Optimisation of tensegrity structures

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March 2024

Abstract

In this project we model tensegrity structures numerically, and wish to minimise their potential energy in order to find a stable resting configuration. We will consider three different cases, namely cable nets, tensegrity domes with fixed nodes and free standing structures. We will implement the BFGS algorithm with a bisection based bracketing method for the strong Wolfe conditions.

1 Introduction

The tensegrity structure is a mechanical structure, first introduced as artworks, but today the range of applications for these structures in engineering have broadened. A tensegrity structure consists of straight elastic bars and elastic cables that are connected at joints, and together they account for the structural integrity and stability of the structure. The tension in the cables is particularly important to the integrity of the structure. The objective of this project is to determine the position X of all the nodes, such that the potential energy of the structure attains a local minima. Under this condition the structure attains a stable resting position, which is why this particular optimisation problem is of interest.

2 Method

To model the configuration of nodes and edges in such structures we use a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of vertices or nodes in the structure, and $\mathcal{E}$ denotes the set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, which can be either a bar or a cable. Further we have the following condition for the edges in the directed graph

$$e_{ij} = (i, j), i < j. \tag{1}$$

Here i and j denotes the numbering of the nodes. The position of node i is given in three dimensions by the following vector $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \in \mathbb{R}^3$. The positions of all the nodes in the structure will be collected in a large vector $X = (x^{(1)}, ..., x^{(N)}), \in \mathbb{R}^{3N}$.

2.1 Model Assumptions

The total energy of the tensegrity structure is given by the expression

$$E(X) = \sum_{e_{ij} \in \mathcal{B}} (E^{bar}_{elast}(e_{ij}) + E^{bar}_{grav}(e_{ij})) + \sum_{e_{ij} \in \mathcal{C}} E^{cable}_{elast}(e_{ij}) + E_{ext}(X), \tag{2}$$

where an edge e_{ij} is either in $\mathcal{B}$ (bars) or $\mathcal{C}$ (cables) belonging to the set of all edges $\mathcal{E}$. The other terms in the expression will be introduced below.

We model both the elastic energy and gravitational energy of the bars in the structure. To simplify our model, we will assume that all bars are made of the same material, and have the same thickness and cross section. These properties will be accounted for through the material parameter $c > 0$. The rest length $l_{ij} > 0$ of the bars may differ. The elastic energy of the bars is given by the following expression:

$$E^{\text{bar}}_{\text{elast}}(e_{ij}) = \frac{c}{2l_{ij}^2}(\|x^{(i)} - x^{(j)}\| - l_{ij})^2. \tag{3}$$

As one can observe from (3), the internal elastic energy is zero when the bar attains it's rest length. Since the bars have mass, we also need to account for their gravitational energy, which is given by

$$E^{\text{bar}}_{\text{grav}}(e_{ij}) = \frac{\rho g l_{ij}}{2}(x_3^{(i)} + x_3^{(j)}) \tag{4}$$

where g is the gravitational acceleration, and ρ the line density of the bar.

We further simplify our model by assuming that the cables are massless, and that they all have the same thickness and are made out of the same material, but may have different rest lengths $l_{ij} > 0$. Since the cables are assumed to have no mass, we only need to account for the elastic energy of the cables, which is given by

$$E^{\text{cable}}_{\text{elast}}(e_{ij}) = \begin{cases} \frac{k}{2l_{ij}^2}(\|x^{(i)} - x^{(j)}\| - l_{ij})^2 & \text{if } \|x^{(i)} - x^{(j)}\| > l_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \le l_{ij} \end{cases}, \tag{5}$$

where $k > 0$ is a material parameter. From (5) one can observe that the cables only attain elastic energy if they are stretched.

Lastly we need to take into account the gravitational energy of the nodes, which are assumed to be point particles with finite mass m_i. Under this assumption we obtain the following expression for the gravitational energy of the nodes

$$E_{\text{ext}}(X) = \sum_{i=1}^{N} m_i g x_3^{(i)} \tag{6}$$

Since the tensegrity structure is in the presence of gravitational force, we have that the problem of minimizing (2) is unbounded below. This is because the total energy of the structure can be decreased by letting all the z-coordinates tend to $-\infty$. This problem can however, be solved by imposing constraints on the problem.

2.2 Possible Model Constraints and Existence of Solution

For the objective function to admit a global solution, we need that it fulfills two conditions, namely that it is lower semi-continuous (lsc) and coercive. Moreover, the set which we minimise over is non-empty, convex and closed.

