0.1 Sets

0.1.1 Definition

A collection of numbers is called a **set**[1], and a number that is part of a set is called an **element**. Sets can contain a finite number of elements and they can contain infinitely many elements.

0.1 Sets

For two numbers a and b, where $a \leq b$, we have that

$[a, b]$	is the set of all numbers greater than or equal to a and less than or equal to b.
$(a, b]$	is the set of all numbers greater than a and less than or equal to b.
$[a, b)$	is the set of all real numbers greater than or equal to a and less than b.

$[a, b]$ is called a **closed interval**, (a, b) is called an **open interval**, and both $(a, b]$ and $[a, b)$ are called **half-open intervals**.

The set that contains only a and b is written as $\{a, b\}$.

That x is an element in a set M is written as $x \in M$.

That x *is not* an element in a set M is written as $x \notin M$.

That the set M consists of the sets M_1 and M_2 is written as $M = M_1 \cup M_2$.

That x is omitted from a set M is written as $M \setminus x$

The language box

$x \in M$ is pronounced "x contained in M".

Many texts use $\langle$ instead of $($ to indicate open (or half-open) intervals.

Note

[1] A set can also be a collection of other mathematical objects, such as functions, but in this book, it is sufficient to consider sets of numbers.

When we define an interval described by a and b hereafter in the book, we take it for granted that a and b are two numbers, and that $a \leq b$.

Example 1

The set of all integers greater than 0 and less than 10 can be written as

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

This set contains 9 elements. 3 is an element in this set, and thus we can write $3 \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

10 is not an element in this set, and thus we can write $10 \notin \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Example 2

In the expression $0 \ ? \ x \ ? \ 1$, replace ? with an inequality symbol so that the expression applies to all $x \in M$, and determine if 1 is contained in M.

a) $M = [0, 1]$

b) $M = (0, 1]$

c) $M = [0, 1)$

Answer

a) $0 \leq x \leq 1$. Furthermore, $1 \in M$.

b) $0 < x \leq 1$. Furthermore, $1 \in M$.

c) $0 \leq x < 1$. Furthermore, $1 \notin M$.

0.2 Names of sets

$\mathbb{N}$	The set of all positive integers[1]
$\mathbb{Z}$	The set of all integers[2]
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers

[1] Does *not* include 0.

[2] Includes 0.

0.1.2 The Symbol for Infinity

The sets in Definition 0.2 contain infinitely many elements. Sometimes we wish to limit parts of an infinite set, and then there arises a need for a symbol that helps symbolize this. ∞ is the symbol for an infinitely large, positive value.

Example

A condition that $x \geq 2$ can be written as $x \in [2, \infty)$.

A condition that $x < -7$ can be written as $x \in (-\infty, -7)$.

The language box

The two intervals in the example above can also be written as $[2, \rightarrow)$ and $(\leftarrow, -7)$.

Note

∞ is not any specific number. Therefore, using the four basic operations alone with this symbol makes no sense.

0.1.3 Domain and Range

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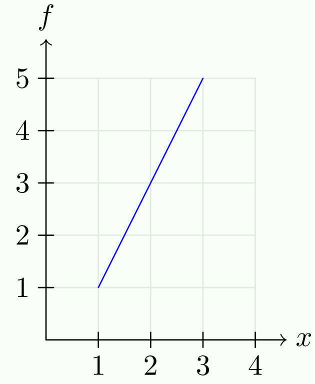
Given a function $f(x)$.

- The set that exclusively contains all the values x can have, is the **domain** of f. This set is written as D_f.
- The set that exclusively contains all the values f can have when $x \in D_f$, is the **range** of f. This set is written as V_f.

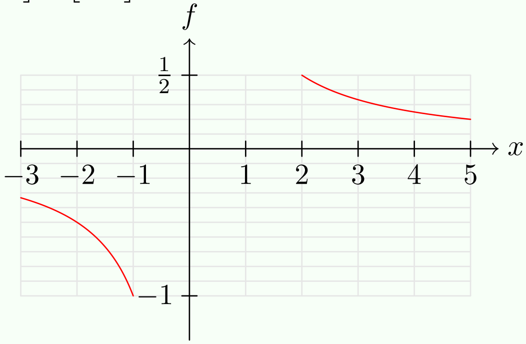
Example 1

Below figure shows $f(x) = 2x + 1$, where $D_f = [1, 3]$. Then $V_f = [3, 7]$.



Example 2

Below figure shows $f(x) = \frac{1}{x}$, where $D_f = [-3, -1] \cup [2, 5]$. Then $V_f = \left[-1, -\frac{1}{3}\right] \cup \left[\frac{1}{5}, \frac{1}{2}\right]$.



Note

The domain of a function is determined by two things; the context in which the function is used, and any values that result in an undefined function expression. In *Example 1* on page 4, the domain is arbitrarily chosen, since the function is defined for all x. In *Example 2*, however, the function is not defined for $x = 0$, so a domain including this value for x would not make sense.

0.2 Conditions

0.2.1 Symbols for Conditions

The symbol $\Rightarrow$ is used to indicate that if one condition is satisfied, then another (or several) condition(s) are also satisfied. For example; in MB we saw that if a triangle is right-angled, then Pythagoras' theorem is valid. We can write this as:

$$\text{the triangle is right-angled} \Rightarrow \text{Pythagoras' theorem is valid}$$

But we also saw that the converse is true; if Pythagoras' theorem is valid, then the triangle must be right-angled. Thus, we can write

$$\text{the triangle is right-angled} \iff \text{Pythagoras' theorem is valid}$$

It is very important to be aware of the difference between $\Rightarrow$ and $\iff$; that condition A satisfied implies condition B satisfied does not necessarily mean that condition B satisfied implies condition A satisfied!

Example 1

$$\text{the square is a square} \Rightarrow \text{the square has four equal sides}$$

Example 2

$$\text{the number is a prime number greater than 2} \Rightarrow \text{the number is an odd number}$$

Example 3

$$\text{the number is an even number} \iff \text{the number is divisible by 2}$$

0.2.2 Functions with Conditions

Functions can have multiple expressions that apply under different conditions. Let us define a function $f(x)$ as follows:

For $x < 1$ the function expression is $-2x + 1$

For $x \geq 1$ the function expression is $x^2 - 2x$

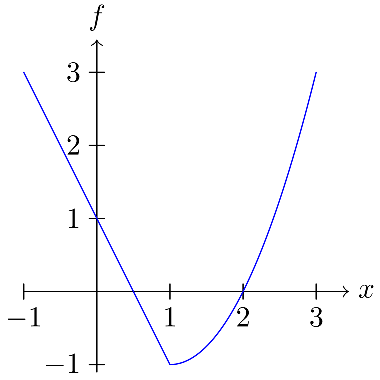


Figure 1: The graph of f on the interval $[-1, 3]$.

This can be written as

$$f(x) = \begin{cases} -2x + 1 & , \quad x < 1 \\ x^2 - 2x & , \quad x \geq 1 \end{cases}$$