Pushdown Automata

Announcements

- Problem Set 5 due this Friday at 12:50PM.
    - Late day extension: Using a 72-hour late day now extends the due date to 12:50PM on **Tuesday, February 19th**.

The Weak Pumping Lemma

- The **Weak Pumping Lemma for Regular Languages** states that

For any regular language L,

There exists a positive natural number n such that

For any $w \in L$ with $|w| \geq n$,

There exists strings x, y, z such that

For any natural number i,

$w = xyz$, w can be broken into three pieces,

$y \neq \varepsilon$ where the middle piece isn't empty,

$xy^iz \in L$ where the middle piece can be replicated zero or more times.

Counting Symbols

- Consider the alphabet $\Sigma = \{ \texttt{0}, \texttt{1} \}$ and the language

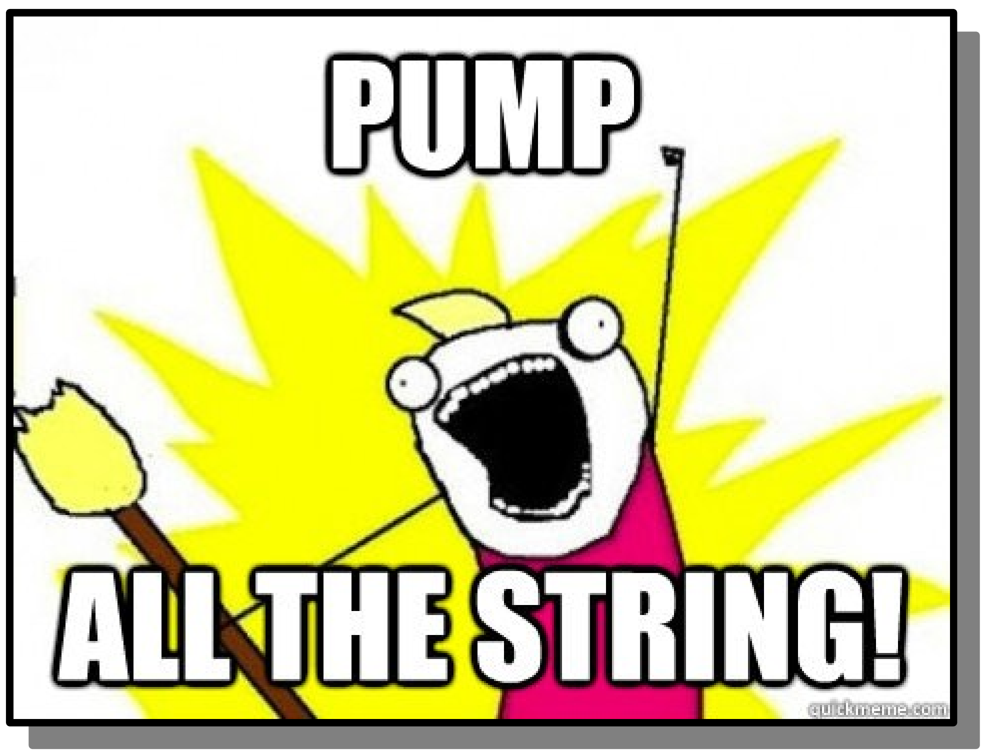
  $L = \{ w \in \Sigma^* \mid w$ contains an equal number of `0`s and `1`s. $\}$

- For example:
  - $\texttt{01} \in L$
  - $\texttt{110010} \in L$
  - $\texttt{11011} \notin L$
- **Question:** Is L a regular language?

The Weak Pumping Lemma

$L = \{ w \in \{0, 1\}^* \mid w$ contains an equal number of 0s and 1s. $\}$

1	0	0	1



An Incorrect Proof

Theorem: L is regular.

Proof: We show that L satisfies the condition of the pumping lemma. Let $n = 2$ and consider any string $w \in L$ such that $|w| \geq 2$. Then we can write $w = xyz$ such that $x = z = \varepsilon$ and $y = w$, so $y \neq \varepsilon$. Then for any natural number i, $xy^iz = w^i$, which has the same number of `0`s and `1`s. Since L passes the conditions of the weak pumping lemma, L is regular. ■

The Weak Pumping Lemma

- The **Weak Pumping Lemma for Regular Languages** states that

For any regular language L,

This says **nothing** about languages that aren't regular!

There exists a positive natural number n such that

For any $w \in L$ with $|w| \geq n$,

There exists strings x, y, z such that

For any natural number i,

$w = xyz$, w can be broken into three pieces,

$y \neq \varepsilon$ where the middle piece isn't empty,

$xy^iz \in L$ where the middle piece can be replicated zero or more times.

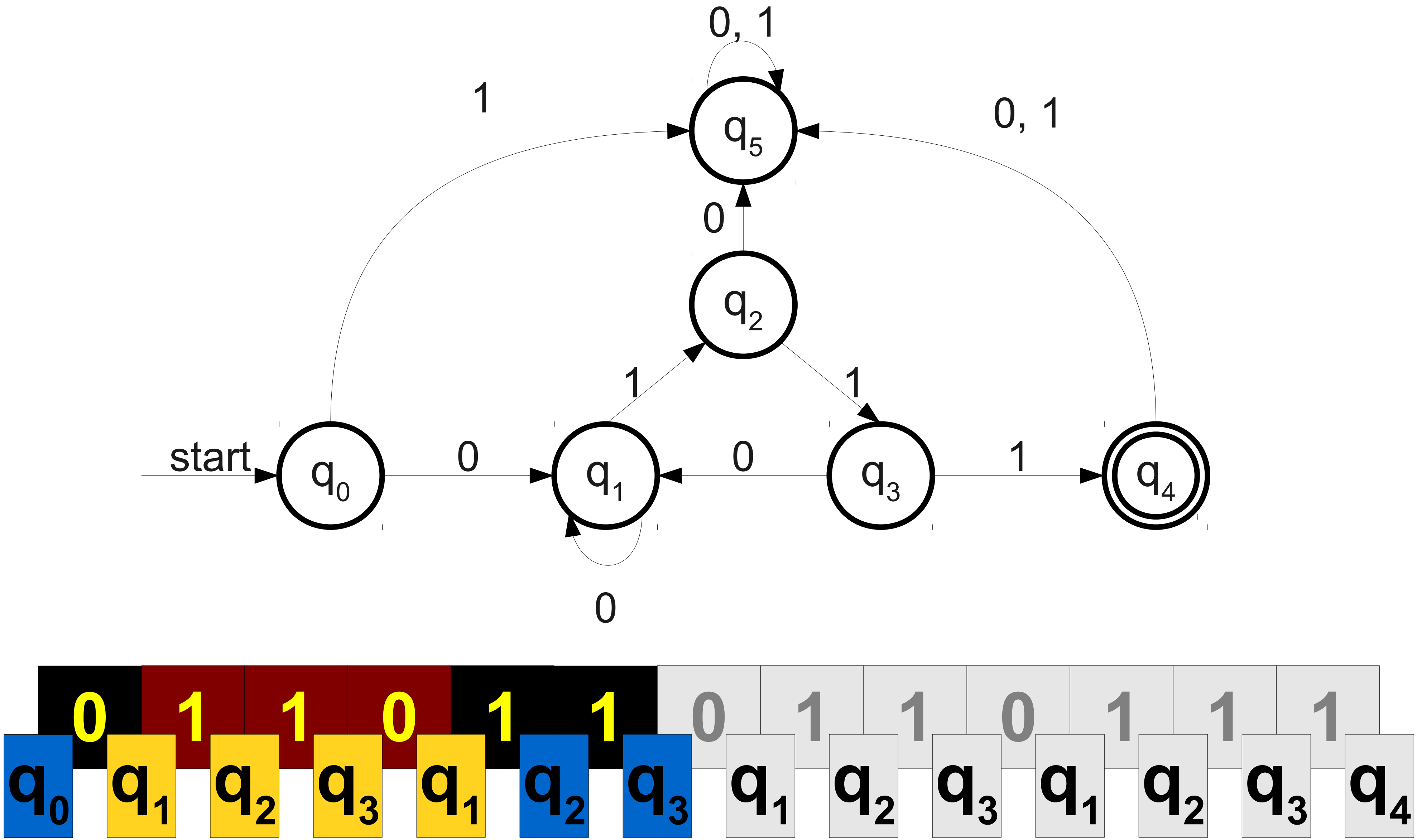
Caution with the Pumping Lemma

- The weak and full pumping lemmas describe a **necessary** condition of regular languages.
    - If L is regular, L passes the conditions of the pumping lemma.
- The weak and full pumping lemmas are not a **sufficient** condition of regular languages.
    - If L is *not* regular, it still might pass the conditions of the pumping lemma!
- If a language fails the pumping lemma, it is definitely not regular.
- If a language passes the pumping lemma, we learn nothing about whether it is regular or not.