First, we have that (2) is continuous, as all the terms are polynomials or terms involving norms that are continuous. The expression for the elastic energy of the cables is continuous at $\|x^{(i)} - x^{(j)}\| = l_{ij}$ as $L(e_{ij}) - l_{ij} \to 0$ when $L(e_{ij}) \to l_{ij}$. Thus our expression for the total energy is lsc.

The first type of constraint that we will consider is the fixation of the positions of some of the nodes, that is

$$x^{(i)} = p^{(i)} \text{ for } i = 1, .., M \text{ for given } p^{(i)} \text{ and } 1 \leq M < N. \tag{7}$$

In practice this means that the variables $x^{(i)}$ for $i = 1, .., M$ can be replaced by constants. This yields a lower dimensional, free optimisation problem. The dimension of the problem then becomes $3(N - M)$.

Let's first consider why the directed graph must be connected in order for our optimisation problem to admit a solution. A function is coercive if the following holds

$$f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\} \text{ is coercive if } f(x) \to +\infty \text{ as } ||x|| \to +\infty.$$

Lets consider a fixed point p and a free node x, and assume that the graph is connected. Thus there exists a set of nodes such that p and x are connected. If the edges between these nodes are finite, the distance between p and x are also finite. If one of the edges in this set has infinite length, we must also have that $||x - p|| \to \infty$. The expressions for the elastic energies (3) and (5) tend to $+\infty$ when the norm does. This would not necessarily be true if the graph is not connected, as there might be no bar or cable between these points. Counteracting the elastic energy is the gravitational energy of bars (4) and nodes (6), but since these expressions are linearly dependent on x_3 they are dominated by large values. Thus we have that the function describing the energy of the tensegrity structure is in fact coercive under the given constraint if the graph is connected. Since we minimise over $\mathbb{R}^3$, a solution to our minimisation problem exists.

Now we consider the constraint given by

$$x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}) \qquad \text{for all } i = 1, \ldots, N. \tag{8}$$

This inequality constraint displays the relationship between the z-coordinate and the ground profile modeled by the continuously differentiable function f. To show that the problem of minimizing (2) wrt. (8) admits a solution, we must show that the set we minimise over is non-empty, convex and closed. Because of the nature of the problem, we assume this condition holds.

Now we want to show that (2) is coercive if the constraint (8) is coercive. Let's first assume that $f(x_1, x_2)$ is not coercive. Then we could move the whole structure infinitely far away from the origin along the ground while the z-coordinate is finite. The energy of the structure would be finite, while the norm of X goes to infinity, meanig $E(X)$ would not be coercive. If f is coercive, $E(X)$ will be coercive as long as we have mass in the structure such that the gravitational energy goes to infinity along with some $x_3^{(i)}$. We have therefore shown that (2) admits a solution if f is coercive.

Now we consider the case $f(x_1, x_2) = 0$. In order to get coercivity of $E(X)$ we could use one of the nodes in the structure as the origin at all times. That is, the position of all other nodes are relative to this one node, such that we still have coerciveness of $E(X)$.

2.3 Cable Nets

In this section we study the situation where all the members of the structure are cables. With the assumption that some of the nodes are fixed, we get the optimisation problem

$$\min_X E(X) = \sum_{e_{ij} \in \varepsilon} E_{elast}^{cable}(e_{ij}) + E_{ext}(X) \text{ s.t. } x^{(i)} = p^{(i)}, i = 1, ..., M. \tag{9}$$

2.3.1 Differentiability

Proposition 1. *The function $E(X)$, given by (9), is in $\mathcal{C}^1$, but not in $\mathcal{C}^2$.*

Proof. In order to show that $E(X)$ given by (9) is $\mathcal{C}^1$ we can find the gradient of $E(X)$. We can notice that E_{ext} is just a polynomial of degree one and therefore in $\mathcal{C}^1$. Thus we only focus on E_{elast}^{cable}. As the sum of continuous functions is continuous, we take the derivative of one arbitrary term with respect to both $x^{(i)}$ and $x^{(j)}$ and obtain that

