0.1 Introduction

0.1 Fractions as rewriting of division

A fraction is a different way of writing a division. In a fraction the dividend is called the *numerator* and the divisor the denominator.

$$1 : 4 = \frac{1 \longleftarrow \text{Numerator}}{4 \longleftarrow \text{Denominator}}$$

The language box

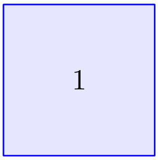
Common ways of saying $\frac{1}{4}$ are[1]

- "one fourth"
- "1 of 4"
- "1 over 4"

[1]We also have the expressions from the language box on page **??**.

Fractions as amounts

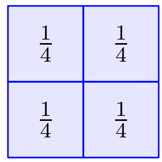
Let us present $\frac{1}{4}$ as an amount. We then think of the number 1 as a box[1]:



$= 1$

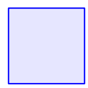
[1]For practical reasons, we choose a unit box larger than the one used in Chapter **??**.

Further, we divide this box into four smaller, equal-sized boxes. The sum of these boxes equals 1.



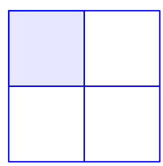
$$= 1$$

One such box equals $\frac{1}{4}$:



$$= \frac{1}{4}$$

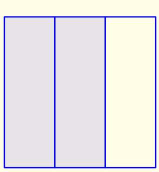
However, if you from a figure only are to see how large a fraction is, the size of 1 must be known, and to make this more apparent we'll also include the "empty" boxes:



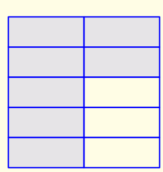
$$= \frac{1}{4}$$

In this way, the blue and the empty boxes tell us how many pieces 1 is divided into, while the blue boxes alone tells how many of these boxes are *actually* present. In other words,

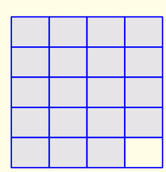
$$\text{number of blue boxes} = \text{numerator}$$
$$\text{number of blue boxes} + \text{number of empty boxes} = \text{denominator}$$



$$= \frac{2}{3}$$



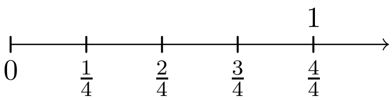
$$= \frac{7}{10}$$



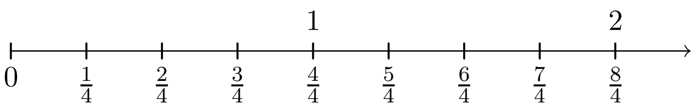
$$= \frac{19}{20}$$

Fractions on the number line

On the number line, we divide the length between 0 and 1 into as many pieces as the denominator indicates. In the case of a fraction with denominator 4, we divide the length between 0 and 1 into 4 equal lengths:



Moreover, fractions larger than 1 are easily presented on the number line:



Numerator and denominator summarized

Although already mentioned, the interpretations of the numerator and the denominator are of such importance that we shortly summarize them:

- The denominator tells how many pieces 1 is divided into.
- The numerator tells how many of these pieces are present.

0.2 Values, expanding and simplifying

0.2 The value of a fraction

The value of a fraction is given by dividing the numerator by the denominator.

Example

Find the value of $\frac{1}{4}$.

