0.1 Factorization

0.1 The Quadratic Identities

For two real numbers a and b, we have

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{(1st square formula)}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \text{(2nd square formula)}$$

$$(a+b)(a-b) = a^2 - b^2 \qquad \text{(3rd square formula)}$$

The language box

$(a+b)^2$ and $(a-b)^2$ are called **complete squares**.

The 3rd square formula is also called the **conjugate formula**.

All the square formulas are *identities*. An **identity** is an equation that is satisfied no matter which values are given to the variables in the equation.

Example 1

Rewrite $a^2 + 8a + 16$ to a complete square.

Answer

$$\begin{aligned} a^2 + 8a + 16 &= a^2 + 2 \cdot 4a + 4^2 \\ &= (a+4)^2 \end{aligned}$$

Example 2

Rewrite $k^2 + 6k + 7$ to an expression where k is part of a complete square.

Answer

$$\begin{aligned} k^2 + 6k + 7 &= k^2 + 2 \cdot 3k + 7 \\ &= k^2 + 2 \cdot 3k + 3^2 - 3^2 + 7 \\ &= (k+3)^2 - 2 \end{aligned}$$

Example 3

Factorize $x^2 - 10x + 16$.

Answer

We start by creating a complete square:

$$\begin{aligned} x^2 - 10x + 16 &= x^2 - 2 \cdot 5x + 5^2 - 5^2 + 16 \\ &= (x-5)^2 - 9 \end{aligned}$$

We note that $9 = 3^2$, and use the 3rd square formula:

$$\begin{aligned} (x-5)^2 - 3^2 &= (x-5+3)(x-5-3) \\ &= (x-2)(x-8) \end{aligned}$$

Thus,

$$x^2 - 10x + 16 = (x-2)(x-8)$$

0.1 The Quadratic Identities (explanation)

The square formulas follow directly from the distributive law in multiplication (see MB).

0.2 The Sum-Product Method

Given $x, b, c \in \mathbb{R}$. If $a_1 + a_2 = b$ and $a_1 a_2 = c$, then

$$x^2 + bx + c = (x + a_1)(x + a_2) \tag{1}$$

Example 1

Factorize the expression $x^2 - x - 6$.

Answer

Since $2(-3) = -6$ and $2 + (-3) = -1$, we have

$$x^2 - 1x - 6 = (x+2)(x-3)$$

Example 2

Factorize the expression $b^2 - 5b + 4$.

Answer

Since $(-4)(-1) = 4$ and $(-4) + (-1) = -5$, we have

$$b^2 - 5b + 4 = (b - 4)(b - 1)$$

Example 3

Solve the inequality

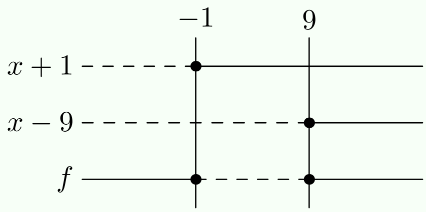
$$x^2 - 8x - 9 \leq 0$$

Answer

Since $1(-9) = -9$ and $1 + (-9) = -8$, we have

$$x^2 - 8x - 9 = (x + 1)(x - 9)$$

We set $f = (x + 1)(x - 9)$, and make a **sign table**:



The sign table shows the following:

- The expression $x + 1$ is negative when $x < -1$, equals 0 when $x = -1$, and is positive when $x > -1$.

- The expression $x - 9$ is negative when $x < 9$, equals zero 0 when $x = 9$, and is positive when $x > 9$.

- Since $f = (x + 1)(x - 9)$,

$$\begin{aligned} &f > 0 \text{ when } x \in [-\infty, -1) \cup (9, \infty] \\ &f = 0 \text{ when } x \in \{-1, 9\} \\ &f < 0 \text{ when } x \in (-1, 9) \end{aligned}$$

Therefore, $x^2 - 8x - 9 \leq 0$ when $x \in [-1, 9]$.