$$\frac{\partial}{\partial x^{(i)}} E_{elast}^{cable}(e_{ij}) := \begin{cases} \frac{k}{l_{ij}^2}(||x^{(i)} - x^{(j)}|| - l_{ij}) \cdot \frac{(x^{(i)} - x^{(j)})}{||x^{(i)} - x^{(j)}||} & \text{if } ||x^{(i)} - x^{(j)}|| > l_{ij}, \\ 0 & \text{if } ||x^{(i)} - x^{(j)}|| \leq l_{ij} \end{cases} \tag{10}$$

and $\frac{\partial}{\partial x^{(j)}} E_{elast}^{cable}(e_{ij})$ is the same with opposite sign. If we take the derivative with respect to any other x, E_{cable}^{elast} will become zero. We look at the case where $||x^{(i)} - x^{(j)}|| > l_{ij}$ and let $||x^{(i)} - x^{(j)}||$ tend to l_{ij}. Then we see that the functions goes to zero, and thus obtain that $E(X)$ is in $\mathcal{C}^1$.

Now we need to check if (9) is in $\mathcal{C}^2$. We can do this by taking the second derivative with respect to $x^{(i)}$ and $x^{(j)}$. Again, we look at the case where $||x^{(i)} - x^{(j)}|| > l_{ij}$. As

$\frac{\partial}{\partial x^{(i)}} E^{cable}_{elast}(e_{ij})$ is a vector with the same expression in each direction, we simplify the second derivative by only looking at the first component, that is

$$\frac{\partial}{\partial x^{(i)}} E^{cable}_{elast}(e_{ij_1}) := \begin{cases} \frac{k}{l_{ij}^2}(||x^{(i)} - x^{(j)}|| - l_{ij}) \cdot \frac{(x_1^{(i)} - x_1^{(j)})}{||x^{(i)} - x^{(j)}||} & \text{if } ||x^{(i)} - x^{(j)}|| > l_{ij}, \\ 0 & \text{if } ||x^{(i)} - x^{(j)}|| \leq l_{ij} \end{cases}. \tag{11}$$

We can use the chain rule to find the second derivative with respect to $x^{(i)}$: let $u := (||x^{(i)} - x^{(j)}|| - l_{ij})$ and $v := \frac{(x_1^{(i)} - x_1^{(j)})}{||x^{(i)} - x^{(j)}||}$. Then

$$\frac{\partial^2 E^{cable}_{elast}(e_{ij_1})}{\partial x^{(i)2}} = (u \frac{\partial^2}{\partial x^{(i)2}} v + v \frac{\partial^2}{\partial x^{(i)2}} u) \frac{k}{l_{ij}^2}$$

By setting $||x^{(i)} - x^{(j)}|| = l_{ij}$, u becomes zero and we get

$$\frac{\partial^2 E^{cable}_{elast}(e_{ij_1}; ||x^{(i)} - x^{(j)}|| = l_{ij})}{\partial x^{(i)2}} = \frac{k}{l_{ij}^2} v \frac{\partial^2}{\partial x^{(i)2}} u = \frac{k}{l_{ij}^2} \frac{(x_1^{(i)} - x_1^{(j)})}{||x^{(i)} - x^{(j)}||^2} (x^{(i)} - x^{(j)}).$$

This is the same derivative as for the case with $\frac{\partial^2}{\partial x^{(j)2}}$. For the mixed derivative, we would get the same answer but with a minus up front.

For these cases to be in $\mathcal{C}^2$, all of them have to become 0. This is only possible if $x_1^i - x_1^j = 0$. We would obtain the same result for the other components, and thus, we get that $E(X)$ only is in $\mathcal{C}^2$ if all points are the same. In that case however, $||x^{(i)} - x^{(j)}|| \leq l_{ij}$. Thus it is not possible for these terms to become zero and (9) is not $\mathcal{C}^2$. $\square$

As a result we can't use Newton's method to minimise (9), as we do not have the hessian.

2.3.2 Convexity

Proposition 2. *$E(X)$ given by (9) is convex.*

Proof. If a function $f : \mathbb{R}^d \to \mathbb{R}$ is convex, the following holds:

