FYS3120 Problem set 12

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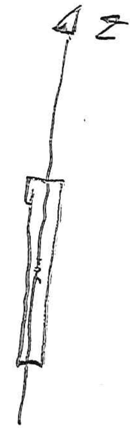

Problem I Exam 2009

a) $I(z,t) = I_0 \cos \frac{\pi z}{a} \cos \omega t$, $-a/2 < z < a/2$

$\lambda(z,0) = 0$

$\vec{j} = I/A \, \hat{k}$, where A is the cross-sectional area, and $\hat{k}$ is the unit-vector in $\hat{z}$-direction

$\rho = \lambda/A$, (because the way I understand the definition of λ, it only acounts for the length of the antenna and not the girth)

We ofcourse assume evenly distributed charge in the antenna.

Now, using the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

$$\Rightarrow \frac{\partial \lambda}{\partial t} + \frac{\partial I}{\partial z} = 0$$

$$\Rightarrow \frac{\partial \lambda}{\partial t} = -\frac{\partial I}{\partial z} = I_0 \frac{\pi}{a} \sin\left(\frac{\pi z}{a}\right)\cos(\omega t)$$

$$\Rightarrow \lambda - \lambda_0 = I_0 \frac{\pi}{a} \sin\frac{\pi z}{a} \int_0^t \cos\omega t'\, dt' = I_0 \frac{\pi}{a\omega} \sin\left(\frac{\pi z}{a}\right)\sin(\omega t)$$

$$\xRightarrow{\lambda_0 = 0} \lambda = I_0 \frac{\pi}{a\omega} \sin\left(\frac{\pi z}{a}\right) \sin(\omega t)$$

b) $$\vec{p}(t) = \int \rho(\vec{r}, t)\, \vec{r}\, d^3\vec{r} = \int \rho(\vec{r}, t)(x, y, z)(0, 0, 1)\, dx\, dy\, dz$$

$$= \int_{-a/2}^{a/2} \frac{\lambda}{A} z\hat{k}\, dx\, dy\, dz$$

$$= \frac{A\hat{k}}{A} I_0 \frac{\pi}{a\omega} \sin\omega t \int_{-a/2}^{a/2} z \sin\frac{\pi z}{a}\, dz$$

$$\gamma = \frac{\pi z}{a} \Rightarrow d\gamma = \frac{\pi}{a} dz \Rightarrow dz = \frac{a}{\pi} d\gamma$$

$$\Rightarrow \vec{p}(t) = \hat{k} I_0 \frac{\pi}{a\omega} \sin(\omega t) \frac{a^2}{\pi^2} \int_{-\pi/2}^{\pi/2} \gamma \sin\gamma\, d\gamma$$

$$= \hat{k} I_0 \frac{a}{\pi\omega} \sin(\omega t)\left[-\sin\gamma\right]_{-\pi/2}^{\pi/2}$$

$$= \hat{k} I_0 \frac{2a}{\pi\omega} \sin(\omega t)$$

$u = \gamma$

$u' = 1$

$v' = \sin\gamma$

$v = -\cos\gamma$

So $P_0 = \frac{2a}{\pi\omega} I_0$

c) $\ddot{p}_{ret} = p_0 \frac{d^2}{dt^2} \sin\left(\omega\left[t - \frac{|r-r'|}{c}\right]\right)\hat{k}$

$r' = 0$, because the rod is in origo

$\Rightarrow \ddot{p}_{ret} = -p_0 \omega^2 \sin\left(\omega\left(t - \frac{r}{c}\right)\right)\hat{k}$

$r = x$, $\hat{n} = \hat{i}$

$B(\vec{r},t) = -\frac{\mu_0}{4\pi r c}\ddot{p}_{ret} \times \hat{n}$

$\Rightarrow B(x,t) = +\frac{\mu_0}{4\pi x c}\omega^2 p_0 \sin\left(\omega\left(t - \frac{x}{c}\right)\right)\hat{i} \times \hat{k}$

$= -\frac{\mu_0}{4\pi x c}\omega^2 p_0 \sin\left(\omega\left(t - \frac{x}{c}\right)\right)\hat{j}$

$E = cB \times \hat{i} = \frac{\mu_0}{4\pi x}\omega^2 p_0 \sin\left(\omega\left(t - \frac{x}{c}\right)\right)\hat{k}$

Paraphrase
To ~~quote~~ the caption of figure 10.2, the E-field oscillates in a fixed direction orthogonal to the propogation direction.

