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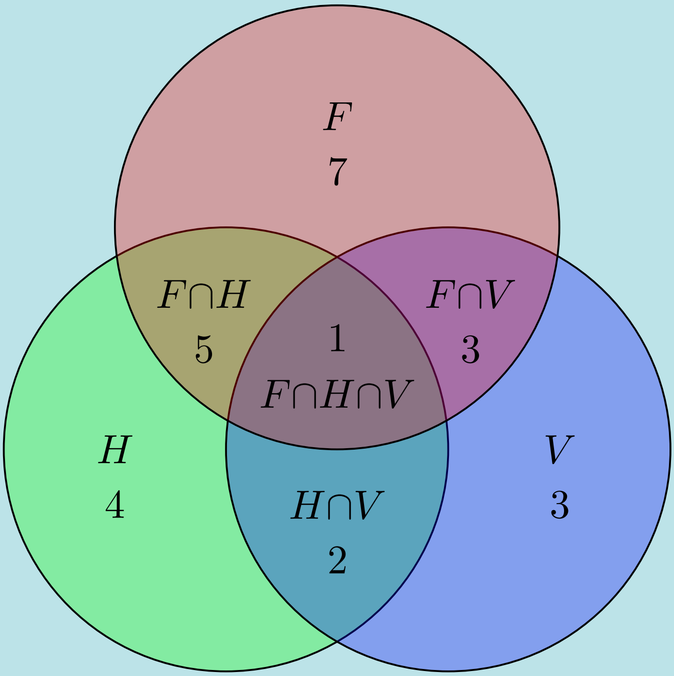
Applied mathematics for elementary school and high school



Sindre Sogge Heggen

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*"Wahrlich es ist nicht das Wissen, sondern das Lernen,
nicht das Besitzen, sondern das Erwerben,
nicht das Da-Seyn, sondern das Hinkommen,
was den grössten Genuss gewährt"*

*"It is not knowing, rather learning,
not possessing, rather obtaining,
not being present, rather reaching there,
which serves the greatest joy."*

— Carl Friedrich Gauss

The text is written in LaTeX and the figures are made using Asymptote.
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14.09.2023

Contents

Chapter 1

Quantities and unit

1.1 Quantities, units, and prefixes

Objects we can measure and describe by numbers are called **quantities**. A quantity usually consists of both a value and a **unit**. In this section we will look at these four units:

unit	abbreviation	unit for
meter	m	length
gram	g	mass
second	s	time
liter	L	volume

Sometimes we have either very large or very small quantities, like the length around the equator, which is ca. 40 075 000 m! For very large or small numbers it is helpful to use a **prefix**. Using the prefix 'kilo', the length around the equator can be written as ca. 40 075 km, where 'km' is an abbreviation for 'kilometer', and 'kilo' meaning '1 000'. Hence, 1 000 meters is 1 kilometer . The prefixes for the powers of ten with integer exponents ranging from -3 to 3 (except 0) are the following:

prefix	abbreviation	value
kilo	k	1 000
hecto	h	100
deka	da	10
desi	d	0.1
centi	c	0.01
milli	m	0.001

The language box

A (potential) **prefix** and a **unit** constitute a **notation**. For example, 9 km has notation 'km', while 9 m has notation 'm'. These quantities have unequal notations but both have 'm' as unit.

We can create a structured way of changing prefixes by placing the prefixes in a horizontal table with a unit added to the place corresponding to the value 1:

1.1 Converting prefixes

When changing the prefix of a quantity we can use this table:

kilo	hecto	deka	unit	desi	centi	milli

The movement of the decimal separator corresponds to the movement from the original prefix to the new prefix.

Example 1

Write 23,4 mL as an amount of L.

Answer

We set 'L' as the unit, and notice that we must move **three cells to the left** to get from mL to L:

kilo	hecto	deka	L	desi	centi	milli

This means the comma separator is to be moved three places to the left:

$$23.4\,\mathrm{mL} = 0.0234\,\mathrm{L}$$

Example 2

Write 30 hg as an amount of cg.

Answer

We set 'g' as the unit, and notice that we must move **four cells to the right** to get from hg to cg:

kilo	hecto	deka	g	desi	centi	milli

This means the comma separator is to be moved four places to the right:

$$30\,\mathrm{mg} = 300\,000\,\mathrm{cg}$$

Example 3

Write 2.7 s as an amount of ms.

Answer

We set 's' as the unit, and notice that we must move **three cells to the right** to get from s to ms:

kilo	hecto	deka	s	desi	centi	milli

This means the comma separator is to be moved three places to the right:

$$2.7\,\text{s} = 2\,700\,\text{ms}$$

1.1 Converting prefixes (forklaring)

Changing prefixes corresponds to multiplying/dividing by 10, 100 etc. (see MB).

As our first example, let us write 3,452 km as an amount of meters. We have

$$\begin{aligned} 3.452\,\text{km} &= 3.452 \cdot 1000\,\text{m} \\ &= 3\,452\,\text{m} \end{aligned}$$

As our second example, let us write 47 mm as an amount of meters. We have

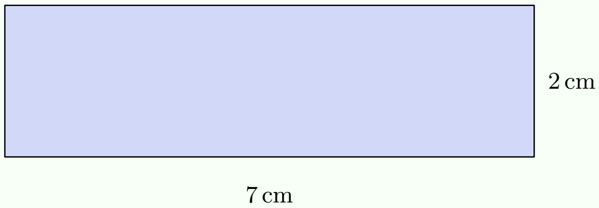
$$\begin{aligned} 47\,\text{mm} &= 47 \cdot \frac{1}{1000}\,\text{m} \\ &= (47 : 1000)\,\text{m} \\ &= 0.047\,\text{m} \end{aligned}$$

1.2 Calculation with quantities

Note: In the examples of this section we use area and volume formulas found in MB.

When performing calculations, notations can be handled in the same way as variables in algebra. [1]. Hence, in the same way as we have $a + a = 2a$, we have $1\,\text{cm} + 1\,\text{cm} = 2\,\text{cm}$, and just like we have $2a \cdot 3a = 6a^2$, we have $2\,\text{cm} \cdot 3\,\text{cm} = 6\,\text{cm}^2$.

Example 1



a) Find the perimeter of the rectangle.

b) Find the area of the rectangle.

Answer

a) The perimeter of the rectangle is

$$7\,\text{cm} + 2\,\text{cm} + 7\,\text{cm} + 2\,\text{cm} = 18\,\text{cm}$$

b) The area of the rectangle is

$$7\,\text{cm} \cdot 2\,\text{cm} = 14\,\text{cm}^2$$

Example 3

A cylinder has radius $4\,\text{m}$ and height $2\,\text{m}$. Find the volume of the cylinder.

Answer

$$\text{base area} = \pi \cdot (4\,\text{cm})^2 = 16\pi\,\text{cm}^2$$

$$\text{volume of the cylinder} = 16\pi\,\text{cm}^2 \cdot 2\,\text{cm} = 32\pi\,\text{cm}^3$$

[1] See MB.

1.3 Proportional quantities

Say it costs[1] 10 kr for 0.5 kg potatoes. If this price is also valid if we buy 0.5 kg additional potatoes, the price for 1 kg potatoes is 20 kr. If this price is valid also if we buy 0,5 kg additional potatoes, the price for 2 kg potatoes is 40 kr. We can put the amount of krones and amount of kilogram potatoes in a table:

kr	10	20	40
kg	0,5	1	2

Moreover, let us divide the price by the weight for each case:

$$\frac{10\,\text{kr}}{0{,}5\,\text{kg}} = 20\,\text{kr/kg} \qquad \frac{20\,\text{kr}}{1\,\text{kg}} = 20\,\text{kr/kg} \qquad \frac{40\,\text{kr}}{2} = 20\,\text{kr/kg}$$

Clearly, the ratio of the price to the weight is the same for all the cases. In that case, we say that the price and the weight are[2] **proportional quantities**. From these two quantities we have also ”made” a new quantity with notation[3] ’kr/kg’. This is expelled ”krones per kilogram”. Therefore, in our example the price per kilogram is 20 kr/kg. Since the price per kilogram is the result of division between two proportional quantities, it is called a **constant of proportionality**.

1.2 Proportional quantities

$$\text{constant of proportionality} = \frac{\text{a quantity}}{\text{another quantity}} \tag{1.1}$$

Applied mathematics includes a great amount of quantities which are constants of proportionality, and in the definition boxes below you will find a selection of these. Note that all these formulas are identical to equation (1.1), only with the names of quantities and units changed.

[1]’kr’ is the Norwegian currency

[2]Also see appendix ?? in MB.

[3]We could also write $\frac{\text{kr}}{\text{kg}}$, but in this case it is more convenient to use / as the symbol for division.

1.3 Price per kilo

Price per kilo yields the ratio of a price (in av given currency) to a weight (in kilograms).

$$\text{price per kilo} = \frac{\text{price}}{\text{weight}} \tag{1.2}$$

Alternatively,

$$\text{price} = \text{price per kilo} \cdot \text{weight} \tag{1.3}$$

The notation for price per kilo is 'currency/kg'.

1.4 Price per liter

Price per liter yields the ratio of a price (in av given currency) to a volume (in liters).

$$\text{price per liter} = \frac{\text{price}}{\text{volume}} \tag{1.4}$$

Alternatively,

$$\text{pris} = \text{literpris} \cdot \text{volum} \tag{1.5}$$

The notation for price per liter is 'currency/L'.

1.5 Speed

Speed yields the ratio of length to time.

$$\text{fart} = \frac{\text{lengde}}{\text{tid}} \tag{1.6}$$

Alternatively,

$$\text{length} = \text{speed} \cdot \text{time} \tag{1.7}$$

Common notations for speed are 'km/h' and 'm/s'.

1.6 Tetthet

Density yields the ratio of weight to volume.

$$\text{density} = \frac{\text{weight}}{\text{volume}} \tag{1.8}$$

Alternatively,

$$\text{weight} = \text{density} \cdot \text{volume} \tag{1.9}$$

Common notations for density are 'kg/m^3' and 'g/cm^3'

1.7 Effect

Effect yields the ratio of energy to time.

$$\text{effect} = \frac{\text{energy}}{\text{time}} \tag{1.10}$$

Alternatively,

$$\text{energy} = \text{effect} \cdot \text{time} \tag{1.11}$$

Common notations for effect are 'J/s' and 'kWh/s'. 'J' is the abbreviation of the energy unit 'Joule'. 'J/s' is the same as 'W', which is the abbreviation of 'Watt'. In 'kWh', 'k' means 'kilo', 'W' means 'Watt' and 'h' means 'hour'.

Note

In (1.2) - (1.8), it is assumed that the quantities on the left side of the equations are *constants*, but that is not always the case. If you let a stone fall from a height, it's speed will obviously not remain the same along the way. By dividing the length by the time it took to travel it, you will find *the constant speed the ball would need have if it were to travel the same distance at the identical time.*

1.4 Regning med forskjellige benevninger

When performing calculations with quantities with notations, it is important to ensure that the notations involved are the same.

Example 1

Calculate $5\,\text{km} + 4\,000\,\text{m}$.

Answer

We must either write $5\,\text{km}$ as an amount of 'm' or $4\,000\,\text{m}$ as an amount of 'km' before we can add the quantities. We choose to write $5\,\text{km}$ as an amount of 'm' (see Rule 1.1):

$$5\,\text{km}{=}5\,000\,\text{m}$$

Now

$$\begin{aligned} 5\,\text{km} + 4\,000\,\text{m} &= 5\,000\,\text{m} + 4\,000\,\text{m} \\ &= 9\,000\,\text{m} \end{aligned}$$

Tips

Calculations involving notations can turn out to be a bit cumbersome. If you have ensured that the notations are the same, further calculations can be performed without them. In *Example 1* above, we could have written

$$5\,000 + 4\,000 = 9\,000$$

However, in a final answer the notation is *absolutely necessary*:

$$5\,\text{km} + 4\,000\,\text{m} = 9\,000\,\text{m}$$

Example 2

Use equation (1.7) to answer the questions.

a) A car drives at 50 km/h. How far will it travel in 3 hours?

b) A car drives at 90 km/h. How far will it travel in 45 minutes?

Answer

a) In equation (1.7), the speed is now 50, and the time is 3, so

$$\text{length} = 50 \cdot 3 = 150$$

That is, the car will travel 150 km.

b) Here, we have two different notations for time involved; 'hours' ('h') and 'minutes' ('min'). Therefore, we must either write the speed as an amount of 'km/min' or the time as an amount of 'h'. We chose to write 'min' as an amount of 'h'[1]:

$$\begin{aligned} 45\,\text{minutes} &= \frac{45}{60}\,\text{hours} \\ &= \frac{3}{4}\,\text{hours} \end{aligned}$$

In equation (1.7), the speed is now 90, and the time is $\frac{3}{4}$, so

$$\text{strekning} = 90 \cdot \frac{3}{4} = 67.5$$

That is, the car will travel 67.5 km in 45 minutes.

[1] Recall that 60 min = 1 h.

Example 3

Use equation (1.2) to answer the questions.

a) 10 kg tomatoes cost 35 kr. what is the price per kilo for the tomatoes?

b) Safran is reckoned to be the world's most expensive spice, with 5 g costing up to 600 kr. In that case, what is the price per kilo for safran?

Answer

a) In equation (1.2), the price is now 35, and the weight is 10, so

$$\text{price per kilo} = \frac{35}{10} = 3.5$$

That is, tomatoes costs 3.5 kr/kg

b) Here, we have two different units for weight involved; 'kg' and 'g'. We write the amount of 'g' as an amount of 'kg' (see Rule 1.1):

$$5\,\text{g} = 0.005\,\text{kg}$$

In equation (1.2), the price is now 600, and the weight is 0.005, so

$$\text{price per kilo} = \frac{600}{0{,}005} = 120\,000$$

That is, safran costs 120 000 kr/kg.

Chapter 2

Statistics

2.1 Introduction

In a **survey** we collect information. This information is often words or numbers, and is called **data**. A collection of data is called a **data set**.

For example, say you ask two people whether they like caviar. The one answers "yes", the other "no". Then "yes" and "no" are the data (answers) you have collected, and {"yes", "no"} is your data set.

Roughly speaking, statistics involves two things; *presenting* and *interpreting* data sets. For both purposes we have some terms which we will study in the following sections, helped by different examples of studies. These examples are found on page 18.

There are no universal laws telling you how to present or interpret data sets, however, you should follow these two guidelines:

- Let it always be clear exactly what you have studied, and what data you have collected.
- Always be aware the methods you use to interpret the data.

The language box

Persons participating in a survey where they are asked to answer questions are called **respondents**.

Survey 1

10 persones tested how many seconds they could hold their breath. These were the results (in seconds):

47 124 61 38 97 84 101 79 56 40

Survey 2

15 persons were asked how many apples they eat during a week. The answers were these:

7 4 5 4 1 0 6 5 4 8 1 6 8 0 14

Survey 3

300 persones where asked to name their favorite animal.

- 46 persons answered tiger
- 23 persons answered lion
- 17 persons answered crocodile
- 91 persons answered dog
- 72 persons answered cat
- 51 persons answered other animals

Survey 4

Mobile phones with smart-functions (app-based) came to the Norwegian market in 2009. The table[1] below shows the total sale of mobile phones during the time period 2009-2014, and the share with and without smart-functions. The numbers express the amount of 1 000 phones.

År	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**
totalt	2 365	2 500	2 250	2 200	2 400	2 100
wtho. sm.f.	1 665	1 250	790	300	240	147
wth. sm.f.	700	1 250	1 460	1 900	2 160	1 953

[1]Numbers imported from medienorge.uib.no.

2.2 Ways of presenting

When presenting data sets, it should be easy to see for others what we have found. This can be accomplished by using frequency tables, bar charts, sector graphs, or line graphs.

2.2.1 Frequency table

In a **frequency table** the data set are organized in a table showing the amount of times each unique answer appears. This amount is called the **frequency**.

Example - Survey 2

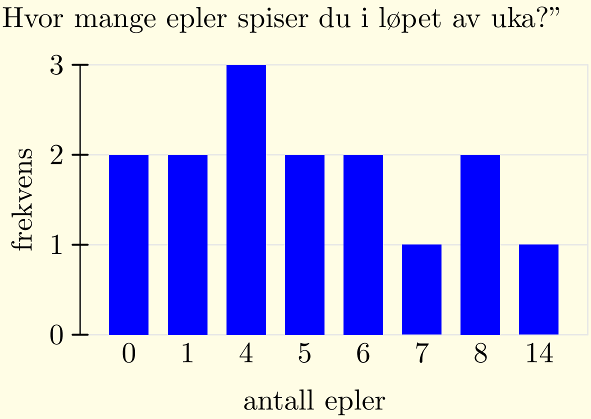
In this survey we have two 0's, two 1's, three 4's, two 5's, two 6's, one 7, two 8's and one 14. In a frequency table we then write

amount of apples	frequency
0	2
1	2
4	3
5	2
6	2
7	1
8	2
14	1

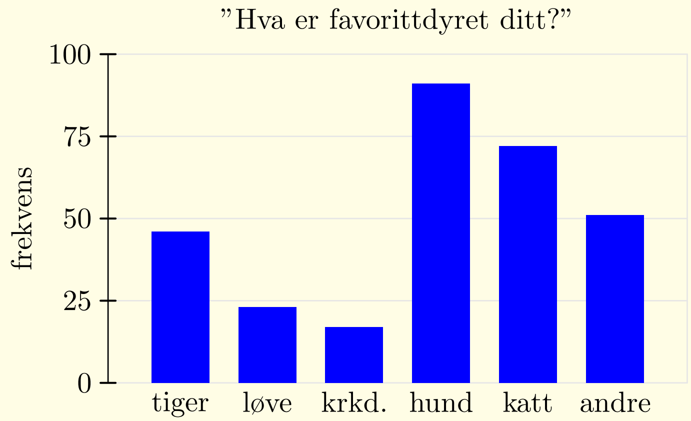
2.2.2 Søylediagram (stolpediagram)

In a bar chart the frequencies are represented by bars.

Survey 2



Survey 3

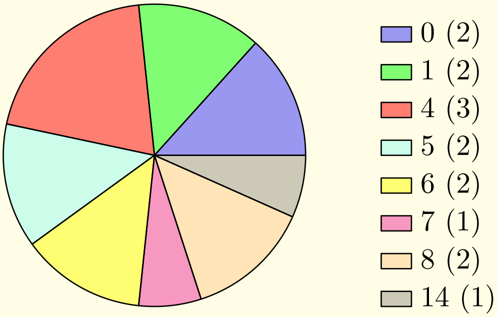


2.2.3 Sektordiagram (kakediagram)

In a sector graph the frequencies are represented by sectors in a circle.