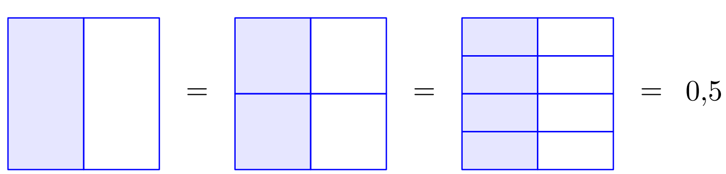
Answer

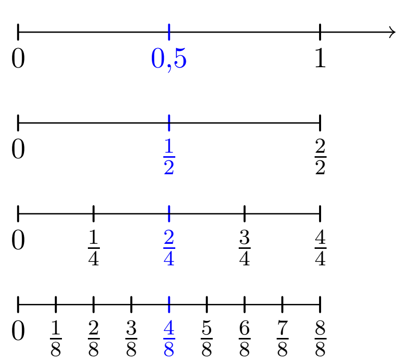
$$\frac{1}{4} = 0.25$$

Fractions with equal value

Fractions can have the same value even though they look different. If you calculate $1 : 2$, $2 : 4$ and $4 : 8$, you will in every case end up with 0.5 as the answer. This means that

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = 0{,}5$$





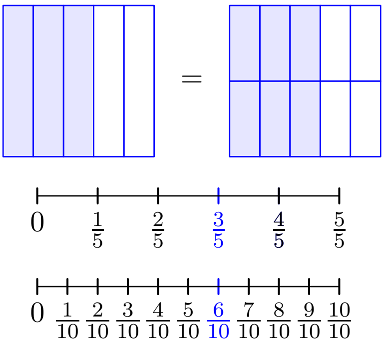
Expanding

The fact that fractions can look different but have the same value, implies that we can change a fraction's look without changing its value. Let's, as an example, change $\frac{3}{5}$ into a fraction of equal value but with denominator 10:

- We can make $\frac{3}{5}$ into a fraction with denominator 10 if we divide each fifth into 2 equal pieces. In that case, 1 is divided into $5 \cdot 2 = 10$ pieces in total.

- The numerator of $\frac{3}{5}$ indicates that there are 3 fifths. When these are divided by 2, they make up $3 \cdot 2 = 6$ tenths. Hence $\frac{3}{5}$ equals $\frac{6}{10}$.

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}$$



Simplifying

Notice that we can also go "the opposite way". We can change $\frac{6}{10}$ into a fraction with denominator 5 by dividing both the numerator and the denominator by 2:

$$\frac{6}{10} = \frac{6 : 2}{10 : 2} = \frac{3}{5}$$

> **0.3 Expanding of fractions**
>
> We can either multiply or divide both the numerator and the denominator by the same number without alternating the fractions value.
>
> Multiplying by a number larger than 1 is called *expanding* the fraction. Dividing by a number larger than 1 is called *simplifying* the fraction.

Example 1

Expand $\frac{3}{5}$ into a fraction with denominator 20.

Answer

Since $5 \cdot 4 = 20$, we multiply both the numerator and the denominator by 4:

$$\begin{aligned} \frac{3}{5} &= \frac{3 \cdot 4}{5 \cdot 4} \\ &= \frac{12}{20} \end{aligned}$$

Example 2

Expand $\frac{150}{50}$ into a fraction with denominator 100.

Answer

Since $50 \cdot 2 = 100$, we multiply both the numerator and the denominator by 2:

$$\begin{aligned} \frac{150}{50} &= \frac{150 \cdot 2}{50 \cdot 2} \\ &= \frac{300}{100} \end{aligned}$$

Example 3

Simplify $\frac{18}{30}$ into a fraction with denominator 5.

Answer

Since $30 : 6 = 5$, we divide both the numerator and the denominator by 6:

$$\begin{aligned} \frac{18}{30} &= \frac{18 : 6}{30 : 6} \\ &= \frac{3}{5} \end{aligned}$$

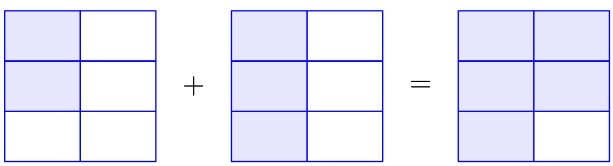
0.3 Addition and subtraction

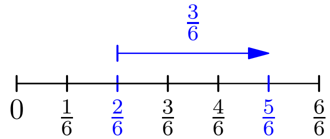
Addition and subtraction of fractions are in large parts focused around the denominators. Recall that the denominators indicate the partitioning of 1. If fractions have equal denominators, they represent amounts of equal-sized pieces. In this case it makes sense calculating addition or subtraction of the numerators. However, if fractions have unequal denominators, they represent amounts of different-sized pieces, and hence addition and subtraction of the numerators makes no sense directly.