L is Not Regular

- The language L can be proven not to be regular using a stronger version of the pumping lemma.
- To see the full pumping lemma, we need to revisit our original insight.

An Important Observation



Pumping Lemma Intuition

- Let D be a DFA with n states.
- Any string w accepted by D that has length at least n must visit some state twice **within its first n characters**.
  - Number of states visited is equal **n + 1**.
  - By the pigeonhole principle, some state is duplicated.
- The substring of w in-between those revisited states can be removed, duplicated, tripled, quadrupled, etc. without changing the fact that w is accepted by D.

The Pumping Lemma

For any regular language L,

There exists a positive natural number n such that

For any $w \in L$ with $|w| \geq n$,

There exists strings x, y, z such that

For any natural number i,

$w = xyz,$	w can be broken into three pieces,
$\lvert xy \rvert \leq n,$	where the first two pieces occur at the start of the string,
$y \neq \varepsilon$	where the middle piece isn't empty,
$xy^iz \in L$	where the middle piece can be replicated zero or more times.

Why This Change Matters

- The restriction $|xy| \leq n$ means that we can limit where the string to pump must be.
- If we specifically craft the first n characters of the string to pump, we can force y to have a specific property.
- We can then show that y cannot be pumped arbitrarily many times.

The Pumping Lemma

$L = \{\ w \in \{0, 1\}^* \mid w$ contains an equal number of 0s and 1s. $\}$

Suppose the pumping length is 4.

$$\underbrace{0\ 0\ 0\ 0}\ 1\ 1\ 1\ 1$$

Since $|xy| \le 4$, the string to pump must be somewhere in here.

$L = \{ w \in \{0, 1\}^* \mid w$ has an equal number of 0s and 1s $\}$

Theorem: L is not regular.

Proof: By contradiction; assume that L is regular. Let n be the length guaranteed by the pumping lemma. Consider the string $w = 0^n 1^n$. Then $|w| = 2n \geq n$ and $w \in L$. Therefore, there exist strings x, y, and z such that $w = xyz$, $|xy| \leq n$, $y \neq \varepsilon$, and for any natural number i, $xy^iz \in L$. Since $|xy| \leq n$, y must consist solely of 0s. But then $xy^2z = 0^{n+|y|}1^n$, and since $|y| > 0$, we have that $xy^2z \notin L$.

We have reached a contradiction, so our assumption was wrong and L is not regular. ■

Summary of the Pumping Lemma

- Using the pigeonhole principle, we can prove the **weak pumping lemma** and **pumping lemma**.
- These lemmas describe essential properties of the regular languages.
- Any language that fails to have these properties cannot be regular.

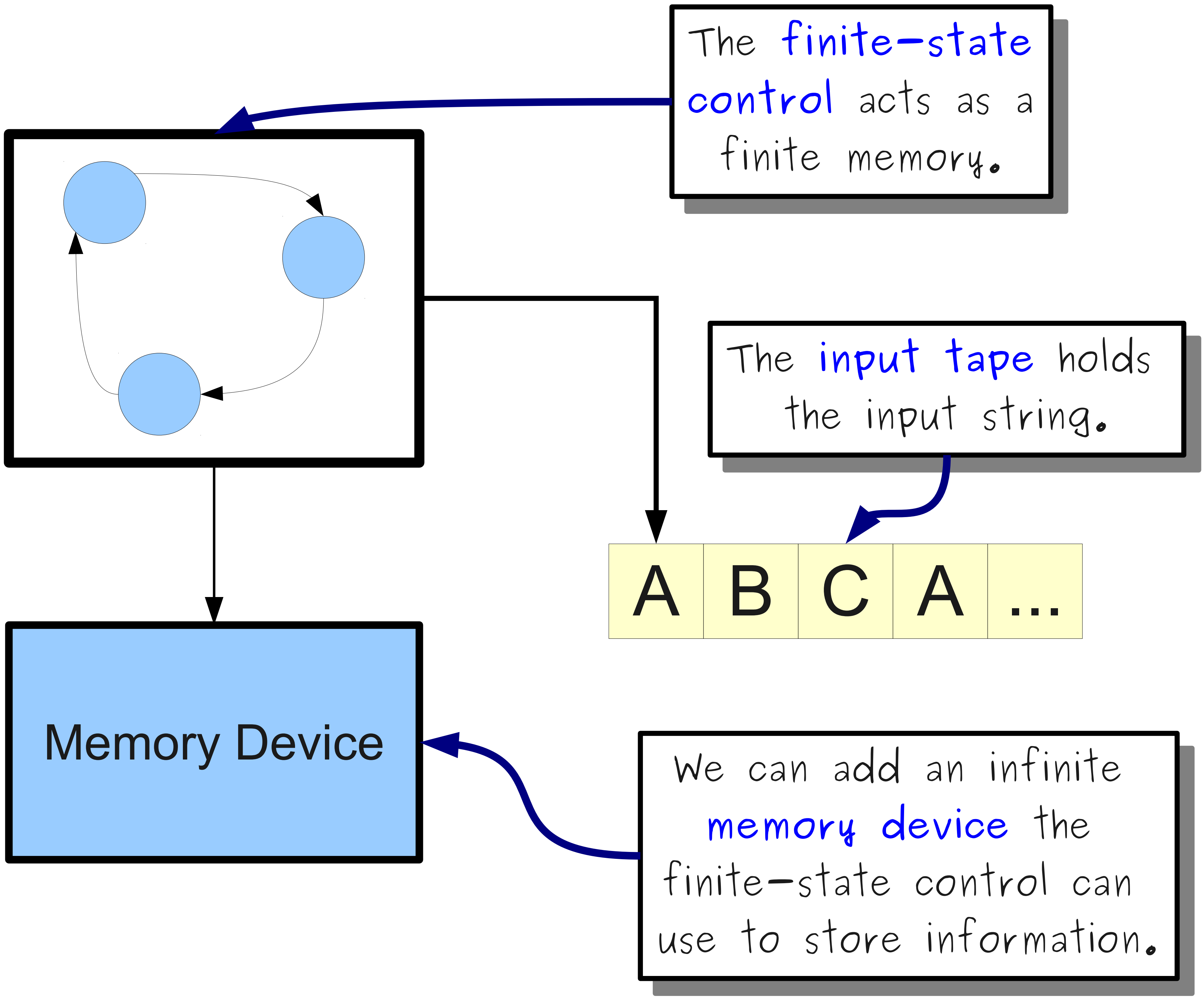
Beyond Finite Automata

Where We Are

- Our study of the **regular languages** gives us an exact characterization of problems that can be solved by finite computers.
- Not all languages are regular.
- How do we build more powerful computing devices?

