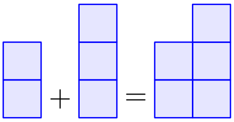
## 0.1 Addition

### Addition with amounts

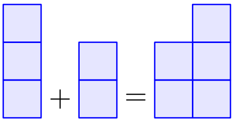
When we have an amount and wish to add more, we use the symbol $+$. If we have 2 and want to add 3, we write

$$2 + 3 = 5$$



The order in which we add have no impact on the results; starting off with 2 and adding 3 is the same as starting off with 3 and adding 2:

$$3 + 2 = 5$$



**The language box**

A calculation involving addition includes two or more *terms* and one *sum*. In the calculation

$$2 + 3 = 5$$

both 2 and 3 are terms while 5 is the sum.

Common ways of saying $2 + 3$ include

- ”2 plus 3”
- ”2 added to 3”
- ”2 and 3 added”

### 0.1 Addition is commutative

The order of the terms has no impact on the sum.

**Example**

$$2+5=7=5+2$$

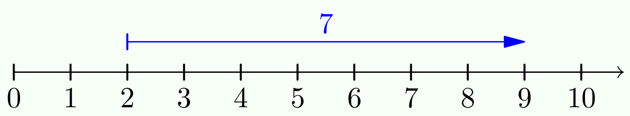
$$6+3=9=3+6$$

### Addition on the number line: moving to the right

On a number line, addition with positive numbers involves moving *to the right*:

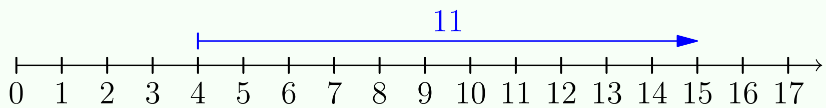
**Example 1**

$$2+7=9$$




**Example 2**

$$4+11=15$$




**Interpretation of $=$**

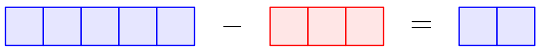
$+$ brings the possibility of expressing numbers in different ways, for example is $5=2+3$ and $5=1+4$. In this context, $=$ means ”has the same value as”. This is also the case regarding subtraction, multiplication and division which we’ll look at in the next three sections.

## 0.2 Subtraction

### Subtraction with amounts

When removing a part of an amount, we use the symbol $-$:

$$5 - 3 = 2$$



**The language box**

A calculation involving subtraction includes one or more *terms* and one *difference*. In the calculation

$$5 - 3 = 2$$

both 5 and 3 are terms while 2 is the difference.

Common ways of saying $5 - 3$ include

- ”5 minus 3”
- ”3 subtracted from 5”

**A new interpretation of 0**

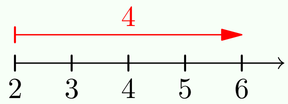
As mentioned earlier in this book, 0 can be interpreted as ”nothing”. However, subtraction brings the possibility of expressing 0 by other numbers, for example $7 - 7 = 0$ and $19 - 19 = 0$. In many practical situations, 0 indicates some form of equilibrium, like two equal but opposite forces.

### Subtraction on the number line: Moving to the left

In *Section 0.1*, we have seen that $+$ (with positive numbers) involves moving *to the right* on the number line. With $-$ it's the opposite, we move *to the left*[1]:

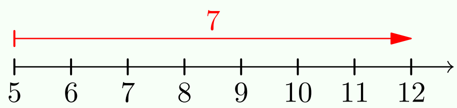
**Example 1**

$$6 - 4 = 2$$




**Example 2**

$$12 - 7 = 5$$




**Notice**

At first it may seem a bit odd moving in the opposite direction of the way in which the arrows point, as in *Example 1* and *2*. However, in *Chapter* **??** this will turn out to be useful.

[1] In figures with number lines, the red colored arrows indicates that you shall start at the arrowhead and move to the other end.

## 0.3 Multiplication

### Multiplication by integers: initial definition

When adding equal numbers, we can use the multiplication symbol `·` to write our calculations more compact:

**Example**

$$4 + 4 + 4 = 4 \cdot 3$$

$$8 + 8 = 8 \cdot 2$$

$$1 + 1 + 1 + 1 + 1 = 1 \cdot 5$$

**The language box**

A calculation involving multiplication includes several *factors* and one *product*. In the calculation

$$4 \cdot 3 = 12$$

both 4 and 3 are factors, while 12 is the product.

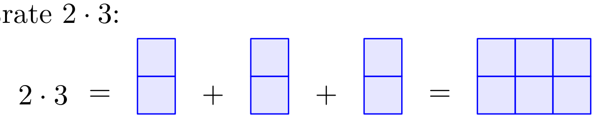
Common ways of saying $4 \cdot 3$ include

- "4 times 3"
- "4 multiplied by 3"
- "4 and 3 multiplied together"

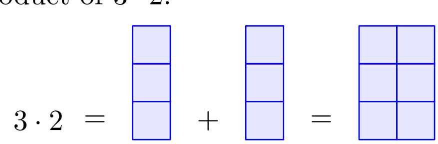
A lot of texts use `×` instead of `·`. In computer programming, `*` is the most common symbol for multiplication.

### Multiplication involving amounts

Let us illustrate $2 \cdot 3$:




Now notice the product of $3 \cdot 2$:



**0.2 Multiplication is commutative**

The order of the factors has no impact on the product.

