SOLUTION TO PROBLEM SET 8 FYS 3140

In this problem set we use the following set of equations:

sine-cosine series($T = 2\pi$): $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{1}$$

$$\text{where:} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \tag{3}$$

sine-cosine series($T = 2l$): $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l}) \tag{4}$$

$$\text{where:} \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{n\pi x}{l})dx \tag{5}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{n\pi x}{l})dx \tag{6}$$

Complex series($T = 2\pi$): $$f(x) = \sum_{n=-\infty}^{n=+\infty} c_n e^{inx} \tag{7}$$

$$\text{where:} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{8}$$

Complex series($T = 2l$): $$f(x) = \sum_{n=-\infty}^{n=+\infty} c_n e^{\frac{in\pi x}{l}} \tag{9}$$

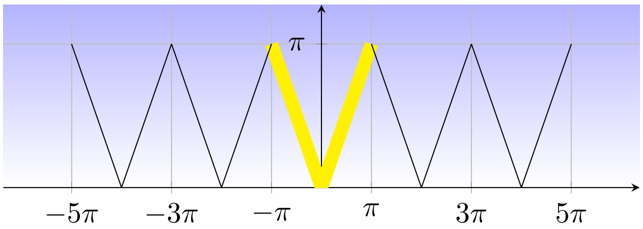
$$\text{where:} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\frac{in\pi x}{l}}dx \tag{10}$$

Parseval's Theorem: $$\overline{|f(x)|^2} = \sum_{n=-\infty}^{n=+\infty} |c_n|^2 \tag{11}$$

or $$\overline{|f(x)|^2} = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}|a_n|^2 + \frac{1}{2}\sum_{n=1}^{\infty}|b_n|^2 \tag{12}$$

Problem 7.5.9
find sine-cosine series of

$$f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$



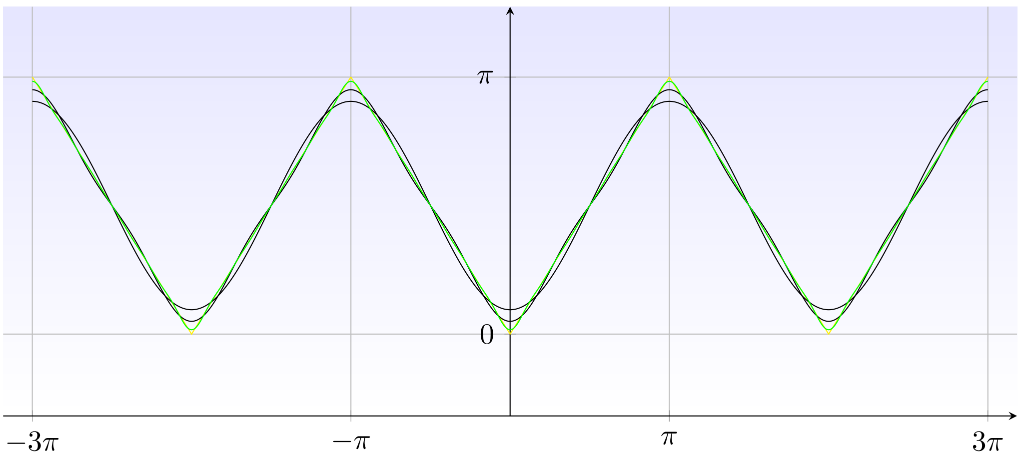
The period is 2π thus use eqn(1-3). $f(x)$ is even function, which means that its Fourier series has no sine component. i.e. $b_n = 0$ for all n. Evaluating eqn(2) we get

$$a_0 = \pi$$

$$a_n = \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$\text{Answer:} \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$$

The following plots show for some of the first few terms in the series.



Problem 7.7.9
-problem to solve: Find complex Fourier series for problem 5.9. Refer to same function and plot.
-use: eqn(7-8)

$$c_n = \frac{1}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$f(x) = \sum_{n=-\infty}^{n=+\infty} \frac{1}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right) e^{inx}$$

Now we can write $e^{inx} = \cos(nx) + i\sin(nx)$ and split the sum into $\sum_{n=-\infty}^{n=+\infty} \to \sum_{n=-\infty}^{n=-1} + (n=0) + \sum_{n=1}^{n=+\infty}$; and using the fact that c_n is real we get

$$c_n + c_{-n} = \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

and

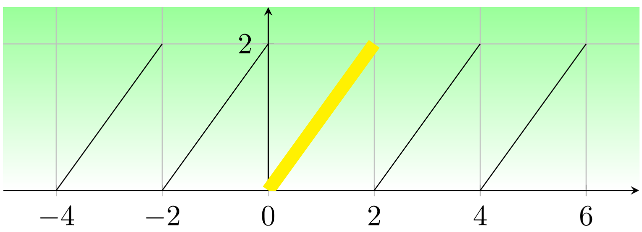
$$f(x) == \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$$

which is identical with series obtained in problem 7.5.9.

