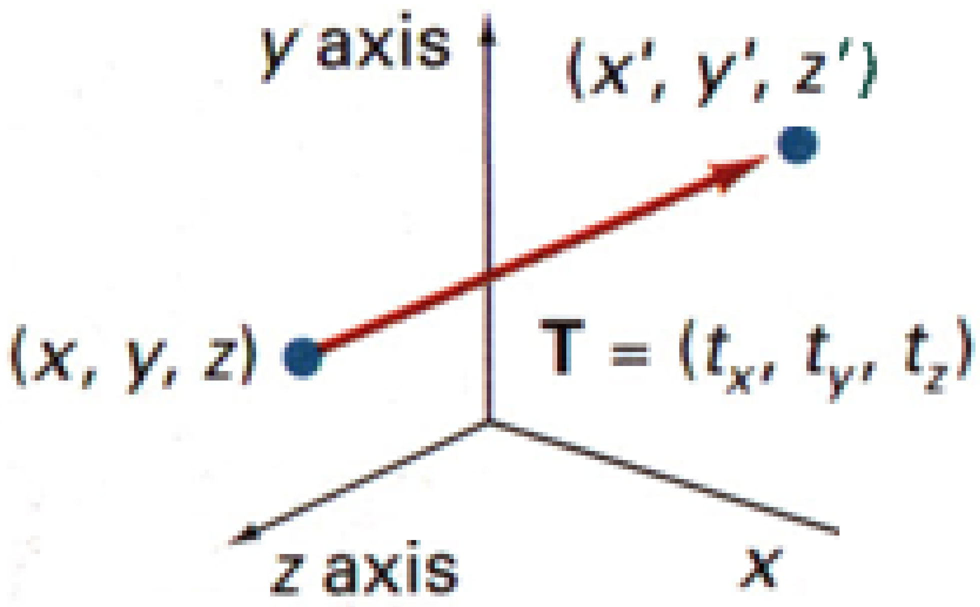
- all concepts can be extended to 3D in a straight forward way
- plus projections 3D → 2D

3D Translation (1)

translation vector (t_x, t_y, t_z)

$$\mathbf{x' = x + t_x, \quad y' = y + t_y, \quad z' = z + t_z}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$\mathbf{P' = T(t_x, t_y, t_z) \cdot P}$$

3D Translation (2)

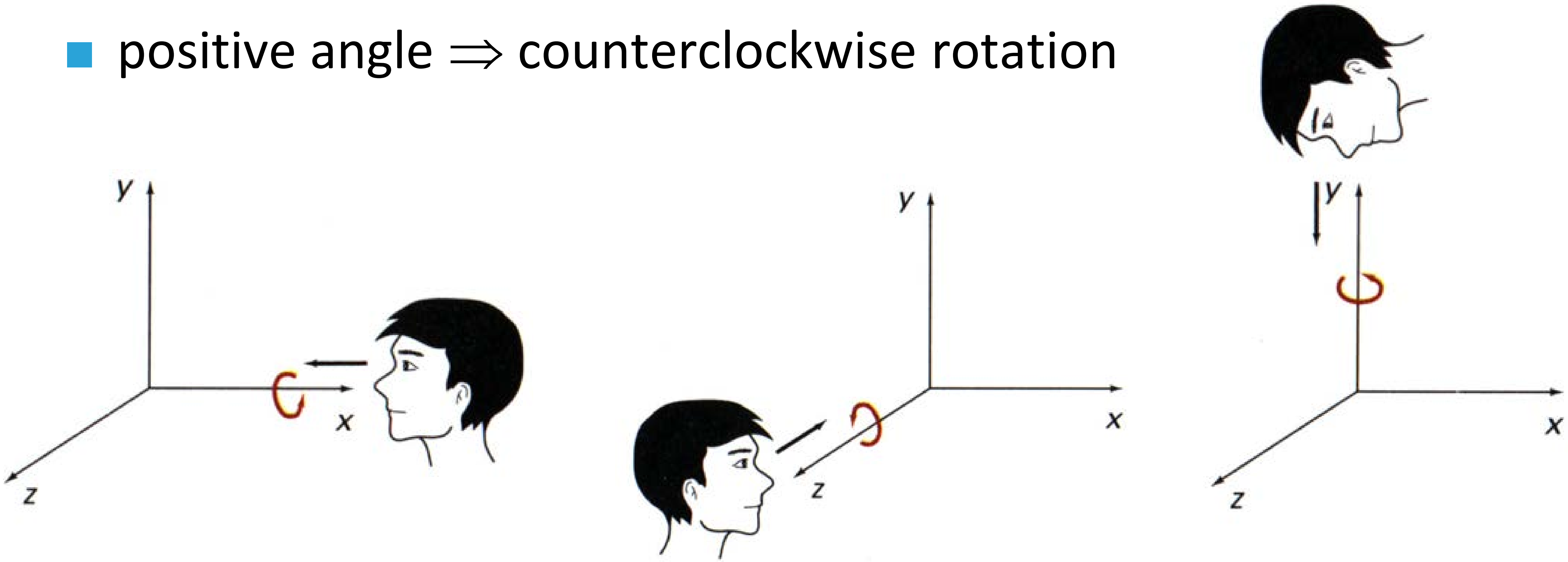
objects are translated by translating boundary points

inverse of translation:

$$\mathbf{T}^{-1}(\mathbf{t}_x, \mathbf{t}_y, \mathbf{t}_z) = \mathbf{T}(-\mathbf{t}_x, -\mathbf{t}_y, -\mathbf{t}_z)$$

3D Rotation: Angle Orientation

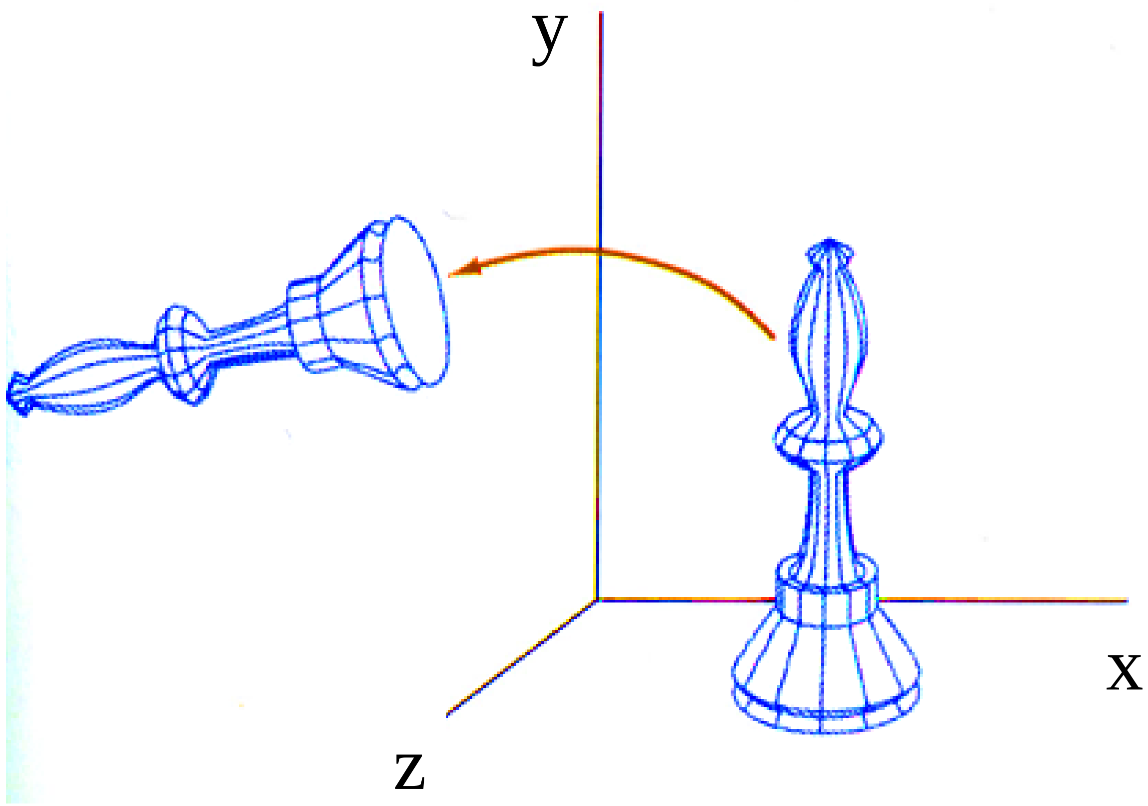
- 3 options for rotation axis
- positive angle $\Rightarrow$ counterclockwise rotation



3D Rotation: Coordinate Axes (z-axis)