Equal denominators

If we, for example, have 2 sixths and add 3 sixths, the sum is 5 sixths:

$$\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$





0.4 Addition/subtraction of fractions with equal denominators

When adding/subtracting fractions with equal denominators, we find the sum/difference of the numerators and keep the denominator.

Example 1

$$\begin{aligned}\frac{2}{7} + \frac{8}{7} &= \frac{2+8}{7} \\ &= \frac{10}{7}\end{aligned}$$

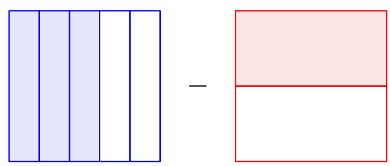
Example 2

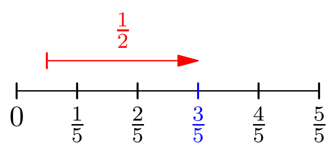
$$\frac{7}{9}-\frac{5}{9}=\frac{7-5}{9}$$
$$=\frac{2}{9}$$

Unequal denominators

Let's examine the calculation[1]

$$\frac{3}{5}-\frac{1}{2}$$



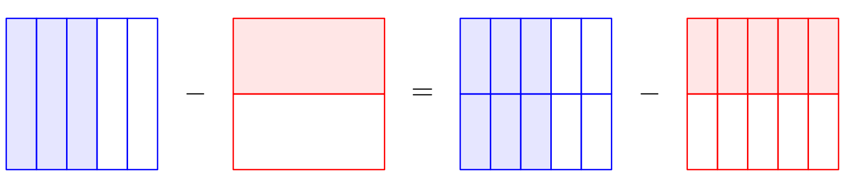


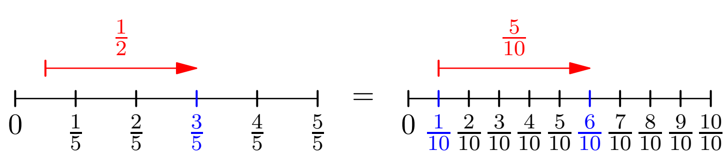
To write the difference as a single fraction, the two terms need to have denominators of equal value. Both of the fractions can have denominator 10:

$$\frac{3}{5}=\frac{3\cdot 2}{5\cdot 2}=\frac{6}{10} \qquad\qquad \frac{1}{2}=\frac{1\cdot 5}{2\cdot 5}=\frac{5}{10}$$

Hence

$$\frac{3}{5}-\frac{1}{2}=\frac{6}{10}-\frac{5}{10}$$





---

[1] Recall that the red-colored arrow indicates that you shall start at the arrowhead and then move to the other end.

Summarized, we have expanded the fractions such that they have denominators of equal value, that is 10. When the denominators are equal, we can calculate the difference of the numerators:

$$\begin{aligned}\frac{3}{5} - \frac{1}{2} &= \frac{6}{10} - \frac{5}{10} \\ &= \frac{1}{10}\end{aligned}$$

0.5 Addition/subtraction of fractions with unequal denominators

When calculating addition/subtraction of fractions, we must expand the fractions such that they have denominators of equal value, and then apply Rule 0.4.

