0.1 Introduction

Simply said, **algebra** is mathematics where letters represent numbers. This makes it easier working with *general* cases. For example, $3 \cdot 2 = 2 \cdot 3$ and $6 \cdot 7 = 7 \cdot 6$ but these are only two of the infinitely many examples of the commutative property of multiplication! One of the aims of algebra is giving *one* example that explains *all* cases, and since our digits (0-9) are inevitably connected to specific numbers, we apply letters to reach this target.

The value of the numbers represented by letters will often vary, in that case we call the letter-numbers *variables*. If letter-numbers on the other hand have a specific value, they are called *constants*.

In Part **??**, we studied calculations through examples with specific numbers, however, most of these rules are *general*; they are valid for all numbers. On page 1 - 4, many of these rules are reproduced in a general form. A good way of getting acquainted with algebra is comparing the rules here presented by the way they are expressed in[1] Part **??**.

0.1 Addition is commutative (??)

$$a + b = b + a$$

Example

$$7 + 5 = 5 + 7$$

0.2 Multiplication is commutative (??)

$$a \cdot b = b \cdot a$$

Example 1

$$9 \cdot 8 = 8 \cdot 9$$

Example 2

$$8 \cdot a = a \cdot 8$$

[1] The number of the rules as found in Part **??** are written inside parentheses.

Multiplication involving letters

When multiplication involves letters, it is common to omit the symbol of multiplication. If a specific number and a letter are multiplied together, the specific number is written first. For example,

$$a \cdot b = ab$$

and

$$a \cdot 8 = 8a$$

We also write

$$1 \cdot a = a$$

In addition, it is common to omit the symbol of multiplication when an expression with parentheses is involved:

$$3 \cdot (a + b) = 3(a + b)$$

0.3 Fractions as rewriting of division (??)

$$a : b = \frac{a}{b}$$

Example

$$a : 2 = \frac{a}{2}$$

0.4 Fractions multiplied by fractions (??)

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example 1

$$\frac{2}{11} \cdot \frac{13}{21} = \frac{2 \cdot 13}{11 \cdot 21} = \frac{26}{231}$$

Example 2

$$\frac{3}{b} \cdot \frac{a}{7} = \frac{3a}{7b}$$

0.5 Division by fractions (??)

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Example 1

$$\frac{1}{2} : \frac{5}{7} = \frac{1}{2} \cdot \frac{7}{5}$$

Example 2

$$\begin{aligned} \frac{a}{13} : \frac{b}{3} &= \frac{a}{13} \cdot \frac{3}{b} \\ &= \frac{3a}{13b} \end{aligned}$$

0.6 Distributive law (??)

$$(a+b)c = ac + bc$$

Example 1

$$(2+a)b = 2b + ab$$

Example 2

$$a(5b-3) = 5ab - 3a$$

0.7 Multiplication by negative numbers I (??)

$$a \cdot (-b) = -(a \cdot b)$$

Example 1

$$\begin{aligned} 3 \cdot (-4) &= -(3 \cdot 4) \\ &= -12 \end{aligned}$$

Example 2

$$\begin{aligned}(-a) \cdot 7 &= -(a \cdot 7)\\ &= -7a\end{aligned}$$

0.8 Multiplication by negative numbers II (??)

$$(-a) \cdot (-b) = a \cdot b$$

Example 1

$$\begin{aligned}(-2) \cdot (-8) &= 2 \cdot 8\\ &= 16\end{aligned}$$

Example 2

$$(-a) \cdot (-15) = 15a$$

Extensions of the rules

One of the strengths of algebra is that we can express compact rules which are easily extended to apply for other cases. Let's, as an example, find another expression of

$$(a + b + c)d$$

Rule 0.6 does not directly imply how to calculate between the expression inside the parentheses and d, but there is no wrong-doing in defining $a + b$ as k:

$$a + b = k$$

Then

$$(a + b + c)d = (k + c)d$$

Now, by Rule 0.6, we have

$$(k + c)d = kd + cd$$

Inserting the expression for k, we have

$$kd + cd = (a + b)d + cd$$

By applying Rule 0.6 once more we can write

$$(a + b)d + cd = ad + bc + cd$$

Then

$$(a + b + c)d = ad + bc + cd$$

Notice! This example is not meant to show how to handle expressions not directly covered by Rule 0.1 - Rule 0.8, but to emphasize why it's always sufficient to write rules with the least amount of terms, factors etc. Usually you apply extension of the rules without even thinking about it, and surely not in such meticulous manner as here provided.

