Solutions to Problem Set 10 FYS3140

Some useful integrals in this PS:

$$
\begin{aligned}
\int_0^\pi \sin\theta \; d\theta &= 2 \\
\int_0^\pi \sin^3\theta \; d\theta &= \frac{4}{3} \\
\int_0^\pi \cos^2\theta \sin\theta \; d\theta &= \frac{2}{3} \\
\int_0^\pi \cos\theta \sin^2\theta \; d\theta &= 0 \\
\int_0^{\pi/2} \cos\phi \; d\phi &= 1 \\
\int_0^{\pi/2} \sin\phi \; d\phi &= 0 \\
\int_0^{\pi/2} \cos^2\phi \; d\phi = \int_0^{\pi/2} \sin^2\phi \; d\phi &= \frac{\pi}{4} \\
\int_0^{\pi/2} \cos\phi \sin\phi \; d\phi &= \frac{1}{2}
\end{aligned}
\tag{1}
$$

Problem 10.4.4

Find the inertia tensor about the origin for a mass of uniform density $= 1$, inside the part of the unit sphere where $x > 0$, $y > 0$. Find the eigenvalues and eigenvectors of the inertia tensor. Since we are in the unit sphere, we will change to spherical coordinates to simplify the limits on our integrals. There is no constraint on z except for the fact that we are in the unit sphere, so $r \in (0, 1)$, $\theta \in (0, \pi)$. The constraint on x and y then gives $\phi \in (0, \pi/2)$. θ is the polar angle and ϕ is the azimuthal angle. We then use the integral expressions on p. 506 in *Boas,* realizing that we only need to calculate the upper triangular part of the inertia matrix by the form of the expressions. (Note: this is not an assumption about symmetry as stated in the warning hint; it simply follows from the formulas for I_{xx}, I_{xy} etc.) Using the integrals as found above, we get

$$
\begin{aligned}
I_{xx} &= \int \left(r^2 - r^2\sin^2\theta\cos^2\phi\right) r^2 \sin\theta \; drd\theta d\phi \\
&= \int r^4 \sin\theta \; drd\theta d\phi - \int r^4 \sin^3\theta \cos^2\phi \; drd\theta d\phi \\
&= \frac{1}{5}\cdot 2 \cdot \frac{\pi}{2} - \frac{1}{5}\cdot\frac{4}{3}\cdot\frac{\pi}{4} \\
&= \frac{2\pi}{15}
\end{aligned}
\tag{2}
$$

where we have written $\int$ when we actually mean $\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi/2}$. Doing this for the components I_{yy}, I_{zz}, I_{xy}, I_{xz} and I_{yz} as well, and using the fact that the tensor is symmetric, we find

$$
I = \begin{pmatrix} 2\pi/15 & -4/30 & 0 \\ -4/30 & 2\pi/15 & 0 \\ 0 & 0 & 2\pi/15 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} 4\pi & -4 & 0 \\ -4 & 4\pi & 0 \\ 0 & 0 & 4\pi \end{pmatrix}
\tag{3}
$$

We find eigenvalues most conveniently by expanding the determinant in the characteristic equation $\det(I-\lambda)=0$ along the third row. We get:

$$(4\pi/30-\lambda)\left((4\pi/30-\lambda)^2-16/30^2\right)=0 \tag{4}$$

which has solutions

$$\lambda=\{\frac{4\pi}{30},\frac{4\pi\pm 4}{30}\} \tag{5}$$

You can find the eigenvectors by computer or similar, they are $v_0=\hat{z}$, $v_\pm=\mp\hat{x}+\hat{y}$, with arbitrary lengths.