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y) \tag{12}$$

for $0 \leq \lambda \leq 1$. As the sum of convex functions is convex, we can check convexity for each term in (9). E_{Ext} is just a sum of linear functions, and thus convex (but not strictly convex). Now we need to check convexity of each term in the sum $\sum_{e_{ij} \in \varepsilon} E^{cable}_{elast}(e_{ij})$. For $||x^{(i)} - x^{(j)}|| \leq l_{ij}$ we have the zero function which is trivial, so we only need to check the case where $||x^{(i)} - x^{(j)}|| > l_{ij}$. The function $f := ||x^{(i)} - x^{(j)}|| - l_{ij}$ is convex because the norm is convex and will stay convex when subtracting a constant. It is, however not strictly convex because of the norm. We want to prove that the square of the convex function f is also convex using the definition of convexity (12):

$$f^2(\lambda x + (1-\lambda)y) \leq \lambda^2 f(x)^2 + 2\lambda(1-\lambda)f(x)f(y) + (1-\lambda)^2 f(y)^2$$

As λ is between zero and one and our function f is positive for all $x^{(i)}, x^{(j)}$ we have that $\lambda^2 \leq \lambda$, $(1-\lambda)^2 \leq (1-\lambda)$ and $f(x)f(y) > 0$, which gives us the following result:

$$f^2(\lambda x + (1-\lambda)y) \leq \lambda f(x)^2 + (1-\lambda)f(y)^2$$

Thus $f^2 = (||x^{(i)} - x^{(j)}|| - l_{ij})^2$ is convex, and it follows that (5) is convex as $\frac{k}{2l_{ij}^2}$ is a positive constant. Thus we have shown that (9) is convex. □

We have proved that the function $E(X)$ (9) is convex. Thus we get the following necessary and sufficient optimality condition:

Theorem 1. *(Necessary and sufficient conditions) Assume that f is convex. Then x^* is a global solution of (9) if and only if $x^* \in \Omega$*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \textit{ for all } x \in \Omega \tag{13}$$

Thus we can say that X^* is a global solution of (9) if and only if $X^* \in \mathbb{R}^{3N}$ and for all $X \in \mathbb{R}^{3N}$. As neither of the terms in (9) are strictly convex, the energy function will not be strictly convex either. This means that the global minimiser is not necessarily unique.

2.4 Tensegrity Domes

Now we study the structure of both cables and bars with the constraint of some fixed nodes. The resulting optimality problem reads

$$\min_X E(X) = \sum_{e_{ij} \in B} (E^{bar}_{elast}(e_{ij}) + E^{bar}_{grav}(e_{ij})) \sum_{e_{ij} \in C} E^{cable}_{elast}(e_{ij}) + E_{ext}(X) \tag{14}$$
$$\text{s.t. } x^{(i)} = p^{(i)}, i = 1, ..., M.$$

2.4.1 Differentiability

We want to verify that (14) is typically not differentiable. This is because of the terms $E^{bar}_{elast}(e_{ij})$. If we differentiate (3) we obtain:

$$\frac{\partial}{\partial x^{(i)}} E^{elast}_{bar} = \frac{c}{l_{ij}^2}(||x^{(i)} - x^{(j)}|| - l_{ij}) \cdot \frac{(x^{(i)} - x^{(j)})}{||x^i - x^j||},$$

which is not defined when $||x^{(i)} - x^{(j)}|| = 0$. This happens when two points that are connected by a bar coincide. In practice, this should not pose a problem as the energy grows rapidly when the bar is being pressed together into one point, and thus any algorithm with reasonable initial values should never end up in this case.

2.4.2 Necessary Conditions

For free optimisation problems the first order necessary conditions states:

Theorem 2. *(Necessary condition) Suppose $f(x)$ is differentiable at x^*. If x^* is a local minimum, $\nabla f(x^*) = 0$.*

Since our objective function is not in $\mathcal{C}^2$, the second order necessary optimality condition does not apply. As previously stated, (14) is usually not differentiable, but since this does not impose any problem in practical situations, we have that the first order necessary condition, namely $\nabla E(X^*) = 0$ still applies in practical situations.

Lemma 3. *The objective function (14) is not convex.*

Proof. E is convex if (12) holds. We do a proof by contradiction and let X and Y be a set of points such that

$$||x^{(i)} - x^{(j)}|| = ||y^{(i)} - y^{(j)}|| = l_{ij} \ \forall i, j \in N.$$

Then let $Y = -X$, and consider the left hand side of (12) under this assumption:

$$E(\lambda X + (1 - \lambda)Y) = E(\lambda X - (1 - \lambda)X).$$

If we let $\lambda = \frac{1}{2}$, we get that

$$E(\frac{1}{2}X - (1 - \frac{1}{2})X) = E(\frac{1}{2}X - \frac{1}{2}X) = E(0) = \sum_{e_{ij} \in B} \frac{c}{2} > 0 \text{ since } c > 0.$$

Now let´s consider the right-hand side of the expression under the given assumptions:

$$\lambda E(X) + (1 - \lambda)E(Y) = \lambda E(X) - (1 - \lambda)E(X) = \frac{1}{2}E(X) - \frac{1}{2}E(X) = 0.$$

$$\lightning \, E(\lambda X + (1 - \lambda)Y) > \lambda E(X) + (1 - \lambda)E(Y)$$

And thus we have that our objective function is not convex. □

In our case this implies that the first order necessary optimality condition is not sufficient.