The Problem

- Finite automata accept precisely the regular languages.
- We may need unbounded memory to recognize context-free languages.
  - e.g. $\{ \texttt{0}^n\texttt{1}^n \mid n \in \mathbb{N} \}$ requires unbounded counting.
- How do we build an automaton with finitely many states but unbounded memory?



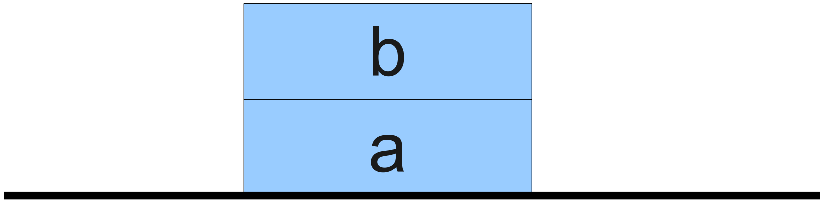
The finite-state control acts as a finite memory.
The input tape holds the input string.
A
B
C
A
...
Memory Device
We can add an infinite memory device the finite-state control can use to store information.

Adding Memory to Automata

- We can augment a finite automaton by adding in a **memory device** for the automaton to store extra information.
- The finite automaton now can base its transition on both the current symbol being read and values stored in memory.
- The finite automaton can issue commands to the memory device whenever it makes a transition.
  - e.g. add new data, change existing data, etc.

Stack-Based Memory

- There are **many** types of memory that we might give to an automaton.
  - We'll see at least two this quarter.
- One of the simplest types of memory is a **stack**.



Stack-Based Memory

- Only the top of the stack is visible at any point in time.
- New symbols may be **pushed** onto the stack, which cover up the old stack top.
- The top symbol of the stack may be **popped**, exposing the symbol below it.

Pushdown Automata

- A **pushdown automaton** (PDA) is a finite automaton equipped with a stack-based memory.
- Each transition
  - is based on the current input symbol and the top of the stack,
  - optionally pops the top of the stack, and
  - optionally pushes new symbols onto the stack.
- Initially, the stack holds a special symbol $\mathbf{Z}_0$ that indicates the bottom of the stack.

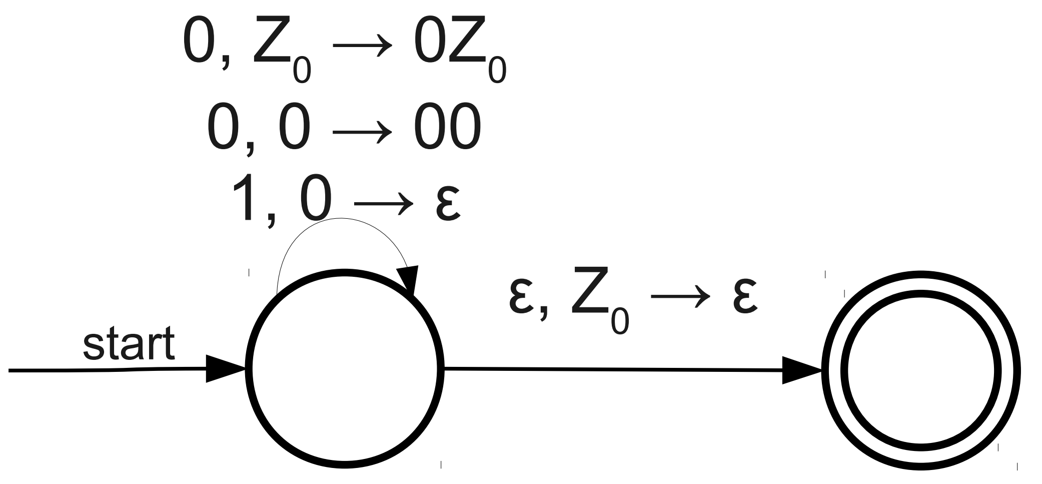
Our First PDA

- Consider the language

  $$L = \{ w \in \Sigma^* \mid w \text{ is a string of balanced digits} \}$$

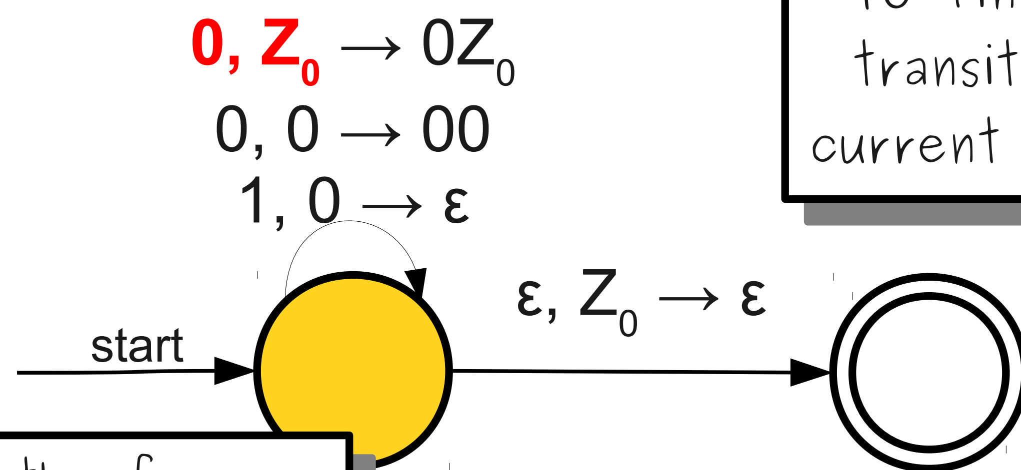
  over $\Sigma = \{$ `0`, `1` $\}$
- We can exploit the stack to our advantage:
  - Whenever we see a `0`, push it onto the stack.
  - Whenever we see a `1`, pop the corresponding `0` from the stack (or fail if not matched)
  - When input is consumed, if the stack is empty, accept.

A Simple Pushdown Automaton



A Simple Pushdown Automaton

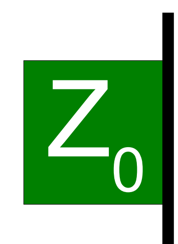
To find an applicable transition, match the current input/stack pair.



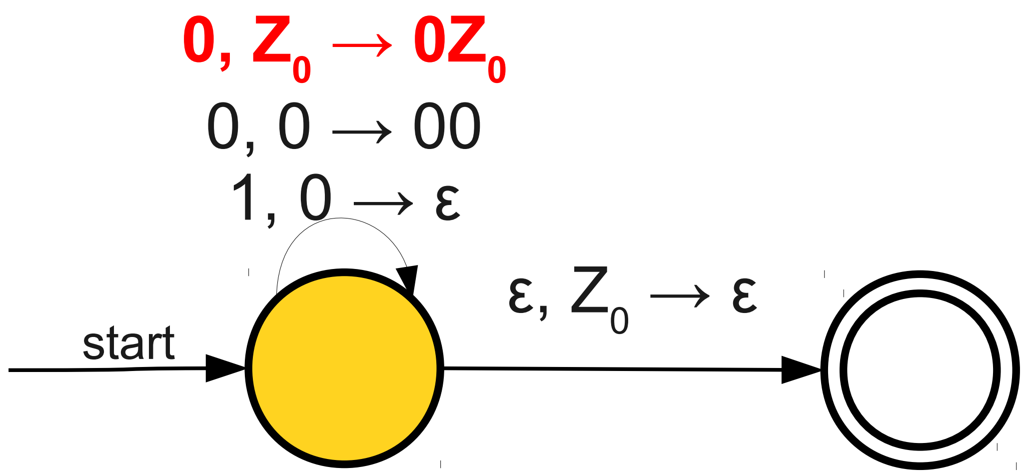
A transition of the form

$$\mathbf{a, b \rightarrow z}$$

Means "If the current input symbol is a and the current stack symbol is b, then follow this transition, pop b, and push the string z.