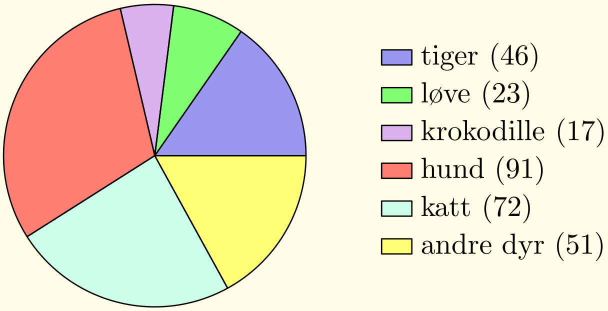
Survey 2

Epler spist i løpet av uka (frekvens i parantes)



Survey 3

Favorittdyr (frekvens i parantes)



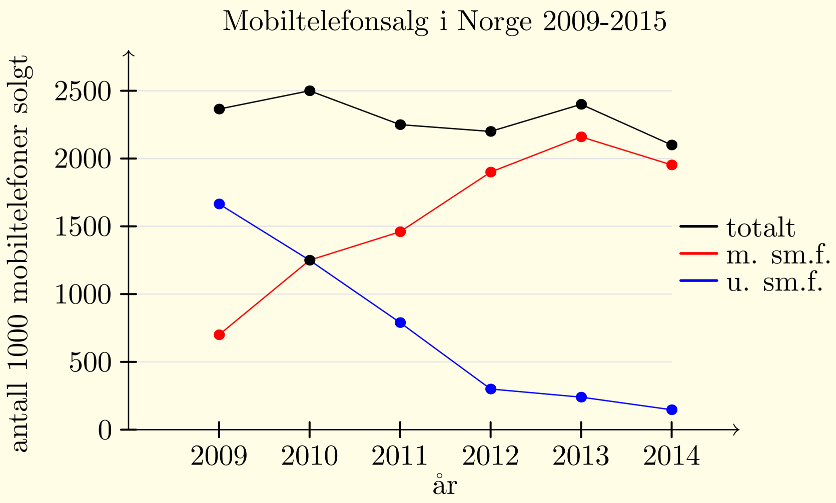
Making a sector graph from scratch

There are a lot of software available that generates sector graphs. However, if you were to make one from scratch, you will need basic knowledge of angles and fraction shares (see MB).

2.2.4 Linjediagram

In a line graph the data is represented as points in a coordinate system, with lines drawn between the points. Line graphs is typically used for describing evolving data.

Survey 4



2.3 Interpretation; central tendencies

In a data set there are often answers which are totally or approximately equal, and which reoccur. This means we can tell something about things that apply to the many; a **central tendeny**. The most common measures of central tendencies are the mode, the mean and the median.

2.3.1 Mode

2.1 Mode

The **mode** is the value occurring the most in the data set.

Example - Survey 2

In this dataset 4 has the highest occurrence (three), so 4 is the mode.

Multiple modes

If multiple values have the highest occurance in the data set, the data set has multiple modes.

2.3.2 Mean

When a data set includes numbers, we can find the sum of their values. When raising the question what the **mean** is, we ask this:

"If all the numbers had equal value, and the sum were still the same, what would the value be?"

The question is answered by the aid of division[1]:

2.2 Gjennomsnitt

$$\text{mean} = \frac{\text{sum of the values of the data set}}{\text{amount of values}}$$

The language box

The mean, as defined here, is also called the **average**. Also, there are multiple types of means. More specifically, the mean from Rule 2.2 is called the **arithmetic mean**.

Example - Survey 1

We sum the values from the data set, and divide by the amount of values:

$$\begin{aligned}
\text{mean} &= \frac{47 + 124 + 61 + 38 + 97 + 84 + 101 + 79 + 56 + 40}{10} \\
&= \frac{727}{10} \\
&= 72.7
\end{aligned}$$

Hence, the 10 participators held their breath for 72.7 seconds on average.

[1]See MB, side 23.

Example - Survey 2

Method 1

$$\begin{aligned}\text{mean} &= \frac{7+4+5+4+1+0+6+5+4+8+1+6+8+0+14}{15} \\ &= \frac{73}{15} \\ &\approx 4.87\end{aligned}$$

Method 2

We expand our frequency table 19 to find the sum of the values from the data set. (we have also included the sum of the frequencies):

Amount of apples	Frequency	amount $\cdot$ frequency
0	2	$0 \cdot 2 = 0$
1	2	$1 \cdot 2 = 2$
4	3	$4 \cdot 3 = 12$
5	2	$5 \cdot 2 = 10$
6	2	$6 \cdot 2 = 12$
7	1	$7 \cdot 1 = 14$
8	1	$8 \cdot 2 = 16$
14	1	$14 \cdot 1 = 14$
sum	15	73

Now

$$\begin{aligned}\text{mean} &= \frac{73}{15} \\ &\approx 4.87\end{aligned}$$

Hence, on average, the 15 respondents ate 4.87 apples a week.

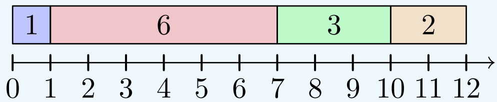
Example - Survey 4

(Calculations omitted. The values are rounded off to the nearest one.)

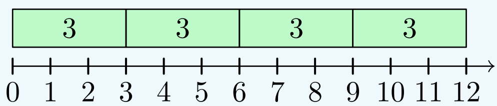
- Mean of total sale of mobiles: 2302
- Mean of sale of mobiles without smart-functions: 732
- Mean of sale of mobiles with smart-functions: 1570

Equal distribution

Note that the mean is about equal distribution. If we have 4 rectangles with respective lengths 1, 6, 3 and 2, their joint length is $1 + 6 + 3 + 2 = 12$.



Therefore, their mean length is $\frac{12}{4} = 3$. Thus, if we could reshape the rectangles so that their lengths were equal, with their joint length unchanged, they would each have length 3.



The average rate of change

Say you go for a jog, and measure your speed three times. Also, say that the data set you end up with is

$$10\,\mathrm{m/s} \qquad 10\,\mathrm{m/s} \qquad 10\,\mathrm{m/s}$$

Then your **average speed** was

$$\frac{10 + 10 + 10}{3}\,\mathrm{m/s} = 10\,\mathrm{m/s}$$

In other words; if your speed is the same alle the time[1], this speed is also you average speed. Consequently, the formula for the speed from Definition 1.5 is also the formula for the average speed. Alternatively stated, it is the formula for the average rate og change of length per time.

2.3 Average rate of change

If we *assume* or *hold* two quantities to be proportional, the proportionality constant from (1.1) is called the **averate rate of change**.

[1]In other words, your speed is *constant*.

Example - Survey 4

- For the years 2009 and 2010, the difference of smartphones sold to the difference of years is

  $$\frac{1\,260 - 700}{2010 - 2009} = \frac{550}{1} = 550$$

  Therefore, between 2009 and 2010 the sale of smartphones have *increased* by 550 000 smartphones per year.

- For the years 2010 and 2014, the difference of smartphones sold to the difference of years is

  $$\frac{1\,953 - 1\,250}{2014 - 2010} = \frac{703}{4} = 175{,}75$$

  Therefore, between 2010 and 2014 the sale of smartphones have *increased* by ca. 176 000 smartphones per year.

- For the years 2013 and 2014, the difference of smartphones sold to the difference of years is

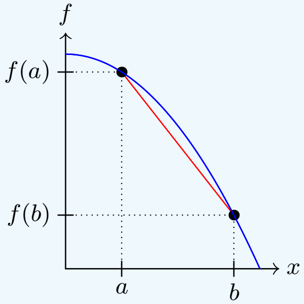
  $$\frac{1\,953 - 2\,160}{2014 - 2013} = \frac{-207}{1} = -207$$

  Therefore, between 2013 and 2014 the sale of smartphones have *decreased* by ca. 207 000 smartphones per year.

The slope of the line through two points

Given a function $f(x)$. In MB we have seen that the slope of the line through the points $(a, f(a))$ og $(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}$$



Comparing this expression with the calculations made on page 27, we realize that the expressions are, in general, identical. Hence, the slope of a line through two points yields the average rate of change between the two points.

2.3.3 Median

2.4 Median

The **median** is the number that ends up in the middle when the data set is arranged in an increasing order.

If the data set has an even amount of values, the median equals the mean of the two values in the middle (after the arranging).

Example - Survey 1

We arrange the data set in an increasing order:

38 40 47 56 61 79 84 97 101 124

The two numbers in the middle are 61 and 79. The mean of these is

$$\frac{61 + 79}{2} = 70$$

Hence, the median is 70.

Example - Survey 2

We arrange the data set in an increasing order:

$$0 \quad 0 \quad 1 \quad 1 \quad 4 \quad 4 \quad 4 \quad \textcolor{red}{5} \quad 5 \quad 6 \quad 6 \quad 7 \quad 8 \quad 8 \quad 14$$

The value in the middle is 5, and thus the median is 5.

Example - Survey 4

(Calculations omitted. The values are rounded off to the nearest one).

- The median for the total sale of mobiles: 2307
- The median for the sale of mobiles without smart-functions: 545
- The median for the sale of mobiles with smart-functions: 1570

2.4 Interpretation; variations

Often there will be large differences between collected data. The collective term for various type of differences is **variation**. The most common measures of variation are variation width, quartile width, variance and standard deviation.

2.4.1 Variation width

2.5 Variasjonsbredde

The difference between the data with the largest value and the data with the smallest value yields the **variation width**.

Example Survey 1

The datas with, respectively, the highest and lowest values are 124 and 38. Thus

$$\text{variation width} = 124 - 38 = 86$$

Example - Survey 2

The datas with, respectively, the highest and lowest values are 14 and 0. Hence

$$\text{variation width} = 14 - 0 = 14$$

Example - Survey 4

- Variation width for mobiles in total:

$$2\,500 - 2\,100 = 400$$

- Variation width e for mobiles without smart functions:

$$1\,665 - 147 = 518$$

- Variation width for mobiles with smart functions:

$$2\,160 - 700 = 1460$$

2.4.2 Quartile width

2.6 Quartile width and upper and lower quartile

The **quartile width** of a data set can be found doing the following:

1. Arrange the data set in increasing order.
2. Divide the arranged data set, on the middle, into two new data sets (if the number of values in the set is even, the median is excluded).
3. Find the respective medians in the two new data sets.
4. Find the difference between the medians from point 3.

Regarding the medians from point 3: The one with the largest value is called the **upper quartile**, and the one with the lowest value is called the **lower quartile**.

Example - Survey 1

1. 38 40 47 56 61 79 84 97 101 124
2. 38 40 47 56 61 79 84 97 101 124
3. The median in the blue set is 47 (lower quartile), and the median in the red set is 97 (upper quartile).

   38 40 47 56 61   79 84 97 101 124

4. Quartile width $= 97 - 47 = 50$

Example - Survey 2

1. 0 0 1 1 4 4 4 5 5 6 6 7 8 8 14
2. 0 0 1 1 4 4 4 5 5 6 6 7 8 8 14
3. The median in the blue set is 1 (lower quartile), and the median in the red set is 7 (upper quartile).

   0 0 1 1 4 4 4   5 6 6 7 8 8 14

4. Quartile width $= 7 - 1 = 6$

Example - Survey 4

(Calculations omitted)

- For mobiles in total the quartile with is 200
- For mobiles without smart functions the quartile width is 1010
- For mobiles with smart functions the quartile width is 703

The language box

The lower quartile, the median and the upper quartile is also called the **1st. quartile**, the **2nd quartile** and the **3rd quartile**, respectively.

2.4.3 Deviations, variance and standard deviation

2.7 Variance

The difference between a value in a data set and the mean of the dataset is called the **deviation** of the value.

The **variance** of a data set can be found in the following way:

1. Square the deviation of every value in the data set, and add these.
2. Divide by the amount of values in the data set.

The **standard deviation** is the square root of the variance.

Example

Given the data set

$$2 \quad 5 \quad 9 \quad 7 \quad 7$$

Then

$$\text{gjennomsnitt} = \frac{2+5+9+7+7}{5} = 6$$

Moreover,

$$\begin{aligned}\text{variance} &= \frac{(2-6)^2+(5-6)^2+(9-6)^2+(7-6)^2+(7-6)^2}{5}\\ &= 5\end{aligned}$$

Then standard deviation $= \sqrt{5} \approx 2.23$.

Example - Survey 1

(Calculations omitted)

The variance is 754.01. The standard deviation is $\sqrt{754.01} \approx 27.46$

Example - Survey 2

We found the mean of the data set on page 25. We expand our table from page 19:

apples	frequency	frequency · squared deviation
0	2	$2 \cdot \left(0 - \frac{73}{15}\right)^2$
1	2	$2 \cdot \left(1 - \frac{73}{15}\right)^2$
4	3	$3 \cdot \left(4 - \frac{73}{15}\right)^2$
5	2	$2 \cdot \left(5 - \frac{73}{15}\right)^2$
6	2	$2 \cdot \left(6 - \frac{73}{15}\right)^2$
7	1	$1 \cdot \left(7 - \frac{73}{15}\right)^2$
8	2	$2 \cdot \left(8 - \frac{73}{15}\right)^2$
14	1	$1 \cdot \left(9 - \frac{73}{15}\right)^2$
sum	15	$189.7\overline{3}$

Hence

$$\text{variance} = \frac{189{,}7\overline{3}}{15} \approx 12.65$$

Thus, the standard deviation is $\sqrt{12.65} \approx 3.57$

Example - Survey 4

(Calculations omitted)

- For mobiles in total the variance is 17 781,25, and the standard deviation ca. 133.4.
- For mobiles without smart functions the variance is $318\,848.\overline{3}$, and the standard deviation ca. 17.87.
- For mobiles with smart functions the variance is $245\,847.91\overline{6}$, and the standard deviation ca. 495,83.

Why squaring when finding the variance?

Let us see what happens if we repeat our calculations from *Example 1* on page 33, but without squaring:

$$\frac{(2-6)+(5-6)+(9-6)+(7-6)+(7-6)}{5} = \frac{2+5+9+7+7}{5} - 6 \quad (2.1)$$

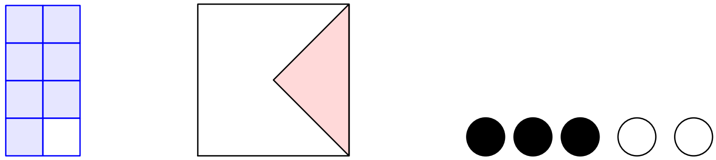
But, by definition, the fraction $\frac{2+5+9+7+7}{5}$ is the mean of the data set, and thus the above expression equals 0. This applies for all data sets, and, in this context, 0 provides no further information. If we however square the deviations, we avoid expressions deemed to equal 0.

Chapter 3

Fractions

3.1 Parts of wholes

In MB, we have seen how fractions are defined by a division of 1. In everyday use we use fractions to describe division of wholes.



(a) *(b)* *(c)*

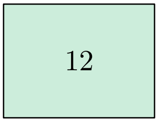
(a) The whole is 8 boxes. $\frac{7}{8}$ of the boxes are blue.

(b) The whole is a square. $\frac{1}{4}$ of the square is red.

(c) The whole is 5 circles. $\frac{3}{5}$ of the circles are black.

Fractions of numbers

Say that the rectangle below has value 12.



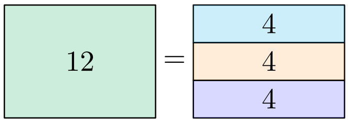
When we say "$\frac{2}{3}$ of 12", we intend to

a) distribute 12 into 3 equal groups.

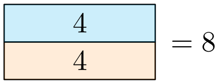
b) find how much 2 of these groups make up.

We have

a) 12 distributed into 3 equal groups is $12 : 3 = 4$.



b) 2 groups with value 4 make up $2 \cdot 4 = 8$.



Hence

$$\frac{2}{3} \text{ av } 12 = 8$$

To find $\frac{2}{3}$ of 12, we divided 12 by 3, and multiplied the quotient by 2. This is the same as multiplying 12 with $\frac{2}{3}$ (see MB).

3.1 The fraction of a number

To find the fraction of a number, we multiply the fraction by the number.

$$\frac{a}{b} \text{ av } c = \frac{a}{b} \cdot c$$

Example 1

Find $\frac{2}{5}$ of 15.

Answer

$$\frac{2}{5} \text{ av } 15 = \frac{2}{5} \cdot 15 = 6$$

Example 2

Find $\frac{7}{9}$ of $\frac{5}{6}$.

Answer

$$\frac{7}{9} \text{ av } \frac{5}{6} = \frac{7}{9} \cdot \frac{5}{6} = \frac{35}{54}$$

The language box

Parts of a whole is also called **shares**.

3.2 Prosent

Fractions are excellent to describe shares because they quickly give an impression of the size. It is for example easy to see (approximately) how much $\frac{3}{5}$ or $\frac{7}{12}$ make up of a cake. However, sometimes it is preferable to quickly determine what share makes up the *most*, and that is easiest to decide if the fractions have the same denominator.

When shares are stated in everyday life, they are often expressed as fractions with denominator 100. Fractions with this denominator are so frequently used that they got their own name.

3.2 Percentage

$$a\% = \frac{a}{100}$$

The language box

% is pronounced *per cent*. The word origins from the latin *per centum*, meaning *per hundreds*.

Example 1

$$43\% = \frac{43}{100}$$

Example 2

$$12{,}7\% = \frac{12{,}7}{100}$$

Note: A bit unfamiliar maybe, but there is nothing wrong with the numerator (or the denominator) being a decimal number.

Example 3

Find the value of

a) 12% b) 19,6% c) 149%

Answer

(See MB for calculations involving division by 100.)

a) $12\% = \dfrac{12}{100} = 0{,}12$

b) $19{,}6\% = \dfrac{19{,}6}{100} = 0{,}196$

c) $149\% = \dfrac{149}{100} = 1{,}49$

Example 4

Write the fraction as percentage.

a) $\dfrac{34}{100}$

b) $\dfrac{203}{100}$

Answer

a) $\dfrac{34}{100} = 34\%$

b) $\dfrac{203}{100} = 203\%$

Example 5

Find 50% of 800.

Answer

By Rule 3.1 and Rule 3.2,

$$50\% \text{ av } 800 = \frac{50}{100} \cdot 800 = 400$$

Example 6

Find 2% av 7.4.

Answer

$$2\% \text{ of } 7.4 = \frac{2}{100} \cdot 7.4 = 0.148$$

Tip

Dividing by 100 being straight forward, we can express percentages as decimal numbers when performing calculations. In the preceding *Example 6* we could have written the following:

$$2\% \text{ av } 7.4 = 0.02 \cdot 7.4 = 0.148$$

Percentage shares

What percentage does 15 make up of 20?

15 equals $\frac{15}{20}$ of 20, so the answer to the question becomes apparent if we expand $\frac{15}{20}$ to a fraction with denominator 100. Since $20 \cdot \frac{100}{20} = 100$, we expand our fraction with $\frac{100}{20} = 5$:

$$\frac{15 \cdot 5}{20 \cdot 5} = \frac{75}{100}$$

So, 15 makes up 75% of 20. We could have got 75 directly writing

$$15 \cdot \frac{100}{20} = 75$$

3.3 The percentage *a* makes up of *b*

$$\text{the percentage } a \text{ makes up of } b = a \cdot \frac{100}{b} \tag{3.1}$$

Example 1

What percentage does 340 make up of 400?

