$3+6+8+10+1=28$

1. a) Nech V, W sú konečnorozmerné vekt. priestory a $f: V \to W$ je lineárne zobrazenie

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Potom $\dim(V) = \dim(Ker(f)) + \dim(Im(f))$

b) $B: \mathbb{R}^3 \longmapsto \mathbb{R}^3$

$B(x_1, x_2, x_3) \longmapsto (x_1 + x_2 + x_3,\ x_1 - x_2,\ -x_1 + x_2)$

$(1,0,0) \longmapsto (1, 1, -1)$

$(0,1,0) \longmapsto (1, -1, 1)$

$(0,0,1) \longmapsto (1, 0, 0)$

matica zobrazenia: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

kernel(B):

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

nech $x_3 = t$

$x_2 + \frac{t}{2} = 0$

$x_2 = \frac{-t}{2}$

$x_1 + \frac{t}{2} = 0$

$x_1 = \frac{-t}{2}$

$Ker(B) = \{(\frac{-t}{2}, \frac{-t}{2}, t), t \in \mathbb{R}\}$

$\dim(Ker(B)) = 1$

Image (B):

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ -1 & 1 & 0 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & b+a \\ 0 & -2 & -1 & b-a \\ 0 & 2 & 1 & c+a \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & b+a \\ 0 & -2 & -1 & b-a \\ 0 & 0 & 0 & c+b \end{array}\right]$$

$c = -b$

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$Im(B) = \{(a, b, -b); a, b \in \mathbb{R}\}$

$\dim(Im(B)) = 2$

máte ortonormalizovať bázy!

ORTONORMALIZUJEM Ker(B) a Im(B)

ak $Ker(B) = \vec{r}_1$,

$$\vec{n}_1 = \frac{\vec{r}_1}{\|\vec{r}_1\|} = \frac{1}{\sqrt{\frac{t^2}{4} + \frac{t^2}{4} + t^2}} \begin{bmatrix} \frac{-t}{2} \\ \frac{-t}{2} \\ t \end{bmatrix} = \frac{1}{\sqrt{\frac{3}{2}t^2}} \begin{bmatrix} \frac{-t}{2} \\ \frac{-t}{2} \\ t \end{bmatrix} =$$

$$= \frac{\sqrt{\frac{2}{3}}}{t} \begin{bmatrix} \frac{-t}{2} \\ \frac{-t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{\frac{2}{3}} \\ -\frac{1}{2}\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ 1 \end{bmatrix}$$

ak $Im(B) = \vec{r}_2 =$

$$\vec{n}_2 = \frac{\vec{r}_2}{\|\vec{r}_2\|} = \frac{1}{\sqrt{a^2 + 2b^2}} \begin{bmatrix} a \\ b \\ -b \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{a^2+2b^2}} \\ \frac{b}{\sqrt{a^2+2b^2}} \\ \frac{-b}{\sqrt{a^2+2b^2}} \end{bmatrix}$$

?!

0

(2.) $\alpha \in \mathbb{R}$; $A_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$: $\begin{bmatrix} 1 & 2 & 1 \\ 0 & \alpha & 0 \\ 1 & 2 & 1 \end{bmatrix}$

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$$(A_\alpha - \lambda I) = \begin{pmatrix} 1-\lambda & 2 & 1 \\ 0 & \alpha-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{pmatrix}$$

$$\det(A_\alpha - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & \alpha-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(\alpha-\lambda)(1-\lambda) - (\alpha-\lambda)$$

$$\begin{matrix} 1-\lambda & 2 & 1 \\ 0 & \alpha-\lambda & 0 \end{matrix}$$

$$= (\alpha-\lambda)\left[(1-\lambda)^2 - 1\right]$$

Charakteristický polynóm:

$$\underbrace{(\alpha-\lambda)}\left[(1-\lambda)^2 - 1\right] = 0$$

$\alpha - \lambda = 0 \Leftrightarrow \alpha = \lambda$

$(1-\lambda)^2 - 1 = 0$

$1 - 2\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 2\lambda = 0$

$\lambda(\alpha - 2) = 0$

$\alpha = 0 \vee \alpha = 2$

našli sme 3 kandidátov:

$(\lambda_1, \lambda_2, \lambda_3) = (\alpha, 0, 2)$

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vlastný vektor pri $\lambda_1 = \alpha$:

$$\begin{bmatrix} 1-\alpha & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1-\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1-\alpha & 2 & 1 & 0 \\ 1 & 2 & 1-\alpha & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{2-\alpha}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

nech $x_3 = s$.