0.2 Powers

$$\text{base} \longrightarrow 2^{3} \longleftarrow \text{exponent}$$

A power is composed by a *base* and an *exponent*. For example, 2^3 is a power with base 2 and exponent 3. An exponent which is a positive integer indicates the amount of instances of the base to be multiplied together, that is

$$2^3 = 2 \cdot 2 \cdot 2$$

0.9 Powers

a^n is a power with base a and exponent n.

If n is a natural number, a^n corresponds to n instances of a multiplied together.

Note: $a^1 = a$

Example 1

$$\begin{aligned} 5^3 &= 5 \cdot 5 \cdot 5 \\ &= 125 \end{aligned}$$

Example 2

$$c^4 = c \cdot c \cdot c \cdot c$$

Example 3

$$\begin{aligned} (-7)^2 &= (-7) \cdot (-7) \\ &= 49 \end{aligned}$$

The language box

Common ways of saying 2^3 are[1]

- "2 (raised) to the power of 3"
- "2 to the third power"

In computer programming, the symbol `^` or the symbols `**` is usually written between the base and the exponent.

Raising a number to the power of 2 is called "squaring" the number.

[1] Attention! The examples illustrate a paradox in the English language; *power* is also a synonym for *exponent*.

Note

The next pages declares rules concerning powers with corresponding explanations. Even though one wish to have these explanations as general as possible, we choose to use, mostly, specific numbers as exponents . Using variables as exponents would lead to less reader-friendly expressions, and it is our claim that the general cases are well illustrated by the specific cases.

0.10 Multiplication by powers

$$a^m \cdot a^n = a^{m+n} \tag{1}$$

Example 1

$$\begin{aligned} 3^5 \cdot 3^2 &= 3^{5+2} \\ &= 3^7 \end{aligned}$$

Example 2

$$\begin{aligned} b^4 \cdot b^{11} &= b^{3+11} \\ &= b^{14} \end{aligned}$$

Example 3

$$
\begin{aligned}
a^5 \cdot a^{-7} &= a^{5+(-7)} \\
&= a^{5-7} \\
&= a^{-2}
\end{aligned}
$$

(See Rule 0.13 regarding how powers with negative exponents can be interpreted.)

0.10 Multiplication by powers (explanation)

Let's study the case

$$a^2 \cdot a^3$$

We have

$$a^2 = 2 \cdot 2$$

$$a^3 = 2 \cdot 2 \cdot 2$$

Hence we can write

$$
\begin{aligned}
a^2 \cdot a^3 &= \overbrace{a \cdot a}^{a^2} \cdot \overbrace{a \cdot a \cdot a}^{a^3} \\
&= a^5
\end{aligned}
$$

0.11 Division by powers

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 1

$$\frac{3^5}{3^2} = 3^{5-2} = 3^3$$

Example 2

$$
\begin{aligned}
\frac{2^4 \cdot a^7}{a^6 \cdot 2^2} &= 2^{4-2} \cdot a^{7-6} \\
&= 2^2 a \\
&= 4a
\end{aligned}
$$

0.11 Division by powers (explanation)

Let's examine the fraction $\frac{a^5}{a^2}$. Expanding the powers, we get

$$\begin{aligned}\frac{a^5}{a^2} &= \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} \\ &= \frac{\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a}{\cancel{a} \cdot \cancel{a}} \\ &= a \cdot a \cdot a \\ &= a^3\end{aligned}$$