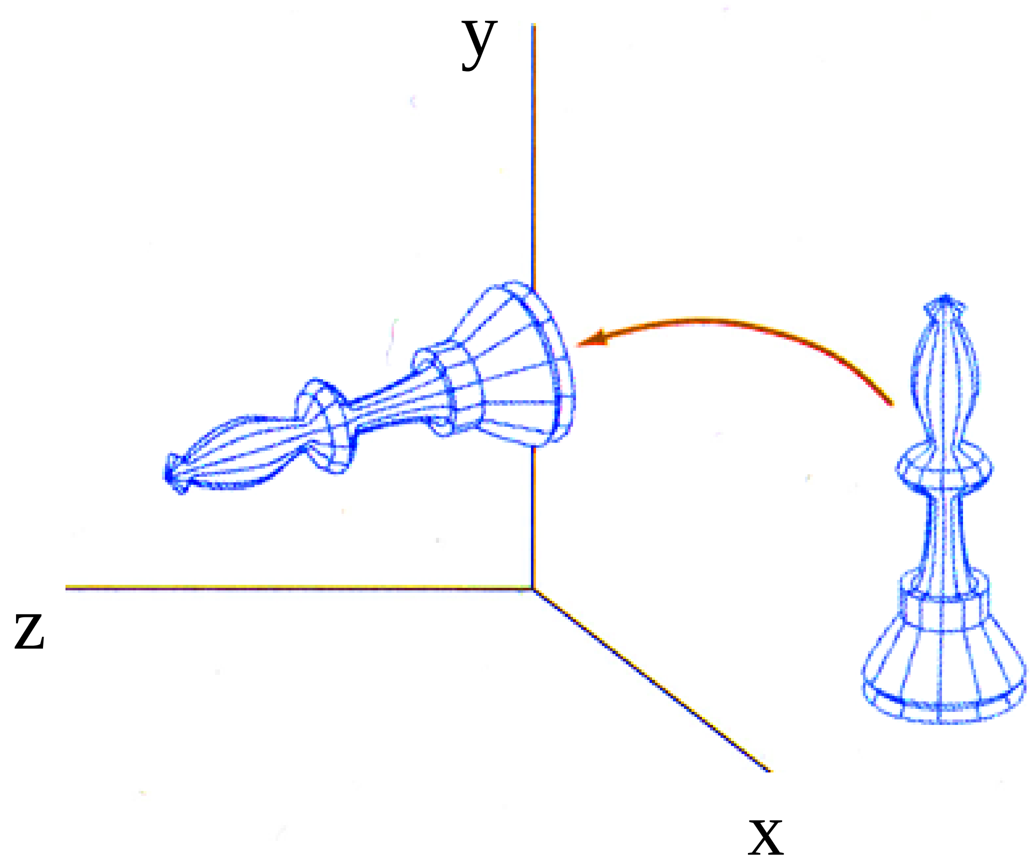
$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$
$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$P' = R_z(\theta) \cdot P$$

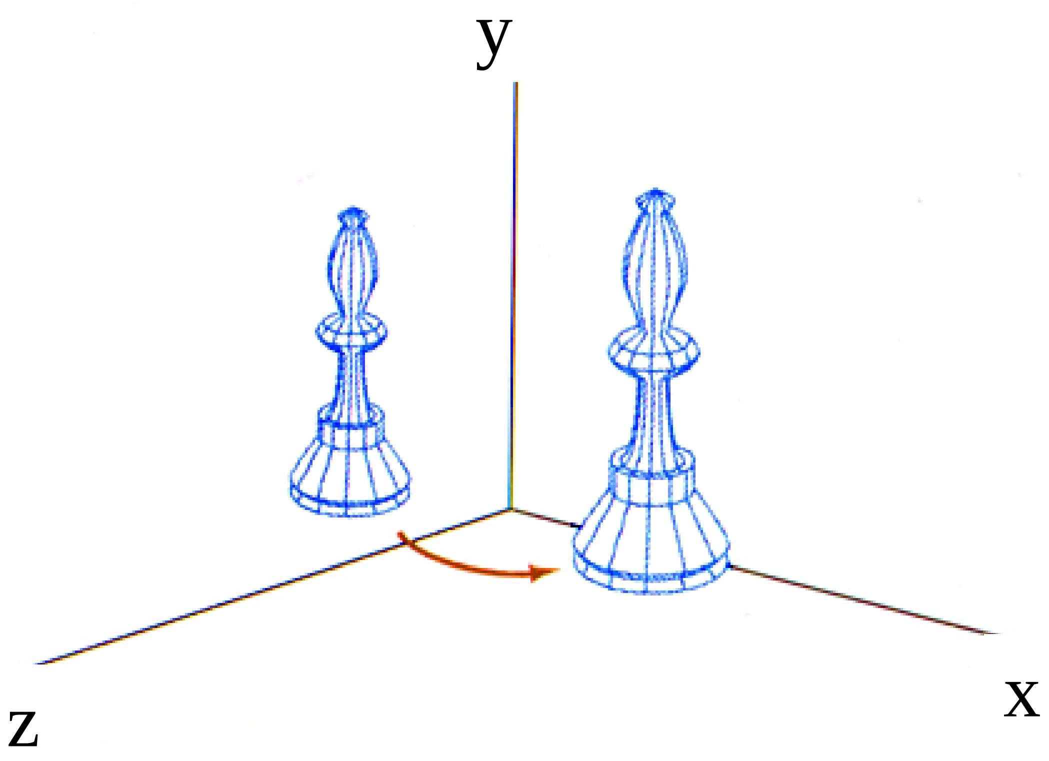
3D Rotation: Coordinate Axes (x-axis)



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_x(\theta) \cdot P$$

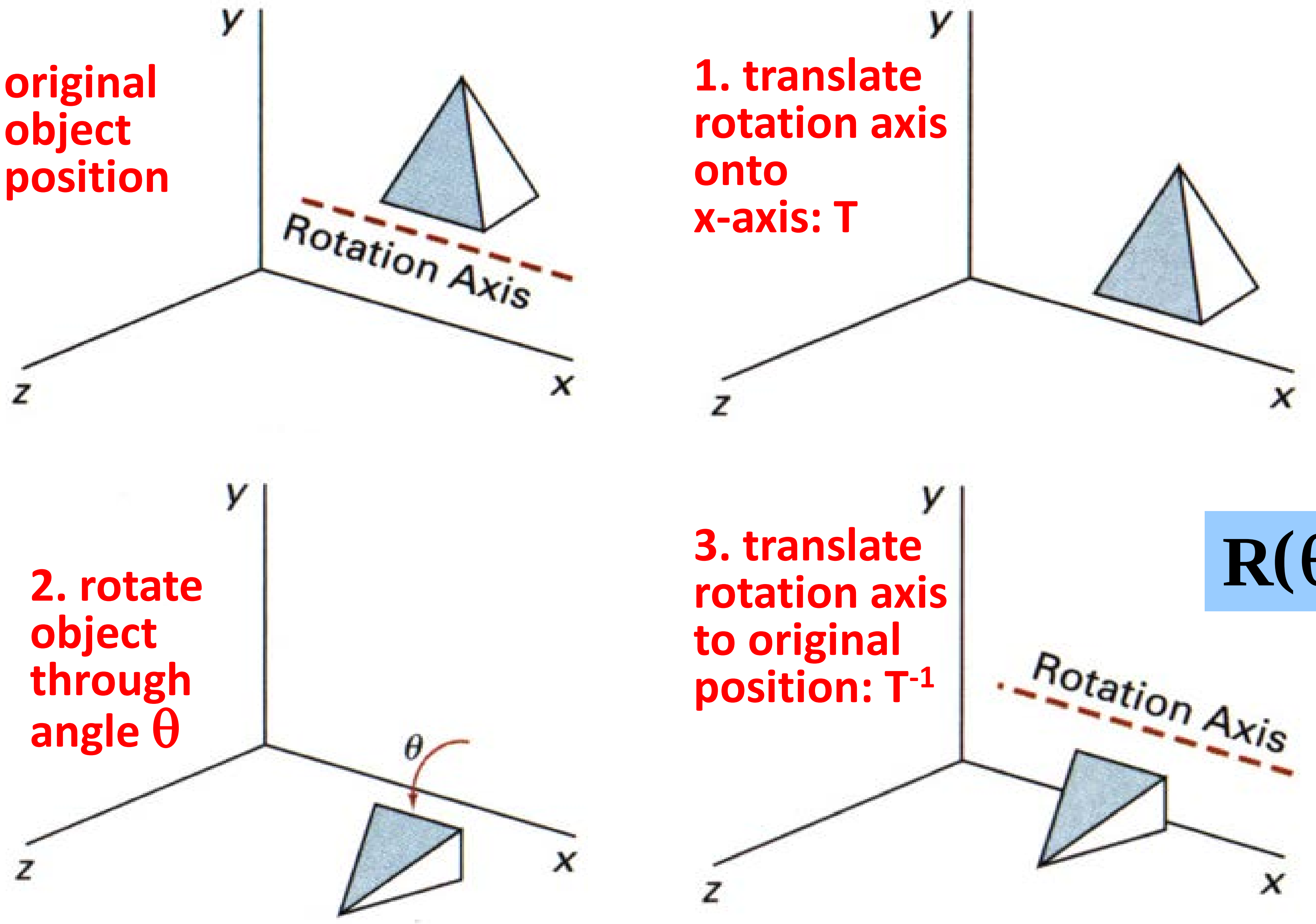
3D Rotation: Coordinate Axes (y-axis)



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_y(\theta) \cdot P$$

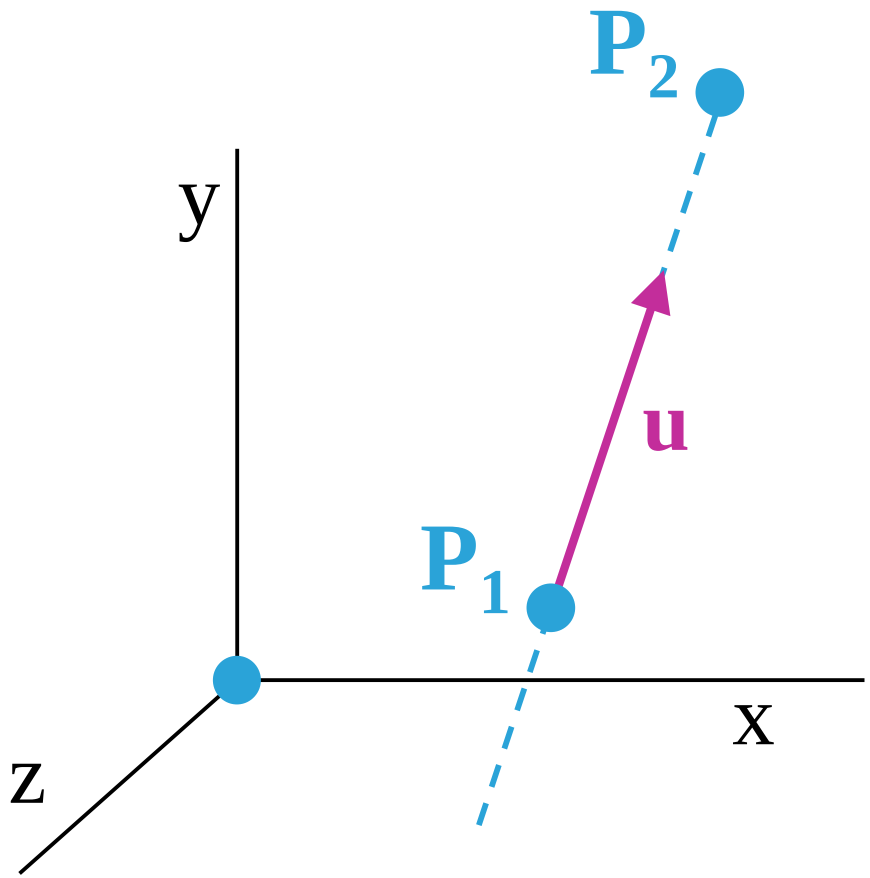
3D Rotation: Axis Parallel to x-Axis



$$\mathbf{R}(\theta) = \mathbf{T}^{-1} \cdot \mathbf{R}_x(\theta) \cdot \mathbf{T}$$