Example 1

Calculate

$$\frac{2}{9} + \frac{6}{7}$$

Both denominators can be transformed into 63 if multiplied by a fitting integer. Therefore, we expand the fractions as follows:

$$\begin{aligned}\frac{2 \cdot 7}{9 \cdot 7} + \frac{6 \cdot 9}{7 \cdot 9} &= \frac{14}{63} + \frac{54}{63} \\ &= \frac{68}{63}\end{aligned}$$

Common denominator

In *Example 1* above, 63 is called a *common denominator* because there exists integers which, when multiplied by the original denominators, results in 63:

$$
\begin{aligned}
9 \cdot 7 &= 63 \\
7 \cdot 9 &= 63
\end{aligned}
$$

Multiplying together the original denominators always results in a common denominator but one can avoid large numbers by finding the *smallest* common denominator. Take, for example,

$$\frac{7}{6} + \frac{5}{3}$$

$6 \cdot 3 = 18$ is a common denominator, but it's worth noticing that $6 \cdot 1 = 3 \cdot 2 = 6$ is too. Hence,

$$
\begin{aligned}
\frac{7}{6} + \frac{5}{3} &= \frac{7}{6} + \frac{5 \cdot 2}{3 \cdot 2} \\
&= \frac{7}{6} + \frac{10}{6} \\
&= \frac{17}{6}
\end{aligned}
$$

Example 2

Calculate

$$\frac{3}{2} - \frac{5}{8} + \frac{10}{4}$$

Answer

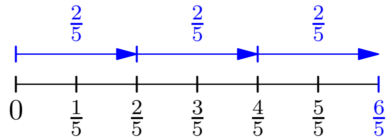
All denominators can be transformed into 8 if multiplied by a fitting integer. Therefore, we expand the fractions as follows:

$$
\begin{aligned}
\frac{3}{2} - \frac{5}{8} + \frac{10}{4} &= \frac{3 \cdot 4}{2 \cdot 4} - \frac{5}{8} + \frac{10 \cdot 2}{4 \cdot 2} \\
&= \frac{12}{8} - \frac{5}{8} + \frac{20}{8} \\
&= \frac{27}{8}
\end{aligned}
$$

0.4 Fractions multiplied by integers

In Section ?? we observed that multiplying by an integer corresponds to repeated addition. Hence, if we are to calculate $\frac{2}{5} \cdot 3$, we can write

$$\begin{aligned} \frac{2}{5} \cdot 3 &= \frac{2}{5} + \frac{2}{5} + \frac{2}{5} \\ &= \frac{2+2+2}{5} \\ &= \frac{6}{5} \end{aligned}$$



Noticing that $2+2+2 = 2 \cdot 3$, we get

$$\begin{aligned} \frac{2}{5} \cdot 3 &= \frac{2 \cdot 3}{5} \\ &= \frac{6}{5} \end{aligned}$$

Multiplication of integers and fractions are also commutative[1]:

$$\begin{aligned} 3 \cdot \frac{2}{5} &= 3 \cdot 2 : 5 \\ &= 6 : 5 \\ &= \frac{6}{5} \end{aligned}$$

0.6 Fractions multiplied by integers

When multiplying a fraction by an integer, we multiply the numerator by the integer.

[1] Recall that $\frac{2}{5}$ corresponds to $2 : 5$.

Example 1

$$\begin{aligned}\frac{1}{3}\cdot 4 &= \frac{1\cdot 4}{3}\\ &= \frac{4}{3}\end{aligned}$$

Example 2

$$\begin{aligned}3\cdot\frac{2}{5} &= \frac{3\cdot 2}{5}\\ &= \frac{6}{5}\end{aligned}$$

An interpretation of multiplying by a fraction

By Rule 0.6 we can make an interpretation of multiplying by a fraction. For example, multiplying 3 by $\frac{2}{5}$ can be interpreted in these two following ways:

- We multiply 3 by 2 and divide by 5:

$$(3\cdot 2):5 = \frac{3\cdot 2}{5} = \frac{6}{5}$$

- We divide 3 by 5 and multiply the quotient by 2:

$$3:5 = \frac{3}{5} \quad , \quad \frac{3}{5}\cdot 2 = \frac{3\cdot 2}{5} = \frac{6}{5}$$