The above calculations are equivalent to writing

$$\begin{aligned}\frac{a^5}{a^2} &= a^{5-2} \\ &= a^3\end{aligned}$$

0.12 The special case of a^0

$$a^0 = 1$$

Example 1

$$1000^0 = 1$$

Example 2

$$(-b)^0 = 1$$

0.12 The special case of a^0 (explanation)

A number divided by itself always equals 1, therefore

$$\frac{a^n}{a^n} = 1$$

From this, and Rule 0.11, it follows that

$$\begin{aligned}1 &= \frac{a^n}{a^n} \\ &= a^{n-n} \\ &= a^0\end{aligned}$$

0.13 Powers with negative exponents

$$a^{-n} = \frac{1}{a^n}$$

Example 1

$$a^{-8} = \frac{1}{a^8}$$

Example 2

$$(-4)^{-3} = \frac{1}{(-4)^3} = -\frac{1}{64}$$

0.13 Powers with negative exponents (explanation)

By Rule 0.12, we have $a^0 = 1$. Thus

$$\frac{1}{a^n} = \frac{a^0}{a^n}$$

By Rule 0.11, we obtain

$$\begin{aligned} \frac{a^0}{a^n} &= a^{0-n} \\ &= a^{-n} \end{aligned}$$

0.14 Fractions as base

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \tag{2}$$

Example 1

$$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$$

Example 2

$$\left(\frac{a}{7}\right)^3 = \frac{a^3}{7^3} = \frac{a^3}{343}$$

0.14 Fractions as base (explanation)

Let's study

$$\left(\frac{a}{b}\right)^3$$

We have

$$\begin{aligned}\left(\frac{a}{b}\right)^3 &= \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \\ &= \frac{a \cdot a \cdot a}{b \cdot b \cdot b} \\ &= \frac{a^3}{b^3}\end{aligned}$$

0.15 Factors as base

$$(ab)^m = a^m b^m \tag{3}$$

Example 1

$$\begin{aligned}(3a)^5 &= 3^5 a^5 \\ &= 243a^5\end{aligned}$$

Example 2

$$(ab)^4 = a^4 b^4$$

0.15 Factors as base (explanation)

Let's use $(a \cdot b)^3$ as an example. We have

$$\begin{aligned}(a \cdot b)^3 &= (a \cdot b) \cdot (a \cdot b) \cdot (a \cdot b) \\ &= a \cdot a \cdot a \cdot b \cdot b \cdot b \\ &= a^3 b^3\end{aligned}$$

0.16 Powers as base

$$(a^m)^n = a^{m\cdot n} \tag{4}$$

Example 1

$$\begin{aligned}\left(c^4\right)^5 &= c^{4\cdot 5}\\ &= c^{20}\end{aligned}$$

Example 2

$$\begin{aligned}\left(3^{\frac{5}{4}}\right)^8 &= 3^{\frac{5}{4}\cdot 8}\\ &= 3^{10}\end{aligned}$$

0.16 Powers as base (explanation)

Let's use $(a^3)^4$ as an example. We have

$$\left(a^3\right)^4 = a^3 \cdot a^3 \cdot a^3 \cdot a^3$$

By Rule 0.10, we get

$$\begin{aligned}a^3 \cdot a^3 \cdot a^3 \cdot a^3 &= a^{3+3+3+3}\\ &= a^{3\cdot 4}\\ &= a^{12}\end{aligned}$$

0.17 n-root

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The symbol $\sqrt{\ }$ is called the *radical sign*. In the case of an exponent equal to $\frac{1}{2}$, it is common to omit 2 from the radical:

$$a^{\frac{1}{2}} = \sqrt{a}$$

Example

By Rule 0.16, we have

$$\begin{aligned}\left(a^b\right)^{\frac{1}{b}} &= a^{b\cdot\frac{1}{b}} \\ &= a\end{aligned}$$

For example is

$$9^{\frac{1}{2}} = \sqrt{9} = 3, \text{ since } 3^2 = 9$$

$$125^{\frac{1}{3}} = \sqrt[3]{125} = 5, \text{ since } 5^3 = 125$$

$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2, \text{ since } 2^4 = 16$$

The language box

$\sqrt{9}$ is called " the square (the 2nd) root of 9"

$\sqrt[3]{8}$ is called "the cube (the 3th) root of 8"

0.3 Irrational numbers

0.18 Irrational numbers

A number which is *not* a rational number, is an irrational number[1].

The value of an irrational number are decimal numbers with infinite digits in a non-repeating manner.

Example 1

$\sqrt{2}$ is an irrational number.

$$\sqrt{2} = 1.414213562373...$$

[1]Strictly speaking, irrational numbers are all *real* numbers which are not rational numbers. But to explain what *real* numbers are, we have to mention *imaginary* numbers, and this we choose not to do in this book.