2.5 Free standing structures

We now want to study a free standing tensegrity structure. This problem is constrained by the equation given in (8). We first formulate the KKT-conditions associated with the problem.

The following is the definition of the Lagrangian

$$\mathcal{L}(X, \boldsymbol{\mu}, \boldsymbol{\lambda}) = E(X) - \sum_{i \in \mathcal{I}} \mu_i g_i(X) - \sum_{j \in \mathcal{E}} \lambda_j h_j(X), \tag{15}$$

where $g_i(X)$ are the inequality constraints, and $h_j(X)$ are the equality constraints. In our case we do not consider any equality constraints, and thus the last term does not apply to our problem. We get the following first order optimality conditions, also known as KKT-conditions.

Theorem 4. *(KKT-conditions for) The objective function (14) is $\in \mathcal{C}^1$ and $g_i(X^*) = f(x_1^{(i)}, x_2^{(i)}) - x_3^{(i)} \in \mathcal{C}^1$. Assume X^* is a local minimum and LICQ holds. Then there exists Lagrange multipliers $\mu_i^* \in \mathbb{R}$, $i \in \mathcal{I}$ st.*
1) $\nabla E(X^*) - \sum_{i\in\mathcal{I}} \mu_i \nabla g_i(X^*) = 0$
2) $\mu_i^* g_i(X*) = 0 \; \forall i \in \mathcal{I}$
3) $\mu_i^* \geq 0 \; \forall i \in \mathcal{I}$
4) $g_i(X^*) \geq 0 \; \forall i \in \mathcal{I}$.

Since the objective function is not convex, the KKT conditions are only necessary.

3 Numerical Experiments

We implemented the BFGS algorithm with a bisection based bracketing method for the strong Wolfe conditions in order to solve the optimisation problems numerically. We chose the BFGS method since our objective function is not in $\mathcal{C}^2$. Thus the Hessian does not exist, and Newtons method can't be applied. The standard stopping criteria used for the BFGS method were a max number of iterations = 100, and for convergence of the norm we used $||\nabla f(x_k) - \nabla f(x_{k+1})|| < 10^{-6}$.

3.1 Cable nets

When modelling the cable net, we used that all $l_{ij} = 3$, all $m_i g = 1/6$ and $k = 3$. When initializing the structure, we used the following nodes:

$$\text{Fixed: } p^{(1)} = [5,5,0],\ p^{(2)} = [-5,5,0],\ p^{(3)} = [-5,-5,0],\ p^{(4)} = [5,-5,0]$$
$$\text{Free: } x^{(5)} = [1,1,1/2],\ x^{(6)} = [-10,20,1/2],\ x^{(7)} = [-1,-1,22],\ x^{(8)} = [1,5,1/2].$$

For this specific problem, the analytical solution of the problem is given from the free nodes

$$x^{(5)} = [2,2,-3/2],\ x^{(6)} = [-2,2,-3/2],\ x^{(7)} = [-2,-2,-3/2],\ x^{(8)} = [2,-2,-3/2].$$

The difference in our solution and the analytical solution were of order 10^{-7}, and it converged linearly.

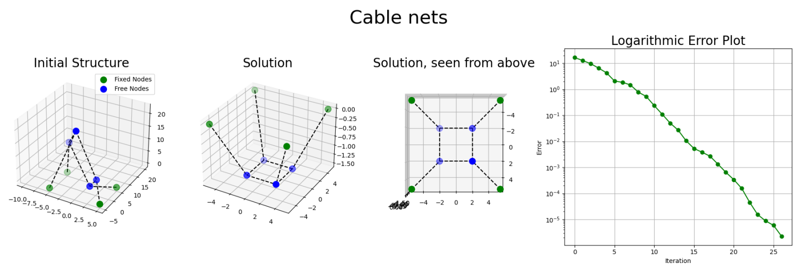

Figure 1: Plots of initial structure, numerical solution, numerical solution seen from above and a logarithmic error plot for the cable net case.