Answer

$$340 \cdot \frac{100}{400} = 85$$

340 makes up 85% of 400.

Example 2

What percentage does 119 make up of 500?

Answer

$$119 \cdot \frac{100}{500} = 23{,}8$$

119 makes up 23.8% of 500.

Tip

Since multiplying by 100 is an easy task, we can omit it from out calculation. In *Example 2* above we could have written

$$\frac{119}{500} = 0.238$$

So, 119 makes up 23.8% of 500. (That is, we arrive at our final answer by simply moving the decimal separator two places, $0.238 \cdot 100 = 23.8$.)

3.2.1 Percentage change; increase or reduction

Increase

The phrase "200 increased by 30%" implies the following:

200 added with 30% av 200.

Therefore

$$\begin{aligned} \text{200 increased with 30\%} &= 200 + 200 \cdot 30\% \\ &= 200 + 60 \\ &= 260 \end{aligned}$$

We note that 200 is present in both our terms in the above equation, so, according to the distributive law[1],

$$\begin{aligned} \text{200 increased by 30\%} &= 200 + 200 \cdot 30\% \\ &= 200 \cdot (1 + 30\%) \\ &= 200 \cdot (100\% + 30\%) \\ &= 200 \cdot 130\% \end{aligned}$$

Consequently,

$$\text{200 økt med 30\%} = \text{130\% av 200}$$

[1]See MB.

Reduction

The phrase ”Reduce 200 by 60%” implies the following:

60% of 200 subtracted from 200

Thus

$$\begin{aligned} \text{200 reduced by 60\%} &= 200 - 200 \cdot 60\% \\ &= 200 - 120 \\ &= 80 \end{aligned}$$

Here as well we notice the presence of 200 in both terms:

$$\begin{aligned} \text{200 reduced by 30\%} &= 200 - 200 \cdot 60\% \\ &= 200 \cdot (1 - 60\%) \\ &= 200 \cdot 40\% \end{aligned}$$

Hence

$$\text{200 reduced by 60\%} = \text{40\% of 200}$$

Percentage change summary

3.4 Percentage change

- When a quantity is reduced by $a\%$, we end up with $(100\% - a\%)$ of the quantity.
- When a quantity is increased by $a\%$, we end up with $(100\% + a\%)$ of the quantity.

Example 1

What is 210 reduced by 70%?

Answer

$100\% - 70\% = 30\%$, so

$$\begin{aligned} \text{210 reduced by 70\%} &= \text{30\% of 210} \\ &= \frac{30}{100} \cdot 210 \\ &= 63 \end{aligned}$$

Example 2

What is 208.9 increased by 124.5%?

Answer

$100\% + 124.5\% = 224.5\%$, so

$$\begin{aligned} 208.9 \text{ increased by } 124.5 &= 224.5\% \text{ of } 208.9 \\ &= \frac{224.5}{100} \cdot 208.9 \end{aligned}$$

The language box

Discount is an amount of money subtracted from a price when an offer is given. This is also called a *cut price*. Discount is usually expressed as an amount of money or as a percentage of the price.

Value added tax (VAT) is a fee added to the price of merchandise and goods. Value added tax is usually expressed as a percentage of the price.

Example 3

In a shop a shirt first cost 500 kr, but is now saled with 40% discount.

What is the new price of the shirt?

Answer

Note: The currency is omitted from the calculations

If we were to pay full price, we would pay 100% of 500. But if we get 40% discount, we only need to pay $100\% - 40\% = 60\%$ of 500:

$$\begin{aligned} 60\% \text{ of } 500 &= \frac{60}{100} \cdot 500 \\ &= 300 \end{aligned}$$

Therefore, with the discount the shirt costs 300 kr.

Example 4

The picture says that the price of a headset is 999.20 kr with VAT *excluded*, and 1 249 with VAT *included*. For headsets the VAT is 25% of the price.

Examine whether the price with VAT included is correct.

Answer

Note: The currency is omitted from the calculations

When VAT is included, we must pay $100\% + 25\%$ of 999.20:

$$
\begin{aligned}
125\% \text{ of } 999.20 &= \frac{125}{100} \cdot 999.20 \\
&= 1249
\end{aligned}
$$

Hence, we have to pay 1249 kr, which is also stated in the picture.

3.2.2 Change factor

On page 43 we increased 200 by 30%, resulting in 130% of 200. In that case we say that the *change factor* is 1.3. On page 44 we reduced 200 by 60%, resulting in 40% of 200. In that case the *change factor* is 0.40.

3.5 Change factor I

When a quantity is changed by $a\%$, the **change factor** is the value of $100\% \pm a\%$.

$+$ is used when increasing, and $-$ is used when reducing.

Example 1

A quantity is increased by 15%. What is the change factor?

Answer

$100\% + 15\% = 115\%$, so the change factor is 1.15.

Example 2

A quantity is reduced by 19,7%. What is the change factor?

Answer

$100\% - 19{,}7\% = 80.3\%$, so the change factor is 0.803

Let us look back at *Example 1* on page 44, where 210 was reduced by 70%. Then the change factor is 0.3. Also,

$$0.3 \cdot 210 = 63$$

Therefore, to find the value of 210 reduced by 70%, we can multiply 210 with the change factor (explain to yourself why!).

3.6 Percentage change using change factor

$$\text{changed original value} = \text{change factor} \cdot \text{original value}$$

Example 1

A merchandise worth 1 000 kr is discounted with 20%.

a) What is the change factor?

b) Find the new price.

Answer

a) Since there is 20% discount, we have to pay $100\% - 20\% = 80\%$ of the original price. Therefore, the change factor is 0.8.

b) We have

$$0.8 \cdot 1000 = 800$$

Hence, the new price is 800 kr .

Example 2

A chocolate costs 9.80 kr, VAT excluded. The VAT on food products is 15%.

a) What is the change factor when finding the price of the chocolate with VAT included?

b) What is the price of the chocolate with VAT included?

Answer

a) With 15% VAT included, we have to pay 100% + 15% = 115% of the price with VAT excluded. Thus, the change factor is 1.15.

b)
$$1.15 \cdot 9.90 = 12.25$$

The chocolate costs 12.25 kr with VAT included.

By rewriting the equation in[1] Rule 3.6, we get an alternative formula for the change factor:

3.7 Change factor II

$$\text{change factor} = \frac{\text{changed original value}}{\text{original value}}$$

Finding the percentage change

When seeking to determine a percentage change, it is important to remember it is about a percentage *of a whole*, which is the reference for the calculations. This whole is the original value.

Let us as an example say that Jakob earned 10 000 kr in 2019, and 12 000 kr in 2020. Then we can raise the question "What was the percentage change of Jakob's salary from 2019 to 2020?".

The question refer to the salary from 2019, which means that 10 000 is our original value. Two ways of finding the percentage change of salary are whe following:

[1]See Chapter 6 regarding how to rewrite equations.

- Jakob's salary changed from $10\,000$ to $12\,000$, a change of $12\,000 - 10\,000 = 2\,000$. Moreover, (see Rule 3.3)

$$\begin{aligned}\text{the percentage } 2\,000 \text{ makes up of } 10\,000 &= 2\,000 \cdot \frac{100}{10\,000} \\ &= 20\end{aligned}$$

  Thus, from 2019 to 2020 Jakob's salary increased by 20%.

- We have

$$\frac{12\,000}{10\,000} = 1{,}2$$

  Hence, from 2019 to 2020 the change factor of Jakob's was 1,2 (see Rule 3.6). This change factor corresponds to a 20% increase (see Rule 3.5), which means his salary increased by 20% during the period.

3.8 Percentage change

$$\text{percantage change} = \frac{\text{changed original value} - \text{original value}}{\text{original value}} \cdot 100$$

A positive/negative percentage change corresponds to a percentage increase/reduction.

Comment

The looks of Rule 3.8 can be a bit intimidating, and is not necessarily easy to remember. However, if you have reached a deep understanding of the topics covered in subsection 3.2.1, you already know how to find a percentage change through a step by step approach without using Rule 3.8. In the following examples we will show both a step by step approach, and the use of Rule 3.8.

Example 1

In 2019, a football team had 20 players. I 2020 the team had 12 players. What percentage of players in 2019 had quit in 2020?

Answer

Firstly, we note that it is the number of players in 2019 that is our original value.

Method 1; step by step approach

The football team went from 20 to 12 players, so $20 - 12 = 8$ players quit. We have

$$\text{the percentage 4 makes up of 20} = 8 \cdot \frac{100}{20} = 40$$

Hence, 40% of the players from 2019 had quit in 2020.

Method 2; using Rule 3.8

We have

$$\begin{aligned}\text{percentage change} &= \frac{12 - 20}{20} \cdot 100 \\ &= -\frac{8}{20} \cdot 100 \\ &= -40\end{aligned}$$

Hence, 40% of the players from 2019 had quit in 2020.

Note: Players quitting involves a *reduction* in number of players. Therefore, we expected the percentage change to be a negative number.

Applying Rule 3.7 and Rule 3.8, we get another formula[1] for the percentage change:

3.9 Percentage change II

$$\text{percentage change} = 100\,(\text{change factor} - 1)$$

[1] You are challenged to arrive at this formula in **Gruble ??**.

Example 1

In 2019 your income was 12 000 kr, and in 2020 your income was 10 000 kr. Find you percentage change of income, with the income of 2019 as reference.

Answer

In this case, 12 000 is our original value. Then, by Rule 3.7,

$$\begin{aligned}\text{change factor} &= \frac{10\,000}{12\,000} \\ &= 0.8\end{aligned}$$

Hence

$$\begin{aligned}\text{percentage change} &= 100(0.8-1) \\ &= 100(-0.2) \\ &= -20\end{aligned}$$

Therefore, the income was *reduced* with 20% in 2020 compared to the income of 2019.

Note

Both Rule 3.8 and Rule 3.9 provides formulas that can be used to find percentage changes. Which of these rules to choose is just a question of preference.

3.2.3 Percentage point

In many situations, we speak of multiple quantities at the same time, and when we use the term *percentage* our sentences can turn out both long and ambiguous if we speak of different original values (references). In order to help simplify such sentences, we have the term **percentage point**.

Say a pair of sunglasses were first sold with a 30% discount of the original price, and a while after that with a 80% discount of the original price. In that case we say that the discount increased by 50 *percentage points*.

$80\% - 30\% = 50\%$

Why can't we say that the discount increased by 50%?

Say that the original price of the sunglasses were 1 000 kr. 30% discount of 1 000 kr yields a cut price of 300 kr. 80% discount of 1000 kr yields a cut price of 800 kr. However, if we increase 300 by 50%, we get $300 \cdot 1.5 = 450$, which is not equal to 800. The reason for this disagreement is that we have two different references:

"The discount was first 30%, and then it increased by 50 percentage points. Then the discount was 80%."

Explanation: 'The discount' is a quantity we calculate with the original price as reference. When we say "percentage point", we imply that **the original price is still the reference** in calculations. When the price is 1 000 kr, we start with a discount of $1\,000\,\text{kr} \cdot 0.3 = 300\,\text{kr}$. When we add 50 *percentage points* to the discount, we cut off an additional 50% of the original price, that is $1\,000\,\text{kr} \cdot 0.5 = 500\,\text{kr}$. In total that is a cut price of 800 kr, which makes up 80% of the original price.

"The discount was first 30%, and then it increased by 50%. Then the discount was 45%."

Explanation: 'The discount" is a quantity we calculate with the original price as a reference, but when calculating the increased discount, **the original discount is the reference**. Since the original price is 1 000 kr, we start with a discount of 300 kr. Moreover,

$$300\,\text{kr increased by } 50\% = 300\,\text{kr} \cdot 1{,}5 = 450\,\text{kr}$$

and

$$\text{the percentage 450 makes up of } 1\,000 = \frac{450}{100} = 45$$

Hence, the new discount is 45%.

In the two (yellow) foregoing textboxes, we calculated the increased discounts using the original price of the sunglasses (1 000 kr). In general, this is not necessary:

- The discount was first 30%, and then it increased by 50 percentage points. Then the discount was

$$30\% + 50\% = 80\%$$

- The discount was first 30%, and then it increased by 50%. Then the discount was

$$30\% \cdot 1.5 = 45\%$$

3.10 Percentage points versus percentage change

$a\%$ increased/reduced by b percentage points $= a\% \pm b\%$.

$a\%$ increased/reduced by $b\% = a\% \cdot (1 \pm b\%)$

Note

The second equation in Rule 3.10 is identical to the equation in Rule 3.6.

Example

One day, 5% of the students in a school were absent. The day after 7.5%, of the students were absent.

a) By how many percentage points did the absence increase?

b) By how many percentages did the absence increase?

Answer

a) $7.5\% - 5\% = 2.5\%$, so the absence increased by 2.5 percentage points.

b) The question is how much the difference of the absence, that is 2.5%, makes up of 5%. By Rule 3.3,

$$\begin{aligned}\text{the percentage } 2.5\% \text{ makes up of } 5\% &= 2.5\% \cdot \frac{100}{5\%} \\ &= 50\end{aligned}$$

Hence, the absence increased by 50%.

Note

In the preceding *Example 1*, asking the question "By how many percentage points did the absence increase?" is the same as asking the question "How many percentage does the difference of absent students make up of the total number of students?".

3.2.4 Repeated percentage change

What is the pattern when a quantity is changed by the same percentage multiple times? As an example, let us start with 2000 and perform a 10% increase 3 consecutive times (see Rule 3.6):

$$\text{value after 1st change} = \overbrace{2000}^{\text{original value}} \cdot 1.10 = 2\,200$$
$$\text{value after 2nd change} = \overbrace{2\,000 \cdot 1.10}^{2\,200} \cdot 1{,}10 = 2\,420$$
$$\text{value after 3rd change} = \overbrace{2\,420 \cdot 1.10 \cdot 1{,}10}^{2\,420} \cdot 1{,}10 = 2\,662$$

The intermediate calculations we have performed may seem a bit unnecessary, but if we exploit the conventions of powers[1] a pattern appears:

$$\text{value after 1st change} = 2\,000 \cdot 1{,}10^1 = 2\,200$$
$$\text{value after 2nd change} = 2\,000 \cdot 1{,}10^2 = 2\,420$$
$$\text{value after 3rd change} = 2\,000 \cdot 1{,}10^3 = 2\,662$$

3.11 Repeated change

$$\text{new value} = \text{original value} \cdot \text{change factor}^{\text{number of changes}}$$

Example 1

Find the new value when 10 000 is increased by 20% 6 consecutive times.

Answer

The change factor is 1.2, so

$$\begin{aligned}\text{new value} &= 10\,000 \cdot 1.2^6 \\ &= 29\,859{,}84\end{aligned}$$

[1] See MB

Example 2

Marion has bought herself a new car worth 300 000 kr, and she expects the value of the car will decrease by 12% each year the next four years. If so, what is the car worth in four years?

Answer

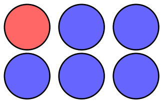
Since the annual reduction is 12%, the change factor is 0.88. The original value is 300 000, and the number of changes is 4:

$$300\,000 \cdot 0.88^4 \approx 179\,908$$

Therefore, Marion expects her car to be worth approximately 179 908 kr in four years.

3.3 Ratio

The **ratio** between two quantities refers to the one quantity divided by the other. If we for example have 1 red circles and 5 blue circles, we say that



$$\text{the ratio of the amount of red circles to the amount of blue circles} = \frac{1}{5}$$

We can also (of course) write the ratio as $1 : 5$, with the value being

$$1 : 5 = 0.2$$

Whether we write a ratio as a fraction or as a division depends on the situation.

3.12 Forhold

$$\text{ratio of } a \text{ to } b = \frac{a}{b}$$

Example 1

In a class there are 10 handball players and 5 football players.

a) What is the ratio of the amount of handball players to the amount of football players?

b) What is the ratio of the amount of football players to the amount of handball players?

Answer

a) The ratio of the amount of handball players to the amount of football players is
$$\frac{10}{5} = 2$$

b) The ratio of the amount of football players to the amount of handball players is
$$\frac{5}{10} = 0.5$$

3.3.1 Scale

In MB, we have studied similar triangles. The fact that the ratio of corresponding sides are equal are also valid for a lot of shapes, such as squares, circles, prisms, spheres etc. From this it follows that small drawings (models) can give us information about real-sized quantities. The number that holds this information is called the **scale**.

3.13 Scale

$$\text{scale} = \frac{\text{a length in a model}}{\text{the corresponding real-sized length}}$$

Example 1

In a drawing of a house, a wall is 6 cm. The real size of the wall is 12 m.

What is the scale of the drawing?

Answer

First we must ensure that the lengts have the same unit[1]. We convert 12 m into an amount of 'cm':

$$12\,\text{m} = 1200\,\text{cm}$$

Now

$$\begin{aligned}\text{scale} &= \frac{6\,\text{cm}}{12\,\text{cm}} \\ &= \frac{6}{12}\end{aligned}$$

Also, we should reduce the fraction

$$\text{scale} = \frac{1}{6}$$

[1] See section 1.4.

Tip

The scale of a map is almost always given as a fraction with numerator 1. In that case one can make the following rules:

$$\text{real-sized length} = \text{length on map} \cdot \text{denominator of the scale}$$

$$\text{length on map} = \frac{\text{real-sized length}}{\text{denominator of the scale}}$$

Example 2

The below map has scale 1 : 25 000.

a) The direct route (the blue) between Helland and Vike is 10.4 cm on the map. What is the real distance between Helland and Vike?

b) The real length of the Tresfjord bridge is approximately 1300 m. How long is the Tresfjord bridge on the map?

Answer

a) Real distance between Helland and Vike $= 10.4\,\text{cm} \cdot 25\,000$

$$= 260\,000\,\text{cm}$$

Hence, the real distance between Helland and Vike is 2.6 km.

b) Map length of the Tresfjord bridge $= \frac{1\,300\,\text{m}}{25\,000} = 0.052\,\text{m}$

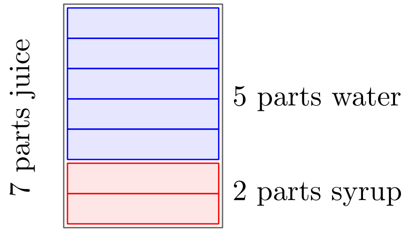
Hence, the map length of the Tresfjord bridge is 5.2 cm.

3.3.2 Mixing ratio

In many situations we want to mix two (or more) to a fitting ratio. If you read the symbol "2 + 5" on a bottle of juice syrup, it means you are supposed to mix syrup and water to the ratio $2:5$. So, if we pour $2\,\mathrm{dL}$ syrup in a can, we must add $5\,\mathrm{dL}$ water in order to make the juice with the correct mixing ratio.

