Solutions to Problem Set 1 FYS3140

Problem 1.1

Ratio test of convergence:

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1 \tag{1}$$

a)

We have $a_n = n(n+1)(z-2i)^n$ and $a_{n+1} = (n+1)(n+2)(z-2i)^{n+1}$. Therefore

$$\lim_{n\to\infty} |\frac{(n+1)(n+2)(z-2i)^{n+1}}{n(n+1)(z-2i)^n}| = \lim_{n\to\infty} |\frac{(n+2)}{n}||(z-2i)| = |z-2i| < 1 \tag{2}$$

which is a circular area of radius 1 centered at the point $(0, 2)$ in the complex plane.

b)

$a_{n+1} = 2^{n+1}(z+i-3)^{2n+2}$, so

$$\lim_{n\to\infty} |\frac{2^{n+1}(z+i-3)^{2n+2}}{2^n(z+i-3)^{2n}}| = \lim_{n\to\infty} |2(z+i-3)^2| = 2|z-(3-i)|^2 < 1 \tag{3}$$

because an absolute value is always positive. Hence we have convergence when

$$|z-(3-i)| < 1/\sqrt{2} \tag{4}$$

which is a circular area of radius $1/\sqrt{2}$ centered at the point $(3, -1)$ in the complex plane.

Problem 1.2

Remember:

$$x + iy = re^{i\theta} = r\left(\cos(\theta) + i\sin(\theta)\right) \tag{5}$$

$$r = \sqrt{x^2+y^2} \quad \tan(\theta) = \frac{y}{x} \quad \text{in same quadrant as the point (x,y)} \tag{6}$$

$$z^{1/n} = r^{1/n}\exp(i(\theta + 2\pi k)/n) \quad k \in \mathbb{Z} \tag{7}$$

a)

$$\sqrt{2}\exp(5i\pi/4) = \sqrt{2}\left(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})\right) = -\sqrt{2}\frac{\sqrt{2}}{2} - i\sqrt{2}\frac{\sqrt{2}}{2} = -1 - i \tag{8}$$

b)

We easily see that $1+i = \sqrt{2}\exp(i\pi/4)$. For the denominator we find $r = \sqrt{x^2+y^2} = \sqrt{3+1} = \sqrt{4} = 2$ and $\theta = -\arctan(1/\sqrt{3}) = -\pi/6$.

$$\begin{aligned}\frac{(1+i)^{48}}{(\sqrt{3}-i)^{25}} &= \frac{\sqrt{2}^{48}\exp(48i\pi/4)}{2^{25}\exp(-25i\pi/6)} = \frac{2^{24}}{2^{25}}\exp(i\pi(12+25/6)) \\ &= \frac{1}{2}(\cos(16\pi+\frac{\pi}{6}) + i\sin(16\pi+\frac{\pi}{6})) = \frac{\sqrt{3}}{4} + i\frac{1}{4}\end{aligned} \tag{9}$$

c)

We write the base in radial form:

$$8i\sqrt{3} - 8 = \sqrt{8^2 + 8^2\cdot\sqrt{3}^2}\exp(i\theta) = 16\exp(i\theta) \tag{10}$$

where

$$\theta = \pi - \arctan(\frac{8\sqrt{3}}{8}) = \pi - \arctan(\sqrt{3}) = 2\pi/3 \tag{11}$$

We find

$$(8i\sqrt{3}-8)^{1/4} = 16^{1/4}\exp(i\left(\frac{2\pi}{12} + \frac{2\pi k}{4}\right)) = 2\exp(i\left(\frac{\pi}{6} + \frac{3\pi k}{6}\right)) \tag{12}$$

The four roots written out are

$$z_0 = 2e^{i\pi/6} \quad z_1 = 2e^{4i\pi/6} \quad z_2 = 2e^{7i\pi/6} \quad z_3 = 2e^{10i\pi/6} \tag{13}$$

d)

8 is written as $8e^{i(0+2\pi k)}$ in polar form. The three cube roots are therefore

$$z_0 = 8^{1/3}e^{2\pi/3} \quad z_1 = 8^{1/3}e^{4\pi/3} \quad z_2 = 8^{1/3}e^{6\pi/3} \tag{14}$$

which can be written in Cartesian form as

$$z_0 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \quad z_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \quad z_2 = 1 \tag{15}$$

ignoring the common factor of $8^{1/3}$. The sum is clearly 0.

