SOLUTION TO PROBLEM SET 5, FYS3140

Problem 5.1 (Residue theory)

a) 14.7.17

Let us substitute $x \to z$ and $\sin x \to e^{iz}$ and start by evaluating the integral in the upper half plane,

$$\int_{-\infty}^{\infty} \frac{x \sin x dx}{x^2 + 4x + 5} \to \oint \frac{z e^{iz} dz}{z^2 + 4z + 5}$$

$$\oint \frac{z e^{iz} dz}{z^2 + 4z + 5} = \oint \frac{z e^{iz} dz}{(z + (i+2))(z - (i-2))}$$

$z = i - 2$ is the only pole in the upper half-plane, and its residue becomes

$$\begin{aligned} R(z = i-2) &= \lim_{z \to (i-2)} (z - (i-2)) \frac{z e^{iz}}{(z + (i+2))(z - (i-2))} \\ &= \frac{(i-2)e^{i(i-2)}}{2i} \\ &= \frac{(i-2)e^{i(i-2)}}{2i} \\ &= \frac{(-2\cos 2 + \sin 2) + i(\cos 2 + 2\sin 2)}{2ie} \end{aligned}$$

and

$$\begin{aligned} \oint \frac{z e^{iz} dz}{(z + (i+2))(z - (i-2))} &= 2\pi i \cdot \left\{ \frac{(-2\cos 2 + \sin 2) + i(\cos 2 + 2\sin 2)}{2ie} \right\} \\ &= \pi \frac{(-2\cos 2 + \sin 2) + i(\cos 2 + 2\sin 2)}{e} \end{aligned}$$

Now the complex integral splits into integral along x-axit and along the semicircle. Along x-axis, $z \to x$ and on the semicircle $z \to \rho e^{i\theta}$

$$\oint \frac{z e^{iz} dz}{z^2 + 4z + 5} = \int_{-\rho}^{\rho} \frac{x e^{ix} dx}{x^2 + 4x + 5} + \int_0^{\pi} \frac{\rho e^{iz} \rho i e^{i\theta} d\theta}{\rho^2 e^{i2\theta} + 4\rho e^{i\theta} + 5}$$

taking the limit $\rho \to \infty$,

$$= \int_{-\infty}^{\infty} \frac{x e^{ix} dx}{x^2 + 4x + 5} + \int_0^{\pi} \frac{e^{iz} \rho^2 i e^{i\theta} d\theta}{\rho^2 e^{i2\theta} + 4\rho e^{i\theta} + 5}$$

The second part of RHS is evaluated as follows,

$$\begin{aligned}
\lim_{\rho\to\infty}\int_0^{\pi}\frac{e^{iz}\rho^2 ie^{i\theta}d\theta}{\rho^2e^{i2\theta}+4\rho e^{i\theta}+5} &= \lim_{\rho\to\infty}\int_0^{\pi}e^{i(\rho\cos\theta+i\rho\sin\theta)}ie^{-i\theta}d\theta \\
&= \lim_{\rho\to\infty} i\int_0^{\pi}e^{i(-\theta+\rho\cos\theta)}e^{-\rho\sin\theta}d\theta \\
&= 0
\end{aligned}$$

where we know that $|e^{iy}| = 1$, and $e^{-\rho\sin\theta}$ vanishes as $\rho\to\infty$ ($\sin\theta$ is positive in upper half plane). You can also apply Jordan's lemma. Therefore,

$$\int_{-\infty}^{\infty}\frac{xe^{ix}dx}{x^2+4x+5} = \pi\frac{(-2\cos 2+\sin 2)+i(\cos 2+2\sin 2)}{e}$$