0.2 The Sum-Product Method (explanation)

We have

$$\begin{aligned}(x+a_1)(x+a_2) &= x^2 + xa_2 + a_1x + a_1a_2 \\ &= x^2 + (a_1+a_2)x + a_1a_2\end{aligned}$$

If $a_1 + a_2 = b$ og $a_1a_2 = c$, then

$$(x+a_1)(x+a_2) = x^2 + bx + c$$

0.2 Quadratic Equations

0.3 Quadratic equation with constant term

Given the equation

$$ax^2 + bx + c = 0 \tag{2}$$

where $a, b, c \in \mathbb{R}$. Then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{abc-formula}$$

If $x = x_1$ and $x = x_2$ are the solutions given by the *abc*-formula, is

$$ax^2 + bx + c = a(x - x_1)(x - x_2) \tag{3}$$

Example 1

a) Solve the equation $2x^2 - 7x + 5 = 0$.

b) Factorize the expression on the left side in task a).

Answer

a) We use the *abc*-formula. Then $a = 2$, $b = -7$ and $c = 5$. Now we get that

$$\begin{aligned} x &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 2 \cdot 5}}{2 \cdot 2} \\ &= \frac{7 \pm \sqrt{49 - 40}}{4} \\ &= \frac{7 \pm \sqrt{9}}{4} \\ &= \frac{7 \pm 3}{4} \end{aligned}$$

Either is

$$x = \frac{7+3}{4} = \frac{10}{4} = \frac{5}{2}$$

or

$$x = \frac{7-3}{4} = 1$$

b) $2x^2 - 7x + 5 = 2(x - 1)\left(x - \frac{5}{2}\right)$

Example 2

Solve the equation

$$x^2 + 3x - 10 = 0$$

Answer

We use the *abc*-formula. Then $a = 1$, $b = 3$, and $c = -10$. Now we get that

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot (-10)}}{2 \cdot 1} \\ &= \frac{-3 \pm \sqrt{9 + 40}}{2} \\ &= \frac{-3 \pm \sqrt{49}}{2} \\ &= \frac{-3 \pm 7}{2} \end{aligned}$$

Thus

$$x = -5 \qquad \vee \qquad x = 2$$

Example 3

Solve the equation

$$4x^2 - 8x + 1 = 0$$

Answer

By the *abc*-formula, we have that

$$\begin{aligned} x &= \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 4 \cdot 1}}{2 \cdot 4} \\ &= \frac{8 \pm \sqrt{64 - 16}}{8} \\ &= \frac{8 \pm \sqrt{48}}{8} \\ &= \frac{8 \pm 4\sqrt{3}}{8} \\ &= \frac{2 \pm \sqrt{3}}{2} \end{aligned}$$

Thus

$$x = \frac{2+\sqrt{3}}{2} \quad \vee \quad x = \frac{2-\sqrt{3}}{2}$$

Quadratic Equations (explanation)

Given the equation

$$ax^2 + bx + c = 0$$

We start by rewriting the equation:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then we make a perfect square and use the conjugate root theorem to factorize the expression:

$$\begin{aligned}
x^2 + \frac{b}{a}x + \frac{c}{a} &= x^2 + 2 \cdot \frac{b}{2a}x + \frac{c}{a} \\
&= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \\
&= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} \\
&= \left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 \\
&= \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)
\end{aligned}$$

The expression above equals 0 when

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \vee \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

0.3 Polynomial Division

0.3.1 Methods

When two given numbers are not divisible by each other, we can use fractions to express the quotient. For example,

$$\frac{17}{3} = 5 + \frac{2}{3} \tag{4}$$

The idea behind (4) is that we rewrite the numerator so that we bring out the part of 17 that is divisible by 3:

$$\frac{17}{3} = \frac{5 \cdot 3 + 2}{3} = 5 + \frac{2}{3}$$

The same reasoning can be applied to fractions with polynomials, and then it's called **polynomial division**.