In the general case, the distinct solutions of the equation

$$re^{i\theta+2\pi ik} = z^n \tag{16}$$

are the complex numbers

$$z_k = r^{1/n}\exp(i\theta/n + 2\pi ik/n) = r^{1/n}\exp(i\theta/n)\exp(2\pi ik/n) \tag{17}$$

where k goes from 0 to $n-1$. The sum of them is

$$S = \sum_{k=0}^{n-1} z_k = r^{1/n}\exp(i\theta/n)\sum_{k=0}^{n-1}\exp(2\pi ik/n) = r^{1/n}\exp(i\theta/n)\sum_{k=0}^{n-1} w^k \tag{18}$$

where $w = \exp(2\pi i/n)$. Now recall the sum of a geometric series

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} \tag{19}$$

Therefore

$$S = r^{1/n} \exp(i\theta/n) \frac{\exp(2\pi i/n)^n - 1}{\exp(2\pi i/n) - 1} = 0 \tag{20}$$

because $\exp(2\pi i/n)^n = \exp(2\pi i) = 1$.

Problem 1.3

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \quad \cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \tag{21}$$

$$\sinh(z) = \frac{1}{2}\left(e^{z} - e^{-z}\right) \quad \cosh(z) = \frac{1}{2}\left(e^{z} + e^{-z}\right) \tag{22}$$

a)

$$\begin{aligned}
\int_0^{2\pi} \sin^2(4x)\, dx &= \int_0^{2\pi} \left(\frac{e^{4ix} - e^{-4ix}}{2i}\right)^2 dx \\
&= -\frac{1}{4}\int_0^{2\pi} \left(e^{8ix} + e^{-8ix} - 2\right) dx \\
&= -\frac{1}{4}\left[\frac{1}{8i}e^{8x} - \frac{1}{8i}e^{-8x} - 2x\right]_0^{2\pi} \\
&= \frac{1}{4} \cdot 2(2\pi) \\
&= \pi
\end{aligned} \tag{23}$$

b)

$$\begin{aligned}
2\sin z \cos z &= 2 \cdot \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \cdot \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
&= \frac{1}{2i}(e^{2iz} + 1 - 1 - e^{-2iz}) \\
&= \sin 2z
\end{aligned} \tag{24}$$

c)

$$\begin{aligned}
\cosh^2 z - \sinh^2 z &= \left(\frac{1}{2}\left(e^{z} + e^{-z}\right)\right)^2 - \left(\frac{1}{2}\left(e^{z} - e^{-z}\right)\right)^2 \\
&= \frac{1}{4}\left(e^{2z} + e^{-2z} + 2 - e^{2z} - e^{-2z} + 2\right) \\
&= \frac{1}{4}(2 + 2) \\
&= 1
\end{aligned} \tag{25}$$

d)

$$\begin{aligned}
\sin\left(i\ln\frac{1-i}{1+i}\right) &= \frac{1}{2i}\left(\exp\left(i^2\ln\frac{1-i}{1+i}\right)-\exp\left(-i^2\ln\frac{1-i}{1+i}\right)\right)\\
&= \frac{1}{2i}\left(\left(\frac{1-i}{1+i}\right)^{-1}-\frac{1-i}{1+i}\right)\\
&= \frac{1}{2i}\left(\frac{1+i}{1-i}-\frac{1-i}{1+i}\right)\\
&= \frac{1}{2i}\cdot\frac{(1+i)^2-(1-i)^2}{|1+i|^2}\\
&= \frac{1}{4i}\left(1+2i-1-1+2i+1\right)\\
&= 1
\end{aligned} \tag{26}$$

e)

$$\begin{aligned}
(-e)^{i\pi} &= (-1\cdot e)^{i\pi}\\
&= (-1)^{i\pi}\cdot e^{i\pi}\\
&= (e^{i\pi})^{i\pi}\cdot(-1)\\
&= -e^{-\pi^2}
\end{aligned} \tag{27}$$

which is a real number.

f)

To show this, we apply tanh to both sides of the equation. The left hand side is simply z; we need to show that the right hand side is also z.