0.5 Fractions divided by integers

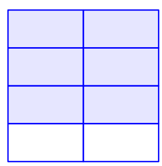
It is now important to recall two things:

- Division can be interpreted as an equal distribution of amounts
- In a fraction, it is the numerator which indicates the amount (the denominator indicates the partitioning of 1)

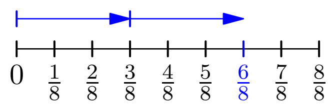
When the numerator is divisible by the divisor

Let's calculate

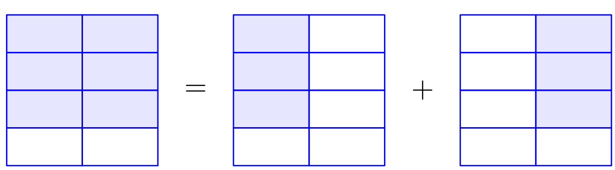
$$\frac{6}{8} : 2$$

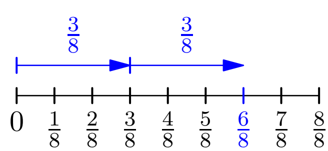


$: 2$



We have 6 eights which are to be equally distributed into 2 groups. This results in $6 : 2 = 3$ eights.





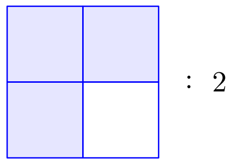
Thus

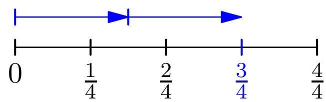
$$\frac{6}{8} : 2 = \frac{3}{8}$$

When the numerator is not divisible by the denominator

What if we are to divide $\frac{3}{4}$ by 2?

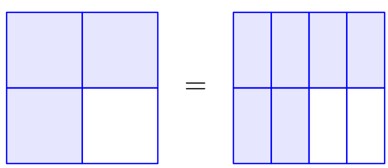
$$\frac{3}{4} : 2$$

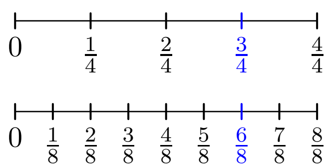




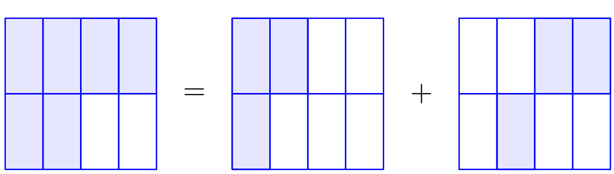
Thing is, we can always expand the fraction such that the numerator becomes divisible by the divisor. Since 2 is the divisor, we expand the fraction by 2:

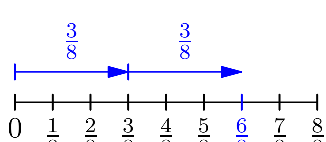
$$\frac{3}{4} = \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8}$$





Now we have 6 eights. 6 eights divided by 2 equals 3 eights:





Hence

$$\frac{3}{4} : 2 = \frac{3}{8}$$

In effect, we have multiplied $\frac{3}{4}$ by 2:

$$\begin{aligned}\frac{3}{4} : 2 &= \frac{3}{4 \cdot 2} \\ &= \frac{3}{8}\end{aligned}$$

0.7 Fractions divided by integers

When dividing a fraction by an integer, we multiply the denominator by the integer.

Example 1

$$\begin{aligned}\frac{5}{3} : 6 &= \frac{5}{3 \cdot 6} \\ &= \frac{5}{18}\end{aligned}$$

Note

At the start of this section we found that

$$\frac{4}{8} : 2 = \frac{2}{8}$$

In that case, there were no need to multiply the denominator by 2, such as Rule 0.7 implies. However, if we do, we have

$$\frac{4}{8} : 2 = \frac{4}{8 \cdot 2} = \frac{4}{16}$$

Now,

$$\frac{2}{8} = \frac{2 \cdot 2}{8 \cdot 2} = \frac{4}{16}$$

Hence, unsurprisingly, the two answers are of equal value.