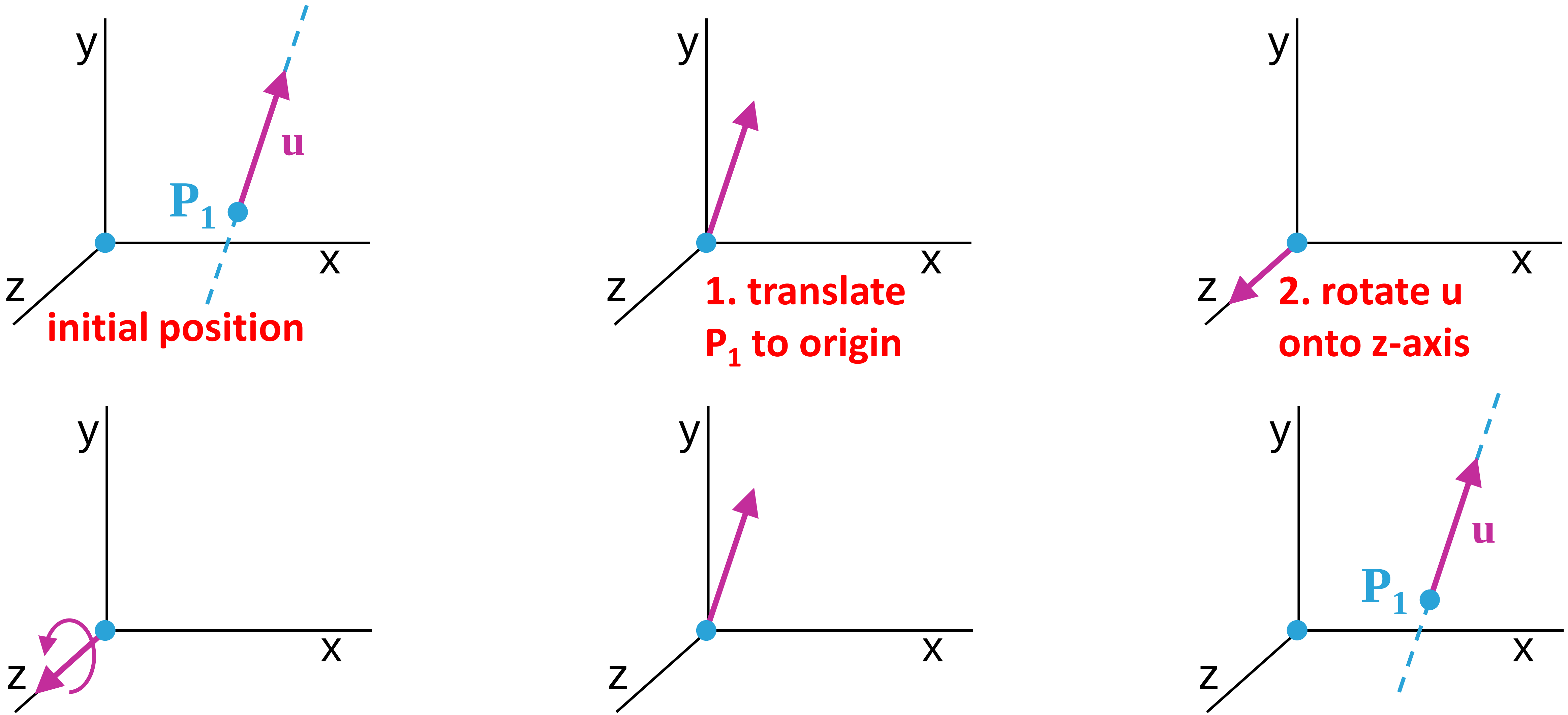
3D Rotation around Arbitrary Axis

an axis of rotation (dashed line) defined with points P_1 and P_2. The direction of the unit axis vector u determines the rotation direction.