3.2 Tensegrity Dome

When considering the tensegrity dome, we used that all $l_{ij} = 10$ for bars, $l_{ij} = 8$ for cables connecting free and fixed nodes, and $l_{ij} = 1$ for cables connecting free nodes. We also used that all $m_i g = 0$ for $i = 5, 6, 7, 8$, $k = 0.1$, $c = 1$ and $g\rho = 0$. When initializing the structure, we used the following nodes:

$$\text{Fixed: } [1, 1, 0], [-1, 1, 0], [-1, -1, 0], [1, -1, 0]$$
$$\text{Free: } [-0.5, 0, 4], [0, 1, 10], [1, 0, 12], [0.5, 0.5, 11].$$

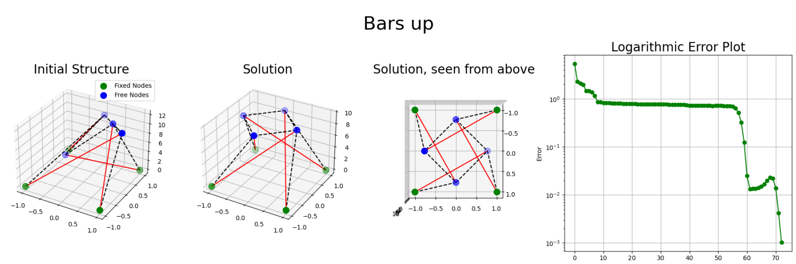

Figure 2: Plots of initial structure, numerical solution, numerical solution seen from above and a logarithmic error plot for the tensegrity dome case.

The difference in our solution and the analytical solution were of order 10^{-4}. it appears that we have linear convergence only around the local minima. This is probably because of the non-convexity of our function. For some other initialization values, the obtained solution appeared upside down. This shows that this problem has at least these two local optima.

3.3 Free Standing Structure

When considering the free-standing structure, we used that all $l_{ij} = 10$ for bars, $l_{ij} = 8$ for "diagonal" cables, $l_{ij} = 2$ for $i, j = 1, 2, 3, 4$, and $l_{ij} = 1$ for $i, j = 5, 6, 7, 8$. We also used $m_i g = 0$, $k = 0.1$, $c = 1$, and $g\rho = (10^{-4})$. In addition, we implemented a penalty function for the constraint presented in (8). When initializing the structure, we used the following free nodes:

$$[1, 1, 1], [-1, 1, 1], [-1, -1, 1], [1, -1, 1], [0.5, 0.5, 10], [-0.5, 0.5, 10], [-0.5, -0.5, 10], [0.5, -0.5, 10].$$

With the ground profile function used, the ground is approximately flat and we obtained

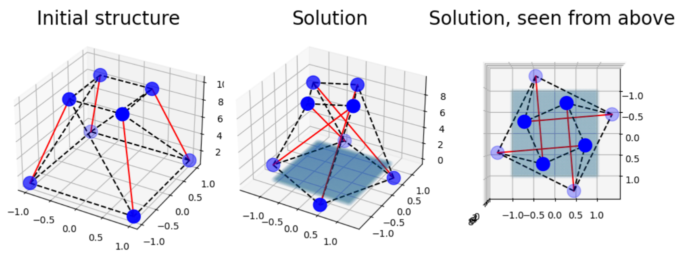

Figure 3: Initial structure to the left and solution found with BFGS to the right, ground profile function $f(x_1, x_2) = 0.1x_1^2 + 0.1x_2^2$.

a similar result as for the tensegrity dome, which is the expected solution.

4 Summary

In this project we studied three different different tensegrity structures. First, we looked at cable-nets with fixed nodes, which was a convex problem and thus we got sufficient conditions and obtained a global minima. In this case we obtained linear convergence.

When we introduced bars, the energy function was no longer convex. Thus we only got necessary conditions and local minima. This was verified as we converged to two different solutions of a test case, given different initial values. The convergence was only linear near the local minima. This is probably because of the non-convexity of the problem.

The last case was the free structure with the inequality constraint that the structure is above the ground. Here the first order necessary conditions were the KKT-conditions, and we only obtained local minima.

In summary, we were able to model the tensegrity structures by optimising their energy functions. Moreover we got a much better understanding of convex and non-convex, constrained optimisation in general.