If you mix juice syrup with water, you get juice :-)

Sometimes we don't care *how much* we are mixing, as long as the mixing ratio is correct. For example, we can mix two full buckets of juice syrup with five full buckets of water, knowing that our mixing ratio is correct, even if we don't know the volume of the bucket. When we only care about the mixing ratio, we use the term **part**. Then we read "2 + 5" on the bottle of juice syrup as "2 parts of syrup to 5 parts of water". Hence, the juice includes $2 + 5 = 7$ parts:



Consequently, 1 part makes up $\frac{1}{7}$ of the juice, the syrup makes up $\frac{2}{7}$ of the juice, and the water makes up $\frac{5}{7}$ of the juice.

3.14 Parts in a mixing ratio

A mix with ratio $a : b$ includes $a + b$ parts.

- 1 part makes up $\frac{1}{a+b}$ of the mix.
- a makes up $\frac{a}{a+b}$ of the mix.
- b makes up $\frac{b}{a+b}$ of the mix.

Example 1

A can with volume $21\,\text{dL}$ is filled with a juice where the mixing ratio of syrup to water is $2 : 5$.

a) How much water is it in the can?

b) How much syrup is it in the can?

Answer

a) In total, the juice includes $2 + 5 = 7$ parts. Since 5 of these are water,

$$\begin{aligned}\text{amount of water} &= \frac{5}{7} \text{ av } 21\,\text{dL} \\ &= \frac{5 \cdot 21}{7}\,\text{dL} \\ &= 15\,\text{dL}\end{aligned}$$

b) We could solve this exercise in a similar way as that of exercise a), but it is better to note that if we have $15\,\text{dL}$ water in a total of $21\,\text{dL}$, then we have $(21 - 15)\,\text{dL} = 6\,\text{dL}$ syrup.

Example 2

In a paint bucket green and red paint is mixed in the ratio $3:7$, and the volume of this mix is $5\,\mathrm{L}$. You wish to change te mixing ratio into $3:11$.

How much red paint do you have to add?

Answer

The mix includes $3+7=10$ parts. Since there is $5\,\mathrm{L}$ in total,

$$\begin{aligned} 1 \text{ part} &= \frac{1}{10} \text{ of } 5\,\mathrm{L} \\ &= \frac{1\cdot 5}{10}\,\mathrm{L} \\ &= 0.5\,\mathrm{L} \end{aligned}$$

When there is 7 parts red paint but we wish 11, we must add 4 parts. So the volume of red paint needed is

$$4\cdot 0{,}5\,\mathrm{L} = 2\,\mathrm{L}$$

We must add $2\,\mathrm{L}$ red paint in order to get a mixing ratio of $3:11$.

Example 3

In a juice the ratio of syrup to water is $3:5$.

How many parts syrup and/or water du you have to add in order to make the ratio $1:4$?

Answer

The fraction we seek, $\frac{1}{4}$, can be written as a fraction with the same numerator the original ratio (that is $\frac{3}{5}$):

$$\frac{1}{4} = \frac{1\cdot 3}{4\cdot 3} = \frac{3}{12}$$

In the original ratio there are 3 parts syrup and 5 parts water. If this is to be changed into 3 parts syrup and 12 parts water, we must add 7 parts water.

Chapter 4

Economics

4.1 Indexes

4.1.1 Introduction

In economy, an **index** is the value of a ratio indicating the change of a quantity. For example, the Norwegian ice cream ”kroneis” cost 0.75 kr when it was released in 1953, while in 2021 it cost 27 kr. Then the ratio of the price in 2021 to the price in 1953 is

$$\frac{\text{price in 2021}}{\text{price in 1953}} = \frac{27}{0.75} = 36$$

In this conjuncture 36 is the index of the price change of ”kroneis” between 1953 and 2021.

4.1.2 Consumer price index and base year

The **Consumer price index** (CPI) is an index indicating a comparative price level of urban merchandise and services, such as

- Food and non-alcoholic beverages
- Alcoholic beverages and tobacco
- Clothing and footwear
- Housing, water, electricity, gas and other fuels
- Furnishings, household equipment and routine maintenance
- Health
- Transport
- Communications
- Recreation and culture
- Education
- Restaurants and hotels

To compare something a reference is needed, and the consumer price index use the price level of the aforementioned wares and servise in 2015 as a reference. Hence, 2015 is called the **base year**[1], with it’s index set to 100.

[1] The year set as the base year changes with time. Before 2015 became the base year, 1998 was.

4.1 Base year

The index of a base year equals 100. 2015 is the base year of the consumer price index.

The below table presents the CPI for the years 2011 - 2020

År	KPI
2020	112,2
2019	110,8
2018	112,2
2017	105,5
2016	103,6
2015	100
2014	97,9
2013	95,9
2012	93,9
2011	93,3

Table 4.1: Consumer price index for the years 2011-2021. Numbers from SSB.

According to the table, we can, for example, read the following:

- Since CPI in 2017 was 105.5, the prices have increased by 5.5% since 2015.
- Since CPI in 2011 was 93.3, the prices were 7,7% lower in 2011 than in 2015.

4.2 Percentage change relative to the base year

$$\text{index} - 100 = \text{percentage change relative to the base year}$$

Example 1

In july 2021 CPI for food was 109,4. How much have the price for food changed compared to the base year?

Answer

$109.4 - 100 = 9.4$. The price of food has increased by 9,4% compared to the base year.

Example 2

In july 2021 CPI for footwear was 98.0. How much have the price for footwear changed compared to the base year?

Answer

$98{,}0 - 100 = -2$. Hence, the price of footwear is reduced by 2% compared to the base year.

4.1.3 Kroneverdi

As mentioned, a "kroneis" cost 0.75 kr in 1953, and 27 kr in 2021. When at two distinct points of time we have to pay a *different* price for the *same* merchandise it will often mean that the[1] **krone value** has changed; *1 kr in 1957 was more worth than 1 kr i 2021.*

The krone value for a given year is defined as 100 divided by the CPI of that year:

4.3 Kroneverdi

$$\text{kroneverdi} = \frac{100}{\text{KPI}}$$

Note: Kroneverdien for basisåret (2015) er 1.

Example 1

CPI in 2012 was 93.9. Calculate the krone value in 2012.

Answer

$$\begin{aligned}\text{krone value in 2012} &= \frac{100}{93.9}\\ &\approx 1.06\end{aligned}$$

This means that 1 kr in 2012 was equivalent to 1,06 kr in the base year.

[1]Here, 'krone' can be substituted by any currency.

4.4 Real value

The real value of an amount of money yields the value of these money in the base year.

$$\text{real value} = \text{original value} \cdot \text{currency value}$$

Example

In 1928, the CPI was 3.2, and in 2020, the CPI was 112.2. Which had the greater real value, 10,000 NOK in 1928 or 350,000 NOK in 2020?

Answer

We have that

$$\text{Krone value in 1928} = \frac{100}{3.2}$$

Thus,

$$\begin{aligned}\text{The value of 10,000 NOK from 1928 in base year} &= 10{,}000\ \text{NOK} \cdot \frac{100}{3.2}\\ &= 312{,}500\ \text{NOK}\end{aligned}$$

Furthermore,

$$\text{Krone value in 2020} = \frac{100}{112.2}$$

Thus,

$$\begin{aligned}\text{The value of 350,000 NOK from 1928 in base year} &= 350{,}000\ \text{NOK} \cdot \frac{100}{112.2}\\ &\approx 311{,}943\ \text{NOK}\end{aligned}$$

Therefore, 10,000 NOK was worth more in 1928 than what 350,000 NOK was worth in 2020.

4.1.4 Real Wages and Nominal Wages

How well off we are depends on how much we earn and what the price level is. Suppose you had an annual salary of 500,000 NOK in both 2020 and 2019. *Table 4.1* then tells us that you had better purchasing power in 2019 because the price level (CPI) was lower than in 2020.

An increase in the price level is the same as a decrease in the value of the currency. This means that if your salary was the same in 2019 and 2020, the *real value* of your salary is higher in 2019 than in 2020. The original salary and the real value of the salary are so commonly used in statistics that they have their own names:

4.5 Real Wages and Nominal Wages

Nominal wages are the wages received in a given year.

Real wages are the real value of the nominal wages.

Example

In 2016, Per earned 450,000 NOK, while in 2012, he earned 420,000 NOK. In 2016, the CPI was 103.6, while in 2012, it was 93.9. In which of these years did Per have better purchasing power?

Answer

To find out in which year Per had better purchasing power, we check which year had the highest real wage[1] (see Rule 4.4):

$$\begin{aligned}\text{Real wage in 2016} &= 450{,}000 \cdot \frac{100}{103.6}\ \text{NOK} \\ &\approx 434{,}363\ \text{NOK} \\ \text{Real wage in 2012} &= 420{,}000 \cdot \frac{100}{93.9}\ \text{NOK} \\ &\approx 447{,}284\ \text{NOK}\end{aligned}$$

Per's real wage was highest in 2012; therefore, he had better purchasing power in that year.

[1]We obtain the CPI values from *Table 1* for this calculation.

4.6 Value Following an Index

A value is said to have *followed an index* if the value and index at two points in time are equal.

$$\frac{\text{value at time 1}}{\text{index at time 1}} = \frac{\text{value at time 2}}{\text{index at time 2}}$$

Example 1

The table below shows an overview of prices recorded in a store for two items at two different points in time.

	2010	**2020**
chocolate	11.00 NOK	13.40 NOK
soda	12.50 NOK	19.00 NOK

In 2010, the CPI was 92.1, and in 2020, the CPI was 112.1. Has the price of any of the items followed the index?

Answer

We have that

$$\frac{\text{price of chocolate in 2010}}{\text{CPI in 2010}} \approx \frac{11.00}{92.1} \approx 0.119$$
$$\frac{\text{price of chocolate in 2020}}{\text{CPI in 2020}} \approx \frac{13.40}{112.1} \approx 0.119$$

Furthermore,

$$\frac{\text{price of soda in 2010}}{\text{CPI in 2010}} \approx \frac{12.50}{92.1} \approx 0.136$$
$$\frac{textpriceofsodain2020}{\text{CPI in 2020}} \approx \frac{19.00}{112.1} \approx 0.169$$

So, it's reasonable to say that the price of chocolate has followed the index, while the price of soda has not.

4.2 Loans and Savings

4.2.1 Loans

Sometimes we don't have enough money to buy what we want and, therefore, need to take out a loan from a bank. The bank then lends us a certain *loan amount* in exchange for us paying it back, along with *interest*, over a specified period. Typically, we pay the bank what is called the *installment amount* along the way, which consists of *principal* and interest. The amount we owe the bank at any given time is called *debt*.

Suppose a bank lends us 100,000 NOK, which is the loan amount. The loan is to be repaid over 5 years, with one installment amount each year, and the interest rate is 10

- **The sum of all installments should equal the loan amount**.

  To simplify our example, we decide to repay the loan with equal installments each year. Since 100,000 NOK should be distributed equally over 5 years, the annual installment should be $\frac{100{,}000}{5}$ NOK = 20,000 NOK.

- **What is paid in installments should be deducted from the debt**.

  The initial debt is 100,000 NOK, but in the first year, we pay 20,000 NOK in installments, so the debt becomes 100,000 NOK $-$ 20,000 NOK = 80,000 NOK. In the second year, we pay another 20,000 NOK, making the debt 80,000 NOK $-$ 20,000 NOK = 60,000 NOK, and so on for the next three years.

- **Interest should be calculated on the debt**.

  Since the debt in the first year is 100,000 NOK, we must pay 100,000 NOK $\times\, 0.1 =$ 10,000 NOK in interest. In the second year, with a debt of 80,000 NOK, we must pay 80,000 NOK $\times\, 0.1 =$ 8,000 NOK in interest, and so on for the next three years.

- **The installment amount is the sum of the principal and interest**.

  From the first and third points, we deduce that:

	1st year	2nd year
Installment amount	20,000 NOK + 10,000 NOK = 30,000 NOK	20,000 NOK + 8,000 N = 28,000 NOK

And so on for the next three years.

- **The loan is complete when the debt is zero, and all interest is paid**.

  If we have paid installments of 20,000 NOK for 5 years, the debt is reduced to zero. If we have also paid all the interest, the loan is complete.

  Note: You always have the right to pay larger installments than originally agreed upon. If you pay off the entire debt, provided that any interest has also been paid, the loan will be terminated.

Amortizing Loans and Annuity Loans

Two common types of loans are *amortizing loans* and *annuity loans*. The loan from the example we just saw is an amortizing loan because the installments are of equal size. If the installment amounts had been equal, it would have been an annuity loan instead. If the loan amount, interest rate, and repayment period are the same, an amortizing loan will always result in lower overall expenses. However, for individuals, annuity loans are very popular because it is easier to plan finances when paying the same amount each time.

Credit Cards

Credit cards are a payment card that works in such a way that if you, for example, use the card to pay 10,000 NOK, you are borrowing the money from the bank. After a period agreed upon with the bank, it will charge interest on your debt. When you pay this debt

is partly up to you, but generally, credit cards have very high-interest rates, so it is wise to pay it off before the interest starts accruing!

4.7 Loans

loan amount	The amount we borrow from the bank.
debt	What we owe the bank at any given time.
interest	Percentage of debt to be paid.
principal	What we pay down on the debt.
	The sum of the principal equals the loan amount.
	New debt = old debt − principal
interest	debt · interest
installment	principal + interest
amortizing loan	A loan where the installments are of equal size.
annuity loan	A loan where the installment amounts are equal.
credit card	A payment card that creates a loan from the bank.

Example 1

From a bank, you borrow 300,000 NOK with a 3**a)** What is the annual principal payment? **b)** What is your debt after paying the third installment amount? **c)** How much do you have to pay in interest for the fourth installment amount? **d)** What is the amount of the fourth installment?

Solution:

a) Since 300,000 NOK is to be paid over 5 years, the annual principal payment is

$$\frac{300{,}000 \text{ NOK}}{5} = 60{,}000 \text{ NOK}$$

b) When the third installment is paid, you have paid three installments. That means your debt is

$$300{,}000 - 60{,}000 \cdot 3 = 300{,}000 - 180{,}000 = 120{,}000 \text{ NOK}$$

So, 120,000 NOK.
c) From the answer to part b), we know that the debt is 180,000 NOK when the fourth installment is to be paid. 3

$$180{,}000 \cdot 0.03 = 5{,}400 \text{ NOK}$$

So, 5,400 NOK.
d) The installment amount is equal to the principal plus interest. Based on the answers to parts a) and c), we know that the fourth installment amount is

$$60{,}000 \text{ NOK} + 5{,}400 \text{ NOK} = 65{,}400 \text{ NOK}$$

So, the fourth installment amount is 65,400 NOK.

Example 2

From a bank, you borrow 100,000 NOK with a 6.4% annual interest rate. The loan is to be repaid as an annuity loan over 5 years, and the bank has calculated that the installment amount will be 24,000 NOK.

Calculate the principal and interest for the first installment amount.

Answer

In the first year, the debt is 100,000 NOK, and you must pay 6.4

$$100{,}000 \cdot 0.064 = 6{,}400 \text{ NOK}$$

So, you must pay 6,400 NOK in interest in the first year.

We have that

$$\text{installment amount} = \text{principal} + \text{interest}$$

So, the principal is

$$\text{principal} = \text{installment amount} - \text{interest} = 24{,}000 \text{ NOK} - 6{,}400 \text{ NOK} = 17{,}600$$

So, you must pay 17,600 NOK in principal in the first year.

4.2.2 Savings; Deposit Interest and Expected Return

Deposit Interest

We have seen that we must pay interest when we borrow money from a bank, but if we instead put money (make a deposit) in a bank, we *earn* interest:

4.8 Deposit Interest

Deposit interest is a percentage increase in the money you have in the bank, repeated over fixed time intervals (monthly, annually, etc.).

Example 1

You deposit 20,000 NOK in a bank that offers a 2% annual savings interest rate. How much money do you have in the bank after 8 years?

Answer

To calculate deposit interest, we can use Rule **??**. Since the interest rate is 2%, the growth factor is 1.02. The original value is 20,000, and the number of changes (time) is 8:

$$20{,}000 \cdot 1.02^8 \approx 23{,}433$$

So, you have approximately 23,433 NOK in the bank after 8 years of saving.

Expected Return

Another way to save money is to invest in a mutual fund. In this case, we talk about *expected return*:

4.9 Expected Return

Expected return specifies an *expected* percentage increase of an investment, repeated over fixed time intervals.

Example 1

You invest 15,000 NOK in a mutual fund that expects a 5% annual return. How much is the investment worth after 8 years with such a return?

Answer

For expected return, we can also use Rule **??**. The growth factor is 1.05, the original value is 15,000, and the number of changes (time) is 8:

$$15{,}000 \cdot 1.05^8 \approx 22{,}162$$

After 8 years, it is expected that the investment is worth 22,162 NOK.

Saving with Deposit Interest or Mutual Funds?

Usually, the expected return on a mutual fund is higher than the deposit interest you get in a bank, but the downside is that expected return does not provide any guarantees. Expected return only indicates the increase experts anticipate. If you're lucky, the increase will be higher; if you're unlucky, it will be lower and may even result in a *reduction* of your investment. In the worst case, although extremely rare, your entire investment may end up being worth 0 NOK.

Deposit interest rates can also change somewhat over time, but they can never lead to a reduction in your investment.

4.3 Taxation

If you have an income, you usually have to pay a portion of that money to the state. This money is called *tax* (and sometimes *duty*). The purpose of tax is to provide the state with the means to offer services to its citizens, such as education, healthcare, and more. Today, taxes are largely calculated by computer systems, but it is your responsibility to ensure that the calculations are correct - and that's why it's important to understand how the tax system works.

Note!

In exam questions and in real life, you will quickly realize that tax systems are presented somewhat differently than in this book. This is because tax rules can change from year to year, and in this book, we have based our explanations on the tax rules of Norway in 2018. The most important thing is not to memorize these specific rules but to learn what is meant by the terms *gross income, deductions, taxable income, social security contribution*, and *net income*.

4.3.1 Gross Income, Deductions, and Taxable Income

Most people have to pay 23% of what is called *taxable income*, which is *gross income* minus *deductions*. Gross income is the salary you receive from your employer, while deductions can be various things. *Personal deductions* and *minimum deductions* are something all taxpayers receive. Additionally, you can receive deductions if you pay *union dues* or have donated money to charitable causes.

Taxable income is sometimes called *taxable basis*.

Union dues are what you pay to be a member of a trade union.

4.10 Gross Income, Deductions, and Taxable Income

$$
\begin{array}{cr}
 & \text{gross income} \\
- & \text{deductions} \\
\hline
= & \text{taxable income} \\
\hline
\end{array}
$$

Example

Magnus's gross income is 500,000 NOK. He receives a 56,000 NOK personal deduction, a 97,600 NOK minimum deduction, and he also pays 1,000 NOK for annual membership in the union *Tekna*.