Problem 10.4.6

Find the inerta tensor for point masses 1 at (1,1,-2) and 2 at (1,1,1). We use the discrete formulas (the ones with summation signs instead of integrals) on p. 506 and obtain

$$I_{xx}=\sum_{i=1}^{2}m_i\left(y_i^2+z_i^2\right)=1\cdot\left(1^2+(-2)^2\right)+2\cdot\left(1^2+1^2\right)=9 \tag{6}$$

etc. The full tensor in matrix form is

$$I=\begin{pmatrix}9&-3&0\\-3&9&0\\0&0&6\end{pmatrix} \tag{7}$$

which has eigenvalues $\lambda=\{6,6,12\}$. The last eigenvalue has an eigenvector $v_{12}=\hat{x}-\hat{y}$ while the eigenspace of the eigenvalue 6 is spanned by two vectors, on the form $v_6=a\hat{x}+a\hat{y}+b\hat{z}$, for instance (1,1,0) and (0,0,1).

Problem 10.5.7

In dealing with the Levi-Civita symbol, we may at any time use the fact that swapping two indices changes the sign, and cyclically permuting the indices does not change the value of the symbol.

a)

$$\epsilon_{ijk}\epsilon_{pjq}=-\epsilon_{jik}\epsilon_{pjq}=\epsilon_{jik}\epsilon_{jpq}=\delta_{ip}\delta_{kq}-\delta_{iq}\delta_{kp} \tag{8}$$

b)

$$\epsilon_{abc}\epsilon_{pqc}=\epsilon_{abc}\epsilon_{cpq}=\epsilon_{cab}\epsilon_{cpq}=\delta_{ap}\delta_{bq}-\delta_{aq}\delta_{bp} \tag{9}$$

Problem 10.5.8

We want to show

$$\begin{cases}\epsilon_{ijk}\epsilon_{ijn} &=2\delta_{kn}\\ \epsilon_{ijk}\epsilon_{ijk} &=6\end{cases} \tag{10}$$

Let us understand what these formulae tell us. Since either symbol can only take the values ±1 or 0 for any specific set of indices, the latter result definitely seems odd. In dealing with tensors, we must remember that we are interested in formulae for general cases of indices, but this is *not* the case when an index is repeated! In the first line for instance, the indices i and j are repeated, meaning that we should consider *all* possible values that i and j can take to give any specific combination of k and n, the two non-repeated indices. Case in point, if we want to look at $k = n = 3$, then both $i = 1$, $j = 2$ *and* $i = 2$, $j = 1$ will give something non-zero. To deal with this, we use Einsteins summation convention, that says that any repeated index should be summed over. This is called contraction. Using Eq. 5.8 in *Boas*, where the index i is already contracted, yields:

$$\epsilon_{ijk}\epsilon_{ijn} = \sum_{j=1}^{3}(\delta_{jj}\delta_{kn} - \delta_{jn}\delta_{kj}) = 3\delta_{kn} - \delta_{kn} = 2\delta_{kn} \tag{11}$$

where we have used the fact that the Kronecker delta picks out one term from a sum,

$$\sum_{i=1}^{N}\delta_{i\alpha}A_{ijk...} = A_{\alpha jk...} \tag{12}$$

much like the Dirac delta picks out one function value from an integral. In our case, the $A_{ijk...}$ was another delta, namely δ_{kj}. The last equation then follows from the first by contraction of the final index-pair:

$$\epsilon_{ijk}\epsilon_{ijk} = \sum_{k=1}^{3}2\delta_{kk} = 2\sum_{k=1}^{3}1 = 2\cdot 3 = 6 \tag{13}$$

Problem 10.5.10

a) Write the vector triple product $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$ in tensor form and show that it is equal to a determinant expression. The dot product is

$$\boldsymbol{A}\cdot\boldsymbol{D} = A_iD_i \tag{14}$$

in tensor notation, and the cross product has components

$$\boldsymbol{D} = \boldsymbol{B}\times\boldsymbol{C} \quad \rightarrow \quad D_i = \epsilon_{ijk}B_jC_k \tag{15}$$

Therefore

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = A_iD_i = A_i\epsilon_{ijk}B_jC_k = A_iB_jC_k\epsilon_{ijk} \tag{16}$$

which is the expression for a 3x3 determinant.

b) There is a spelling error in the book here, we are to write Eq. 3.3 of Chapter 6 in tensor form. Using a), we find

$$\begin{aligned}\epsilon_{ijk}A_jB_kC_i &= A_iB_jC_k\epsilon_{ijk} \\ &= C_i\epsilon_{ijk}A_jB_k \\ &= -\epsilon_{ijk}A_jC_kB_i\end{aligned} \tag{17}$$

The first line follows from the cyclic permutation $jki \rightarrow ijk$, the second line is the same as the LHS of the first line, and the third line follows from the swapping of indices i and k, leaving a minus sign.