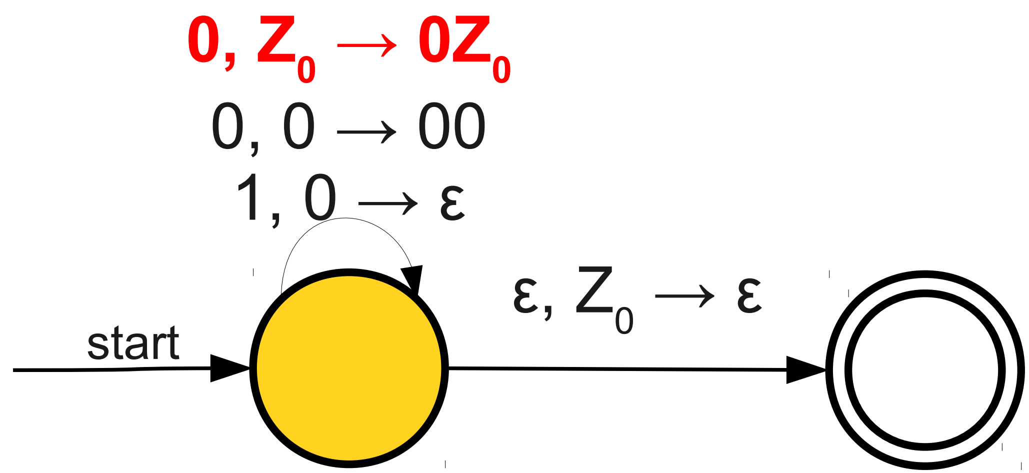


A Simple Pushdown Automaton



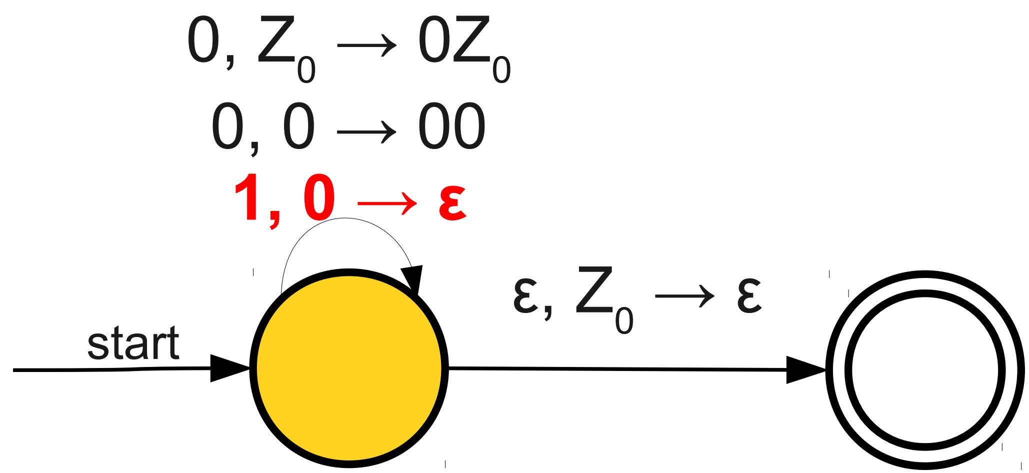
If a transition reads the top symbol of the stack, it <u>always</u> pops that symbol (though it might replace it)

A Simple Pushdown Automaton

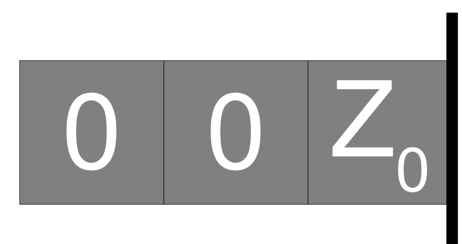


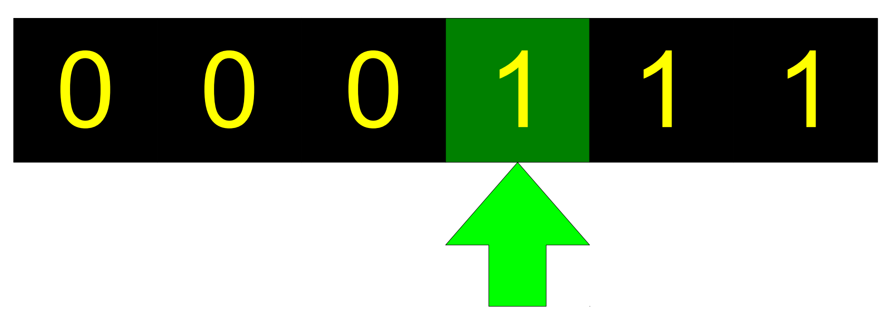
Each transition then pushes some (possibly empty) string back onto the stack. Notice that the leftmost symbol is pushed onto the top.

A Simple Pushdown Automaton

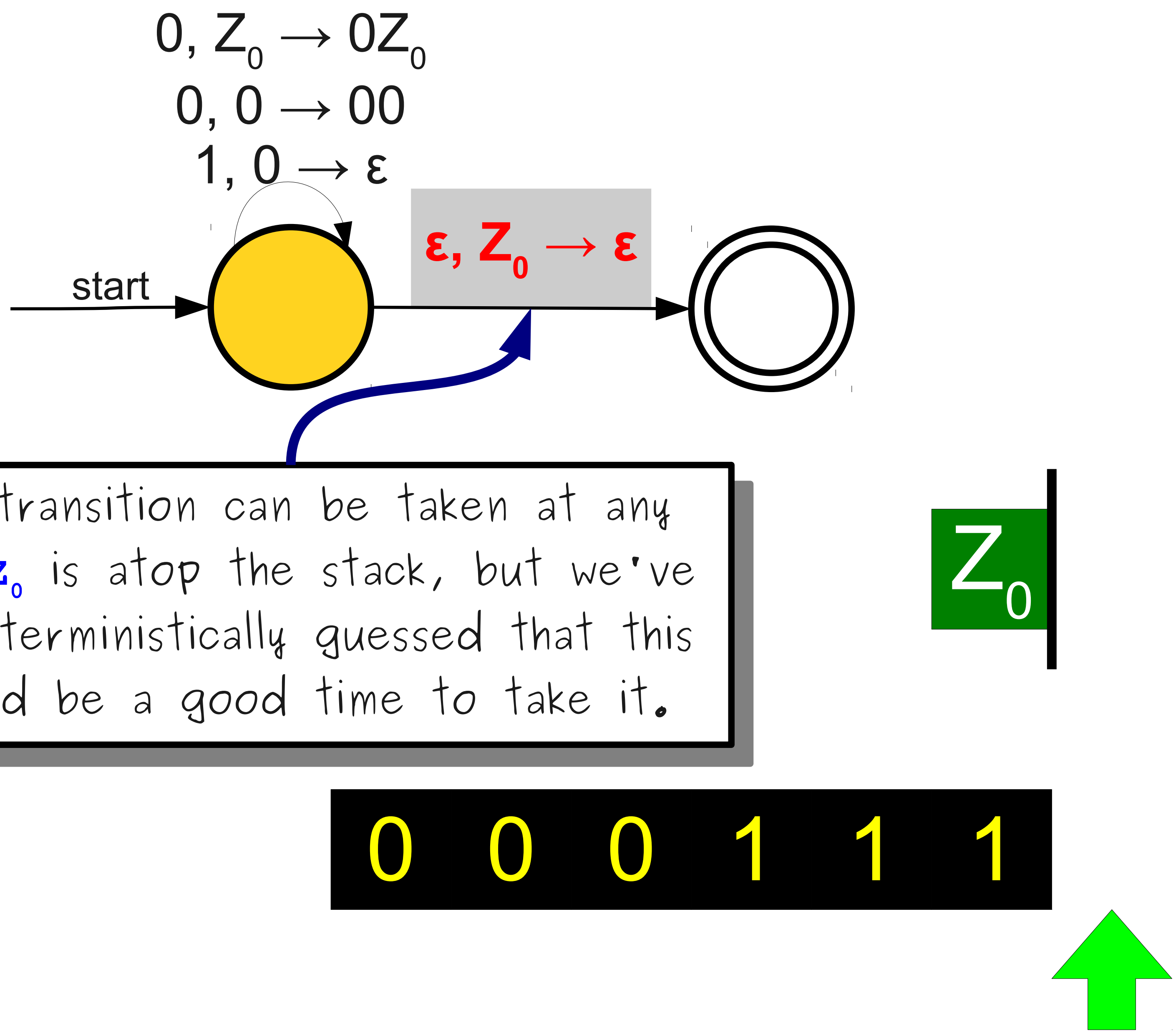


We now push the string ε onto the stack, which adds no new characters. This essentially means "pop the stack."