How much does Magnus have to pay if he is taxed at 23

Answer

We start by calculating the taxable income, which is the gross income minus the deductions:

	500,000	gross income
$-$	56,000	personal deduction
$-$	97,600	minimum deduction
$-$	1,000	union dues
$=$	345,400	taxable income

Example

Jonas and his grandmother, Line, both have a salary of 150,000 NOK. Jonas is 18 years old, and Line is 71 years old.

a) How much does Jonas have to pay in social security tax?

b) How much does Line have to pay in social security tax?

Answer

a) Since Jonas is between 17 and 69 years old, he has to pay 8.2

$$150{,}000 \cdot 0.082 = 12{,}300$$

So, Jonas has to pay 12,300 NOK in social security tax.
b) Since Line is over 69 years old, she has to pay 5.1

$$150{,}000 \cdot 0.051 = 7{,}650$$

So, Line has to pay 7,650 NOK in social security tax.

Example

If you earn 550,000 NOK, the calculation of progressive tax is as follows:

Trinn 1	As the entire salary is over 237,900 NOK, you must pay tax on $(237{,}900 - 169{,}000)\,NOK = 68{,}900\,NOK$. The tax for trinn 1 is then $68{,}900\,NOK \cdot 0.014 \approx 965\,NOK$.
Trinn 2	Since 550,000 NOK is over 237,900 NOK but below 598,050 NOK, you must pay tax on $(550{,}000 - 237{,}900)\,NOK = 312{,}100\,NOK$. The tax for trinn 2 is then $312{,}100\,NOK \cdot 0.033 \approx 10{,}299\,NOK$.
Total	In total, you must pay $965\,NOK + 10{,}299\,NOK = 11{,}264\,NOK$ in progressive tax.

4.3.2 Accounting

In a budget, you list *expected* incomes and expenses, while in an *accounting* record, you list *actual* incomes and expenses. The difference between the budget and accounting is called the *deviation*. For the deviation, it is common to calculate 'accounting − budget' for incomes and results, while for expenses, you calculate 'budget − accounting'. This is because we want positive numbers if the incomes are higher than expected, and negative numbers if the expenses are higher than expected.

Example

In the example from the previous subsection (**??**), we set up a monthly budget for Lisa. In March, it turned out that these were her actual incomes and expenses:

- She didn't work as much as she had planned. Net pay was 3,500 NOK.
- She spent 4,200 NOK on food.
- She received 4,360 NOK in away-from-home grant.
- In birthday gifts, she received a total of 2,000 NOK.
- She spent about 3,600 NOK on clothing, leisure activities, etc.

Set up an accounting record for Lisa's month of March.

Answer

Incomes	Budget	Accounting	Deviation
Salary	4,000	3,500	−500
Grant	4,360	4,360	0
Birthday Gifts	0	2,000	2,000
Total	8,360	9,860	2,000
Expenses			
Food	4,500	4,200	300
Clothing, Leisure, etc.	1,200	3,600	−2,400
Total	5,700	7,800	1,900
Result	2,660	2,060	−600

So, Lisa ended up with a surplus of 2,060 NOK, but 600 NOK less than expected based on the budget.

Chapter 5

Probability

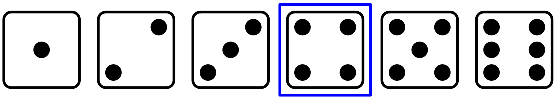
5.1 Fundamental Principle

The fundamental principle behind probability theory is that we inquire about the number of **favorable outcomes** in a set of **possible outcomes**. The probability of an **event** is then determined as a ratio between these two.

5.1 Probability of an Event

$$\text{probability of an event} = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$$

When we roll a dice, we call 'getting a four' an event. Since a dice has six different sides, there are six possible outcomes.



If we want to 'get a four,' there is only 1 out of these 6 outcomes that gives us what we want, so

$$\text{probability of 'getting a four'} = \frac{1}{6}$$

To avoid long expressions, we often use single letters to represent an event. Instead of writing 'getting a four,' we can use the letter F, and to indicate that we are talking about the probability of an event, we use the letter P.

P comes from the word *probability*.

When we write $P(S)$, this means 'the probability of getting a four':

$$P(S) = \frac{1}{6}$$

What about the opposite, i.e., the probability of *not* getting a four? To express that something is the opposite of an event, we put a line over the name. So, we write the event 'not getting a four' as $\bar{F}$. 'Not getting a four' is the same as 'getting *either* a one, a two, a three, a five, *or* a six,' which means this event has 5 favorable outcomes. That means

$$P(\bar{S}) = \frac{5}{6}$$

5.2 Symbols for Probability

$P(A)$ is the probability that event A occurs. A and $\bar{A}$ are opposite events. $P(\bar{A})$ is the probability that A *does not* occur, and vice versa.

Note!

In general, it's a good practice to simplify fractions when possible. However, in probability theory, it often pays off to leave fractions in their current form. Therefore, you will notice that many fractions in the upcoming sections could have been simplified.

5.2 Events with and without common outcomes

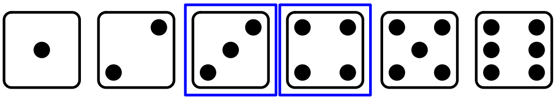
5.2.1 Events without common outcomes

Let's call the event 'rolling a three' (on a dice) as T. The event 'rolling a three *or* a four' is then written as $T \cup F$.

The symbol $\cup$ is called *union*.

There are 2 out of 6 sides on a dice that show a three *or* a four, so the probability of 'rolling a three *or* a four' is therefore $\frac{2}{6}$:

$$P(F \cup S) = \frac{2}{6}$$



We get the same answer by adding $P(F)$ and $P(S)$:

$$P(T \cup F) = P(T) + P(F) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

Finding $P(T \cup F)$ by summing $P(T)$ and $P(F)$ can be done because T and F do not have any *common outcomes*. This is because no sides of the dice show *both* a three and a four.

5.3 Events without common outcomes

For two events A and B without common outcomes, we have

$$P(A \cup B) = P(A) + P(B)$$

Example

You draw a ball from a bowl containing one red, two blue, and one green ball. What is the probability of drawing a red *or* a blue ball?

Answer

We call the events 'drawing a red ball' as R and the event 'drawing a blue ball' as B.

- There are a total of 4 possible outcomes (balls).
- Since all the balls have only one color, there are no common outcomes between events R and B.
- The probability of drawing a red ball is

$$P(R) = \frac{1}{4}$$

- The probability of drawing a blue ball is

$$P(B) = \frac{2}{4}$$

The probability of drawing a red *or* a blue ball is therefore

$$\begin{aligned} P(R \cup B) &= P(R) + P(B) \\ &= \frac{1}{4} + \frac{2}{4} \\ &= \frac{3}{4} \end{aligned}$$

5.2.2 The sum of all probabilities is 1

Let's consider rolling a die and consider both 'rolling a four' and 'not rolling a four' as favorable events. We have previously seen that $P(F) = \frac{1}{6}$, $P(\bar{F}) = \frac{5}{6}$, and that F and $\bar{F}$ have no common outcomes. Using Rule 5.3, we then have

$$\begin{aligned} P(F \cup \bar{F}) &= P(F) + P(\bar{F}) \\ &= \frac{1}{6} + \frac{5}{6} \\ &= 1 \end{aligned}$$

Either F happens, or it doesn't. And if it doesn't, then $\bar{F}$ happens. If we say that *both* F and $\bar{F}$ are favorable events, we are saying that all possible outcomes are favorable, and then Rule 5.4 gives a probability of 1.

5.4 The sum of all probabilities

The sum of the probabilities of all possible events is always equal to 1.

An event A and its complementary event $\bar{A}$ will together always constitute all events. From Rule 5.4, we have

$$\begin{aligned} P(A) + P(\bar{A}) &= 1 \\ P(A) &= 1 - P(\bar{A}) \end{aligned}$$

5.5 Complementary events

For an event A,

$$P(A) = 1 - P(\bar{A})$$

Example

In a class of 25 students, there are 12 girls and 13 boys. One student will be randomly selected to participate in a mathematics competition.

a) What is the probability that a boy is selected?

b) What is the probability that a boy is *not* selected?

Answer

We call the event 'a boy is selected' for G.

a) The probability that a boy is selected is

$$P(G) = \frac{13}{25}$$

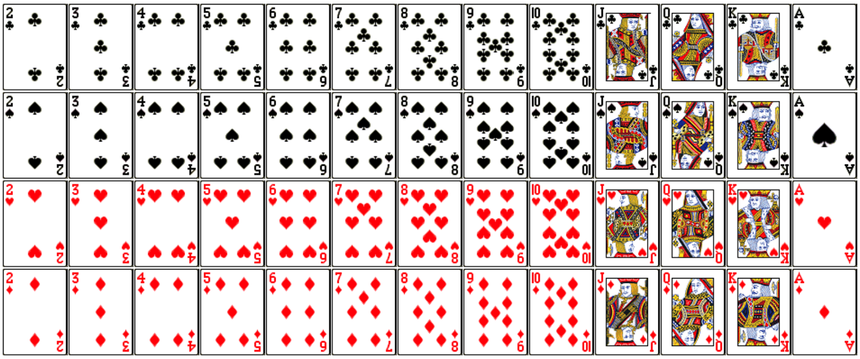
b) The probability that a boy is *not* selected is

$$\begin{aligned} P(\bar{G}) &= 1 - P(G) \\ &= 1 - \frac{13}{25} \\ &= \frac{12}{25} \end{aligned}$$

Note: The event 'a boy is *not* selected' is the same as 'a girl is selected'.

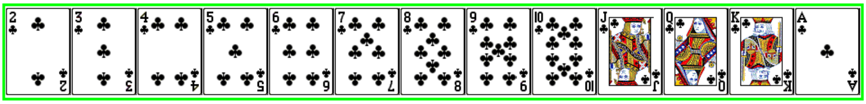
5.2.3 Common Outcomes

Sometimes, two events can have **common outcomes**. Let's look at a standard deck of 52 cards divided into the suits of spades, hearts, diamonds, and clubs. Cards such as the jack, queen, king, and ace are called *honour cards*.

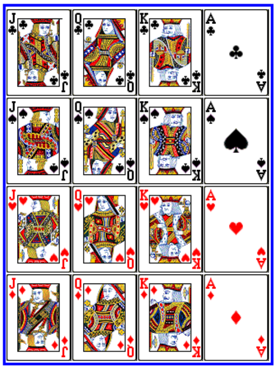


Imagine drawing a card from a shuffled deck. We want to find the probability of 'drawing a club card *or* an honour card'. We start by counting the favorable outcomes for club cards, and find that there are 13 of them.

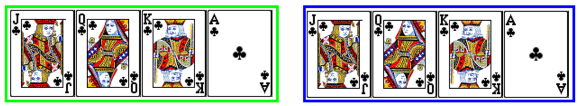
A card like the club king is a club card, but it's also an honour card, so it's both; *both* a club card *and* an honour card.



Next, we count the favorable outcomes for honour cards, and find that there are 16 of them.

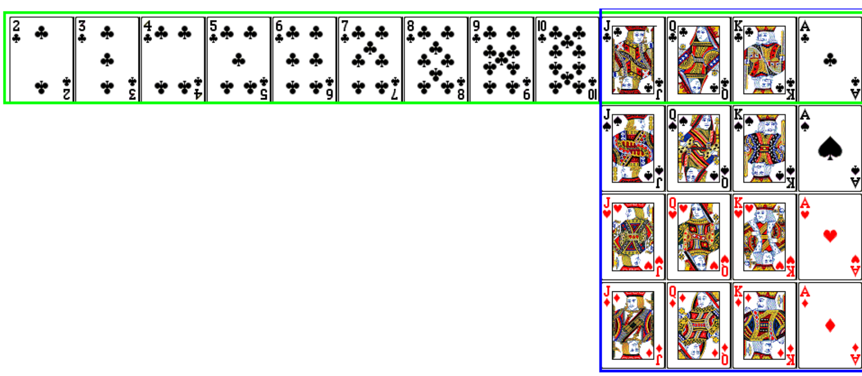


In total, we've counted $13 + 16 = 29$ favorable outcomes, but now we encounter a problem. When we found all the club cards, we counted cards like the club jack, queen, king, and ace. These same four cards were also counted when we found all the honour cards, which means we've counted the same cards twice!



For example, there aren't two club aces in a deck, so to calculate the number of cards that meet the requirement of being either a club or an honour card, we must subtract the number of cards we've counted twice:

$$13 + 16 - 4 = 25$$



Let K be the event 'drawing a club card' and H be the event 'drawing an honour card'. Since there are 25 cards that are either club cards *or* honour cards out of a total of 52 cards, we have

$$P(K \cup H) = \frac{25}{52}$$

Since we have 13 club cards and 16 honour cards, we further have

$$P(K) = \frac{13}{52} \text{ and } P(H) = \frac{16}{52}$$

We've seen that four cards are *both* clubs *and* honour cards; this is represented as

The symbol $\cap$ is called *intersection*.

$$K \cap H = 4$$

We then say that K and H have 4 common outcomes.
Furthermore, we have

$$P(K \cap H) = \frac{4}{52}$$

Now that we've found $P(K), P(H)$, and $P(K \cup H)$, we can find $P(K \cap H)$ again as follows:

$$\begin{aligned} P(K \cup H) &= P(K) + P(H) - P(K \cap H) \\ &= \frac{13}{52} + \frac{16}{52} - \frac{4}{52} \\ &= \frac{25}{52} \end{aligned}$$

5.6 Events with common outcomes

For two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note

If you apply Rule 5.6 to two events without common outcomes, you end up with Rule 5.3.

Example

In a class of 20 people, 7 play football, and 10 play handball. Out of these, 4 play both football and handball. If one person is randomly chosen from the class, what is the probability that this person plays football *or* handball?

Answer

Let F be the event 'plays football' and H be the event 'plays handball'.

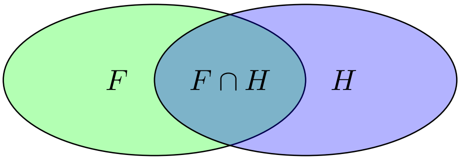
- The probability that a person plays football is
$$P(F) = \frac{7}{20}$$
- The probability that a person plays handball is
$$P(H) = \frac{10}{20}$$
- The probability that a person plays *both* football and handball is
$$P(F \cap H) = \frac{4}{20}$$

Therefore, the probability that a person plays football *or* handball is

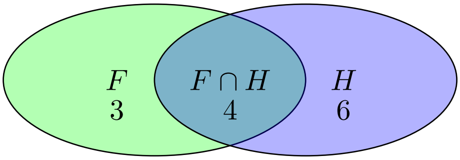
$$\begin{aligned} P(F \cup H) &= P(F) + P(H) - P(F \cap H) \\ &= \frac{7}{20} + \frac{10}{20} - \frac{4}{20} \\ &= \frac{13}{20} \end{aligned}$$

5.3 Venn Diagrams

The purpose of a Venn diagram is to create a visual representation that illustrates the count of distinct outcomes and shared outcomes. Let's use the example on page [page number] to construct such a diagram. For a class where some students play football, some play handball, and some play both, we can create a Venn diagram as shown below:



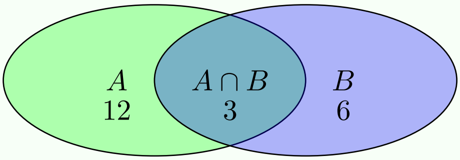
The green ellipse represents those who play football (F), and the blue ellipse represents those who play handball (H). Since some students play both sports ($F \cap H$), we've depicted the ellipses with a slight overlap. We know that 7 students play football, 10 play handball, and 4 of them do both, which is illustrated as follows:



The diagram now shows that 3 students play football only, 6 play handball only, and 4 play both football and handball. So, a total of 7 students play football, and 10 play handball.

Example 1

In a class of 31 students, 15 students take the bus to school, and 9 students take a boat. Out of these, 3 students take both the bus and the boat.
a) Create a Venn diagram that illustrates the given information.
b) If one person is randomly selected from the class, what is the probability that this person takes either the bus or the boat to school?
Solution:
a) Since 3 students take both the bus and the boat, there are 15 - 3 = 12 students who take only the bus, and 9 - 3 = 6 students who take only the boat. We can represent this using a Venn diagram:



b) To find the probability of selecting a student who takes either the bus or the boat, we need to calculate the total number of students who take either mode of transportation. From the diagram, we see that there are 8 students who take only the bus, 3 who take only the boat, and 3 who take both. So, there are $8 + 3 + 3 = 14$ students who take either the bus or the boat. Since there are 31 students in total, the probability is:

$$P(\text{Bus or Boat}) = \frac{14}{31}$$

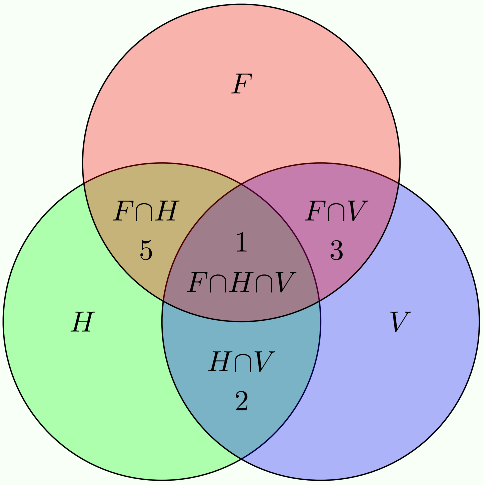
Example 2

In a class of 29 students, we have the following information:

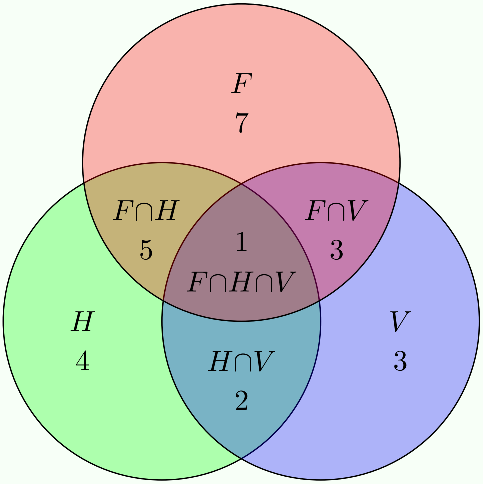
- 16 students play football.
- 12 students play handball.
- 7 students play volleyball.
- 5 students play both football and handball but not volleyball.
- 3 students play both football and volleyball but not handball.
- 2 students play both handball and volleyball but not football.
- 1 student plays all three sports.

a) Create a Venn diagram that describes the distribution of the three sports in the class.
b) One student is randomly selected from the class. What is the probability that this student plays either football, handball, or volleyball?
c) The selected student turns out to play football. What is the chance that this student also plays handball?