$$\begin{aligned}
\tanh\left(\frac{1}{2}\ln\frac{1+z}{1-z}\right) &= \frac{\sinh\left(\frac{1}{2}\ln\frac{1+z}{1-z}\right)}{\cosh\left(\frac{1}{2}\ln\frac{1+z}{1-z}\right)}\\
&= \frac{2}{2}\,\frac{\exp\left(\frac{1}{2}\ln\frac{1+z}{1-z}\right)-\exp\left(-\frac{1}{2}\ln\frac{1+z}{1-z}\right)}{\exp\left(\frac{1}{2}\ln\frac{1+z}{1-z}\right)+\exp\left(-\frac{1}{2}\ln\frac{1+z}{1-z}\right)}\\
&= \frac{\sqrt{\frac{1+z}{1-z}}-\sqrt{\frac{1-z}{1+z}}}{\sqrt{\frac{1+z}{1-z}}+\sqrt{\frac{1-z}{1+z}}}
\end{aligned} \tag{28}$$

Now we multiply the numerator and the denominator by $\sqrt{(1+z)(1-z)}$:

$$\begin{aligned}
\tanh\left(\frac{1}{2}\ln\frac{1+z}{1-z}\right) &= \frac{\sqrt{\frac{1+z}{1-z}}-\sqrt{\frac{1-z}{1+z}}}{\sqrt{\frac{1+z}{1-z}}+\sqrt{\frac{1-z}{1+z}}} \\
&= \frac{\sqrt{(1+z)^2}-\sqrt{(1-z)^2}}{\sqrt{(1+z)^2}+\sqrt{(1-z)^2}} \\
&= \frac{1+z-1+z}{1+z+1-z} \\
&= \frac{2z}{2} \\
&= z
\end{aligned} \tag{29}$$

An alternative approach is to derive an expression for $\tanh^{-1}$ from the definition of tanh:

$$\begin{aligned}
\tanh z &= \frac{\sinh z}{\cosh z} \\
&= \frac{e^z - e^{-z}}{e^z + e^{-z}} \\
&= \frac{e^{2z}-1}{e^{2z}+1}
\end{aligned} \tag{30}$$

Setting $w = \tanh z$ such that $z = \tanh^{-1} w$, we find

$$\begin{aligned}
w &= \frac{e^{2z}-1}{e^{2z}+1} \\
we^{2z} + w &= e^{2z} - 1 \\
e^{2z}(w-1) &= -1 - w \\
e^{2z} &= \frac{-1-w}{w-1} \\
2z &= \ln\frac{1+w}{1-w} \\
z &= \frac{1}{2}\ln\frac{1+w}{1-w} \\
\tanh^{-1} w &= \frac{1}{2}\ln\frac{1+w}{1-w}
\end{aligned} \tag{31}$$

Extra problem 2.17.30

The expression

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{32}$$

holds for complex x as well as real ones. We find

$$e^{x(1+i)} = \sum_{n=0}^{\infty} \frac{x^n(1+i)^n}{n!} = 1 + x(1+i) + \frac{1}{2}x^2(1+i)^2 + \frac{1}{6}x^3(1+i)^3 + \cdots \tag{33}$$

The next step is to write $1+i$ in polar form: the point in the complex plane is $(1,1)$ and the form is therefore $1+i = \sqrt{2}e^{i\pi/4}$. That gives us

$$e^{x(1+i)} = \sum_{n=0}^{\infty} \frac{x^n 2^{n/2} e^{ni\pi/4}}{n!} = 1 + 2^{1/2}xe^{i\pi/4} + x^2 e^{i\pi/2} + \frac{2^{1/2}}{3}x^3 e^{3i\pi/4} + \cdots \tag{34}$$

Notice that for even n, the exponential part is either purely real or purely imaginary.

Next, let's look at $e^x \cos x$. We have

$$e^x \cos x = \frac{1}{2} e^x \left(e^{ix} + e^{-ix}\right) = \frac{1}{2}\left(e^{x(1+i)} + e^{x(1-i)}\right) \tag{35}$$

Since $1 - i = \sqrt{2} e^{-i\pi/4}$, we find

$$e^x \cos x = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n 2^{n/2} \left(e^{ni\pi/4} + e^{-ni\pi/4}\right)}{n!} = \sum_{n=0}^{\infty} \frac{x^n 2^{n/2}}{n!} \cos(n\pi/4) \tag{36}$$

Since $\cos(\pi/2 + k\pi) = 0$ for all $k \in \mathbb{Z}$, we see that the series for $e^x \cos x$ does not have terms with $n = 2$, $n = 6$, $n = 10$ etc. The proof for $e^x \sin x$ follows in a similar fashion. The series expression is

$$e^x \sin x = \sum_{n=0}^{\infty} \frac{x^n 2^{n/2}}{n!} \sin(n\pi/4) \tag{37}$$