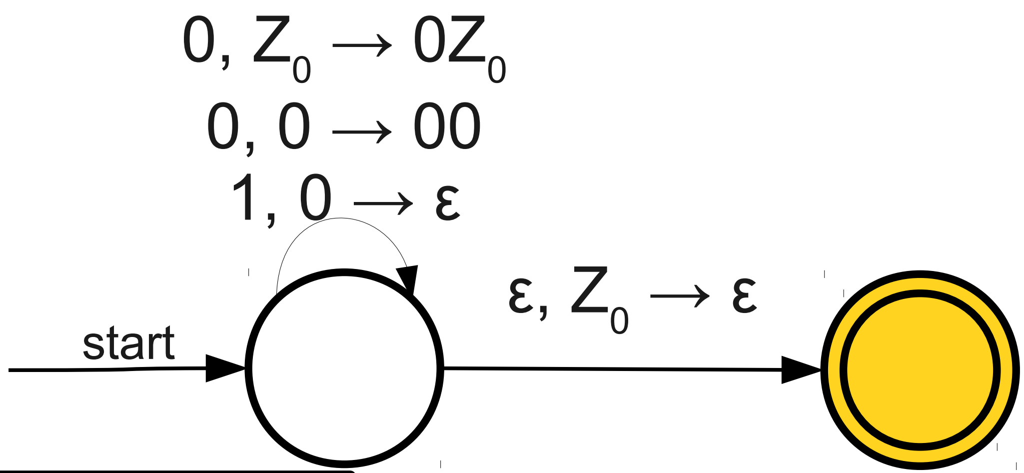


A Simple Pushdown Automaton



This transition can be taken at any time Z_0 is atop the stack, but we've nondeterministically guessed that this would be a good time to take it.

A Simple Pushdown Automaton



0 0 0 1 1 1

The Language of a PDA

- Given a PDA P and a string w, P accepts w iff there is some series of choices such that when P is run on w, it ends in an accepting state.
  - The stack can contain any number of symbols when the machine accepts.
- The **language of a PDA** is the set of strings that the PDA accepts:

$$\mathscr{L}(\boldsymbol{P}) = \{\ \boldsymbol{w} \in \boldsymbol{\Sigma}^* \mid \boldsymbol{P} \textbf{ accepts } \boldsymbol{w}\ \}$$

- If P is a PDA where $\mathscr{L}(P) = L$, we say that P **recognizes** L.

A Note on Terminology

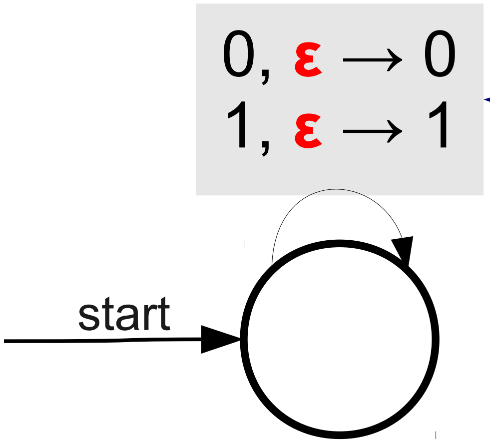
- Finite automata are highly standardized.
- There are many equivalent but different definitions of PDAs.
- The one we will use is a slight variant on the one described in Sipser.
  - Sipser does not have a start stack symbol.
  - Sipser does not allow transitions to push multiple symbols onto the stack.
- Feel free to use either this version or Sipser's; the two are equivalent to one another.

A PDA for Palindromes

- A **palindrome** is a string that is the same forwards and backwards.
- Let Σ = {`0`, `1`} and consider the language

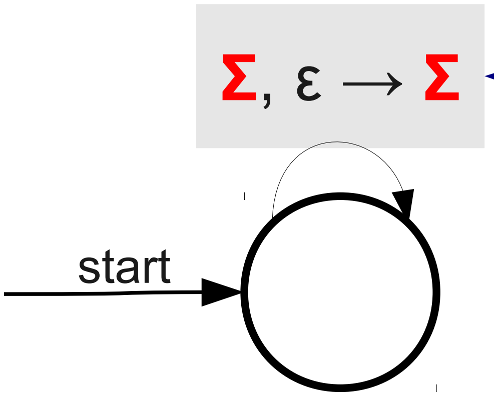
  $$PALINDROME = \{ w \in \Sigma^* \mid w \text{ is a palindrome } \}.$$
- How would we build a PDA for *PALINDROME*?
- ***Idea***: Push the first half of the symbols on to the stack, then verify that the second half of the symbols match.
- ***Nondeterministically*** guess when we've read half of the symbols.
- This handles even-length strings; we'll see a cute trick to handle odd-length strings in a minute.

A PDA for Palindromes



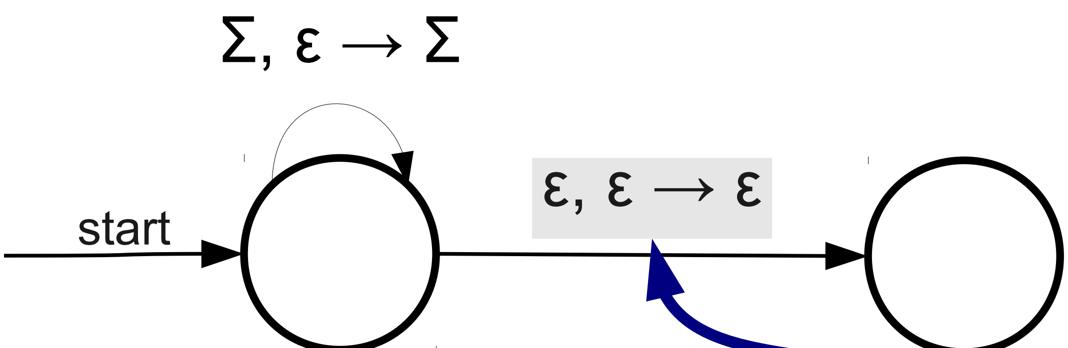
This transition indicates that the transition does not pop anything from the stack. It just pushes on a new symbol instead.