Example 1

Perform polynomial division on the expression

$$\frac{2x^2 + 3x - 4}{x + 5}$$

Answer

Method 1

We do the following steps; starting with the highest power of x in the numerator, we create expressions that are divisible by the denominator.

$$\begin{aligned}
\frac{2x^2 + 3x - 4}{x + 5} &= \frac{2x(x+5) - 10x + 3x - 4}{x + 5} \\
&= 2x + \frac{-7x - 4}{x + 5} \\
&= 2x + \frac{-7(x+5) + 35 - 4}{x + 5} \\
&= 2x - 7 + \frac{31}{x + 5}
\end{aligned}$$

Method 2 (See the calculation under the points)

i) We observe that the term with the highest order of x in the dividend is $2x^2$. This expression can be obtained by multiplying the dividend by $2x$. We write $2x$ to the right of the equals sign, and subtract $2x(x+5) = 2x^2 + 10x$.

ii) The difference from point ii) is $-7x - 4$. We can bring out the term with the highest order of x by multiplying the dividend by -7. We write -7 to the right of the equals sign, and subtract $-7(x+5) = -7x - 35$.

iii) The difference from point iii) is 31. This is an expression that has a lower order of x than the dividend, and thus we write $\frac{31}{x+5}$ to the right of the equals sign.

$$
\begin{array}{l}
(2x^2 + 3x - 4) : (x+5) = 2x - 7 + \dfrac{31}{x+5} \\
-\underline{(2x^2 + 10x)} \\
\qquad - 7x - 4 \\
\qquad - \underline{(-7x - 35)} \\
\qquad\qquad\quad 31
\end{array}
$$

Example 2

Perform polynomial division on the expression

$$\frac{x^3 - 4x^2 + 9}{x^2 - 2}$$

Answer

Method 1

$$\begin{aligned}
\frac{x^3 - 4x^2 + 9}{x^2 - 2} &= \frac{x(x^2 - 2) + 2x - 4x^2 + 9}{x^2 - 2} \\
&= x + \frac{-4x^2 + 2x + 9}{x^2 - 2} \\
&= x + \frac{-4(x^2 - 2) - 8 + 2x + 9}{x^2 - 2} \\
&= x - 4 + \frac{2x + 1}{x^2 - 2}
\end{aligned}$$

Method 2

$$\begin{array}{l}
(x^3 - 4x^2 + 9) : (x^2 - 2) = x - 4 + \dfrac{2x + 1}{x^2 - 2} \\
-\underline{(x^3 - 2x)} \\
\qquad -4x^2 + 2x + 9 \\
\quad -\underline{(-4x^2 + 8)} \\
\qquad\qquad 2x + 1
\end{array}$$

Example 3

Perform polynomial division on the expression

$$\frac{x^3 - 3x^2 - 6x + 8}{x - 4}$$

Answer

Method 1

$$\begin{aligned}
\frac{x^3 - 3x^2 - 6x + 8}{x - 4} &= \frac{x^2(x-4) + 4x^2 - 3x^2 - 6x + 8}{x - 4} \\
&= x^2 + \frac{x^2 - 6x + 8}{x - 4} \\
&= x^2 + \frac{x(x-4) + 4x - 6x + 8}{x - 4} \\
&= x^2 + x + \frac{-2x + 8}{x - 4} \\
&= x^2 + x - 2
\end{aligned}$$

Method 2

$$\begin{array}{l}
(x^3 - 3x^2 - 6x + 8) : (x - 4) = x^2 + x - 2 \\
-\underline{(x^3 - 4x^2)} \\
\qquad\qquad x^2 - 6x + 8 \\
\qquad - \underline{(-x^2 + 4x)} \\
\qquad\qquad\quad -2x + 8 \\
\qquad\qquad - \underline{(-2x + 8)} \\
\qquad\qquad\qquad\quad 0
\end{array}$$

0.3.2 Divisibility and Factors

The examples on pages 9 - 12 point to some important relationships that apply to general cases:

0.4 Polynomial Division

Let A_k denote a polynomial A of degree k. Given the polynomial P_m, then there exist polynomials Q_n, S_{m-n}, and R_{n-1}, where $m \geq n > 0$, such that

$$\frac{P_m}{Q_n} = S_{m-n} + \frac{R_{n-1}}{Q_n} \tag{5}$$

The language box

If $R_{n-1} = 0$, we say that P_m is **divisible** by Q_n.

