A Guide to Stable Population

Kristin E. Bietsch

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Abstract

Hello! Presented in the following pages is a quick guide to stable population theory and equations. Stable population is a distinguished branch in the noble field of demography. This topic is the cap on your demographic methods training at Princeton. I hope this proves useful while study for your general exam. Remember, while this packet is prepared to aid your studies, it is no replacement for reading Preston Chapter 7 at least three times. Take heart, you are all on your way to becoming demographers! Your loyal preceptor, Kristin

1 An Introduction to the Idea of a Stable Population

1.1 The Three Conditions of Stable Population

A stable population is formed when three conditions prevail for a long time period

1. Age-specific fertility rates are constant
2. Age-specific death rates are constant
3. Age-specific rates of net migration are zero

How long is a long time? Preston suggests that these conditions must prevail for a period as long as the maximum age to which anyone survives. The key is that a population "forgets its past", a concept known as ergodicity, where the population takes on features entirely determined by the regime of vital rates to which it is exposed.

1.2 Stable versus Stationary

A stationary population is a special case of a stable population. A stable population has a constant growth rate, birth rate, death rate, and age composition. If the birth rate equals the death rate, the population will be stationary: zero growth. That's it, it's that simple!

2 $Lotka's\ r$ and Stable Population Equations

Note: The following is done for a one-sex population.

2.1 Arriving at $Lotka's\ r$

We will start midway through his explanation. For a population where rates have been constant **for at least 50 years**:

$$B(t) = \int_0^t B(t-a)p(a)m(a)da \tag{1}$$

What this means: We have the number of births a years ago, we survive them forward, then we multiply these survivors by the maternity function to find the number of births to these people at time t.

This is a **homogeneous integral equation** and can be solved by trial and error (just like when we run iterations in precept!).

Lotka found that an **exponential birth series** solved the equation:

$$B(t) = Be^{\rho t} \tag{2}$$

Where:

- There is now the unknown ρ (rho)

Substituting Equation (2) into Equation (1), which gives you:

$$Be^{\rho t} = \int_0^t Be^{\rho(t-a)}p(a)m(a)da \tag{3}$$

The $Be^{\rho t}$ from both sides cancel:

$$1 = \int_0^t e^{-\rho a}p(a)m(a)da \tag{4}$$

We are almost there! Now we just need to solve for the unknown: ρ. We need a value of ρ so that the equation equals 1. This is where iterations and loops come in handy! The unique value of ρ which solves this equation is called r, and it is the **intrinsic growth rate**. It is intrinsic to the $m(a)$ and $p(a)$ schedules that produced it.

For the final equation, we can simplify things a bit. Before childbearing begins and after it ends, the maternity function, $m(a)$, will be 0. Therefore, we can restrict our integral to the beginning of childbearing, (α), and to the end of childbearing, (β).

And Finally:

$$1 = \int_{\alpha}^{\beta} e^{-ra} p(a) m(a) da \tag{5}$$

Now that we have found how to calculate $Lotka's\ r$, we can move on to two useful topics concerning our stable population.

2.2 Age Composition in a Stable Population

The age composition in a stable population is determined by two pieces of demographic information:

1. The lifetable
2. The growth rate in the annual number of births

In a stable population, the **number of people** age a at time t:

$$N(a,t) = B(t) * e^{-ra} * p(a) \tag{6}$$

Where:

- $N(a,t)$ is the number of people age a alive at time t
- $B(t)$ is the number of births at time t
- r is the growth rate (remember this is constant)
- $p(a)$ is probability of surviving till age a

What this means: $B(t) * e^{-ra}$ is telling us how many births there were a years ago. Since the population is growing at a constant rate, the number of birth now times negative growth (because we are essentially going back in time) tells us how many births there were when this group was born. Taking this number of births, we survive them to present using $p(a)$ to arrive at the current number of people age a.

Dividing Equation (6) by the total population size gives the **proportion of the population** age a at time t:

$$c(a) = b * e^{-ra} * p(a) \tag{7}$$

Where:

- $c(a)$ is the proportion of the population that is age a
- b is the birth rate (remember this is constant)
- r is the growth rate (remember this is constant)

- $p(a)$ is probability of surviving till age a

What this means: Remember, age distribution is constant in a stable population, so there is no need for a specified time t. You now can see that the age composition of the population is determined by fertility, b, mortality, $p(a)$, and the growth rate, r.