$$u = \frac{P_2 - P_1}{|P_2 - P_1|} = (a, b, c)$$

3D Rotation around Arbitrary Axis - Overview



initial position

1. translate P_1 to origin

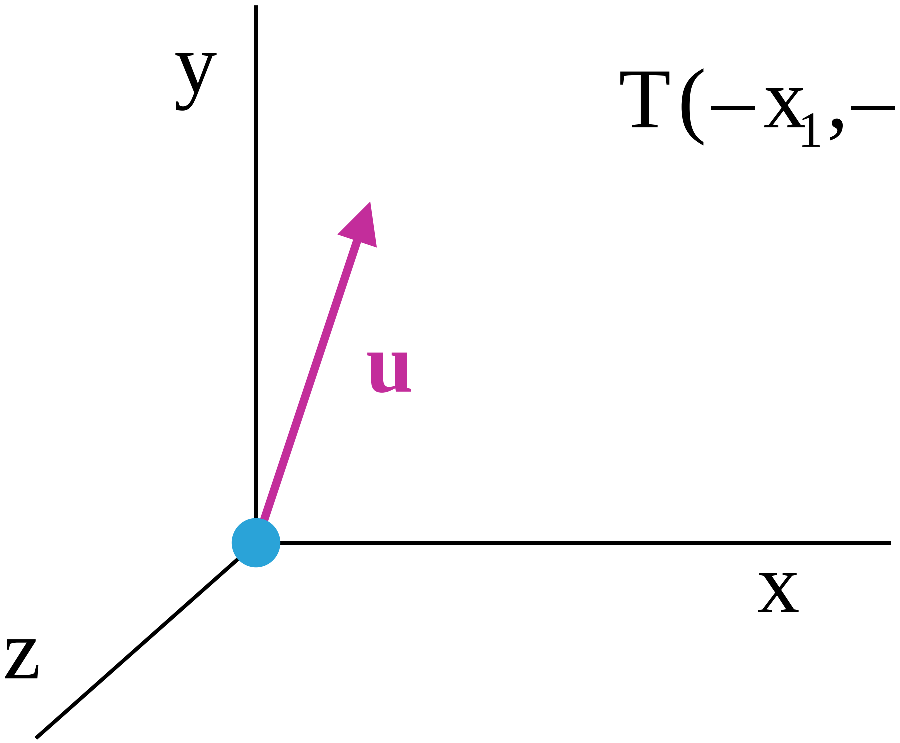
2. rotate u onto z-axis

3. rotate object around z-axis

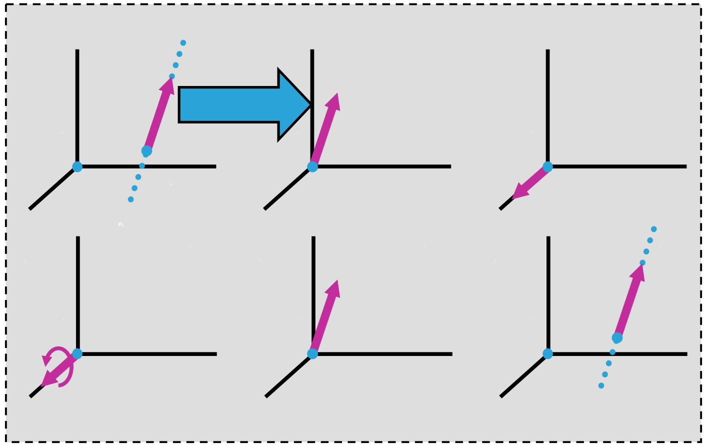
4. rotate axis to original orientation

5. translate axis to original position

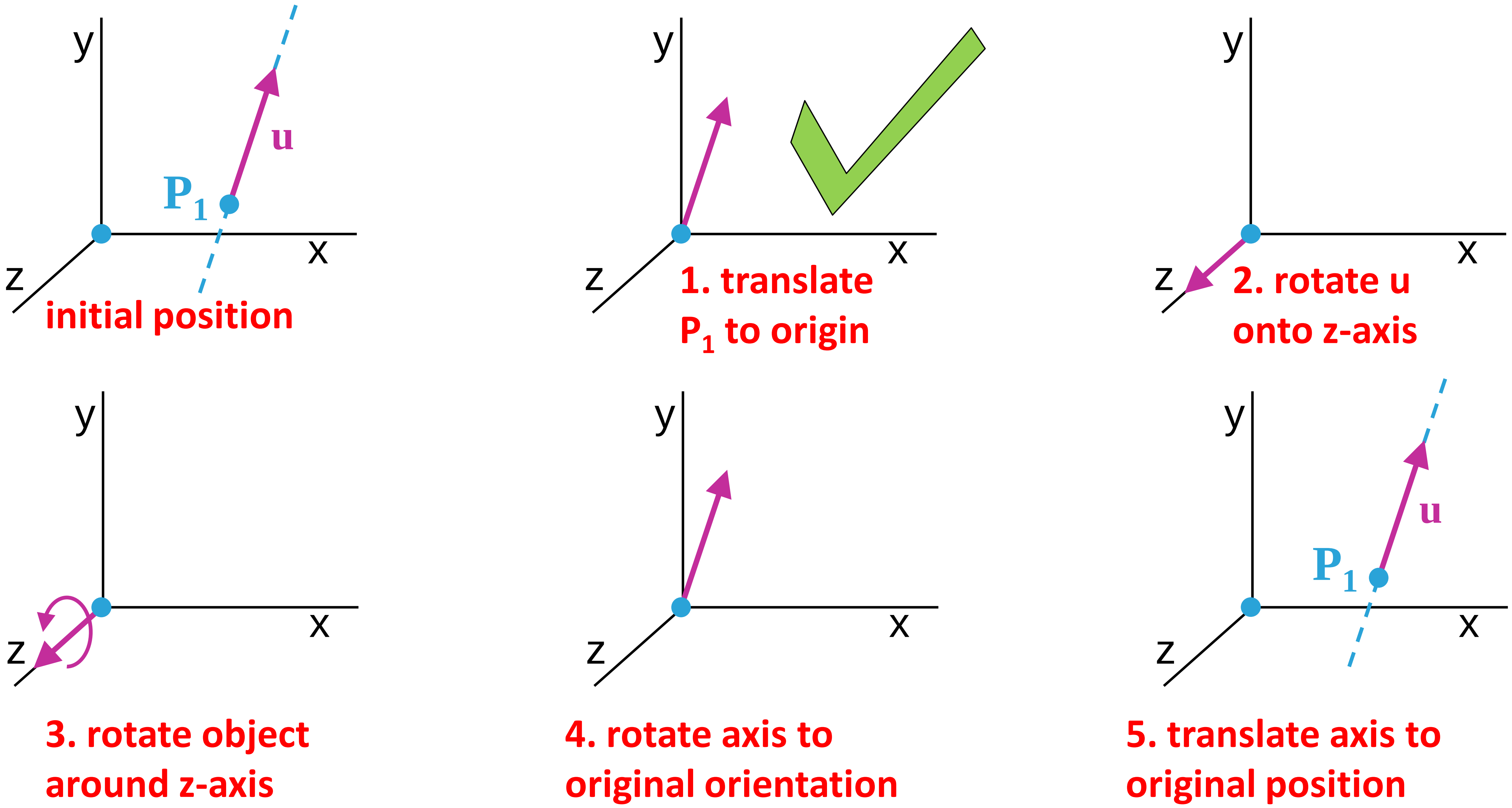
step 1: translation $T(-x_1, -y_1, -z_1)$



$$T(-x_1,-y_1,-z_1) = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D Rotation around Arbitrary Axis – After Step 1

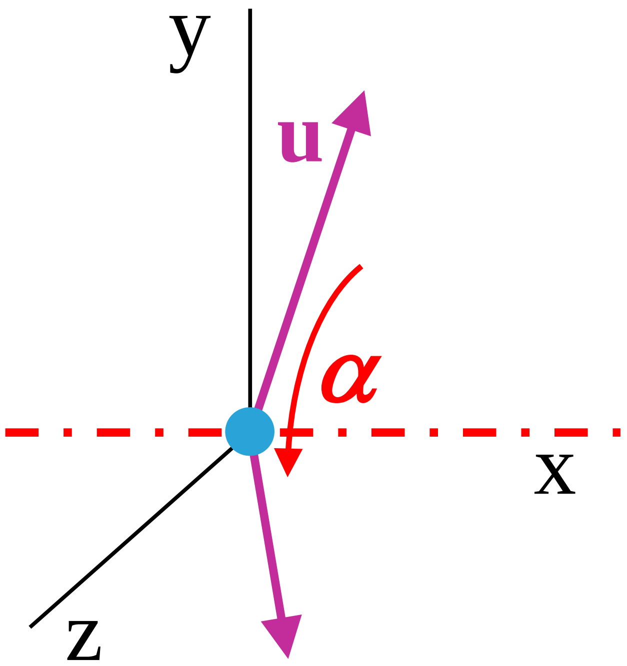


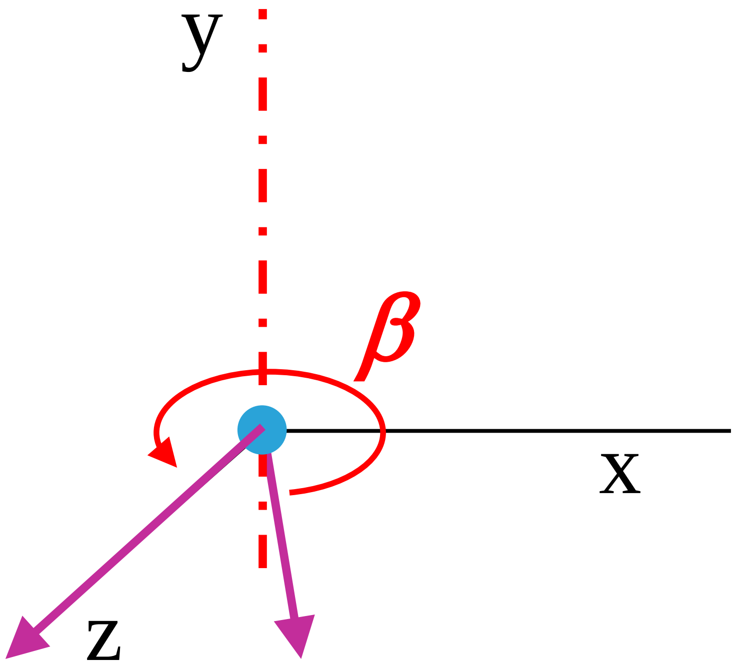
3D Rotation around Arbitrary Axis – Step 2

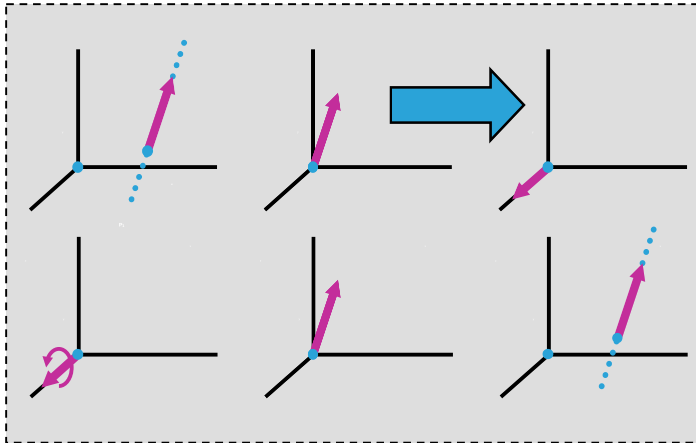
step 2: rotation so that u coincides with z-axis (done with 2 rotations)

step 2a: $R_x(a)$: u $\rightarrow$ xz-plane

step 2b: $R_y(b)$: u $\rightarrow$ z-axis







3D Rotation around Arbitrary Axis – Step 2

step 2a:

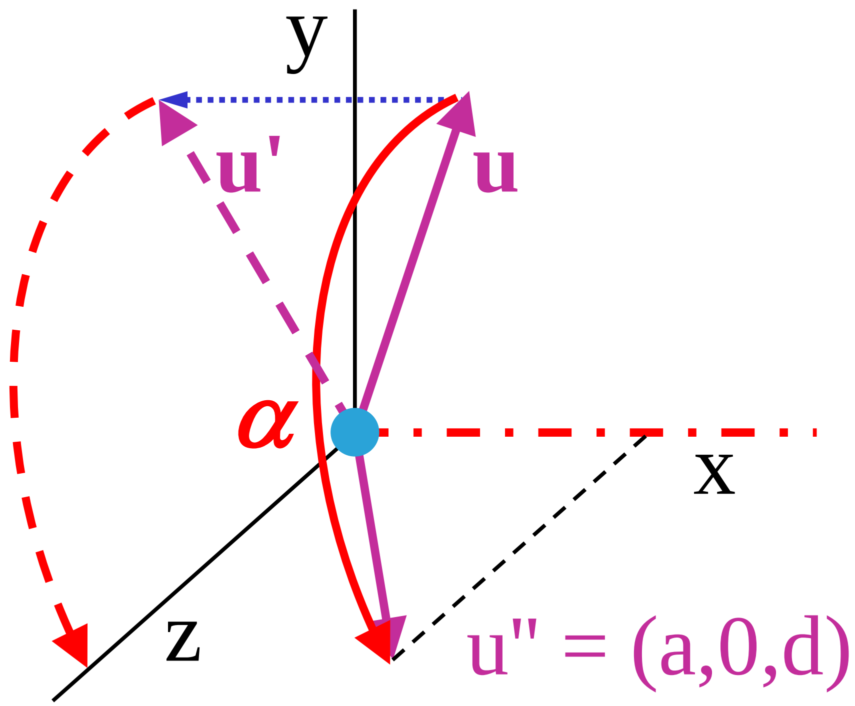
$$u = (a, b, c)$$

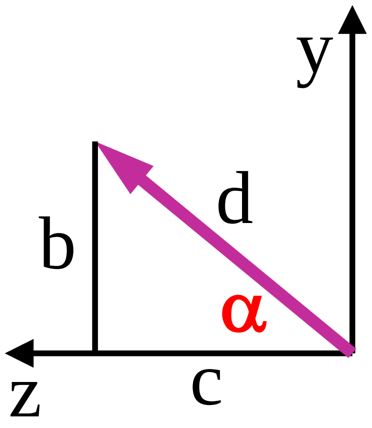
$$u' = (0, b, c)$$

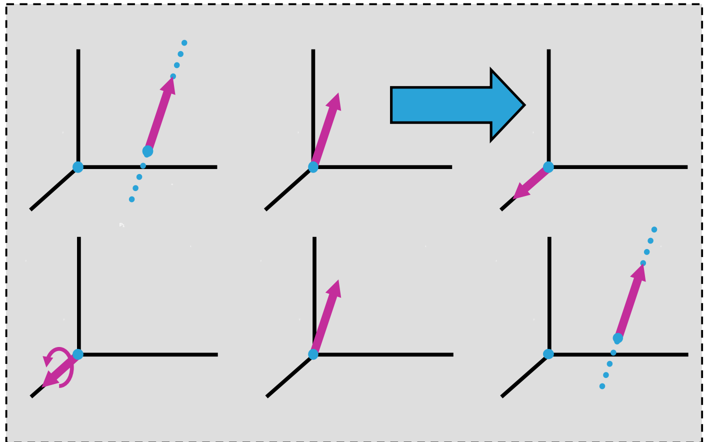
$$|u'| = d = \sqrt{b^2 + c^2}$$

$$\boxed{\cos\alpha = c/d}$$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$







step 2b:

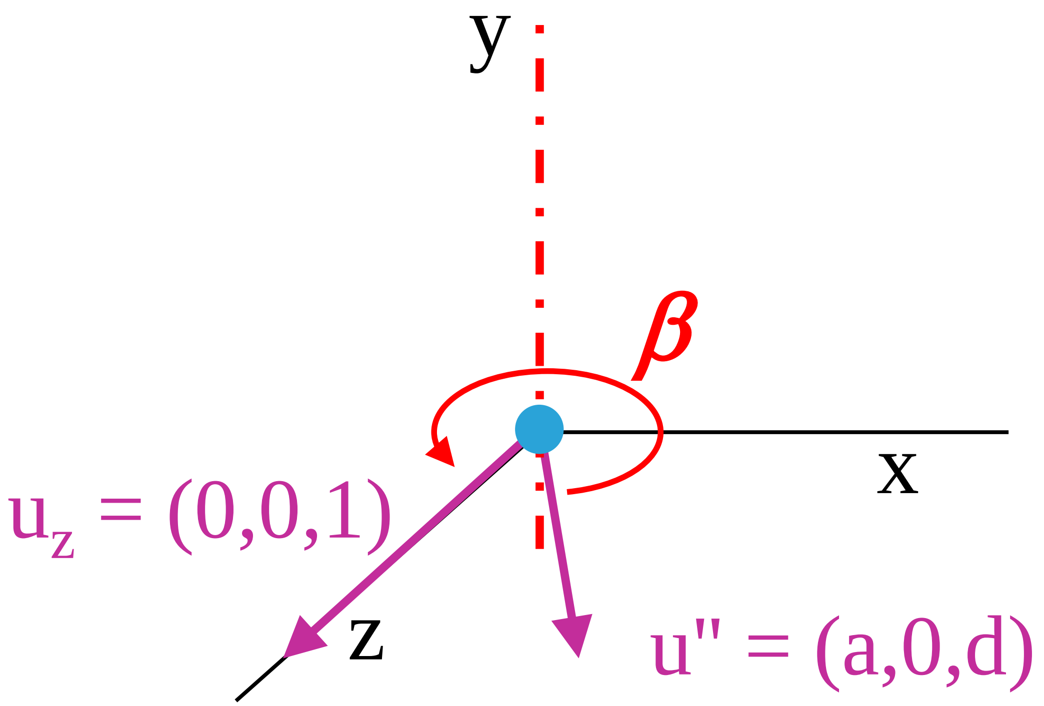
$$u' = (0,b,c)$$

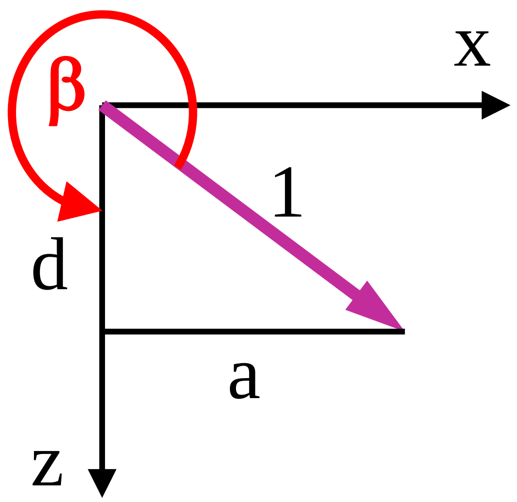
$$|u'| = d$$

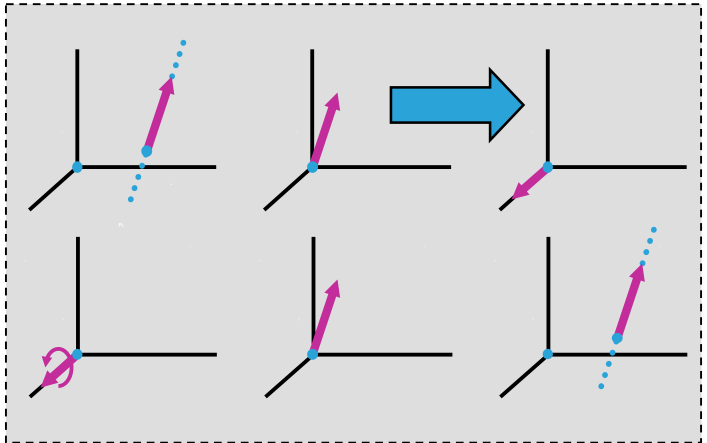
$$u'' = (a,0,d)$$

$\boxed{\begin{array}{l}\cos\beta = d \\ \sin\beta = -a\end{array}}$

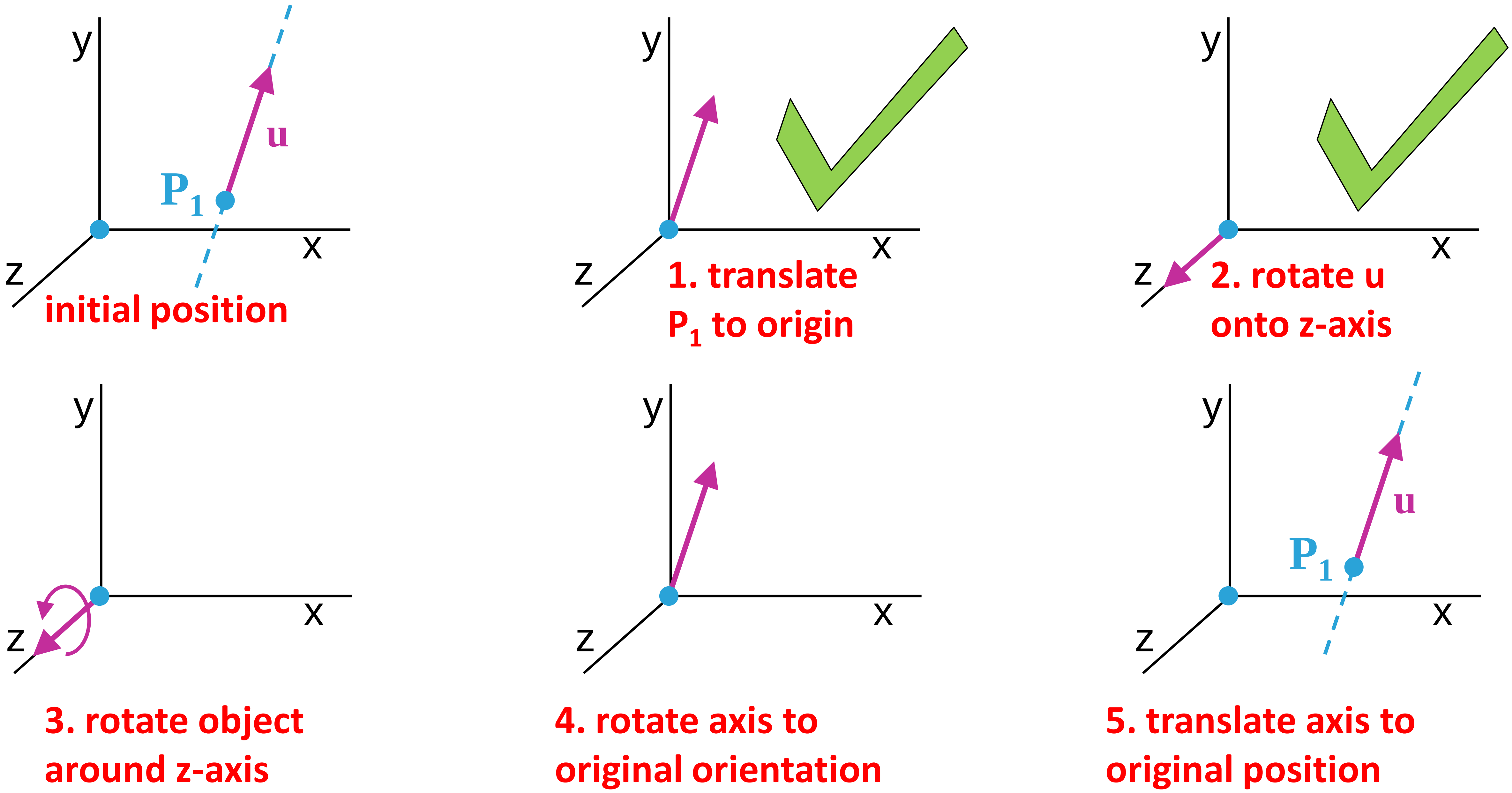
$$R_y(\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$