0.6 Fractions multiplied by fractions

We have seen that[1] multiplying a number by a fraction involves multiplying the number by the numerator and then dividing the product by the denominator. Let us apply this to calculate

$$\frac{5}{4} \cdot \frac{3}{2}$$

Firstly, we multiply $\frac{5}{4}$ by 3, then we divide the resulting product by 2. By Rule 0.6, we have

$$\frac{5}{4} \cdot 3 = \frac{5 \cdot 3}{4}$$

And by Rule 0.7, we get

$$\frac{5 \cdot 3}{4} : 2 = \frac{5 \cdot 3}{4 \cdot 2}$$

Hence

$$\frac{5}{4} \cdot \frac{3}{2} = \frac{5 \cdot 3}{4 \cdot 2}$$

0.8 Fractions multiplied by fractions

When multiplying a fraction by a fraction, we multiply numerator by numerator and denominator by denominator.

Example 1

$$\begin{aligned} \frac{4}{7} \cdot \frac{6}{9} &= \frac{4 \cdot 6}{7 \cdot 9} \\ &= \frac{24}{63} \end{aligned}$$

Example 2

$$\begin{aligned} \frac{1}{2} \cdot \frac{9}{10} &= \frac{1 \cdot 9}{2 \cdot 10} \\ &= \frac{9}{20} \end{aligned}$$

[1] Look at the text box titled "*An interpretation of multiplying by a fraction*" on page 12.

0.7 Cancelation of fractions

When the numerator and the denominator are of equal value, the fractions value always equals 1. For example, $\frac{3}{3} = 1$, $\frac{25}{25} = 1$ etc. We can exploit this fact to simplify expressions involving fractions.

Let us simplify the expression

$$\frac{8 \cdot 5}{9 \cdot 8}$$

Since $8 \cdot 5 = 5 \cdot 8$, we can write

$$\frac{8 \cdot 5}{9 \cdot 8} = \frac{5 \cdot 8}{9 \cdot 8}$$

And, as recently seen (Rule 0.8), we have

$$\frac{5 \cdot 8}{9 \cdot 8} = \frac{5}{9} \cdot \frac{8}{8}$$

Since $\frac{8}{8} = 1$,

$$\begin{aligned} \frac{5}{9} \cdot \frac{8}{8} &= \frac{5}{9} \cdot 1 \\ &= \frac{5}{9} \end{aligned}$$

When multiplication is exclusively present in a fraction, you can always shuffle the way we did in the above expressions. However, when you have understood the outcome of the shuffling, it is better to apply *cancellation*. You then draw a line across two and two equal factors, thus indicating that they constitute a fraction which equals 1. Hence, our most recent example can be simplified to

$$\frac{\cancel{8} \cdot 5}{9 \cdot \cancel{8}} = \frac{5}{9}$$

0.9 Cancellation of factors

When multiplication is exclusively present in a fraction, we can cancel pair of equal factors in numerator and denominator.

Example 1

Cancel as many factors as possible in the fraction.

$$\frac{3 \cdot 12 \cdot 7}{7 \cdot 4 \cdot 12}$$

Answer

$$\frac{3 \cdot \cancel{12} \cdot \cancel{7}}{\cancel{7} \cdot 4 \cdot \cancel{12}} = \frac{3}{4}$$

Example 2

Simplify the fraction $\frac{12}{42}$.

Answer

We notice that 6 is a factor of both 12 and 42, and thus

$$\begin{aligned} \frac{12}{42} &= \frac{\cancel{6} \cdot 2}{\cancel{6} \cdot 7} \\ &= \frac{2}{7} \end{aligned}$$

Example 3

Simplify the fraction $\frac{48}{16}$.