A PDA for Palindromes



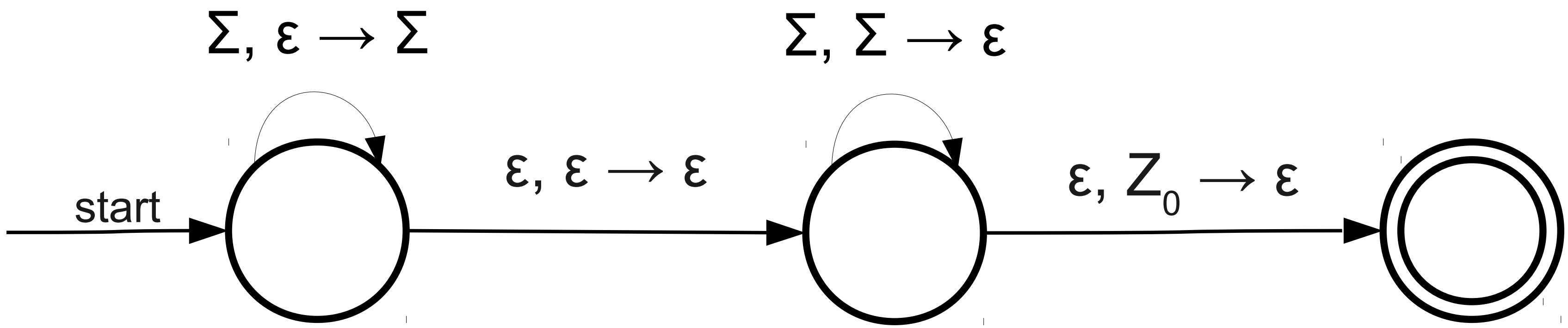
The Σ here refers to the same symbol in both contexts. It is a shorthand for "treat any symbol in Σ this way"

A PDA for Palindromes

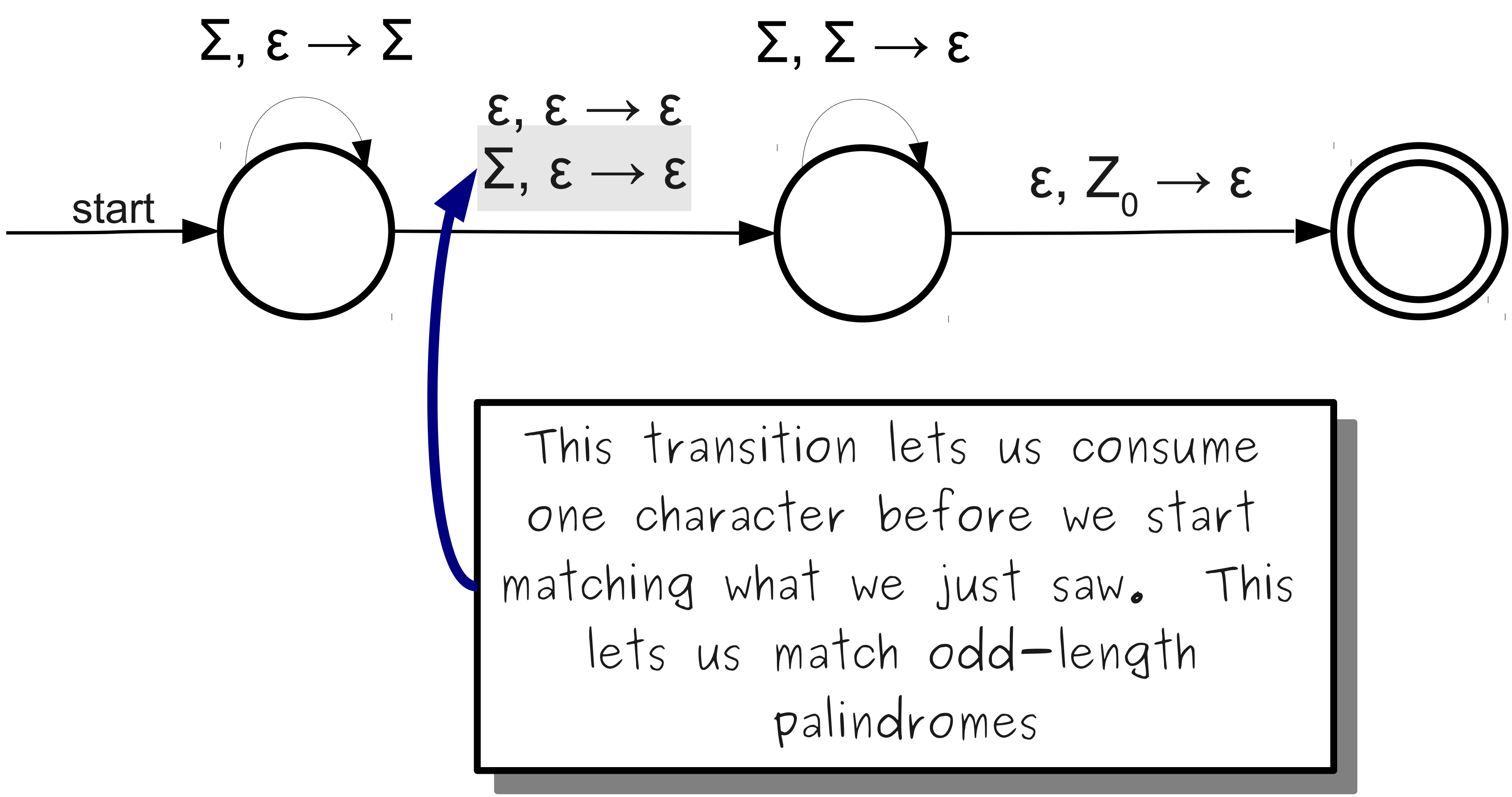


This transition means "don't consume any input, don't change the top of the stack, and don't add anything to a stack. It's the equivalent of an ε-transition in an NFA.

A PDA for Palindromes

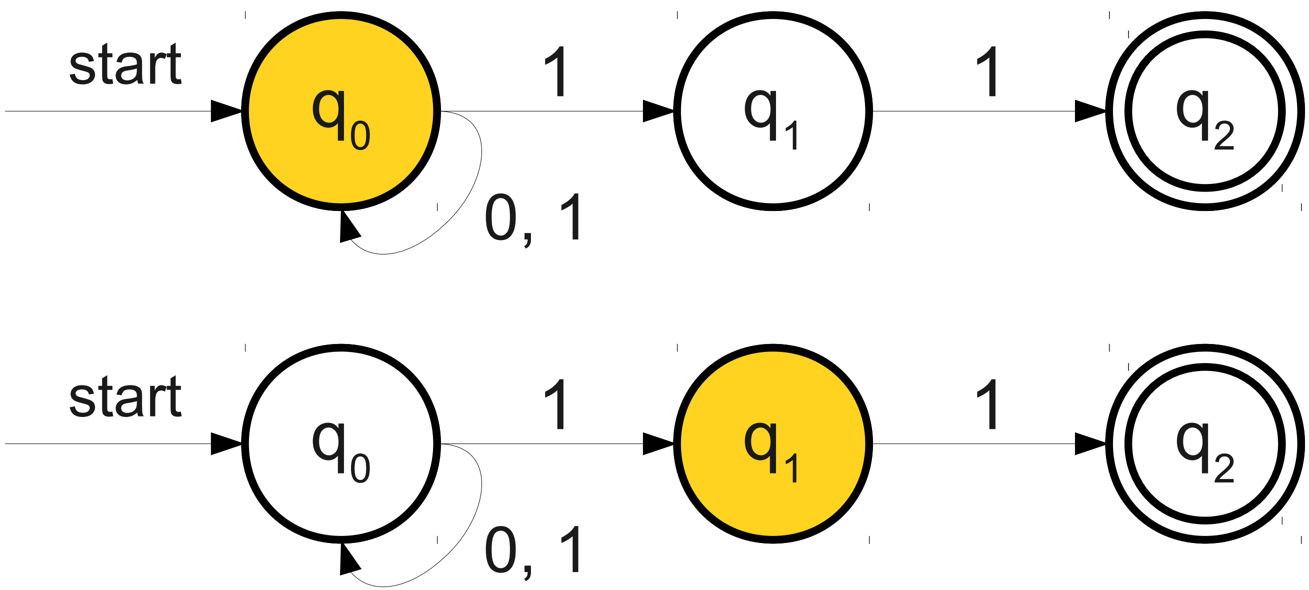


A PDA for Palindromes



A Note on Nondeterminism

- In an NFA, we could interpret nondeterminism as being in multiple states simultaneously.
- This is only possible because NFAs have no extra storage.