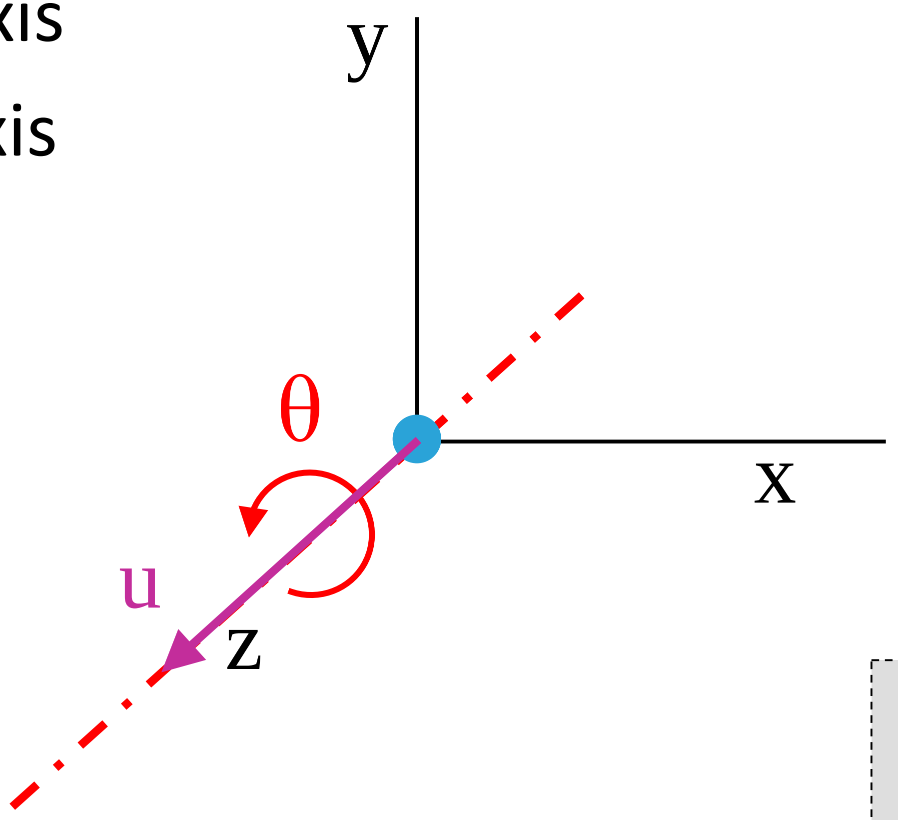
3D Rotation around Arbitrary Axis – After Step 2

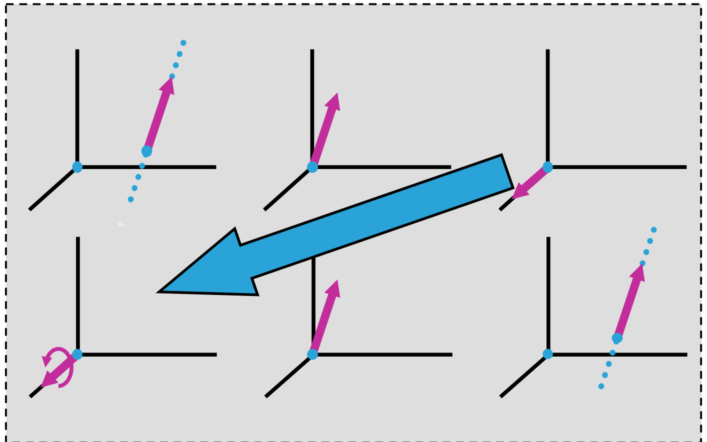


3D Rotation around Arbitrary Axis – Step 3

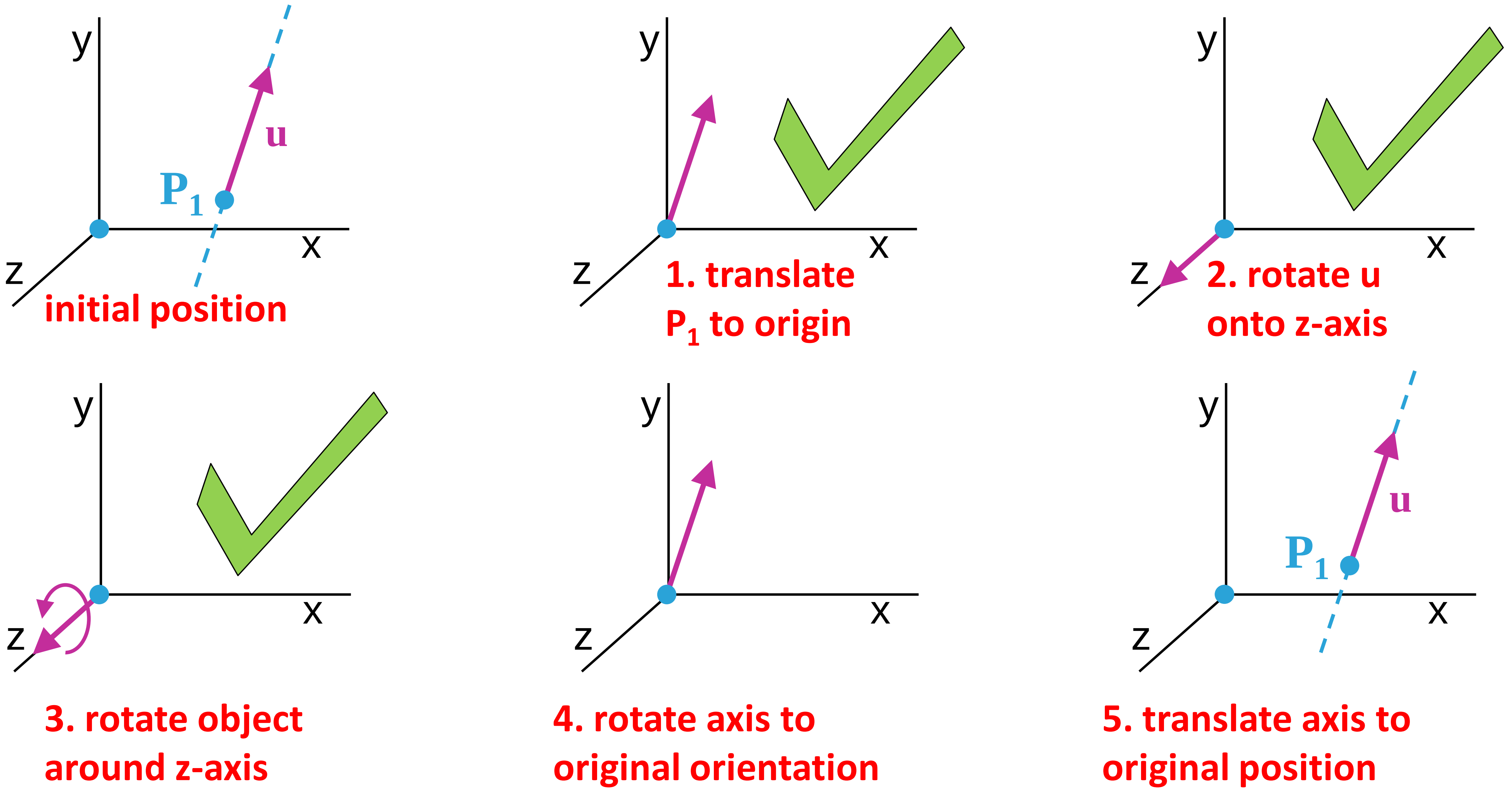
step 3:

- u is aligned with z-axis
- rotation around z-axis





3D Rotation around Arbitrary Axis – After Step 3



3D Rotation around Arbitrary Axis

step 4: undo rotations of step 2

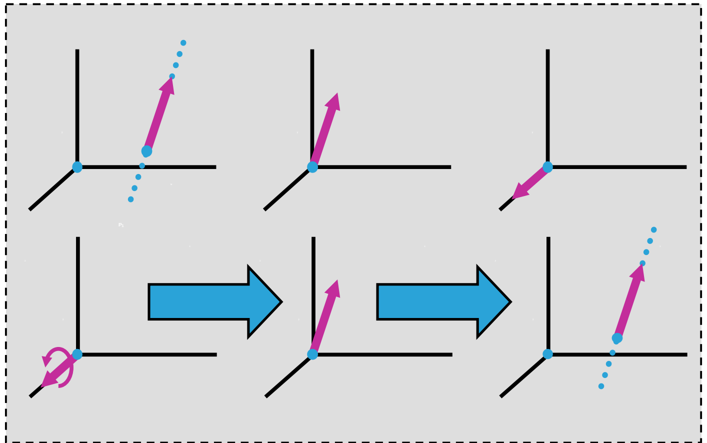
step 5: undo translation of step 1

$$R(\theta) = T^{-1}(P_1) \cdot R_x^{-1}(\alpha) \cdot R_y^{-1}(\beta) \cdot \mathbf{R_z(\theta)} \cdot R_y(\beta) \cdot R_x(\alpha) \cdot T(P_1)$$

steps: 5 4a 4b 3 2b 2a 1

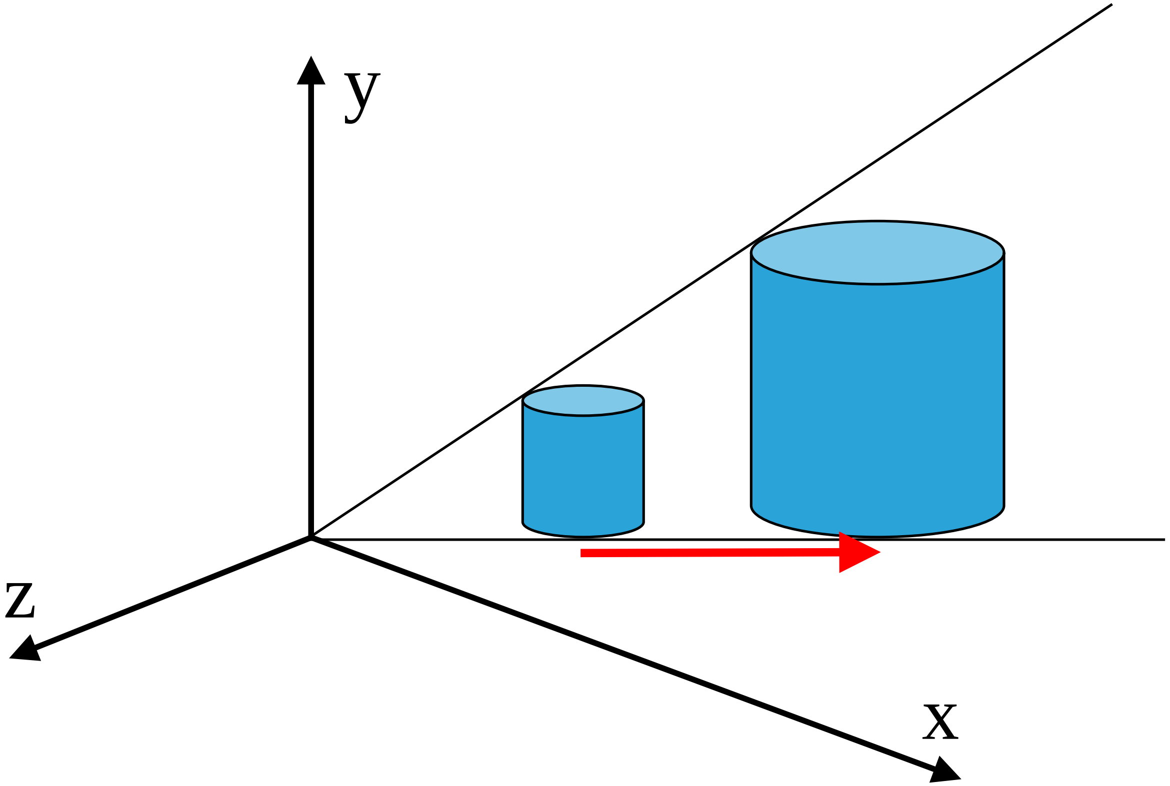
inverse of rotation:

$$R_x^{-1}(\theta) = R_x(-\theta) = R_x^T(\theta)$$



3D Scaling with respect to Origin

doubling the size of an object also moves the object farther away from the origin