**Example**

$$3 \cdot 4 = 12 = 4 \cdot 3$$

$$6 \cdot 7 = 42 = 7 \cdot 6$$

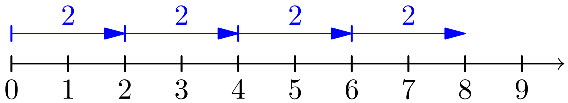
$$8 \cdot 9 = 72 = 9 \cdot 8$$

## Multiplication on the number line

We can also use the number line to calculate multiplications. In the case of $2 \cdot 4$ we can think like this:

"$2 \cdot 4$ means moving 2 places to the right, 4 times."

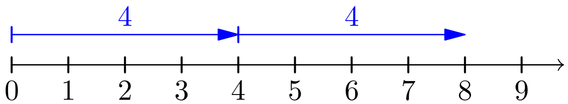
$$2 \cdot 4 = 8$$




We can also use the number line to prove to ourselves that multiplication is commutative:

"$4 \cdot 2$ means moving 4 places to the right, 2 times."

$$4 \cdot 2 = 8$$




## Final definition of multiplication by positive integers

It may be the most intuitive to interpret "2 times 3" as "3, 2 times". Then it follows:

$$\text{"2 times 3"} = 3 + 3$$

In this section we introduced $2 \cdot 3$, that is "2 times 3", as $2 + 2 + 2$. With this interpretation, $3 + 3$ corresponds to $3 \cdot 2$, but the fact that multiplication is a commutative operation (*Rule 0.2*) ensures that the one interpretation does not exclude the other; $2 \cdot 3 = 2 + 2 + 2$ and $2 \cdot 3 = 3 + 3$ are two expressions of same value.

### 0.3 Multiplication as repeated addition

Multiplication involving a positive integer can be expressed as repeated addition.

**Example 1**

$$4 + 4 + 4 = 4 \cdot 3 = 3 + 3 + 3 + 3$$

$$8 + 8 = 8 \cdot 2 = 2 + 2 + 2 + 2 + 2 + 2 + 2$$

$$1 + 1 + 1 + 1 + 1 = 1 \cdot 5 = 5$$

**Notice**

The fact that multiplication with positive integers can be expressed as repeated addition does not exclude other expressions. There's nothing wrong with writing $2 \cdot 3 = 1 + 5$.

## 0.4 Division

$:$ is the symbol for division. Division has three different interpretations:

**0.4 The three interpretations of division**

- **Distribution of amounts**

  $12 : 3 =$ ”The number in each group when evenly distributing 12 into 3 groups”

- **Number of equal terms**

  $12 : 3 =$ ”The number of 3’s added to make 12”

- **The inverse operation of multiplication**

  $12 : 3 =$ ”The number which yields 12 when multiplied by 3”

**The language box**

A calculation involving division includes a *dividend*, a *divisor* and a *quotient*. In the calculation

$$12 : 3 = 4$$

12 is the dividend, 3 is the divisor and 4 is the quotient.

Common ways of saying $12 : 3$ include

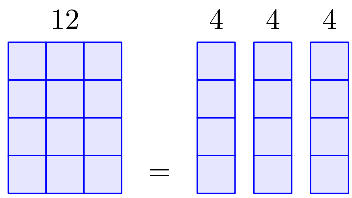
- ”12 divided by 3”
- ”12 to 3”

In a lot of contexts, $/$ is used instead of $:$, especially in computer programming.

Sometimes $12 : 3$ is called ”the *ratio* of 12 to 3”.

## Distribution of amounts

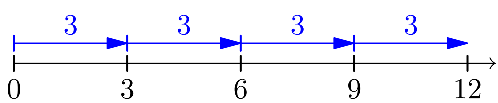
The calculation $12 : 3$ tells that we shall distribute 12 into 3 equal groups:




We observe that each group contains 4 boxes, which means that

$$12 : 3 = 4$$

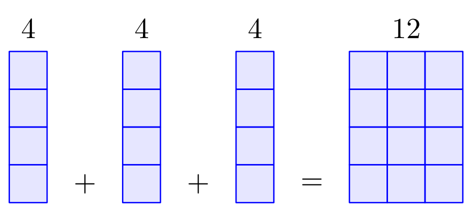
## Number of equal terms



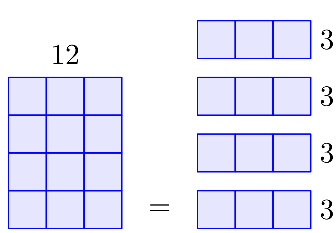

12 equals the sum of 4 instances of 3, that is $12 : 3 = 4$.

## The inverse operation of multiplication

We have just seen that if we divide 12 into 3 equal groups, we get 4 in each group. Hence $12 : 3 = 4$. The sum of these groups makes 12:




However, this is the same as multiplying 4 by 3, in other words: If we know that $4 \cdot 3 = 12$, we also know that $12 : 3 = 4$. As well we know that $12 : 4 = 3$.



**Example 1**

Since $6 \cdot 3 = 18$,

$$18 : 6 = 3$$

$$18 : 3 = 6$$

**Example 2**

Since $5 \cdot 7 = 35$,

$$35 : 5 = 7$$

$$35 : 7 = 5$$