Problem 7.8.16

find sine-cosine series of

$$f(x) = x, \quad 0 < x < 2$$



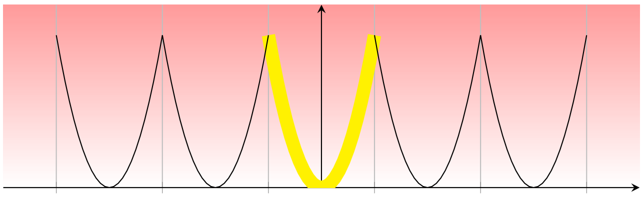
Since the period is $T = 2 \neq 2\pi$ we use eqns(4-6). We now substitute $l = 1$ in the equations and apply integration by part

$$\begin{aligned}
a_0 &= \frac{1}{1}\int_0^2 x\cos(\frac{0\pi x}{1})dx = 2 \\
a_n &= \frac{1}{1}\int_0^2 x\cos(\frac{n\pi x}{1})dx = 0 \\
b_n &= \frac{1}{1}\int_0^2 x\sin(\frac{n\pi x}{1})dx = -\frac{2}{n\pi} \\
\therefore f(x) &= 1 - \frac{2}{n\pi}\sum_1^{\infty}\frac{\sin n\pi x}{n}
\end{aligned}$$

Problem 7.9.9

find sine-cosine series of

$$f(x) = x^2, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$



It is even function hence no sine component in the Fourier series.

$$a_0 = 2\int_{-1/2}^{1/2} x^2 dx = \frac{1}{6}$$

$$a_n = 2\int_{-1/2}^{1/2} x^2 \cos(2n\pi x)dx = -\frac{(-1)^n}{(n\pi)^2}$$

$$\underline{f(x) = \frac{1}{12} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(2n\pi x)}$$

Problem 7.11.6

Find the value of the following series based on problem (7.9.9):

$$\sum_{n=1}^{\infty}\frac{1}{n^4}$$

using eqn(12) we have

$$\overline{|f(x)|^2} = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}|a_n|^2 + \frac{1}{2}\sum_{n=1}^{\infty}|b_n|^2$$

where $f(x) = x^2$, $a_0 = \frac{1}{6}$, $a_n = -\frac{(-1)^n}{(n\pi)^2}$, $b_n = 0$ then

$$\overline{|f(x)|^2} = \frac{\int_{-1/2}^{1/2}(x^2)^2 dx}{\int_{-1/2}^{1/2} dx} = \frac{1}{80}$$

$$|a_n|^2 = \left|-\frac{(-1)^n}{(n\pi)^2}\right|^2 = \frac{1}{(n\pi)^4}$$

eqn(5) then becomes

$$\frac{1}{80} = (\frac{1}{12})^2 + \frac{1}{2\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

rearranging and solving for $\sum_{n=1}^{\infty}\frac{1}{n^4}$ we get

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=2\pi^4\left(\frac{1}{80}-\frac{1}{144}\right)=\underline{\frac{\pi^4}{90}}$$

Extra problems
Problem 7.9.15
Given $f(x)=x$ for $0<x<1$, the even odd functions can be sketched as follows

$$f_{even}(x)=\begin{cases}-x & -1<x<0\\ x & 0<x<1\end{cases}$$

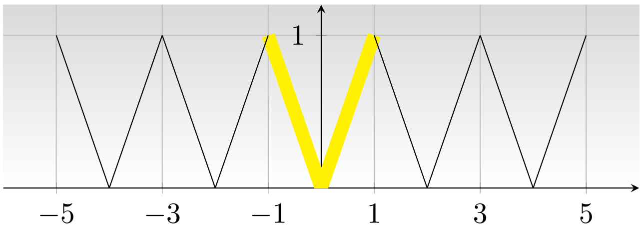


Figure 1: We know that for even function $b_n=0$. using eqn(4-6) where $l=1$, we get $a_0=1$, $a_n=\int_{-1}^{0}-x\cos(n\pi x)dx+\int_{0}^{1}x\cos(n\pi x)dx=2\frac{(-1)^n-1}{(n\pi)^2}$ and finally $\underline{\mathrm{f}_c(x)=\frac{1}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}((-1)^n-1)\frac{\cos(n\pi x)}{n^2}}$.

$$f_{odd}(x)=x\quad -1<x<1$$

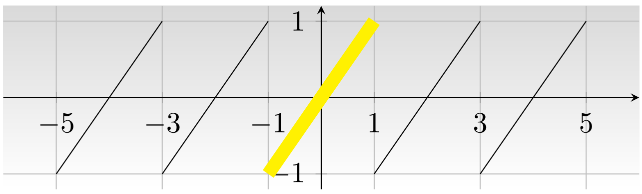


Figure 2: We know that for odd function $a_n=0$. Using eqn(4-6) where $l=1$ we get $b_n=\int_{-1}^{1}x\sin(n\pi x)dx=-2\frac{(-1)^n}{n\pi}$, and $\mathrm{f}_s(x)=\underline{-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\pi x)}{n}}$.