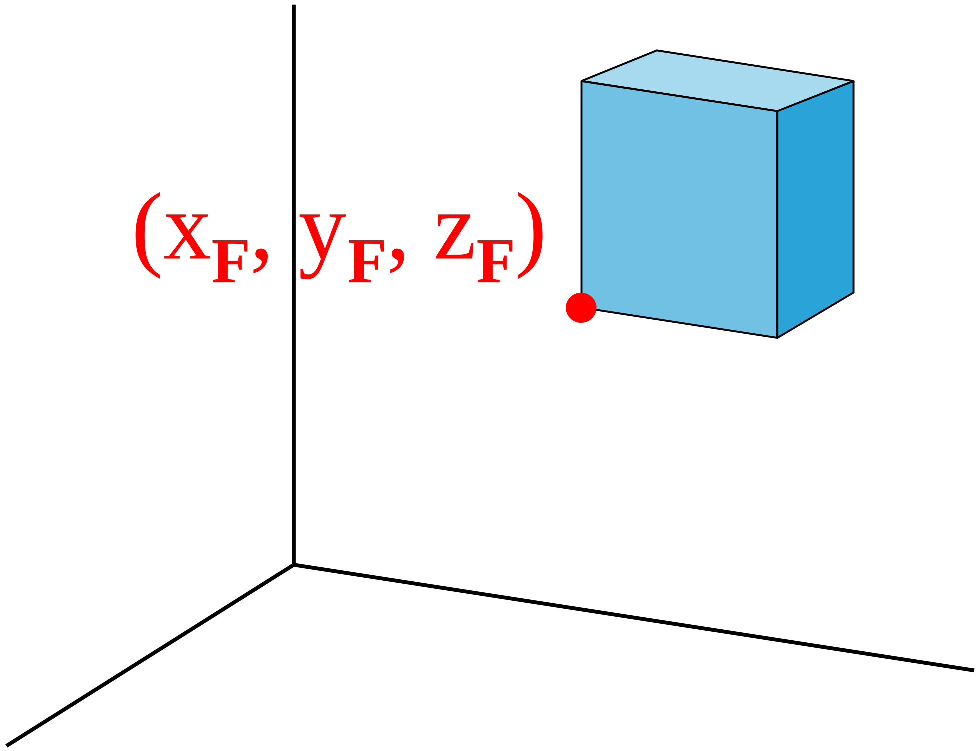


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = S \cdot P$$

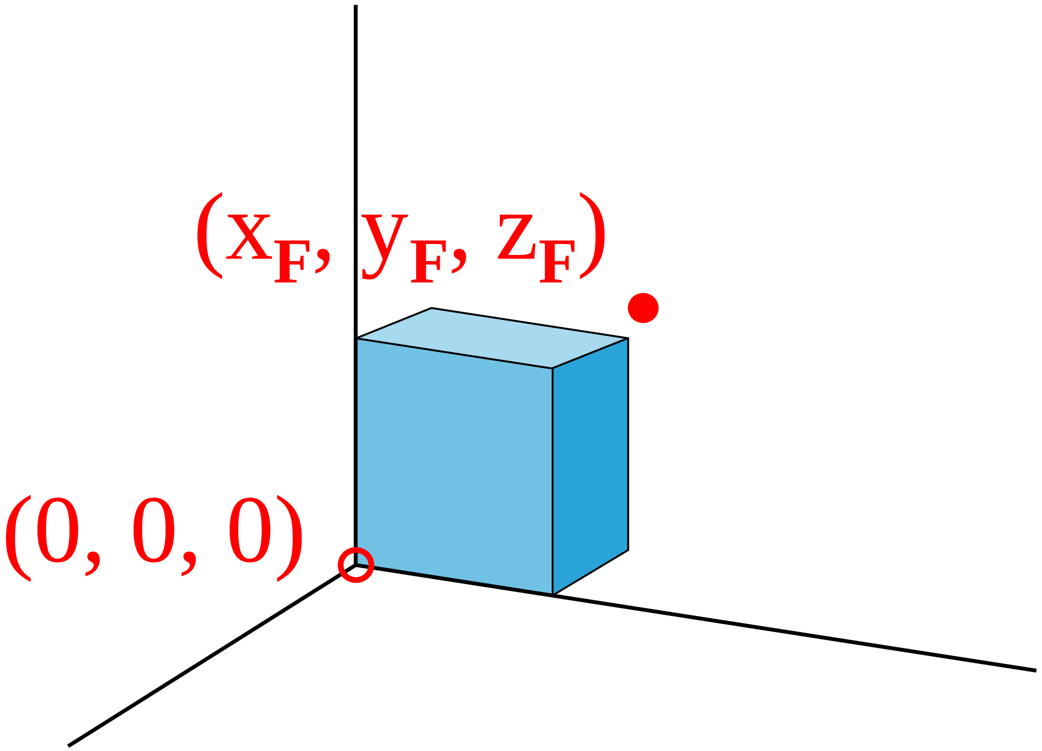
3D Scaling with other Fixed Point

$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$



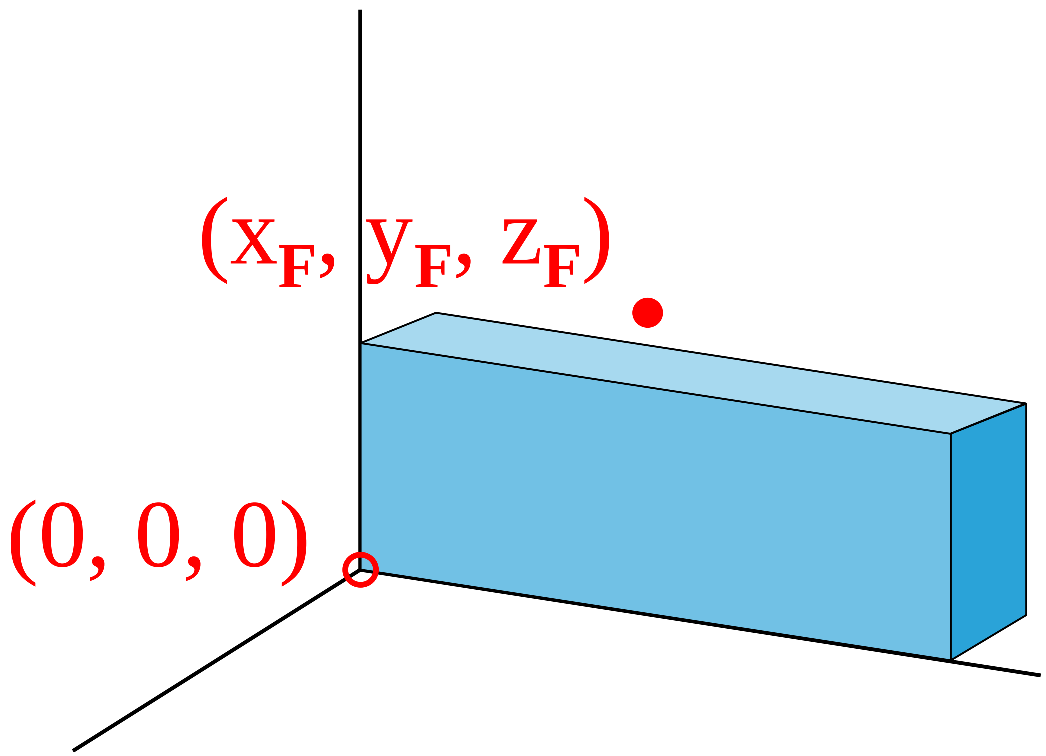
3D Scaling with other Fixed Point

$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$



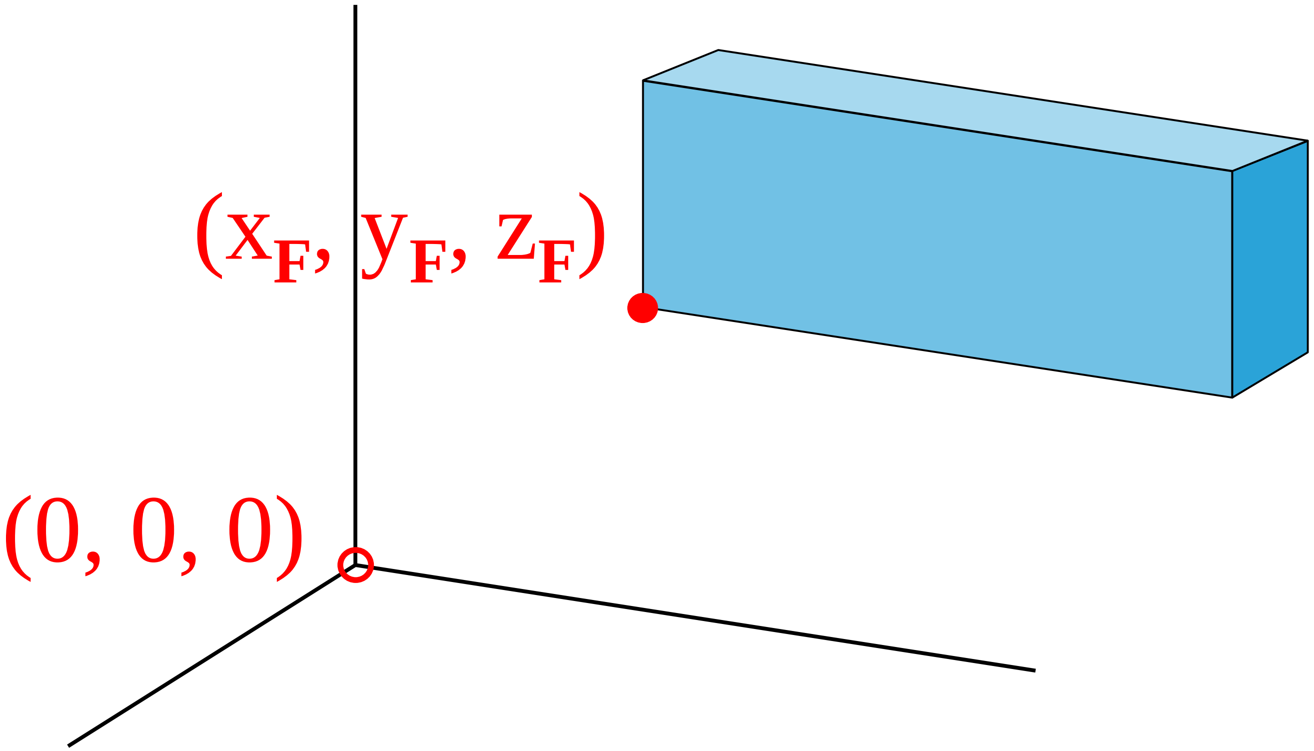
3D Scaling with other Fixed Point

$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$

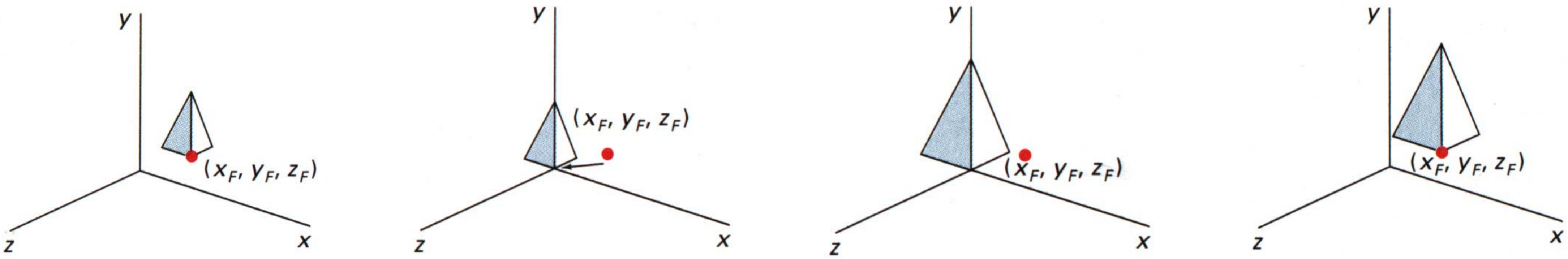


3D Scaling with other Fixed Point

$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$