$x_2 + s \cdot \left(\frac{2-\alpha}{2}\right) = 0$

$2x_2 = -s(2-\alpha)$

$x_2 = \frac{-s(2-\alpha)}{2}$

$x_1 = s$

$\mathrm{Ker}(A_\alpha - \alpha_1 I) = \{(s, \frac{-s(2-\alpha)}{2}, s); s \in \mathbb{R}\}$

napr. $\underbrace{(1, \frac{\alpha-2}{2}, 1)}$

ale $\alpha = 2$, tento je rovnaký

vlastný vektor pri $\alpha_2 = 0$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

nech $x_3 = s$

$\alpha x_2 = 0$

$x_2 = 0$

$x_1 = -s$

$\mathrm{Ker}(A_\alpha - \alpha_2 I) = \{(s, 0, s); s \in \mathbb{R}\}$

napr. $(-1, 0, 1)$

vlastný vektor pri α_3:

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$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & \alpha-2 & 0 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & \alpha-2 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right] \cdot \frac{1}{\alpha-2} \Big\} \cdot (-1) \sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

nech $x_3 = s$

$x_2 = 0$

$x_1 = s$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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$\mathrm{Kernel}(A_\alpha - \alpha_3 I) = \{(s, 0, s); s \in \mathbb{R}\}$

napr. $(1, 0, 1)$ ako tento

príklad bázy: $X = \begin{bmatrix} 1 & -1 & 1 \\ \frac{\alpha-2}{2} & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ a $[A_\alpha]_{XX} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\to$ tu vznikne diagonálna matica pre $\forall \alpha; \alpha \in \mathbb{R}$

Išli ste na to dobre, ale nedotiahli ste to.

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(3.)

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Dokazujeme $T^2 = 2T;\ T\vec{v} = \lambda\vec{v} \Rightarrow \lambda \in \{0,2\}$

$$T^2\vec{v} = T(T\vec{v}) = T(\lambda\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$$

$$2T\vec{v} = 2(T\vec{v}) = 2(\lambda\vec{v}) = 2\lambda\vec{v}$$

$$T^2 = 2T \Leftrightarrow \lambda^2\vec{v} = 2\lambda v$$

$\vec{v} \neq \vec{0}$

$$\lambda^2\vec{v} - 2\lambda\vec{v} = 0$$

$$\lambda\vec{v}(\lambda - 2) = 0 \Leftrightarrow \lambda = 0 \vee \lambda = 2$$

QED.

(4.) V : priestor vybavený skal. súčinom $\langle\circ,\circ\rangle$ DANIEL KRIŠTOF

$S: V \mapsto V$

$$\langle\circ,\circ\rangle_S := \langle S\circ, S\circ\rangle \quad \text{je tiež skal. súčin?}$$

Dokazujem cez vlastnosti skal. súčinu:

1) SYMETRIA: $\langle x,y\rangle = \langle y,x\rangle$:

$$\langle x,y\rangle_S \overset{\text{def.}}{=} \langle Sx, Sy\rangle = \langle Sy, Sx\rangle = \langle y,x\rangle_S$$

2) LINEARITA V ARGUMENTOCH: $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$

$\langle ax, y\rangle = a\langle x,y\rangle$

$$\langle S(x+y), Sz\rangle = \langle Sx+Sy, Sz\rangle =$$

$$= \langle Sx, Sz\rangle + \langle Sy, Sz\rangle = \langle x,z\rangle_S + \langle y,z\rangle_S$$

3) NEZÁPORNOSŤ (DEFINITNOSŤ) $\Rightarrow$ $\langle x,x\rangle \geq 0$ pre $\forall x \in V$

$\langle x,x\rangle = 0$ pre $x = \vec{0}$

$$\langle x,x\rangle_S \geq 0 \;;$$

$$\langle x,x\rangle_S = \langle Sx, Sx\rangle \geq 0 \quad \text{z def.}$$

a z pôvodného výrazu: $\langle Sx, \cancel{Sy}\rangle = 0 \iff Sx = 0$

Sx

$\underline{\underline{x = 0}}$

teda $\langle\circ,\circ\rangle_S$ je skalárny súčin.

QED

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5.) a) Vlastný vektor $\vec{u}$ lineárnej transformácie A je také $\vec{u} \neq 0$, že existuje λ:

$$A\vec{u} = \lambda \vec{u},$$

kde λ je vlastné číslo zobrazenia A.

b)