Example 1

Investigate whether the polynomials are divisible by $x - 3$.

a) $P(x) = x^3 + 5x^2 - 22x - 56$

b) $K(x) = x^3 + 6x^2 - 13x - 42$

Answer

a) By polynomial division, we find that

$$\frac{P}{x-2} = x^2 + 8x + 2 - \frac{50}{x-2}$$

Thus, P is *not* divisible by $x - 3$.

b) By polynomial division, we find that

$$\frac{K}{x-2} = x^2 + 9x + 14$$

Thus, K is divisible by $x - 3$.

0.5 Factors in Polynomials

Given a polynomial $P(x)$ and a constant a. Then we have that

$$P \text{ is divisible by } x - a \Longleftrightarrow P(a) = 0 \tag{6}$$

If this is true, there exists a polynomial $S(x)$ such that

$$P = (a - x)S \tag{7}$$

Example 1

Given the polynomial

$$P(x) = x^3 - 3x^2 - 6x + 8$$

a) Show that $x = 1$ solves the equation $P = 0$.

b) Factorize P.

Answer

a) We investigate $P(1)$:

$$\begin{aligned} P(1) &= 1^3 - 3 \cdot 1^2 - 6 \cdot 1 + 8 \\ &= 0 \end{aligned}$$

Thus, $P = 0$ when $x = 1$.

b) Since $P(1) = 0$, $x - 1$ is a factor in P. By polynomial division, we find that

$$P = (x - 1)(x^2 - 2x - 8)$$

Since $2(-4) = -8$ and $-4 + 2 = -2$, we have

$$x^2 - 2x - 8 = (x + 2)(x - 4)$$

This means that

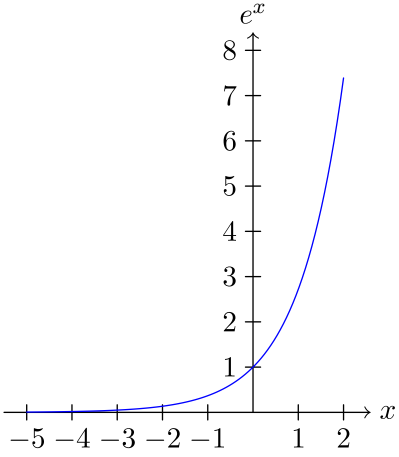
$$P = (x - 1)(x + 2)(x - 4)$$

0.4 Euler's Number

Euler's Number is a constant of such significant importance in mathematics that it has been given its own letter; e. The number is irrational[1], and the first ten digits are

$$e = 2.718281828...$$

The most fascinating properties of this number become apparent when investigating the function $f(x) = e^x$. This is an exponential function of such importance that it is simply known as **the exponential function**. This function will be examined more closely in Appendix ?? and Chapter ??.



[1] And transcendental.

0.5 Logarithms

In MB, we looked at powers, which consist of a base and an exponent. A **logarithm** is a mathematical operation relative to a number. If a logarithm is relative to the base of a power, the operation will result in the exponent.

The logarithm relative to 10 is written $\log_{10}$. For example,

$$\log_{10} 10^2 = 2$$

Furthermore, for example,

$$\log_{10} 1000 = \log_{10} 10^3 = 3$$

Consequently, we can write

$$1000 = 10^{\log_{10} 1000}$$

With the power rules as a starting point (see MB), many rules for logarithms can be derived.