3D Scaling with other Fixed Point

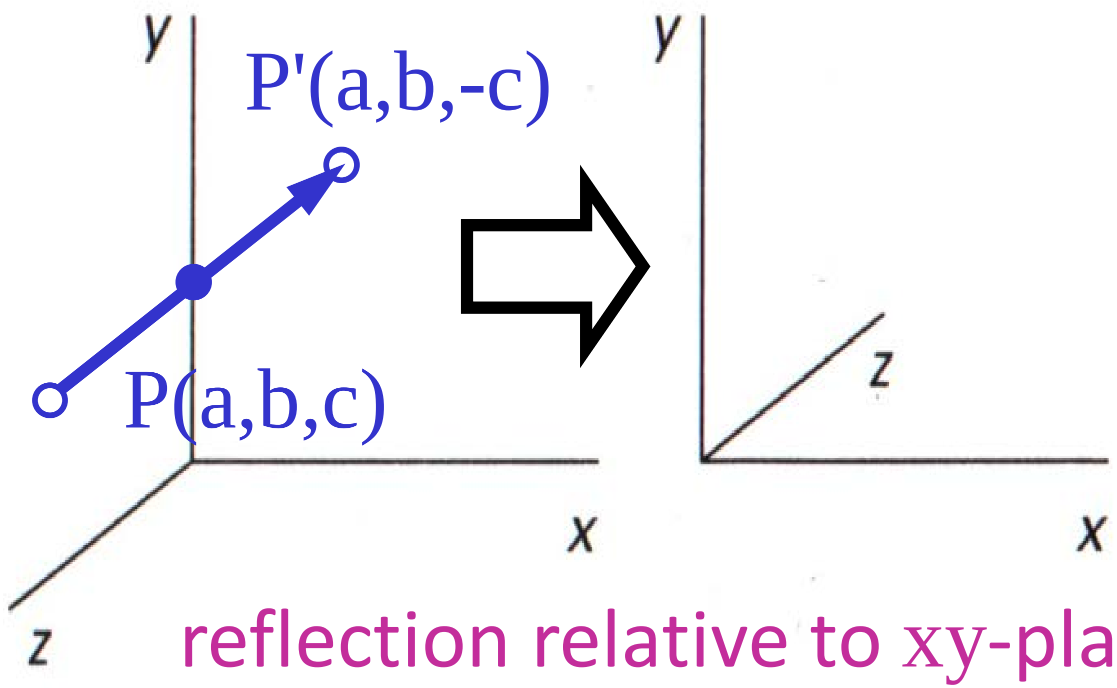


$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$

$$\begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_F \\ 0 & s_y & 0 & (1-s_y)y_F \\ 0 & 0 & s_z & (1-s_z)z_F \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reflection with respect to

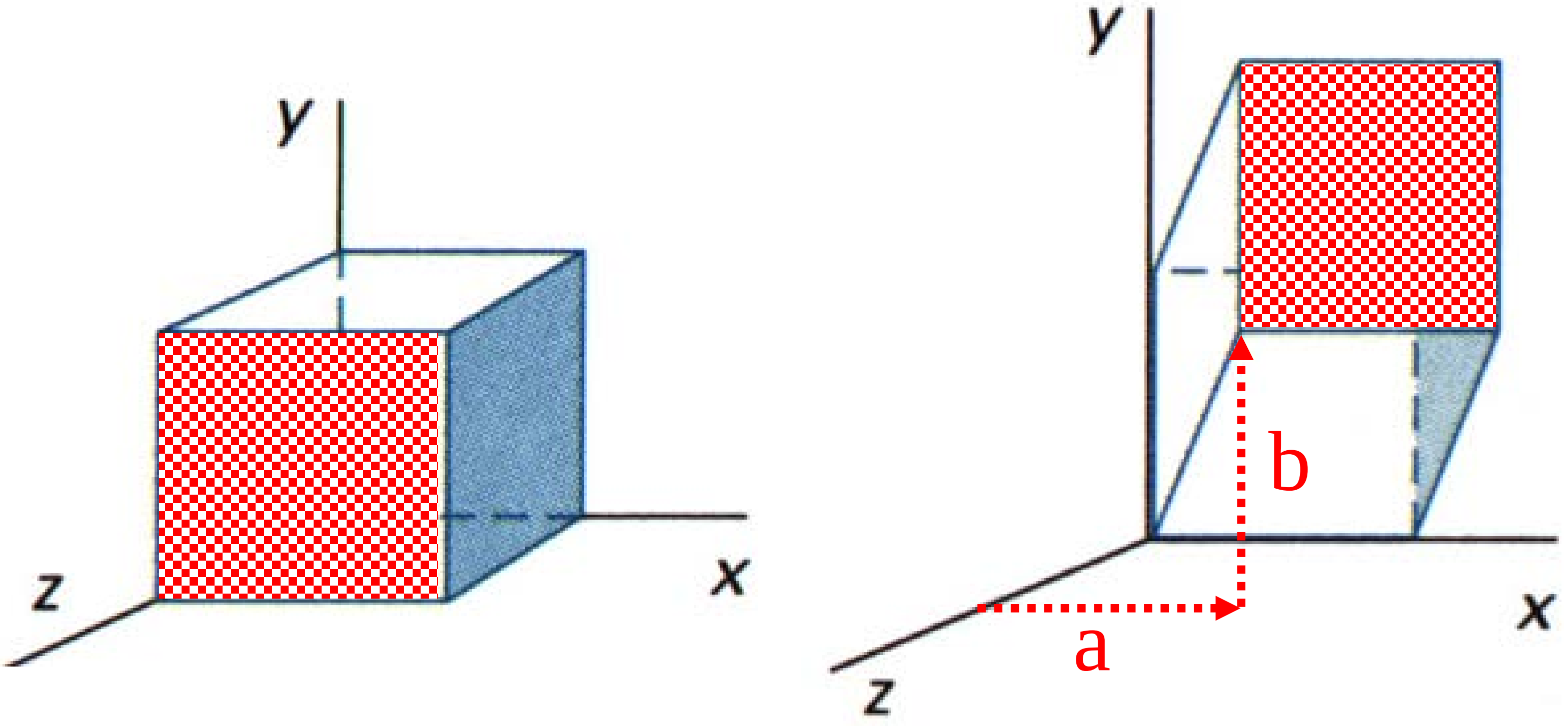
- point
- line (180° rotation)
- plane, e.g., xy-plane: RF_z



reflection relative to xy-plane

$$RF_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

example: shear relative to z-axis with a=b=1



$$SH_z = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$