Solution:
a) Let F represent 'plays football,' H represent 'plays handball,' and V represent 'plays volleyball.' When creating a Venn diagram, it's a good idea to fill in the shared outcomes first. Based on the fourth to seventh points, we can draw the following:



Next, we see that there are $16-5-1-3=7$ students who play football only, $12-5-1-2=4$ who play handball only, and $9-3-1-2=3$ who play volleyball only:



b) From the diagram, we can see that there are $8 + 5 + 1 + 3 + 4 + 2 + 3 = 26$ unique students who play one or more of

the sports. The probability of selecting one of these 26 students from a class of 29 is $\frac{26}{29}$.
c) Reading from the diagram, out of the total 16 students who play football, there are $5 + 1 = 6$ who also play handball. Therefore, the probability that the selected student, who plays football, also plays handball is $\frac{6}{16} = \frac{3}{8}$.

5.3.1 Cross Table

When it comes to two events, we can also set up a *cross table* to get an overview. Let's say that in a school with 300 students, milk and apples are distributed to the students who want them during lunch. Furthermore, 220 of the students receive milk, while 250 receive apples. Among them, 180 receive both milk and apples. If we let M mean *receives milk* and E mean *receives apples*, our cross table will initially look like this:

	M	$\bar{M}$	Total
E			
$\bar{E}$			
Total			

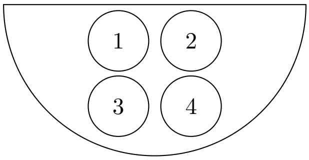
Then we fill in the table based on the information we have:

- Receives *both* milk and apples: $M \cap E = 180$
- Receives milk but not apples: $M \cap \bar{E} = 220 - 180 = 40$
- Receives apples but not milk: $E \cap M = 250 - 180 = 70$
- Receives neither milk nor apples: $\bar{M} \cap \bar{E} = 300 - 180 - 40 - 70 = 10$

	M	$\bar{M}$	Total
E	180	70	250
$\bar{E}$	40	10	50
Total	220	80	300

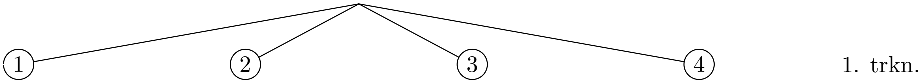
5.4 Repeated Draws

5.4.1 Permutations

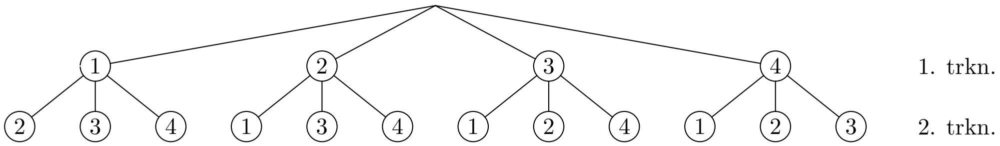


Let's say we have a bowl with four balls numbered from 1 to 4. In an experiment, we draw one ball at a time until we have drawn three balls. For example, if we first draw ball 2, then ball 4, and then ball 3, we get the *permutation* 2 4 3.

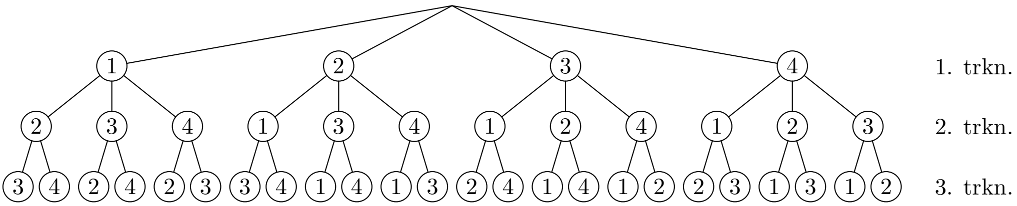
How many different permutations can we get? Let's create a figure to help us find the answer. In the first draw, there are 4 balls to choose from, so we can say that we have 4 paths to take. We can either draw ball 1, ball 2, ball 3, or ball 4:



We remove the ball we draw from the bowl and draw again for the second time. For each of the 4 paths we could take in the first draw, we now have 3 new paths to take. So, we have a total of $3 \cdot 4 = 12$ paths we can take.



We also remove the second ball we draw from the bowl, and for each of the 12 paths from the second draw, we now have 2 new possible paths to take. Therefore, the total number of paths (permutations) is $12 \cdot 2 = 24$.



We could also express this calculation as:

$$4 \cdot 3 \cdot 2 = 24$$

Rule for the product of permutations: When we perform multiple draws in succession, we find all possible permutations by multiplying the number of possible outcomes in each draw.

Example 1: Out of the 29 letters in the alphabet, we want to create a 3-letter word. We accept words that have no meaning, but each letter can only be used once in the word.

How many words can we create?

Solution: First, we have 29 letters to draw from, then 28 letters, and finally 27 letters. Therefore, the number of permutations is given as

$$\underbrace{29}_{\substack{\text{possible outcomes}\\ \text{1st draw}}} \cdot \underbrace{28}_{\substack{\text{possible outcomes}\\ \text{2nd draw}}} \cdot \underbrace{27}_{\substack{\text{possible outcomes}\\ \text{3rd draw}}} = 21\,924$$

So, we can create 21 924 different words.

Example 2: We flip a coin four times in a row. How many permutations do we have?

Solution: Each time we flip a coin, we have two possible outcomes. Therefore, the number of permutations is given as

$$2 \cdot 2 \cdot 2 \cdot 2 = 16$$

Combinations

In everyday language, the word "combinations" is often used instead of "permutations," but in probability theory, combinations and permutations have different meanings. The key difference is that permutations take order into account, while combinations do not.

For example, if we want to form a two-letter word using the letters A, B, and C, and we allow the reuse of letters, then we have 9 possible permutations:

$$AA, AB, AC, BB, BA, BC, CC, CA, CB$$

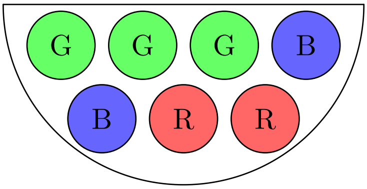
Combinations, on the other hand, refer to a unique combination when order is not considered. For example, both AB and BA are the same combination. In this case, we have 6 combinations:

$$AA, AB, AC, BB, BC, CC$$

I can help you translate the provided Norwegian text to English while preserving LaTeX commands and not translating words starting with a backslash. Here's the translation:
"'latex

5.5 Probability in Repeated Draws

Imagine we have a bowl with seven balls. Three of them are green, two are blue, and two are red. Let's say we first draw one ball from the bowl, and then another. What is the probability of drawing two green balls?



If we let G represent 'drawing a green ball,' we can write this probability as $P(GG)$. To find an answer, we start by determining the number of *favorable* permutations we have. Since there are 3 favorable outcomes in the first draw and 2 in the second draw, we have $3 \cdot 2 = 6$ favorable permutations. We choose from 7 balls in the first draw and 6 balls in the second draw. Therefore, the number of *possible* permutations is $7 \cdot 6 = 42$. So, the probability of getting two green balls is

$$P(GG) = \frac{3 \cdot 2}{7 \cdot 6} = \frac{6}{42} = \frac{1}{7} \tag{5.1}$$

Let's also find the probability of getting a green ball in each draw separately. In the first draw, we have 3 green balls out of a total of 7 balls, so

$$P(G) = \frac{3}{7}$$

In the second draw, it is assumed that a green ball has been picked in the first draw and is therefore out of the bowl. We now have 2 out of 6 balls that are green:

$$P(G|G) = \frac{2}{6}$$

The symbol $|$ means '*given that... has happened.*' $P(G|G)$ is a shorthand for 'the probability of drawing a green ball, *given that* a green ball has been drawn.'

If we multiply the probability from the first draw by the probability from the second draw, the calculation is the same as in equation (5.1):

$$P(GG) = \frac{3}{7} \cdot \frac{2}{6} = \frac{6}{42} = \frac{1}{7}$$

5.7 Probability in Repeated Draws

The probability of event A happening, *given that* event B has occurred, is written as $P(A|B)$.

The probability of events happening in sequence, such as A first, then B, then C, and so on (...), is given by

$$P(ABC...) = P(A) \cdot P(B|A) \cdot P(C|AB) \cdot ...$$

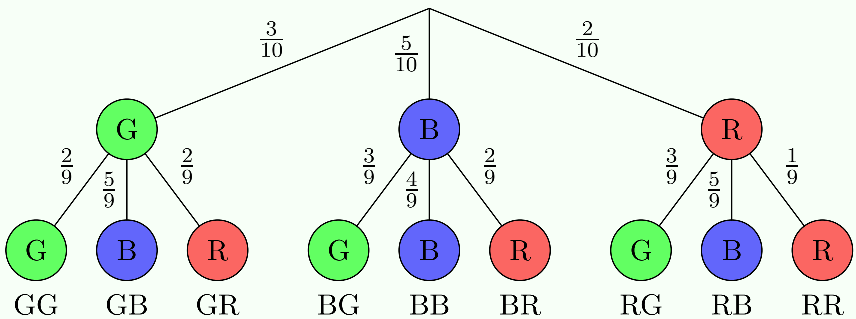
Example

In a bowl, there are 10 balls: three green, two blue, and five red. You draw two balls from the bowl. Let G, B, and R denote 'drawing a blue ball,' 'drawing a green ball,' and 'drawing a red ball,' respectively.

a) Draw a choice tree that outlines the permutations of B, G, and R that you can obtain. **b)** What is the probability that you draw two red balls? **c)** What is the probability that you draw one blue and one green ball? **d)** What is the probability that you draw *at least* one blue *or* one green ball?

Solution:

a)



b) From our choice tree, we see that

$$\begin{aligned} P(RR) &= \frac{2}{10} \cdot \frac{1}{9} \\ &= \frac{2}{90} \\ &= \frac{1}{45} \end{aligned}$$

c) Both permutations GB and BG result in one blue and one green ball. The probability for each of them is

$$\begin{aligned} P(GB) &= \frac{3}{10} \cdot \frac{5}{9} \\ &= \frac{15}{90} \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} P(BG) &= \frac{5}{10} \cdot \frac{3}{9} \\ &= \frac{1}{6} \end{aligned}$$

The probability for GB *or* BG is the sum of $P(GB)$ and $P(BG)$:

$$\begin{aligned} P(GB \cup BG) &= P(GB) + P(BG) \\ &= \frac{1}{6} + \frac{1}{6} \\ &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$

d) To answer this question, we can, of course, add the probability of the permutations $GG, GB, GR, BG, BB, BR, RG$, and RB, but we save a lot of work by noting this: Getting *at least* one blue or one green ball is the opposite of getting *only* red balls. We found the probability of this, getting two red balls, in part b). Using Rule **??**, we have that

$$\begin{aligned} P(\bar{R}) &= 1 - P(R) \\ &= 1 - \frac{1}{45} \\ &= \frac{45}{45} - \frac{1}{45} \\ &= \frac{44}{45} \end{aligned}$$

So, the probability of getting *at least* one blue or one green ball is $\frac{44}{45}$.

Chapter 6

Equations, formulas, and functions

6.1 Finding Quantities

Equations, formulas, and functions are concepts that appear in various contexts but ultimately revolve around the same idea; *they express relationships between quantities*. Most of the boxes in this book contain a formula. For example, Rule **??** contains a formula for 'scale'. When the other quantities are known, it's just a matter of plugging them into the formula to find the 'scale'. We can say that we then find the 'scale' *directly*. If you've worked on problems from the previous chapters, you've already practiced finding quantities *directly*.
In this section, we will look at finding quantities *indirectly*. By that, we mean that at least one of the following applies:

- We need to solve an equation to find the unknown quantity.
- Based on a description of a situation, we need to set up a formula that includes the unknown quantity.

Note

In this section, only examples are provided, and no rules are given. That's because we use rules we've covered in the chapters on equations and functions in MB. The only difference is that here, we are dealing with quantities with units.

Example 1

For a taxi ride, the following costs apply:

- You have to pay 50 kr regardless of how far you are driven.
- In addition, you pay 15 kr for each kilometer you are driven.

a) Set up an expression for how much the taxi ride costs for each kilometer you are driven.

b) What is the cost of a taxi ride for 17 km?

Answer

a) There are two unknown quantities here: 'the cost of the taxi ride' and 'the number of kilometers driven.' The relationship between them is as follows:

$$\text{cost of the taxi ride} = 50 + 15 \cdot \text{number of kilometers driven}$$

b) Now we have:

$$\text{cost of the taxi ride} = 50 + 15 \cdot 17 = 305$$

So, the cost of the taxi ride is 305 kr.

Tips

By letting single letters represent quantities, you can shorten expressions. Let k represent 'cost of the taxi ride' and x represent 'number of kilometers driven.' Then the expression from *Example 1* above becomes:

$$k = 50 + 15x$$

Additionally, you can use the notation for functions:

$$k(x) = 50 + 15x$$

Example 2

Imagine that your class wants to go on a class trip that costs a total of 11,000 kr. To cover the expenses, you have already raised 2,000 kr, and the rest will be obtained through a lottery. For each ticket sold, you earn 25 kr.

a) Create an equation for how many tickets the class must sell to afford the class trip.

b) Solve the equation.

Answer

a) Let's start by thinking about the situation in words:

$$\text{money already raised} + \text{number of tickets} \cdot \text{money per ticket} = \text{cost of the trip}$$

The only quantity we don't know about is 'number of tickets.' We replace[1] 'number of tickets' with x and substitute the values of the other quantities into the equation:

$$2{,}000 + x \cdot 25 = 11{,}000$$

b)

$$
\begin{aligned}
25x &= 11{,}000 - 2{,}000 \\
25x &= 9{,}000 \\
\frac{25x}{25} &= \frac{9{,}000}{25} \\
x &= 360
\end{aligned}
$$

[1] We do this only to make it easier for us to write.

Example 3

A group of friends wants to chip in for a car that costs 50,000 kr, but it's uncertain how many people will participate in the pooling.
a) Call 'number of people participating in the pooling' P and 'expense per person' U, and create a formula for U.
b) Find the expense per person if 20 people participate.

Answer

a) Since the cost of the car will be divided among the number of people participating in the pooling, the formula becomes:

$$U = \frac{50{,}000}{P}$$

b) We substitute P with 20 and find:

$$
\begin{aligned}
U &= \frac{50{,}000}{20} \\
&= 2{,}500
\end{aligned}
$$

So, the expense per person is 2,500 kr.

Example 4

A sports club is planning a trip that requires a bus ride. They receive offers from two bus companies:

- **Bus Company 1**

  The class pays 10,000 kr regardless and 10 kr per kilometer.

- **Bus Company 2**

  The class pays 4,000 kr regardless and 30 kr per kilometer.

For what distance driven do both bus companies offer the same price?

Answer

We introduce the following variables:

- x = number of kilometers driven
- $f(x)$ = price for Bus Company 1
- $g(x)$ = price for Bus Company 2

Then we have:

$$
\begin{aligned}
f(x) &= 10x + 10{,}000 \\
g(x) &= 30x + 4{,}000
\end{aligned}
$$

The bus companies offer the same price when:

$$
\begin{aligned}
f(x) &= g(x) \\
10x + 10{,}000 &= 30x + 6{,}000 \\
4{,}000 &= 20x \\
x &= 200
\end{aligned}
$$

So, the bus companies offer the same price if the sports club plans to drive 200 km.

Example 5

Ohm's Law states that the current I through a metallic conductor (at constant temperature) is given by the formula:

$$I = \frac{U}{R}$$

where U is voltage and R is resistance.

a) Rewrite the formula as a formula for R.

b) If the current is $2\,\mathrm{A}$ and the voltage is $12\,\mathrm{V}$, what is the resistance?

Answer

a) We isolate R on one side of the equation:

$$\begin{aligned}
I \cdot R &= \frac{U \cdot \cancel{R}}{\cancel{R}} \\
I \cdot R &= U \\
\frac{\cancel{I} \cdot R}{\cancel{I}} &= \frac{U}{I} \\
R &= \frac{U}{I}
\end{aligned}$$

b) Using the formula from a):

$$\begin{aligned}
R &= \frac{U}{I} \\
&= \frac{12}{2} \\
&= 6
\end{aligned}$$

So, the resistance is $6\,\Omega$.

Example 6

Given a temperature T_C measured in degrees Celsius (°C). The temperature T_F measured in degrees Fahrenheit (°F) is given by the formula:

$$T_F = \frac{9}{5} \cdot T_C + 32$$

a) Rewrite the formula as a formula for T_C.

b) If a temperature is measured as 59°F, what is the temperature in °C?

Answer

a) We isolate T_C on one side of the equation:

$$\begin{aligned}
T_F &= \frac{9}{5} \cdot T_C + 32 \\
T_F - 32 &= \frac{9}{5} \cdot T_C \\
5(T_F - 32) &= \cancel{5} \cdot \frac{9}{\cancel{5}} \cdot T_C \\
5(T_F - 32) &= 9T_C \\
\frac{5(T_F - 32)}{9} &= \frac{\cancel{9}T_C}{\cancel{9}} \\
\frac{5(T_F - 32)}{9} &= T_C
\end{aligned}$$

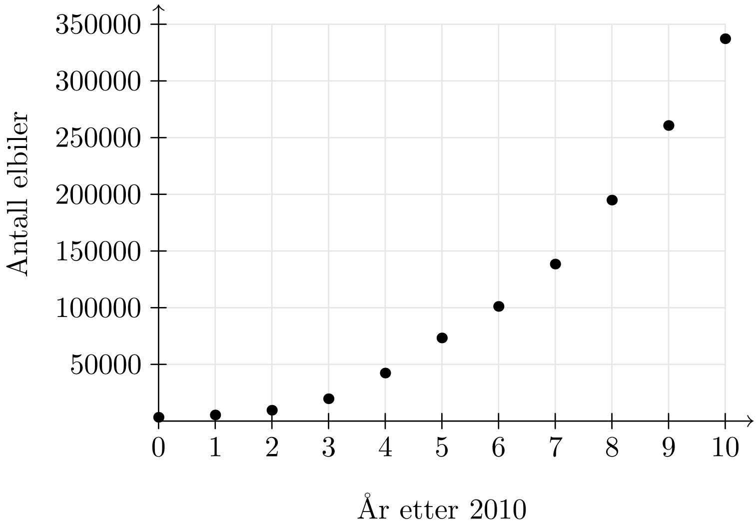
b) Using the formula from a):

$$\begin{aligned}
T_C &= \frac{5(59 - 32)}{9} \\
&= \frac{5(27)}{9} \\
&= 5 \cdot 3 \\
&= 15
\end{aligned}$$

So, 59°**F** is equivalent to 15°**C**.

6.2 Regression

Trying to describe how something will *develop* is one of the most important applications of functions. If we have a dataset that describes past events, we can try to find the function that best fits the dataset. This is called *regression*.
The graph below shows[1] the number of electric cars in Norway after 2010.