A Note on Nondeterminism

- In a PDA, if there are multiple nondeterministic choices, you **cannot** treat the machine as being in multiple states at once.
  - Each state might have its own stack associated with it.
- Instead, there are multiple parallel copies of the machine running at once, each of which has its own stack.

A PDA for Arithmetic

- Let Σ = { `int`, `+`, `*`, `(`, `)` } and consider the language

  $ARITH$ = { $w \in \Sigma^*$ | w is a legal arithmetic expression }

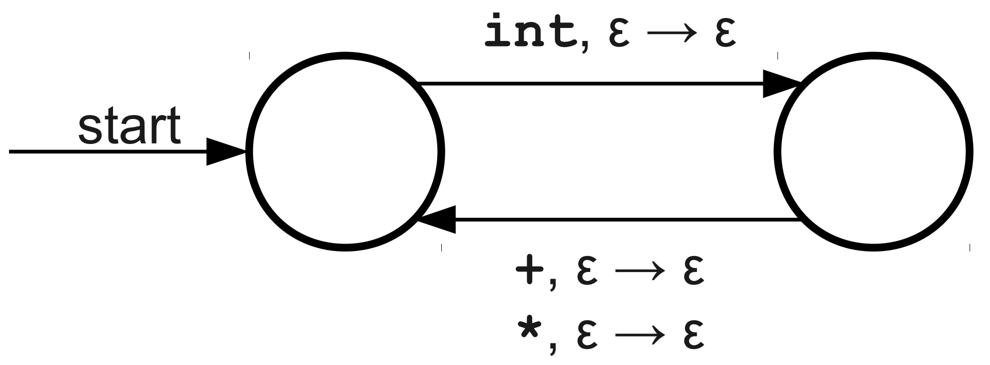
- Examples:

  `int + int * int`

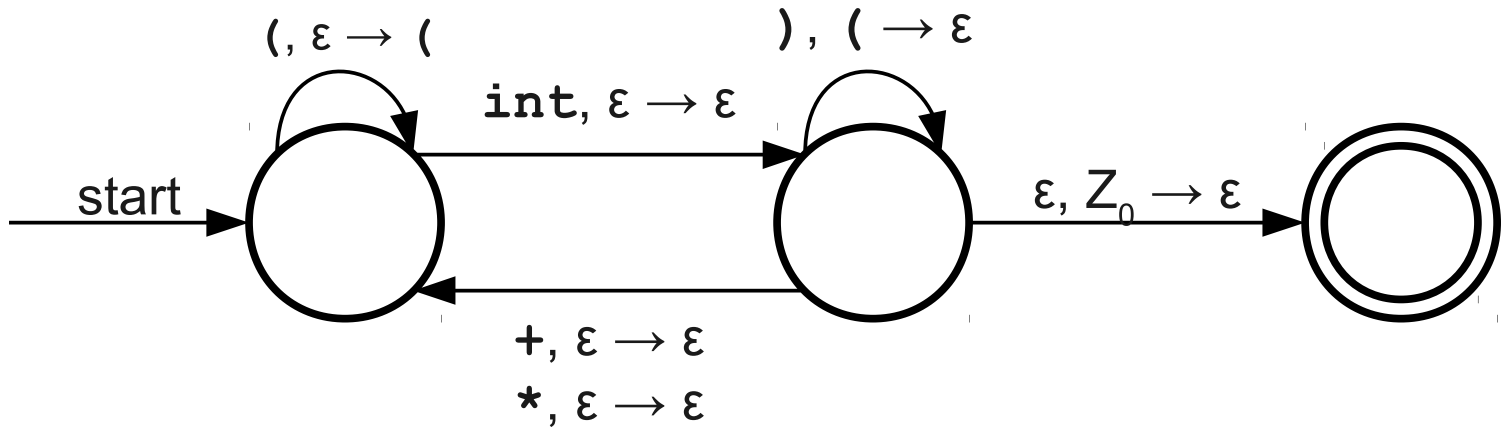
  `((int + int) * (int + int)) + (int)`

- Can we build a PDA for *ARITH*?

A PDA for Arithmetic



A PDA for Arithmetic



The Power of PDAs

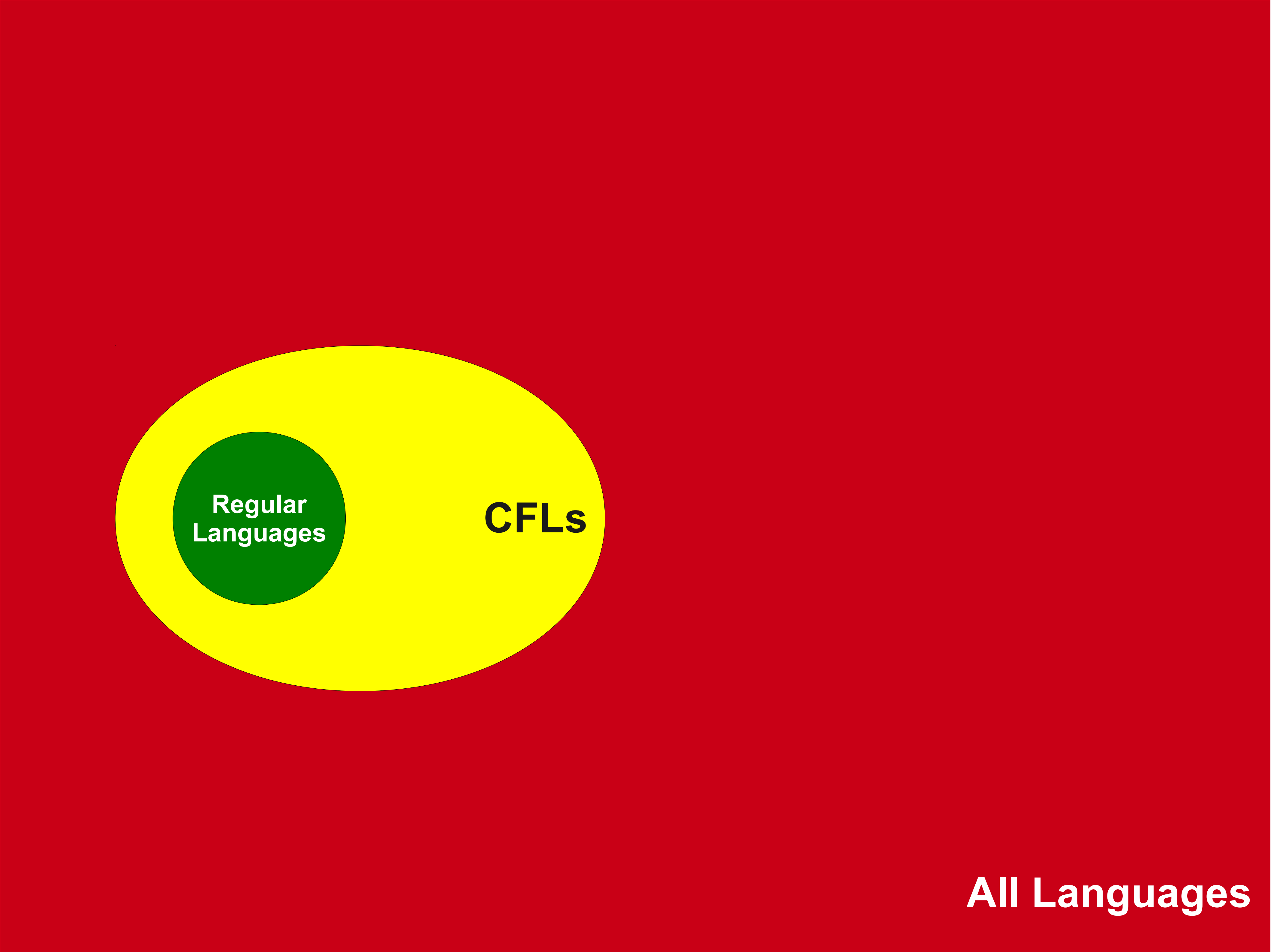
Classes of Languages

- Recall: A language is **regular** iff there is a DFA, NFA, or regular expression for it.
- A language is called **context-free** iff there is a PDA for it.
  - More on that terminology next time.
- We have seen at least one language (palindromes) that is context-free but not regular.
- How do these classes relate to one another?

