0.1 Powers

$$\text{base} \longrightarrow 2^3 \longleftarrow \text{exponent}$$

A power is composed by a *base* and an *exponent*. For example, 2^3 is a power with base 2 and exponent 3. An exponent which is a positive integer indicates the amount of instances of the base to be multiplied together, that is

$$2^3 = 2 \cdot 2 \cdot 2$$

0.1 Powers

a^n is a power with base a and exponent n.

If n is a natural number, a^n corresponds to n instances of a multiplied together.

Note: $a^1 = a$

Example 1

$$\begin{aligned} 5^3 &= 5 \cdot 5 \cdot 5 \\ &= 125 \end{aligned}$$

Example 2

$$c^4 = c \cdot c \cdot c \cdot c$$

Example 3

$$\begin{aligned} (-7)^2 &= (-7) \cdot (-7) \\ &= 49 \end{aligned}$$

The language box

Common ways of saying 2^3 are[1]

- "2 (raised) to the power of 3"
- "2 to the third power"

In computer programming, the symbol `^` or the symbols `**` is usually written between the base and the exponent.

Raising a number to the power of 2 is called "squaring" the number.

[1] Attention! The examples illustrate a paradox in the English language; *power* is also a synonym for *exponent*.

Note

The next pages declares rules concerning powers with corresponding explanations. Even though one wish to have these explanations as general as possible, we choose to use, mostly, specific numbers as exponents . Using variables as exponents would lead to less reader-friendly expressions, and it is our claim that the general cases are well illustrated by the specific cases.

0.2 Multiplication by powers

$$a^m \cdot a^n = a^{m+n} \tag{1}$$

Example 1

$$\begin{aligned} 3^5 \cdot 3^2 &= 3^{5+2} \\ &= 3^7 \end{aligned}$$

Example 2

$$\begin{aligned} b^4 \cdot b^{11} &= b^{3+11} \\ &= b^{14} \end{aligned}$$

Example 3

$$\begin{aligned} a^5 \cdot a^{-7} &= a^{5+(-7)} \\ &= a^{5-7} \\ &= a^{-2} \end{aligned}$$

(See Rule 0.5 regarding how powers with negative exponents can be interpreted.)

0.2 Multiplication by powers (explanation)

Let's study the case

$$a^2 \cdot a^3$$

We have

$$a^2 = 2 \cdot 2$$

$$a^3 = 2 \cdot 2 \cdot 2$$

Hence we can write

$$\begin{aligned} a^2 \cdot a^3 &= \overbrace{a \cdot a}^{a^2} \cdot \overbrace{a \cdot a \cdot a}^{a^3} \\ &= a^5 \end{aligned}$$

0.3 Division by powers

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 1

$$\frac{3^5}{3^2} = 3^{5-2} = 3^3$$

Example 2

$$\begin{aligned} \frac{2^4 \cdot a^7}{a^6 \cdot 2^2} &= 2^{4-2} \cdot a^{7-6} \\ &= 2^2 a \\ &= 4a \end{aligned}$$

0.3 Division by powers (explanation)

Let's examine the fraction $\frac{a^5}{a^2}$. Expanding the powers, we get

$$\begin{aligned}\frac{a^5}{a^2} &= \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} \\ &= \frac{\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a}{\cancel{a} \cdot \cancel{a}} \\ &= a \cdot a \cdot a \\ &= a^3\end{aligned}$$