2.3 The Crude Birth Rate in a Stable Population

Remember Equation (6)? If we integrate both sides, we have two expressions for the size of total population.

$$\int_0^\omega N(a,t) = B(t)\int_0^\omega e^{-ra}p(a)da \tag{8}$$

Where:

- ω (omega) is the highest age

We can rearrange this to get:

$$\frac{B(t)}{\int_0^\omega N(a,t)} = \frac{1}{\int_0^\omega e^{-ra}p(a)da} \tag{9}$$

What this means: These equations both show the crude birth rate! How does this work? You have the births at a particular time, and you are dividing them by the total population size. Therefore, remember that in a stable population, the crude birth rate can be written in terms of the growth rate and the survival schedule:

$$b = \frac{1}{\int_0^\omega e^{-ra}p(a)da} \tag{10}$$

And the crude birth rate is constant over time!

2.4 Equations in Discrete Notation

2.4.1 Lotka's r

$$1 = \sum_{a=\alpha,5}^{\beta-5} e^{-r(a+2.5)}\frac{{}_5L_a}{l_0}{}_5m_a \tag{11}$$

2.4.2 Crude Birth Rate

$$b = \frac{1}{\sum_{a=0,5}^{\omega} e^{-r(a+2.5)}\frac{{}_5L_a}{l_0}} \tag{12}$$

2.4.3 Age Structure

$${}_5c_a = be^{-r(a+2.5)}\frac{{}_5L_a}{l_0} \tag{13}$$

3 The Stable Equivalent Population

Every population has an underlying stable population that would emerge if its current rates remained unchanged. The stable equivalent population would have birth rate, death rate, age distribution, and growth rate intrinsic to the populations $m(a)$ and $p(a)$ schedules. **Important! It is not influenced by the actual age distribution**. If the $m(a)$ and $p(a)$ schedules remain constant for long enough, the stable equivalent population is the actual population.

4 The Relationship Between NRR and r

First:

$$NRR = e^{rT} \tag{14}$$

Where:

- T is the mean length of a generation
- T is the length of time (in years) that it will take for a population growing at rate r to increase by a factor of NRR
- T can be approximated as the average of the mean age at childbearing in the stable population (A_B) and the mean age of childbearing in a cohort subject to the $m(a)$ and $p(a)$ schedules (μ)

Rearranging things a bit:

$$r = \frac{ln(NRR)}{T} \tag{15}$$

Now remember Coale's approximation:

$$NRR = GRR * p(A_M) \tag{16}$$

Where:

- GRR is the gross reproduction rate- the mean number of daughters that would be born to a cohort of women subject to the $m(a)$ schedule
- $p(A_M)$ is the probability of surviving from birth to the mean age of child-bearing

Now, if there sex ratio at birth is constant across ages of mothers, then:

$$NRR = TFR * \frac{1}{1 + SRB} * p(A_M) \tag{17}$$

Now if you substitute Equation (17) into Equation (15):

$$r = \frac{ln(TFR) + ln(SRB) + ln(p(A_M))}{T} \tag{18}$$

THIS IS A REALLY IMPORTANT FORMULA, REMEMBER IT!

- This formula indicates that mortality and fertility levels have essentially separate influences on the intrinsic growth rate (because they are additive here)
  - To predict the effect of a change in fertility, you do not need to know the level of mortality

Also important:

$$\Delta r = \frac{ln(\frac{TFR(2)}{TFR(1)})}{T} \tag{19}$$

The change of the intrinsic growth rate depends only on the proportionate decline in the TFR and not the absolute decline

- A decline in the TFR from 3 to 2 will have exactly the same impact as a decline from 6 to 4
- In terms of its impact on long-term growth rates, fertility reduction clearly has increasing returns

5 Conclusions

In this packet I did not cover **The Effects of Changes in Fertility and Mortality** (Preston 7.6) and **Population Momentum** (Preston 7.7) which are covered in great detail in Preston, and which you should spend a **LOT** of time studying! **Actually, make sure you study everything!** Like really, stop reading this paragraph and go study!