Problem 10.5.11

We find straight away

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{A}) = A_iB_jA_k\epsilon_{ijk} \tag{18}$$

But by renaming and swapping the indices i and k on the RHS, we find

$$A_iB_jA_k\epsilon_{ijk} = A_kB_jA_i\epsilon_{kji} = -A_kB_jA_i\epsilon_{ijk} = -A_iB_jA_k\epsilon_{ijk} \tag{19}$$

and something that is equal to minus itself is zero.

Problem 10.5.13fgh

f) Prove

$$\boldsymbol{\nabla} \cdot (\phi \boldsymbol{V}) = \phi(\boldsymbol{\nabla} \cdot \boldsymbol{V}) + \boldsymbol{V} \cdot (\boldsymbol{\nabla}\phi) \tag{20}$$

LHS:

$$\nabla_i(\phi V)_i = \frac{\partial}{\partial x_i}(\phi V)_i = \frac{\partial \phi}{\partial x_i}V_i + \phi\frac{\partial V_i}{\partial x_i} = (\nabla\phi)_i V_i + \phi\nabla_i V_i \tag{21}$$

RHS:

$$\phi\nabla_i V_i + V_i(\nabla\phi)_i = \text{LHS} \tag{22}$$

g) Prove

$$\boldsymbol{\nabla} \times (\phi \boldsymbol{V}) = \phi(\boldsymbol{\nabla} \times \boldsymbol{V}) - \boldsymbol{V} \times (\boldsymbol{\nabla}\phi) \tag{23}$$

Component i of LHS:

$$[\boldsymbol{\nabla} \times (\phi \boldsymbol{V})]_i = \epsilon_{ijk}\nabla_j(\phi V)_k = \epsilon_{ijk}\left(V_k(\nabla_j\phi) + \phi(\nabla_j V_k)\right) = \epsilon_{ijk}V_k(\nabla_j\phi) + \phi\epsilon_{ijk}\nabla_j V_k \tag{24}$$

Component i of RHS:

$$[\phi(\boldsymbol{\nabla} \times \boldsymbol{V}) - \boldsymbol{V} \times (\boldsymbol{\nabla}\phi)]_i = \phi\epsilon_{ijk}\nabla_j V_k - \epsilon_{ijk}V_j(\nabla_k\phi) \tag{25}$$

By interchanging j and k on the last term and then swapping them only on the ϵ for a minus sign, we recover the LHS.

h) Prove

$$\boldsymbol{\nabla} \cdot (\boldsymbol{U} \times \boldsymbol{V}) = \boldsymbol{V} \cdot (\boldsymbol{\nabla} \times \boldsymbol{U}) - \boldsymbol{U} \cdot (\boldsymbol{\nabla} \times \boldsymbol{V}) \tag{26}$$

LHS:

$$\nabla_i\left(\boldsymbol{U} \times \boldsymbol{V}\right)_i = \nabla_i\epsilon_{ijk}U_j V_k = \epsilon_{ijk}\left(V_k(\nabla_i U_j) + U_j(\nabla_i V_k)\right) \tag{27}$$

RHS:

$$\begin{aligned} V_k\left(\boldsymbol{\nabla} \times \boldsymbol{U}\right)_k - U_j(\boldsymbol{\nabla} \times \boldsymbol{V})_j &= V_k\epsilon_{kij}\nabla_i U_j - U_j\epsilon_{jki}\nabla_k V_i \\ &= V_k\epsilon_{ijk}\nabla_i U_j - \epsilon_{jik}U_j\nabla_i V_k \\ &= V_k\epsilon_{ijk}\nabla_i U_j + \epsilon_{ijk}U_j\nabla_i V_k \\ &= \text{LHS} \end{aligned} \tag{28}$$