The above calculations are equivalent to writing

$$\begin{aligned}\frac{a^5}{a^2} &= a^{5-2} \\ &= a^3\end{aligned}$$

0.4 The special case of $\boldsymbol{a^0}$

$$a^0 = 1$$

Example 1

$$1000^0 = 1$$

Example 2

$$(-b)^0 = 1$$

0.4 The special case of $\boldsymbol{a^0}$ (explanation)

A number divided by itself always equals 1, therefore

$$\frac{a^n}{a^n} = 1$$

From this, and Rule 0.3, it follows that

$$\begin{aligned}1 &= \frac{a^n}{a^n} \\ &= a^{n-n} \\ &= a^0\end{aligned}$$

0.5 Powers with negative exponents

$$a^{-n} = \frac{1}{a^n}$$

Example 1

$$a^{-8} = \frac{1}{a^8}$$

Example 2

$$(-4)^{-3} = \frac{1}{(-4)^3} = -\frac{1}{64}$$

0.5 Powers with negative exponents (explanation)

By Rule 0.4, we have $a^0 = 1$. Thus

$$\frac{1}{a^n} = \frac{a^0}{a^n}$$

By Rule 0.3, we obtain

$$\begin{aligned} \frac{a^0}{a^n} &= a^{0-n} \\ &= a^{-n} \end{aligned}$$

0.6 Fractions as base

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \tag{2}$$

Example 1

$$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$$

Example 2

$$\left(\frac{a}{7}\right)^3 = \frac{a^3}{7^3} = \frac{a^3}{343}$$

0.6 Fractions as base (explanation)

Let's study

$$\left(\frac{a}{b}\right)^3$$

We have

$$\begin{aligned}\left(\frac{a}{b}\right)^3 &= \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\\ &= \frac{a\cdot a\cdot a}{b\cdot b\cdot b}\\ &= \frac{a^3}{b^3}\end{aligned}$$

0.7 Factors as base

$$(ab)^m = a^m b^m \tag{3}$$

Example 1

$$\begin{aligned}(3a)^5 &= 3^5a^5\\ &= 243a^5\end{aligned}$$

Example 2

$$(ab)^4 = a^4b^4$$

0.7 Factors as base (explanation)

Let's use $(a\cdot b)^3$ as an example. We have

$$\begin{aligned}(a\cdot b)^3 &= (a\cdot b)\cdot(a\cdot b)\cdot(a\cdot b)\\ &= a\cdot a\cdot a\cdot b\cdot b\cdot b\\ &= a^3b^3\end{aligned}$$

0.8 Powers as base

$$(a^m)^n = a^{m \cdot n} \tag{4}$$

Example 1

$$\begin{aligned}\left(c^4\right)^5 &= c^{4\cdot 5}\\ &= c^{20}\end{aligned}$$

Example 2

$$\begin{aligned}\left(3^{\frac{5}{4}}\right)^8 &= 3^{\frac{5}{4}\cdot 8}\\ &= 3^{10}\end{aligned}$$

0.8 Powers as base (explanation)

Let's use $\left(a^3\right)^4$ as an example. We have

$$\left(a^3\right)^4 = a^3 \cdot a^3 \cdot a^3 \cdot a^3$$

By Rule 0.2, we get

$$\begin{aligned}a^3 \cdot a^3 \cdot a^3 \cdot a^3 &= a^{3+3+3+3}\\ &= a^{3\cdot 4}\\ &= a^{12}\end{aligned}$$

0.9 n-root

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The symbol $\sqrt{\ }$ is called the *radical sign*. In the case of an exponent equal to $\frac{1}{2}$, it is common to omit 2 from the radical:

$$a^{\frac{1}{2}} = \sqrt{a}$$

Example

By Rule 0.8, we have

$$\begin{aligned}\left(a^b\right)^{\frac{1}{b}} &= a^{b\cdot\frac{1}{b}}\\ &= a\end{aligned}$$

For example is

$$9^{\frac{1}{2}} = \sqrt{9} = 3, \text{ since } 3^2 = 9$$

$$125^{\frac{1}{3}} = \sqrt[3]{125} = 5, \text{ since } 5^3 = 125$$

$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2, \text{ since } 2^4 = 16$$

The language box

$\sqrt{9}$ is called " the square (the 2nd) root of 9"

$\sqrt[3]{8}$ is called "the cube (the 3th) root of 8"

0.2 Irrational numbers

0.10 Irrational numbers

A number which is *not* a rational number, is an irrational number[1].

The value of an irrational number are decimal numbers with infinite digits in a non-repeating manner.

Example 1

$\sqrt{2}$ is an irrational number.

$$\sqrt{2} = 1.414213562373...$$

[1]Strictly speaking, irrational numbers are all *real* numbers which are not rational numbers. But to explain what *real* numbers are, we have to mention *imaginary* numbers, and this we choose not to do in this book.