Answer

We notice that 16 is a factor of 48, and thus

$$\begin{aligned} \frac{48}{16} &= \frac{3 \cdot \cancel{16}}{\cancel{16}} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

Note: If all factors are canceled in the numerator or the denominator, 1 takes their place.

Simplifying by prime factorization

It's not necessarily easy to notice common factors, as we did in *Example 2* and *Example 3* above. To be sure you have not overlooked any common factors, you can always prime factorize (see page **??**) both the numerator and the denominator. For example,

$$\begin{aligned}\frac{12}{42} &= \frac{\cancel{2}\cdot 2\cdot \cancel{3}}{\cancel{2}\cdot \cancel{3}\cdot 7}\\ &= \frac{2}{7}\end{aligned}$$

Fractions simplify calculations

The decimal number 0.125 can be written as the fraction $\frac{1}{8}$. The calculation

$$0.125\cdot 16$$

is, for the most of us, rather strenuous to carry out. However, exploiting the nature of fractions, we have

$$\begin{aligned}0.125\cdot 16 &= \frac{1}{8}\cdot 16\\ &= \frac{2\cdot \cancel{8}}{\cancel{8}}\\ &= 2\end{aligned}$$

”Cancelling zeros”

A number such as 3000 equals $3 \cdot 10 \cdot 10 \cdot 10$, while 700 equals $7 \cdot 10 \cdot 10$. Hence, we can simplify $\frac{3000}{700}$ like this:

$$\begin{aligned}\frac{3000}{700} &= \frac{3 \cdot \cancel{10} \cdot \cancel{10} \cdot 10}{7 \cdot \cancel{10} \cdot \cancel{10}} \\ &= \frac{3 \cdot 10}{7} \\ &= \frac{30}{7}\end{aligned}$$

In practice, this is the same as ”cancelling zeros”:

$$\frac{30\cancel{00}}{7\cancel{00}} = \frac{30}{7}$$

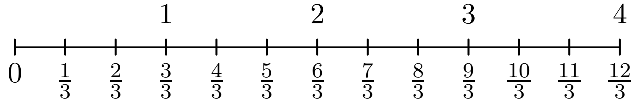
Aware! Zeros are the only digits we can ”cancel” this way. For example, $\frac{123}{13}$ cannot be simplified in any way. Also, we can only ”cancel” zeros which are right-most situated, e.g. we cannot ”cancel” zeros in the fraction $\frac{101}{10}$.

0.8 Division by fractions

Division by studying the number line

Let's calculate $4 : \frac{2}{3}$. Since the fraction have denominator 3, it could be wise to transform also 4 into a fraction with denominator 3.

$$4 = \frac{12}{3}$$

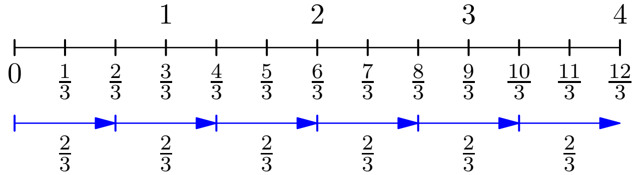


Recall that one of the interpretations of $4 : \frac{2}{3}$ is

"The number of $\frac{2}{3}$'s added to make 4."

By studying a number line, we find that 6 instances of $\frac{2}{3}$ added together equals 4. Hence

$$4 : \frac{2}{3} = 6$$



A general method

We can't study the number line every time we are to divide by a fraction, so here we shall find a general method, again with $4 : \frac{2}{3}$ as our example. In this case, we apply the following interpretation of division:

$$4 : \frac{2}{3} = \text{"The number to multiply } \tfrac{2}{3} \text{ by to make 4."}$$