Equating the real and imaginary parts of the two sides, we get

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{xe^{ix}dx}{x^2+4x+5} &= \int_{-\infty}^{\infty}\frac{x\cos x dx}{x^2+4x+5}+i\int_{-\infty}^{\infty}\frac{x\sin x dx}{x^2+4x+5} \\
&= \pi\frac{(-2\cos 2+\sin 2)+i(\cos 2+2\sin 2)}{e} \\
&\Rightarrow \int_{-\infty}^{\infty}\frac{x\sin x dx}{x^2+4x+5} = \frac{\pi}{e}(\cos 2+2\sin 2)
\end{aligned}$$

b) 14.7.24

We use same procedure here as the previous problem. Making substitutions $x\to z$ and $\sin x\to e^{iz}$, we start by evaluating the following complex integral in the upper half-plane. i.e

$$\int_{-\infty}^{\infty}\frac{x\sin\pi x}{1-x^2}dx \to \oint\frac{ze^{i\pi z}dz}{1-z^2}$$

here the poles $z=1$ and $z=-1$ are at the boundary. Residue method is not generally correct when multiple poles are present at the boundary. Thus, I will evaluate the integral directly by excluding the poles from our region by creating small circles of radius r about each pole.

$$\oint\frac{ze^{i\pi z}dz}{1-z^2} = 0 \qquad (*)$$

Now we let the small circles to vanish by taking the limit $r\to 0$.
For semicircle around $z=1$ the following substitutions will be made: $z = 1+re^{i\theta}$; $dz = rie^{i\theta}d\theta = i(z-1)d\theta$. When $r\to 0$; then $z\to 1$; and $e^{i\pi z}\to -1$. the integral around the small semicircle about $z=1$ is thus

$$
\begin{aligned}
\lim_{r\to 0}\int_{C'} \frac{ze^{i\pi z}dz}{(1-z)(1+z)} &= \int_{C'} \frac{i}{2}d\theta \\
&= \int_{\pi}^{0} \frac{i}{2}d\theta \\
&= \frac{-i\pi}{2}
\end{aligned}
$$

similarly, for semicircle around $z = -1$, we make substitutions: $z = -1 + re^{i\theta}$; $dz = rie^{i\theta}d\theta = i(z+1)d\theta$. When $r \to 0$; then $z \to -1$; and $e^{i\pi z} \to -1$, hence

$$
\lim_{r\to 0}\int_{C'} \frac{ze^{i\pi z}dz}{(1-z)(1+z)} = \frac{-i\pi}{2}
$$

Now we resolve the integral in eqn (*) into six components, i.e. along x-axis; along the two small circles of radius r; and along the larger semicircle of radius ρ. i.e.

$$
\begin{aligned}
\oint \frac{ze^{i\pi z}dz}{1-z^2} &= \overbrace{\int_{-\rho}^{-1-r} \frac{xe^{i\pi x}dx}{1-x^2} + \int_{-1+r}^{1-r} \frac{xe^{i\pi x}dx}{1-x^2} + \int_{1+r}^{\rho} \frac{xe^{i\pi x}dx}{1-x^2}}^{1} \\
&+ \underbrace{\oint_{C'_1} \frac{ze^{i\pi z}dz}{1-z^2} + \oint_{C'_{-1}} \frac{ze^{i\pi z}dz}{1-z^2}}_{2} \\
&+ \underbrace{\oint_{C'_{big}} \frac{ze^{i\pi z}dz}{1-z^2}}_{3} \\
&= 0
\end{aligned}
$$

when we take the limit as $r \to 0$ and $\rho \to \infty$, integral (1) gives $\int_{-\infty}^{\infty}()$; integral (2) is $-i\pi$; and integral (3) vanishes by the Jordan's lemma or the technique we used in the previous problem. Thus,

$$
\oint \frac{ze^{i\pi z}dz}{1-z^2} = \int_{-\infty}^{\infty} \frac{xe^{i\pi x}dx}{1-x^2} - i\pi = 0
$$

$$
\Rightarrow \int_{-\infty}^{\infty} \frac{xe^{i\pi x}dx}{1-x^2} = i\pi
$$

by equating real and imaginary parts of the two sides, we get

$$
\int_{-\infty}^{\infty} \frac{x\sin \pi x}{1-x^2}dx = \pi
$$

c)