Regular and Context-Free Languages

Theorem: Any regular language is context-free.

Proof Sketch: Let L be any regular language and consider a DFA D for L. Then we can convert D into a PDA for L by converting any transition on a symbol **`a`** into a transition **`a`**, $\varepsilon \to \varepsilon$ that ignores the stack. This new PDA accepts L, so L is context-free. ■-*ish*


Regular Languages
CFLs
All Languages

Refining the Context-Free Languages

NPDAs and DPDAs

- With finite automata, we considered both deterministic (DFAs) and nondeterministic (NFAs) automata.
- So far, we've only seen nondeterministic PDAs (or **NPDAs**).
- What about deterministic PDAs (**DPDAs**)?

DPDAs

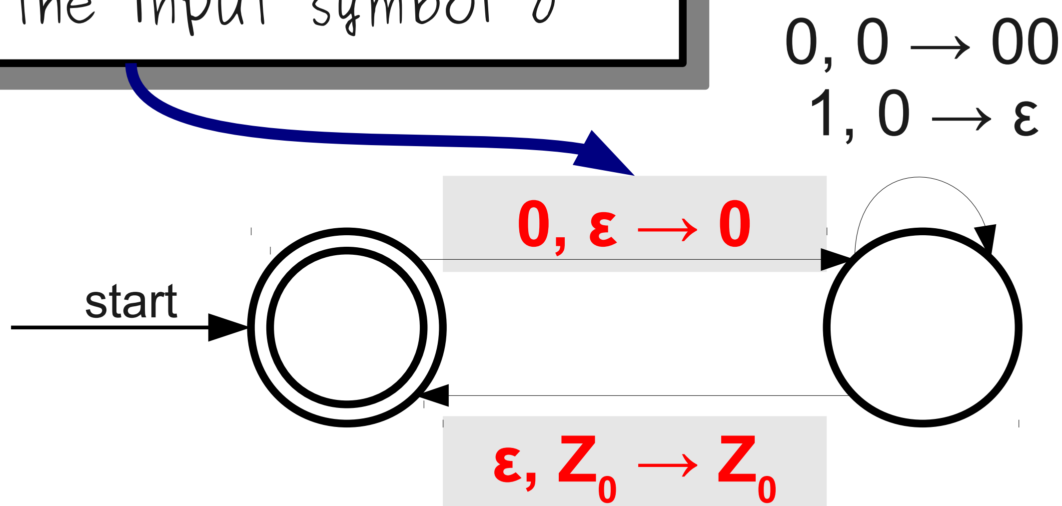
- A **deterministic pushdown automaton** is a PDA with the extra property that

  For each state in the PDA, and for any combination of a current input symbol and a current stack symbol, there is **at most** one transition defined.

- In other words, there is ***at most*** one legal sequence of transitions that can be followed for any input.
- This does ***not*** preclude ε-transitions, as long as there is never a conflict between following the ε-transition or some other transition.
- However, there can be ***at most*** one ε-transition that could be followed at any one time.
- This does ***not*** preclude the automaton "dying" from having no transitions defined; DPDAs can have undefined transitions.

Is this a DPDA?

This ε–transition is allowable because no other transitions in this state use the input symbol 0



This ε–transition is allowable because no other transitions in this state use the stack symbol Z_0.

Why DPDAs Matter

- Because DPDAs are deterministic, they can be simulated efficiently:
  - Keep track of the top of the stack.
  - Store an **action/goto table** that says what operations to perform on the stack and what state to enter on each input/stack pair.
  - Loop over the input, processing input/stack pairs until the automaton rejects or ends in an accepting state with all input consumed.
- If we can find a DPDA for a CFL, then we can recognize strings in that language efficiently.

If we can find a DPDA for a CFL, then we can recognize strings in that language efficiently.

Can we guarantee that we can always find a DPDA for a CFL?

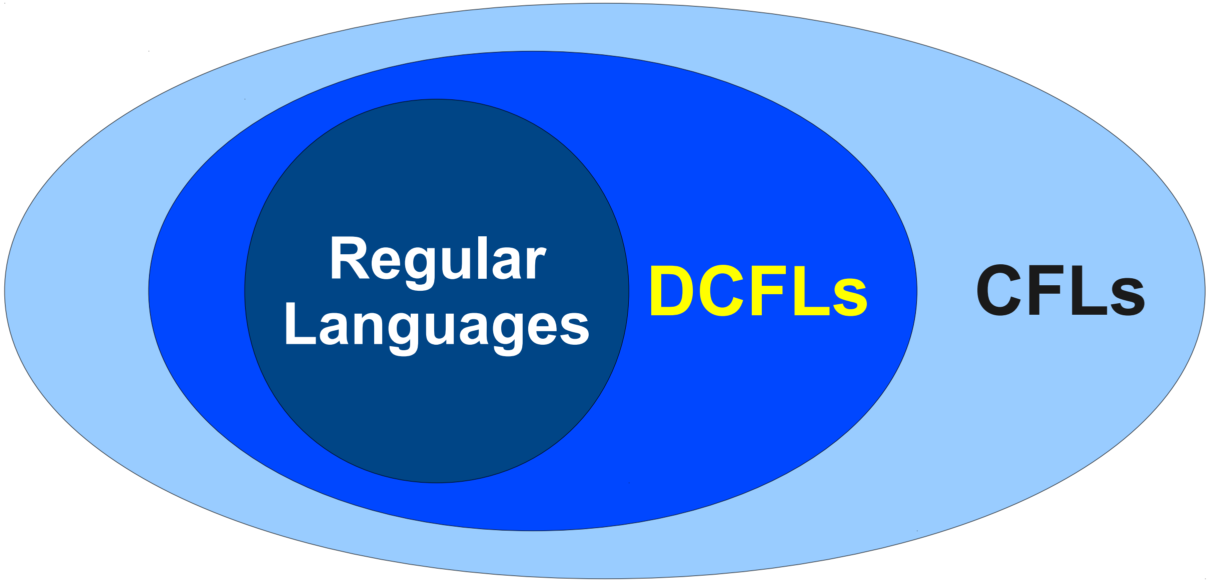
The Power of Nondeterminism

- When dealing with finite automata, there is no difference in the power of NFAs and DFAs.
- However, when dealing with PDAs, there are CFLs that can be recognized by NPDAs that **cannot** be recognized by DPDAs.
- Simple example: The language of palindromes.
  - How do you know when you've read half the string?
- NPDAs are **more powerful** than DPDAs.

Deterministic CFLs

- A context-free language L is called a **deterministic context-free language** (DCFL) if there is some DPDA that recognizes L.
- Not all CFLs are DCFLs, though many important ones are.
  - Balanced parentheses, most programming languages, etc.

Why are all regular languages DCFLs?



Separating DCFLs and CFLs

- It is *extremely difficult* to prove that a given CFL is not a DCFL.
- Challenge problem:

Prove that the language of all palindromes over $\Sigma = \{0, 1\}$ is not deterministic context-free.

Summary

- Automata can be augmented with a memory storage to increase their power.
- PDAs are finite automata equipped with a stack.
- PDAs accept precisely the context-free languages, which are a strict superset of the regular languages.
- Deterministic PDAs are strictly weaker than nondeterministic PDAs.

Next Time

- **Context-Free Grammars**
  - A different formalism for context-free languages.
- **The Limits of CFLs**
  - What problems cannot be solved by PDAs?