0.6 Logarithms

Let $\log_a$ denote the logarithm relative to $a > 0$. For $m \in \mathbb{R}$, then

$$\log_a a^m = m \tag{8}$$

Alternatively, we can write

$$m = a^{\log_a m} \tag{9}$$

Example 1

$$\log_5 5^9 = 9$$

Example 2

$$3 = 8^{\log_8 3}$$

The language box

$\log_{10}$ is often written as $\log$, while $\log_e$ is often written as $\ln$ or (!) $\log$. When using digital aids to find logarithm values, it is therefore important to check what the base is. In this book, we shall write $\log_e$ as $\ln$.

The logarithm with e as the base is called the **natural logarithm**.

Example 3

$$\log 10^7 = 7$$

Example 4

$$\ln e^{-3} = -3$$

0.7 Logarithm Rules

Note: The logarithm rules are here given by the natural logarithm. The same rules will apply by replacing ln with $\log_a$, and e with a, for $a > 0$.

For $x, y > 0$, we have that

$$\ln e = 1 \tag{10}$$

$$\ln 1 = 0 \tag{11}$$

$$\ln(xy) = \ln x + \ln y \tag{12}$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y \tag{13}$$

For a number y and $x > 0$, is

$$\ln x^y = y \ln x \tag{14}$$

Example 1

$$\ln\left(ex^5\right) = \ln e + \ln x^5 = 1 + 5\ln x$$

Example 2

$$\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$$

Logarithm Rules (explanation)

Equation (10)

$$\ln e = \ln e^1 = 1$$

Equation (11)

$$\ln 1 = \ln e^0 = 0$$

Equation (12)

For $m, n \in \mathbb{R}$, we have that

$$\begin{aligned} \ln e^{m+n} &= m + n \\ &= \ln e^m + \ln e^n \end{aligned}$$

We set[1] $x = e^m$ and $y = e^n$. Since $\ln e^{m+n} = \ln(e^m \cdot e^n)$, then

$$\ln(xy) = \ln x + \ln y$$

Equation (13)

By examining $\ln a^{m-n}$, and by setting $y = a^{-n}$, the explanation is analogous to that given for equation (12).

Equation (14)

Since $x = e^{\ln x}$ and[2] $\left(e^{\ln x}\right)^y = e^{y \ln x}$, we have that

$$\begin{aligned} \ln x^y &= \ln e^{y \ln x} \\ &= y \ln x \end{aligned}$$

---

[1]It is taken for granted here that all positive numbers different from 0 can be expressed as a power.

[2]See power rules in MB.

0.6 Explanations

0.4 Polynomial Division (explanation)

Given the polynomials

P_m, where ax^m is the term with the highest degree

Q_n, where bx^n is the term with the highest degree

Then we can write

$$P_m = \frac{a}{b}x^{m-n}Q_n - \frac{a}{b}x^{m-n}Q_n + P_m$$

We set $U = -\frac{a}{b}x^{m-n}Q_n + P_m$, and note that U necessarily has a degree lower or equal to $m - 1$. Further, we have that

$$\frac{P_m}{Q_n} = \frac{a}{b}x^{m-n} + \frac{U}{Q_n} \tag{15}$$

Let's call the first and the second term on the right side in (15) respectively a "power term" and a "remaining fraction". By following the procedure that led us to (15), we can also express $\frac{U}{Q_n}$ by a "power term" and a "remaining term". This "power term" will have a degree less or equal to $m - 1$, while the numerator in the "remaining term" will have a degree less or equal to $m - 2$. By applying (15) we can continually create new "power terms" and "remaining terms" until we have a "remaining term" with a degree of $n - 1$.

0.5 Factorization of Polynomials (explanation)

P is divisible by $x - a \Rightarrow P(a) = 0$.

For a polynomial S, we have from (5) that

$$\frac{P}{x-a} = S$$
$$P = (x-a)S$$

Then obviously $x = a$ is a solution for the equation $P = 0$.

P is divisible by $x - a \Leftarrow P(a) = 0$.

From (5), there exists a polynomial S and a constant R such that

$$\frac{P}{x-a} = S + \frac{R}{x-a}$$
$$P = (x-a)S + R$$

Since $P(a) = 0$, $0 = R$, and then P is divisible by $x - a$.