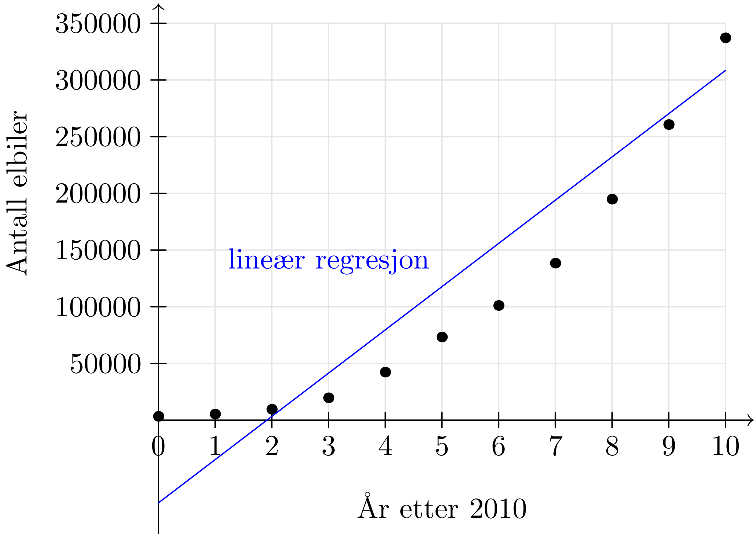
Now we want to find a function that

(i) closely approximates each point,

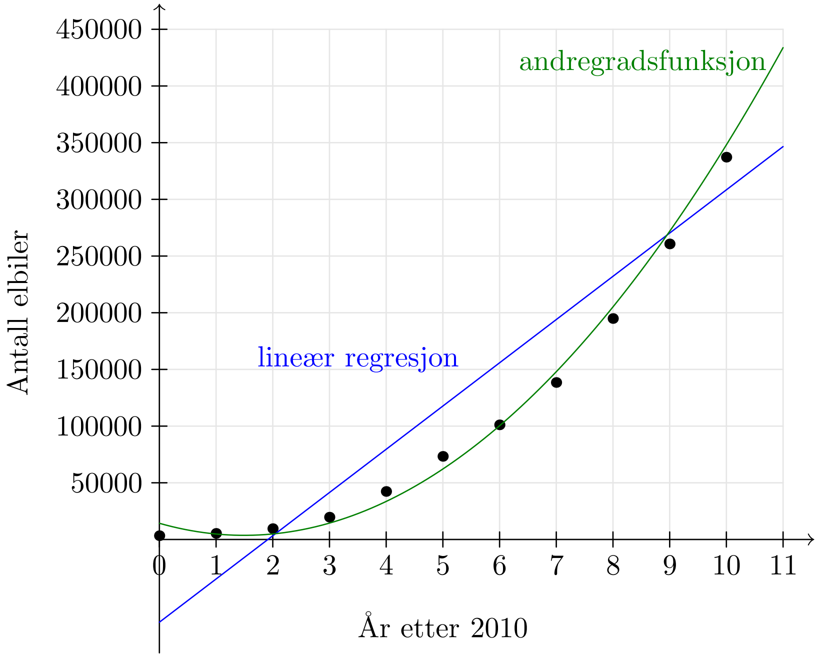
(ii) has a graph that fits the situation we are modeling.

If we perform regression with a linear function in GeoGebra (see page **??**), we get this graph:

[1]Data obtained from elbil.no



If we also perform regression with a quadratic function, we get the following result:

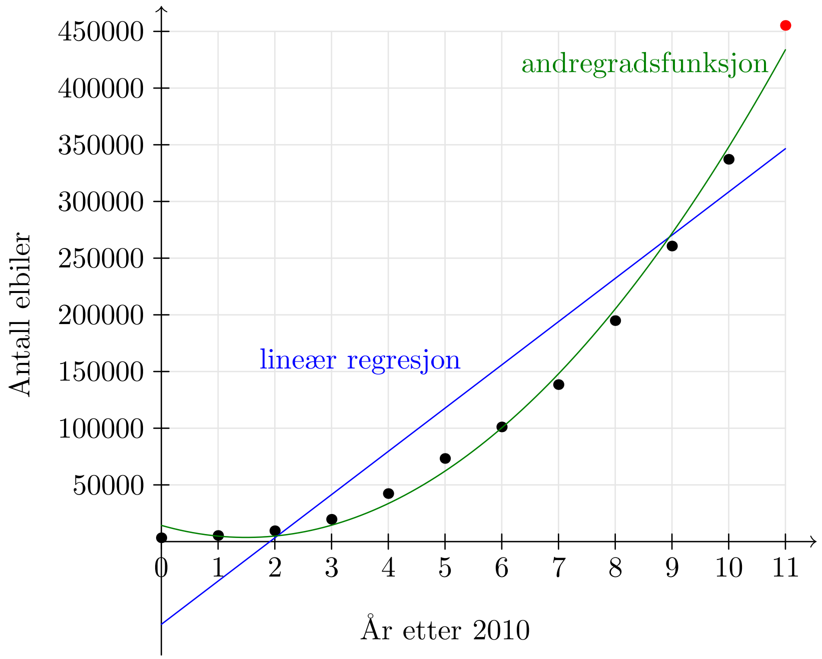


In the figure above, we can note that:

- Both models (functions) behave incorrectly at the beginning. The linear function starts with a negative number of cars, while the quadratic function starts with a decrease in the number of cars from year 0 to year 1.

- The graph of the quadratic function fits the data points much better than the graph of the linear function.

  If we had assumed that the linear function provided a good description of the number of electric cars going forward, we could have read from the graph that the number of electric cars in 2021 was approximately 350,000. If we had assumed the same about the quadratic function, we could have read from the graph that the number of electric cars in 2021 was just over 425,000. The correct answer is that the number of electric cars in 2021 was 455,271.



Chapter 7

Digital tools

A crucial part of mastering digital tools is understanding the basics of **programming**. Programming is about giving instructions to a computer. In this manner, computers can perform calculations, produce images, animations, games, and much more. To give instructions, we use different **programming languages**, and there's a myriad of languages to choose from. In Norwegian schools, the most commonly used languages are Scratch, Python, and JavaScript[2]. There's a vast array of free resources for learning programming languages, including:

- code.org (general coding)
- w3schools.com (general coding)
- scratch.mit.edu (Scratch)
- microbit.org (coding with micro:bit)
- espensklasserom.co (Coding in Scratch, micro:bit, etc.)
- kidsakoder.no (coding in Scratch, micro:bit, Python, etc.)

Have you already reached a high level as a programmer and feel confident with data types, functions, classes, etc.? Then, the language Rust is recommended. Many consider this to be the successor to C++ and similar languages.

[2]Admittedly in a block-based version when coding with micro:bit.

7.1 Introduction to Python

Python is a programming language for **text-based coding**. This means that the actions we want to be executed must be coded as text. The file containing all the code is referred to as a **script**. The visible result of running the script is termed **output**[1]. There are various ways to run one's script; for example, one can use an online compiler like programmiz.com.

7.1.1 Object, Type, Function, and Expression

Our first script consists of just one line of code:

```python
print("Hello world!")
```

Utdata

```
Hello world!
```

In the upcoming sections, the terms **object**, **type**, **function**, and **expression** will frequently be discussed.

- Almost everything in Python is objects. In the above script, both `print()` and `"Hello world"` are objects.
- Objects come in different types. `print()` is of the `function` type, while `"Hello world"` is of the `str` type[2]. The operations that can be executed with various objects depend on their types.
- Functions can accept **arguments** and then perform operations. In the script above, the `print()` function takes the argument `"Hello world"` and displays the text as output.
- Expressions have strong similarities with functions, but they don't accept arguments.

[1] **Output** in English.

[2] 'str' is an abbreviation for the English word 'string'.

Assignment and Calculation

Text and numbers can be seen as some of the smallest building blocks (objects). Python has one type for text and two types for real numbers:

`str`	text
`int`	integer
`float`	decimal

It is usually useful to give our objects names. We do this by writing the name followed by `=` and the object. **Comments** are text that is not treated as code. We can write comments by starting the sentence with `#`.

```
hei = "hei" # hei er av typen str. Legg merke til "
              ved start og slutt
a = 3 # a er av typen int
b = 2.8 # b er av typen float
c = 2. # c=2.0, og er av typen float
d = .7 # d=0.7, og er av typen float
e = -5 # e er av typen int
f = -0.01 # f er av typen float
```

With Python, we can of course perform classic arithmetic operations:

```python
a = 5
b = 2

print("a+b = ", a+b);
print("a-b = ", a-b);
print ("a*b = " ,a*b);
print ("a/b = " ,a/b);
print("a**b = " ,a**b); # potens med grunntall a og eksponent b
print ("a//b = ", a//b); # a/b rundet ned til nærmeste heltall
print ("a%b = ", a%b); # resten til a//b
```

Utdata

```
a+b = 7
a-b = 3
a*b = 10
a/b = 2.5
a**b = 25
a//b = 2
a%b = 1
```

The functions `str()`, `int()` and `float()` can be used to convert objects to types `int` or `float`:

```
s = "2"
b = 3
c = 2.0

b_s = str(b) # b omgjort til str
c_s = str(c) # c omgjort til str
print(b_s+c_s)

s_i = int(s)
print(s_i*b)

s_f = float(s)
print(s_f*b)
```

Utdata

```
32.0
6
6.0
```

One important thing to be aware of is that `=` in Python *does not* mean the same as $=$ in mathematics. While $=$ can be translated to ”equals”, we can say that `=` can be translated to ’is assigned to’.

```
a = 5 # a ER nå 5
print(a)
a = a+1 # a ER nå det a VAR, + 1
print(a)
```

Utdata

```
5
6
```

For an object to add itself and another value is so common in programming that Python has its own operator for it:

```
a = 5 # a ER nå 5
a+= 1 # Samme som å skrive a = a+1
print(a)
```

Utdata

```
5
6
```

Although computers are extremely fast at performing calculations, they have a limitation that is important to be aware of: rounding errors. One reason for this is that computers can only use a certain number of decimals to represent numbers. Another reason is that computers use the binary system. There are many values that we can write exactly in the decimal system that cannot be written exactly in the binary system. To address this, we can use the `round()` function:

```
a = 8.3/10
print(a) # avrundingsfeil, da vi skulle hatt a = 0.83

a = round(a, 2) # runder av a til tall med to
    desimaler
print(a)
```

Utdata

```
0.8300000000000001
0.83
```

7.1.2 Custom Functions

Using the method `def`, you can create your own functions. A function can perform actions, and it can **return** one or more objects. It can also accept arguments. The code we write inside a function is only executed if we **call** the function.

```
# a er en funksjon som ikke tar noen argumenter.
# Legg merke til 'def' først og ':' til slutt.
# Kodelinjene som hører til funksjonen må stå med innrykk
def a():
  print("Hei, noen kalte visst på funksjon a?")


# b er en funksjon som tar argumentet 'test'
def b(tekst):
  print("Hei. Noen kalte på funksjon b. Argumentet som ble gitt var: ", tekst)

# c er er funksjon som tar argmunentene a og b
# c returnerer et objekt
def c(a, b):
  return a+b

b("Hello!") # vi kaller på b med argumentet "hello"

d = c(2,3) # Vi kaller på a med argumentene 2 og 3

print(d)

# merk at teksten gitt i a ikke blir printet, fordi vi ikke har kalt på a.
```

Utdata

```
Hi. Someone called function b. The argument given was:
Hello!
5
```

7.1.3 Boolean Values and Conditions

The values `True` and `False` are called **boolean values**. These will be the result when we check if objects are equal or different. To check this, we have the **comparative operators**:

operator	meaning
`==`	is equal to
`!=`	is *not* equal to
`>`	is greater than
`>=`	is greater than, or equal to
`<`	is less than
`<=`	is less than, or equal to

```python
a = 5
b = 4

print(a == b)
print(a != b)
print(a > b)
print(a < b)
```

Utdata

```
False
True
True
False
```

In addition to the comparative operators, we can use the **logical operators** `and`, `or`, and `not`.

```python
a = 5
b = 4
c = 9

print(a == b and c > a)
print(a == b or c > a)
print(not a == b)
```

Utdata

```
False
True
True
```

> **The language box**
>
> Checks that use both comparative and logical operators will henceforth be called **conditions**.

7.1.4 Expressions `if`, `else`, and `elif`

When we want to perform actions only *if* a condition is true (`True`), we use the expression `if` in front of the condition. The code we write indented under the `if` line will only be executed if the condition evaluates to `True`.

```python
a = 5
b = 4
c = 9

if c > b: # legg merke til kolon (:) til slutt
  print("Jepp, c er større enn b")

if a > c: # legg merke til kolon (:) til slutt
  print("Denne teksten kommer ikke i output, siden vilkåret er False")
```

Utdata

```
Yep, c is greater than b
```

If you first want to check if a condition is true, and then perform actions if it's *not*, you can use the expression `else`:

```python
a = 5
c = 9

if a > c: # legg merke til kolon (:) til slutt
  print("Denne teksten kommer ikke i output, siden vilkåret er False")

else: # legg merke til kolon (:) til slutt
  print("Men denne kommer, fordi vilkåret i if-linja over var False")
```

Utdata

```
But this comes because the condition in the if-line above was False
```

The expression `else` only considers (and doesn't make sense without) the `if` expression right above it. If we want actions to be performed *only* if no previous `if` expressions produced any result, we must use[1] the expression `elif`. This is an `if` expression that takes effect if the `if` expression above did *not* take effect.

```python
a = 2

if a > 3:
  print("Denne linja printes ikke, vilkåret er False")

elif a < 1: #Siden if uttrykket over ikke ga utslag, sjekkes vilkåret b < 1
  print("Denne linja printes ikke, vilkåret er False")

else:
  print("Nå er vi sikre på at 1 < b < 3")
```

Utdata

```
Now we are sure that 1 < b < 3
```

Note

When working with numbers, some conditions you expect to be `True` might turn out to be `False`. This often deals with rounding errors, as mentioned on page 118.

[1] `elif` is a shortcut for `else if`, which can also be used.

7.1.5 Lists

Lists can be used to collect objects. The objects in the list are called the **elements** of the list.

```
strings = ["98", "99", "100"]
floats = [1.7, 1.2]
ints = [96, 97, 98, 99, 100]
mixed = [1.7, 96, "100"]
empty = []
```

The elements in lists are **indexed**. The first object has index 0, the second object has index 1, and so on:

```
strings = ["98", "99", "100"]
floats = [1.7, 1.2]
ints = [96, 97, 98, 99, 100]
mixed = [1.7, 96, "100"]
empty = []
```

Utdata

```
96
99
98
```

Using the built-in function `append()` we can add an object to the end of the list. This is an **in-built function**[1], which we write at the end of the list name, preceded by a dot.

```
min_liste = []
print(min_liste)

min_liste.append(3)
print(min_liste)

min_liste.append(7)
print(min_liste)
```

Utdata

```
[]
[3]
[3, 7]
```

[1] In short, it means that only certain types of objects can use this function.

With the `pop()` function, we can retrieve an object from the list.

```python
min_liste = [6, 10, 15, 19]

a = min_liste.pop() # a = det siste elementet i listen
print("a =",a)
print("min_liste =",min_liste)

a = min_liste.pop(1) # a = elementet med indeks 1
print("a =",a)
print("min_liste =",min_liste)
```

Utdata

```
a = 19
min_liste = [6, 10, 15]
a = 10
min_liste = [6, 15]
```

Explain to yourself

What's the difference between writing `a = min_liste[1]` and `a = min_liste.pop(1)`?

With the `sort()` function, we can sort the elements in the list.

```python
heltall =  [9, 0, 8, 3, 1, 7, 4]
bokstaver = ['c', 'a', 'b', 'e', 'd']

heltall.sort()
bokstaver.sort()

print(heltall)
print(bokstaver)
```

Utdata

```
[0, 1, 3, 4, 7, 8, 9]
['a', 'b', 'c', 'd', 'e']
```

With the `count()` function, we can count repeated elements in the list.

```python
heltall =  [2, 7, 2, 2, 2]
frukt = ['banan', 'eple', 'banan']

antall_toere = heltall.count(2)
antall_sjuere = heltall.count(7)
antall_bananer = frukt.count('banan')
antall_appelsiner = frukt.count('appelsin')

print(antall_toere)
print(antall_sjuere)
print(antall_bananer)
print(antall_appelsiner)
```

Utdata

```
4
1
2
0
```

With the `len()` function, we can find the number of elements in a list, and with the `sum()` function, we can find the sum of lists with numbers as elements.

```python
heltall =  [2, 7, 2, 2, 2]
frukt = ['banan', 'eple', 'banan']

print(len(heltall))
print(len(frukt))
print(sum(heltall))
```

Utdata

```
5
3
15
```

With the `in` expression, we can check if an element is in a list.

```python
heltall = [1, 2, 3]

print(1 in heltall)
print(0 in heltall)
```

Utdata

```
True
False
```

7.1.6 Loops; `for` and `while`

`for` loop

For objects containing multiple elements, we can use `for` loops to perform actions for each element. The actions must be written with an indentation after the `for` statement:

```python
min_liste = [5, 10, 15]

for number in min_liste:
  print(number)
  print(number*10)
  print("\n") # lager et blankt mellomrom

```

Utdata

```
5
50

10
100

15
150
```

The language box

Going through each element in (for example) a list is called "iterating over the list".

Often, it's desired to iterate over the integers $0, 1, 2$ and so forth. For this, we can use `range()`:

```python
ints = range(3)

for i in ints:
  print(i)

```

Utdata

```
0
1
2
```

`while` loop

If we want actions to be performed until a condition is met, we can use a `while` loop:

```
a = 1

while a < 5:
  print(a)
  a += 1
```

Utdata

```
1
2
3
4
```

7.1.7 `input()`

We can use the `input()` function to enter text while the script is running:

```
1 innskrevet_tekst = input("Skriv inn her: ")
2 print(innskrevet_tekst)
```

The text written inside `input()` in the script above is the text we want displayed before the text to be entered. Line 2 of this code will not execute until text is entered.

```
1 innskrevet_tekst = input("Skriv inn her: ")
2 print(innskrevet_tekst)
```

Utdata

```
Enter text here: OK
OK
```

The object provided by an `input()` function will always be of type `str`. One must always ensure to convert objects to the correct type:

```
print("La oss regne ut a*b")
a_str = input("a = ")
b_str = input("b = ")
a = float(a_str)
b = float(b_str)
print("a*b = ", a*b)
```

Utdata

```
Let's calculate a*b
a = 3.7
b = 4
a*b = 14.8
```

7.1.8 Error Messages

Claim: All programmers will experience that the script does not run because we haven't written the code correctly. This is called a **syntax error**. With a syntax error, you will be informed about which line the error is on and what the error is. The most common errors are:

- **Forgetting indentation when using methods like `def`, `for`, `while`, and `if`**

```
a = 472
b = 98

if a*b > 48000:
print("a*b er større enn 48000")
```

Utdata

```
line 5, in <module>
print("a*b is greater than 48000")
^
IndentationError: expected an indented block after
'if' statement on line 4
```

- **Performing operations on types where it doesn't make sense**

```python
b = "98"
b_opphøyd_i_andre = b**2
```

Utdata

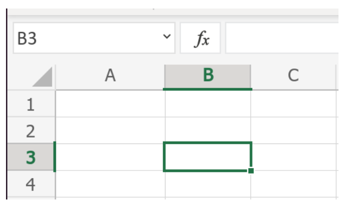
```
line 2, in <module>
b_raised_to_second = b**2
TypeError: unsupported operand type(s) for ** or
pow(): 'str' and 'int'
```

7.2 Spreadsheets

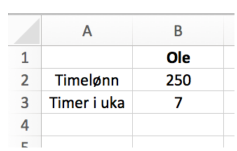
In this book, we base our discussions on Microsoft's software, Excel. There are other good spreadsheets available in the market, such as Google Sheets and Libre Office Calc. These three mentioned spreadsheets are very similar in design and in the functions they offer.