We begin the search of this number by multiplying $\frac{2}{3}$ by the number which results in the product equal to 1. This number is the *inverted fraction* of $\frac{2}{3}$, namely $\frac{3}{2}$:

$$\frac{2}{3} \cdot \frac{3}{2} = 1$$

Now we only have to multiply by 4 to make 4:

$$\frac{2}{3} \cdot \frac{3}{2} \cdot 4 = 4$$

Therefore, to make 4 we must multiply $\dfrac{2}{3}$ by $\dfrac{3}{2} \cdot 4$. Consequently,

$$\begin{aligned} 4 : \frac{2}{3} &= \frac{3}{2} \cdot 4 \\ &= 6 \end{aligned}$$

0.10 Fractions divided by fractions

When dividing a number by a fraction, we multiply the number by the inverted fraction.

Example 1

$$\begin{aligned} 3 : \frac{2}{9} &= 3 \cdot \frac{9}{2} \\ &= \frac{27}{2} \end{aligned}$$

Example 2

$$\begin{aligned} \frac{4}{3} : \frac{5}{8} &= \frac{4}{3} \cdot \frac{8}{5} \\ &= \frac{32}{15} \end{aligned}$$

Example 3

$$\begin{aligned}\frac{3}{5} : \frac{3}{10} &= \frac{3}{5} \cdot \frac{10}{3} \\ &= \frac{30}{15}\end{aligned}$$

In this case we should also observe that the fraction can be simplified:

$$\begin{aligned}\frac{30}{15} &= \frac{2 \cdot \cancel{15}}{\cancel{15}} \\ &= 2\end{aligned}$$

Note: Canceling factors along the way saves the labor of working with large numbers:

$$\begin{aligned}\frac{3}{5} \cdot \frac{10}{3} &= \frac{\cancel{3} \cdot 2 \cdot \cancel{5}}{\cancel{5} \cdot \cancel{3}} \\ &= 2\end{aligned}$$

0.9 Rational numbers

0.11 Rational numbers

Any number which can be expressed as a fraction is a *rational number*.

Note

Rational numbers is a collective name of

- **Integers**

  For example $4 = \frac{4}{1}$.

- **Decimal numbers with a finite number of digits**

  For example $0{,}2 = \frac{1}{5}$.

- **Decimal numbers with infinite digits in a repeating manner**

  For example [1] $0.08\bar{3} = \frac{1}{12}$.

---

[1] $\bar{3}$ indicates that 3 repeats infinite. Another way of expressing this is by using ... . That is, $0.08\bar{3} = 0.08333333...$

0.12 Mixed numbers

If we add an integer with a fraction where the numerator has a lower value than the denominator, we have a **mixed number**

Example 1

Three different mixed numbers:

$$2 + \frac{5}{7} \qquad 8 + \frac{2}{7} \qquad \frac{1}{10} + 4$$

Note

In a lot of texts you will find numbers like those from *Example 1* written like this:

$$2\frac{5}{7} \qquad 8\frac{2}{7} \qquad 4\frac{1}{10}$$

Example 2

Write the fraction $\frac{17}{3}$ as a mixed number.

Answer

We observe that 17 is the numerator and 3 the denominator. The largest number we can multiply by 3 without getting a product larger than 17 is 5, so

$$\begin{aligned}\frac{17}{3} &= \frac{5 \cdot 3 + 2}{3} \\ &= \frac{5 \cdot \cancel{3}}{\cancel{3}} + \frac{2}{3} \\ &= 5 + \frac{2}{3}\end{aligned}$$

Example 3

Write the mixed number $3 + \frac{4}{5}$ as a fraction.

Answer

We have $3 = \frac{3}{1}$, so

$$3 + \frac{4}{5} = \frac{3}{1} + \frac{4}{5}$$

Moreover[1],

$$\begin{aligned}\frac{3}{1} + \frac{4}{5} &= \frac{3 \cdot 5}{1 \cdot 5} + \frac{4}{5} \\ &= \frac{15}{5} + \frac{4}{5} \\ &= \frac{19}{5}\end{aligned}$$

[1] See Rule ??.