The poles are $z_1 = -1$ and $z_2 = 2$, both are inside $|z| = 3$, and both are simple poles. We find

$$\begin{aligned}\oint \frac{\cos(z-1)\ dz}{(z+1)(z-2)} &= 2\pi i\,(R(-1)+R(2)) \\ &= 2\pi i\left(\frac{\cos(-2)}{-3}+\frac{\cos(1)}{3}\right) \\ &= \frac{2\pi i}{3}\,(\cos 1 - \cos 2)\end{aligned} \tag{1}$$

d)

The poles are at $z_1 = 1$ and $z_2 = -1$, which is clear from rewriting the integral as

$$\oint \frac{e^{-z}\ dz}{(z+1)^2(z-1)^2} \tag{2}$$

The poles are of second order, so

$$R(1) = \lim_{z\to 1}\frac{d}{dz}\left(\frac{e^{-z}}{(z+1)^2}\right) = \frac{-e^{-1}\cdot 2^2 - 2e^{-1}\cdot 2}{2^4} = \frac{-e^{-1}}{2} = \frac{-1}{2e} \tag{3}$$

and

$$R(-1) = \lim_{z\to -1}\frac{d}{dz}\left(\frac{e^{-z}}{(z-1)^2}\right) = \frac{-e^{1}\cdot 2^2 - 2e^{1}\cdot(-2)}{2^4} = 0 \tag{4}$$

so

$$\oint \frac{e^{-z}\ dz}{(z+1)^2(z-1)^2} = 2\pi i\cdot\frac{-1}{2e} = \frac{-i\pi}{e} \tag{5}$$

Problem 5.2

a)

Solve

$$dy + (2xy - xe^{-x^2})dx = 0 \tag{6}$$

This is equivalent to

$$y' + 2xy = xe^{-x^2} \tag{7}$$

we identify $P(x) = 2x$ and $Q(x) = xe^{-x^2}$. We have

$$\mu(x) = e^{\int P(x)\ dx} = e^{\int 2x\ dx} = e^{x^2} \tag{8}$$

which gives us the new differential equation

$$(y\mu(x))' = \mu(x)xe^{-x^2} = x \tag{9}$$

with the solution

$$y(x) = \left(\frac{1}{2}x^2 + C\right)e^{-x^2} \tag{10}$$

b)

Solve

$$y' + y\cos x = \sin 2x \tag{11}$$

We identify $P(x) = \cos x$ and $Q(x) = \sin 2x$. We find

$$\mu(x) = e^{\int P(x)\ dx} = e^{\int \cos x\ dx} = e^{\sin x} \tag{12}$$

The new differential equation reads

$$(y\mu(x))' = \mu(x)\sin 2x = e^{\sin x}\sin 2x = 2e^{\sin x}\sin x\cos x \tag{13}$$

By integrating over x the LHS becomes $y(x)\mu(x)$. The RHS becomes

$$\begin{aligned}\int e^{\sin x}\sin 2x\ dx &= \int e^{\sin x} 2\sin x\cos x\ dx \\ &= 2\int ue^{u}\ du \\ &= 2e^{\sin x}\sin x - 2e^{\sin x} + C \\ &= 2e^{\sin x}(\sin x - 1) + C\end{aligned} \tag{14}$$

and therefore the solution is

$$y(x) = 2\sin x - 2 + Ce^{-\sin x} \tag{15}$$

c)