7.2.1 Introduction

When you open a spreadsheet, you'll see a table where *rows* are numbered with numbers (1, 2, 3 etc.), while *columns* are indexed with letters (A, B, C etc.). How the rows and columns are used is crucial for understanding Excel. In the figure below, we have highlighted what we call *cell B3*. This is the cell where *row 3 and column B intersect*. (Also, notice that B3 is highlighted in the top left of the figure).



In each cell, we can input both numbers and text. Let's say Ole has a job with an hourly wage of 250 kr, and he works 7 hours a week. We can input this information into Excel as shown:



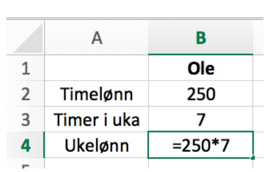
	A	B
1		**Ole**
2	Timelønn	250
3	Timer i uka	7
4		

7.2.2 Calculations

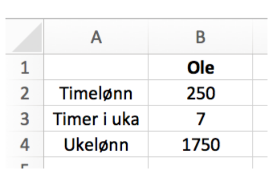
Now, we want to determine Ole's weekly wage. The weekly wage is given by the formula

$$\text{weekly wage} = \text{hourly wage} \cdot \text{hours per week}$$

To perform a calculation in a spreadsheet, you start by typing = in the cell. In cell B4, we find Ole's weekly wage by typing =250*7.

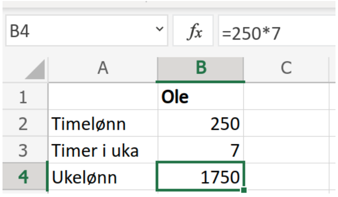


	A	B
1		**Ole**
2	Timelønn	250
3	Timer i uka	7
4	Ukelønn	=250*7



	A	B
1		**Ole**
2	Timelønn	250
3	Timer i uka	7
4	Ukelønn	1750

When we press the enter key, the result, 1750, is displayed in the cell. If you want to see the formula we used, you can double click on the cell, or look at the *input field* (top right in the figure below).

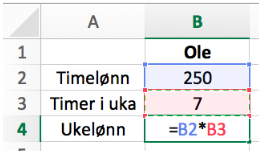


	A	B	C
1		**Ole**	
2	Timelønn	250	
3	Timer i uka	7	
4	Ukelønn	1750	

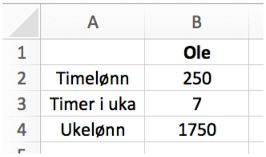
Note: The input field can also be used to input numbers and text in the cell.

7.2.3 Cell References

Perhaps Excel's most essential feature is *cell references*. In short, this means that we use cells instead of numbers when making calculations. In the previous section, we calculated Ole's wage by multiplying 250 (hourly wage) by 7 (hours per week). By using cell references, we could have done this instead:
The number corresponding to the hourly wage (250) is in cell B2, while the number corresponding to hours (35) is in cell B3. To multiply the numbers in these cells, we can type `=B2*B3`:

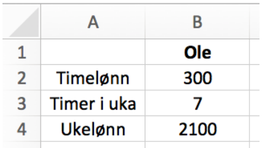


	A	B
1		**Ole**
2	Timelønn	250
3	Timer i uka	7
4	Ukelønn	=B2*B3



	A	B
1		**Ole**
2	Timelønn	250
3	Timer i uka	7
4	Ukelønn	1750

One advantage of using cell references is that it becomes much easier to correct mistakes that have been made. Say, for instance, it should have been 300 instead of 250 in B3. If we change B3, the result in B4 will adjust accordingly:



	A	B
1		**Ole**
2	Timelønn	300
3	Timer i uka	7
4	Ukelønn	2100

Note: You can also click on the cells you want to use in your formulas, as shown here.

7.2.4 Copying and Locking Cells

Copying cells is a method that prevents you from writing the same formulas over and over again. We now want to create a sheet suitable for the following information:

- The hourly wages for Ole, Dole, and Doffen are 300 kr, 200 kr, and 500 kr, respectively.
- All three work 7 days a week.
- We want to calculate the total hours they work and their total weekly wages.

We start by setting up this spreadsheet:

	A	B	C	D
1		**Ole**	**Dole**	**Doffen**
2	Timelønn	300	200	500
3	Timer i uka			
4	Ukelønn			

Here, we have only filled in the information that is *unique* for Ole, Dole, and Doffen, precisely because the other cells either contain the same numbers or use the same calculation method. For cells that are not unique, we should use the copying features, as shown in this video. Here is a brief description of what is done:

1. Since all three work 7 hours, we type 7 in cell B4. Then, we copy by clicking the mouse pointer right at the bottom right corner of B4 and drag *across* to C2 and D2.
2. Since the calculation method for weekly wages is the same for all three, we type it (with cell references) in B4, and copy it *across* to cell C4 and D4.
3. The calculation method for the sum of hours and the sum of weekly wages is also the same; hence, we type it in cell E3 and copy it *down* to E4.

The result is:

	A	B	C	D	E
1		**Ole**	**Dole**	**Doffen**	
2	Timelønn	300	200	500	**Sum**
3	Timer i uka	7	7	7	21
4	Ukelønn	2100	1400	3500	7000

	A	B	C	D	E
1		Ole	Dole	Doffen	
2	Timelønn	300	200	500	Sum
3	Timer i uka	=7	=7	=7	=B3+C3+D3
4	Ukelønn	=B2*B3	=C2*C3	=D2*D3	=B4+C4+D4

From what we've seen in the video and the figures above, we can take away two general rules:

1. Every time you copy a formula one cell *across*, the columns in the formula will increase by one letter in the alphabet. (A becomes B, B becomes C, etc.)

2. Every time you copy a formula one cell *down*, the rows in the formula will increase by 1 (1 becomes 2, 2 becomes 3, etc.).

Locking of cells

When copying cells, it's essential to watch out for cells you want to use in all copies, because these cells must be *locked*. Let's say for instance that Huey, Dewey, and Louie all work 48 weeks a year. To find their annual salary, we must multiply the weekly wage of each of them by 48.

Again, we note that the calculation method to find the annual salary is the same for all three. However, if we use cell B8 in a formula and copy as we have done so far, the letter B will change in the formulas. To avoid this, we write `$` in front of `B` in the formula – this ensures that the column letter doesn't change, even if we copy the formula. This is shown in this video, and the result looks like this:

	A	B	C	D	E
1	Arbeidsuker	48			
2					
3		Ole	Dole	Doffen	
4	Timelønn	300	200	500	Sum
5	Timer i uka	7	7	7	21
6	Ukelønn	2100	1400	3500	7000
7	Årslønn	100800	67200	168000	

	A	B	C	D	E
1	Arbeidsuker	48			
2					
3		Ole	Dole	Doffen	
4	Timelønn	300	200	500	Sum
5	Timer i uka	=7	=7	=7	=B5+C5+D5
6	Ukelønn	=B4*B5	=C4*C5	=D4*D5	=B6+C6+D6
7	Årslønn	=$B1*B6	=$B1*C6	=$B1*D6	

To lock a cell *downwards*, we must place the dollar sign in front of the row number, for instance `B$1`.

7.2.5 Other useful functions

Videos

- Sum across and sum down
- Adjust column width
- Insert row
- Formula view
- Convert to percentage
- Change the number of decimals
- Sort in ascending/descending order
- Create a bar chart
- Create a pie chart
- Create a line chart

Commands (written with `=` in front).

- `SUM(cell1:cell2)`

  Sums all the values from cell1 to cell2.
- `AVERAGE(cell1:cell2)`

  Finds the average for all values from cell1 to cell2.
- `MEDIAN(cell1:cell2)`

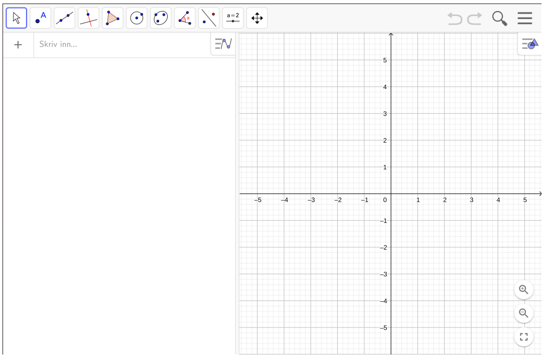
  Finds the median for all values from cell1 to cell2.
- `VAR.P(cell1:cell2)`

  Finds the variance for all values from cell1 to cell2.

7.3 GeoGebra

7.3.1 Introduction

When you open GeoGebra, you see a picture like this:



The field labeled "Skriv inn" is called the *input field*. This field, along with the blank field below, constitutes the *algebra field*. The coordinate system on the right is called the *graphics field*.

7.3.2 Entering points, functions, and lines

Points

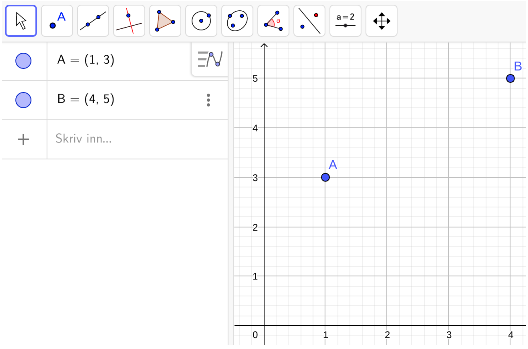
Suppose we want the points (1,3) and (4,5) to appear in the graphics field. In the input field, we then write

```
(1,3)
```

and

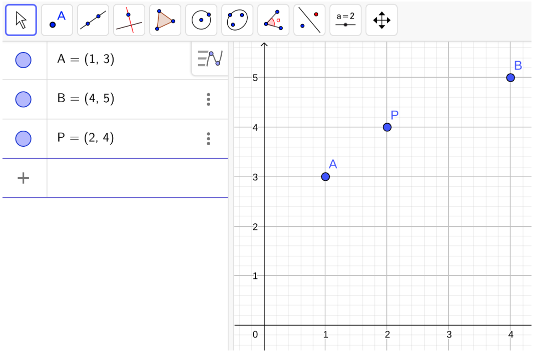
```
(4,5)
```

GeoGebra then names the points A and B, and plots them in the graphics field:



If we want to set a point's name ourselves, we can write, for instance,

```
P=(2,4)
```



Functions

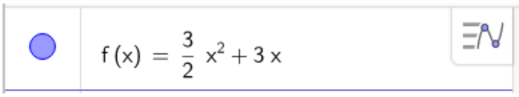
Suppose we have the function

$$f(x) = \frac{3}{2}x^2 + 3x$$

To use $f(x)$ in GeoGebra, we write:

```
3/2*x^2+3x
```

When we do not give the function a name, GeoGebra will automatically name the function f. In the algebra field, we therefore get



In the graphics field, we get the graph of f.

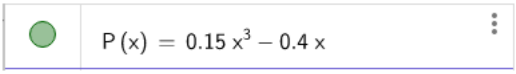
If instead we have the function

$$P(x) = 0.15x^3 - 0.4x$$

there are two things to be aware of. The first is that *all decimals must be written using a period instead of a comma* in GeoGebra. The second is that we want to give the function the name $P(x)$. We then write

```
P(x) = 0.15x^3 - 0.4x
```

and get



Note!

You can never name functions $y(x)$ in GeoGebra. y can only be used when entering expressions for a straight line, i.e., $y = ax + b$, where a and b are two arbitrary numbers.

Horizontal and Vertical Lines

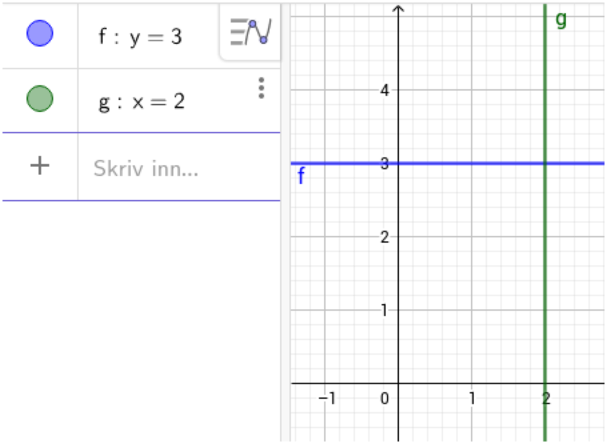
If we want to create a line that runs horizontally through the value 3 on the y-axis and a line that runs vertically through the value 2 on the x-axis, we write:

```
y = 3
```

and

```
x = 2
```

This gives us the following figure:



7.3.3 Finding the Value of Functions and Lines

Functions

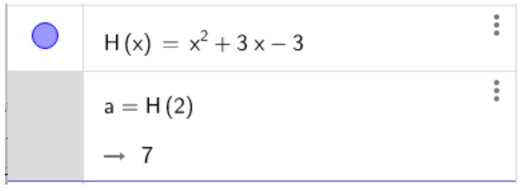
Suppose we have the function

$$H(x) = x^2 + 3x - 3$$

If we want to know what $H(2)$ is, we write:

```
H(2)
```

which results in this:



From this, we know that $H(2) = 7$.

Lines

It is strongly recommended that you use function expressions when dealing with lines in GeoGebra, but in some cases, you cannot avoid lines in the form $y = ax + b$.

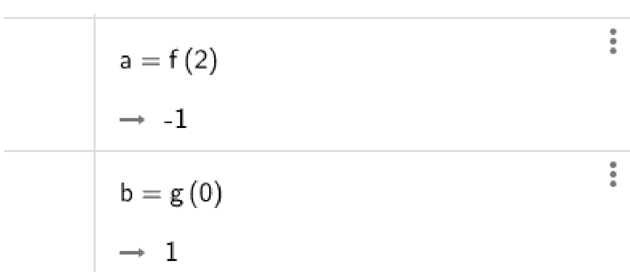
Consider the two lines

$$y = x - 3$$

$$y = -2x + 1$$

We enter these into GeoGebra, and get:

If we now want to find out what the value of $y = x - 3$ is when $x = 2$, we need to note that GeoGebra has named this line f. The answer we are looking for is then obtained by writing $f(2)$. If we also want to know what $y = -2x + 1$ is when $x = 0$, we write $g(0)$:



7.3.4 Buttons and Commands

Graphics Field

Buttons are selected from dropdown menus on the toolbar. The numbering of the menus is from the left.

Creates a new point. (Menu no. 1)

 Creates a line between two points. (Menu no. 2)

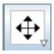
 Finds the maximum and minimum points of a function. (Menu no. 2)

 Finds the zeros of a function. (Menu no. 2)

 Finds the intersection point between two objects. (Menu no. 3)

 Creates the vector between two points (Menu no. 3)

Creates a text box. (Menu no. 10)

 Moves the graphics field. Changes the value distance if pointing at the axes. (Menu no. 10)

Shortcut Keys

	Description	PC	Mac
$\sqrt{}$	square root	`alt + r`	`alt + r`
π	pi	`alt + p`	`alt + p`
∞	infinity	`alt + u`	`alt + ,`
$\otimes$	cross product	`alt + shift + 8`	`ctrl + shift + 8`
e	euler's number	`alt + e`	`alt + e`
$^{\circ}$	degree symbol ($\frac{\pi}{180}$)	`alt + o`	`alt + o`

Videos

- Find the zeros of a graph
- Find the local minimum (or maximum) of a graph
- Find the intersection points of two functions
- Adjust axes
- Change thickness, color, etc. on graph
- Draw graph on a given interval
  In the video, we draw $f(x) = x^2 - 3x + 2$ on the interval $0 \leq x \leq 5$.
- Draw a line between two points
  Notice what is done towards the end of the video to get the familiar expression $y = ax + b$.
- Perform regression
  In the video, we have previously entered the numbers in the table below, which shows the electric car sales in Norway the number of years after 2010. These numbers were also used in section 6.2.

  Regression is performed with a line, a quadratic function, and a 4th-degree function.

number of years	electric cars
0	3347
1	5381
2	9565
3	19678
4	42356
5	73312
6	101126
7	138477
8	194900
9	260688
10	337201
11	455271

Command List

Note: Many of the commands have their own buttons, as shown in the videos above.

- abs(<x>)
  Gives the length of x (a number, a line segment, etc.). Alternatively, you can write |x|.

- Line(<Point>, <Point>)
  Gives the line between two points.

- ExtremalPoint(<Function>, <Start>, <End>)
  Finds local maxima and minima for a function over a specified interval.

- Function(<Function>, <Start>, <End>)
  Draws a function within a specified interval.

- Polygon(<Point>, ..., <Point>)
  Draws the polygon between the given points.

- Zeroes(<Function>, <Start>, <End>)
  Provides the zeroes of a function within a specified interval.

- LinReg(<List>)
  Uses linear regression to fit points given in a list.

- ExpReg(<List>)
  Uses regression with an exponential function to fit points given in a list.

- PolyReg(<List>, <Degree>)
  Uses regression with a polynomial of a given degree to fit points provided in a list.

- PotReg(<List>)
  Uses regression with a power function to fit points given in a list.

- Intersection(<Object>, <Object>)
  Finds the intersection points of two objects (functions, lines, etc.)

Appendix

Appendix A: Functions

7.1 Domain

The domain of a function $f(x)$ is the x-values for which $f(x)$ is valid.

7.2 Range

The range of a function $f(x)$ is all the values $f(x)$ can take. The range is determined by the function's expression and its domain.

7.3 Proportional Quantities

Given a constant a and two variables x and y. If

$$a = \frac{x}{y}$$

then x and y are **proportional** quantities.

7.4 Inversely Proportional Quantities

Given a constant a and two variables x and y. If

$$a = xy$$

then x and y are **inversely proportional** quantities.

7.5 Polynomial Functions

A polynomial function is a function consisting of a sum of powers with positive exponents and a variable as a base.

Polynomial functions have sub-titles determined by the highest exponent in the function expression. For constants a, b, c, and d, and a variable x, we have

function expression	function name
$ax + b$	1st degree function (linear)
$ax^2 + bx + c$	2nd degree function (quadratic)
$ax^3 + bx^2 + cx + d$	3rd degree function (cubic)

The language box

The validity domain of a function states the x-values for which the function makes sense in the practical situation it's used in.