Solve

$$y'\cos x + y = \cos^2 x \tag{16}$$

In canonical form it is

$$y' + \frac{1}{\cos x}y = \cos x \tag{17}$$

and we identify $P(x) = 1/\cos x$ and $Q(x) = \cos x$. The integrating factor is

$$\mu(x) = e^{\int P(x)dx} = e^{\int 1/\cos x\ dx} \tag{18}$$

The integral is

$$\begin{aligned}\int \frac{1}{\cos x}\ dx &= \int \frac{1}{\cos x}\frac{du}{\cos x} \\ &= \int \frac{1}{\cos^2 x}\ du \\ &= \int \frac{1}{1-u^2}\ du \\ &= \frac{1}{2}\int \frac{1}{1+u} + \frac{1}{1-u}\ du \\ &= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| \\ &= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|\end{aligned} \tag{19}$$

where we have used the change of variable $u = \sin x$ and done partial fraction decomposition,

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right) \tag{20}$$

Now that we have the integral, the integrating factor becomes

$$\mu(x) = e^{\int P(x)dx} = \left(\frac{1+\sin x}{1-\sin x}\right)^{1/2} \tag{21}$$

since both numerator and denominator are positive semidefinite. Finally, the differential equation becomes

$$\begin{aligned} y(x)\mu(x) &= \int \mu(x)R(x)\ dx \\ &= \int \left(\frac{1+\sin x}{1-\sin x}\right)^{1/2} \cos x\ dx \\ &= \int \left(\frac{1+u}{1-u}\right)^{1/2} du \\ &= \int (1+u)\frac{1}{\sqrt{1-u^2}}\ du \end{aligned} \tag{22}$$

We do integration by parts with $v = (1+u)$, $w' = 1/\sqrt{1-u^2}$ which gives

$$\begin{aligned} y(x)\mu(x) &= (1+u)\arcsin(u) - \int \arcsin(u)\ du \\ &= (1+u)\arcsin(u) - \sqrt{1-u^2} - u \cdot \arcsin(u) + C \\ &= (1+\sin x)\arcsin(\sin x) - \sqrt{1-\sin^2 x} - \sin x \cdot \arcsin(\sin x) + C \\ &= \arcsin(\sin x) - |\cos x| + C \end{aligned} \tag{23}$$

Note that we *cannot* replace $\arcsin(\sin x)$ with x here! The function $f(x) = \arcsin(\sin x)$ behaves as x from $x = 0$ to $x = \pi/2$, like $-x$ from $x = \pi/2$ to $x = \pi$ etc. It has the property that

$$\frac{d}{dz}(\arcsin(\sin x)) = \frac{\cos x}{|\cos x|} \tag{24}$$

Finally the solution is

$$y(x) = \left(\frac{1-\sin x}{1+\sin x}\right)^{1/2}(\arcsin(\sin x) - |\cos x| + C) = \frac{1-\sin x}{|\cos x|}(\arcsin(\sin x) - |\cos x| + C) \tag{25}$$

Note that it is not defined for $x = \pi/2 + k\pi$, because the differential equation in canonical form was not defined at those points. The original differential equation demands that

$$y(\pi/2 + k\pi) = 0 \tag{26}$$

Problem 5.3

a)

Solve the differential equation

$$4y'' + 12y' + 9y = 0 \tag{27}$$

The auxilliary equation is

$$4\lambda^2 + 12\lambda + 9 = 0 \tag{28}$$

which has two coinciding solutions

$$\lambda_1 = \lambda_2 = -\frac{3}{2} \tag{29}$$

The solution is therefore

$$y(x) = C_1 e^{-3x/3} + C_2 x e^{-3x/2} \tag{30}$$

b)

The auxilliary equation is

$$\lambda^2 - 4\lambda + 13 = 0 \tag{31}$$

which has complex solutions

$$\lambda_1 = 2 + 3i \qquad \lambda_2 = 2 - 3i \tag{32}$$

The solution is therefore

$$y(x) = e^{2x}\left(C_1 e^{3ix} + C_2 e^{-3ix}\right) = e^{2x}\left(A\cos(3x) + B\sin(3x)\right) \tag{33}$$