So it is linear

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a)

integrating

$$\frac{d}{dt}\vec{P} = \vec{F} \Rightarrow \vec{P} = \vec{F}t$$

The force on a particle in E-field is $q\vec{E}$, with q the charge of the particle

$$\Rightarrow \vec{P} = q\vec{E}_{field}\,t$$

$$E_{energy} = \sqrt{p^2c^2 + m^2c^4} = \sqrt{q^2|\vec{E}|^2t^2c^2 + m^2c^4}$$

$$= mc^2\sqrt{1 + \left(\frac{qE}{mc}\right)^2 t^2}$$

2b) We normally have the relation $E = \gamma m c^2$ and comparing with our expression in a) we find

$\gamma(t) = \sqrt{1 + \left(\frac{qE}{mc}\right)^2 t^2}$. I am not completely sure why I am allowed to do this, but it gives me the right answer so....

using $\frac{dt}{d\tau} = \gamma$

$$T_0 \int_0^{T_0} d\tau = \int_0^{t_0} \frac{dt}{\gamma} = \int_0^{t_0} \frac{dt}{\sqrt{1 + k^2 t^2}} = \underbrace{\frac{1}{k} \operatorname{asinh}(k t_0)}_{\text{Rottmann p. 130}}$$

$k = \frac{eE}{m_0 c}$, as used in the Problem set

$$\Rightarrow t_0 = \frac{1}{k} \sinh(k \tilde{T}_0)$$

2c) $a_0^2 = A^2 = |\vec{A}|^2 - A^{0\,2}$

$$A = \left(c \frac{d\gamma}{d\tau}, \frac{1}{m} \frac{d\vec{P}}{d\tau} \right)$$

$$\frac{d\gamma}{d\tau} = \gamma \frac{d\gamma}{dt} = \gamma \frac{k^2 t}{\sqrt{1 + k^2 t^2}} = k^2 t, \quad \frac{dP}{d\tau} = \gamma \frac{dP}{dt} = \gamma e E$$

from a)

$$a_0^2 = |\vec{A}|^2 - A^{0\,2} = \left(\frac{\gamma e \vec{E}}{m}\right)^2 - (k^2 c t)^2$$

$$kc = \frac{eE}{m}$$

So

$$a_0^2 = \gamma^2 k^2 c^2 - k^4 c^2 t^2 = (1 + k^2 t^2) k^2 c^2 - k^4 c^2 t^2$$

$$= k^2 c^2$$

$$\Rightarrow P = \frac{\mu_0 e^2}{6\pi c} k^2 c^2 = \frac{\mu_0 e^2}{6\pi} k^2 c$$

Finding the energy:

$$E = P t_0 = \frac{\mu_0 e^2}{6\pi} k^2 c \sinh(k\tau_0)$$

Problem 3

a) $\vec{p}(t) = q\ell\left(\cos(\omega t)\hat{e}_x + \sin(\omega t)\hat{e}_y\right)$

$\ddot{p}_{ret} = -\omega^2 p_{ret}$

$\vec{n} = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}$

$$B = \frac{\mu_0}{4\pi r c}\ddot{p}_{ret}\times\vec{n} = \frac{\mu_0}{4\pi r c}(-\omega^2)q\ell\left[\cos(\omega(t-\tfrac{r}{c}))\hat{n}\times\hat{i} + \sin(\omega(t-\tfrac{r}{c}))\hat{n}\times\hat{j}\right]$$

$\hat{n}\times\hat{i} = -\sin\theta\sin\varphi\,\hat{k} + \cos\theta\,\hat{j}$

$\hat{n}\times\hat{j} = \sin\theta\cos\varphi\,\hat{k} - \cos\theta\,\hat{i}$

$$\Rightarrow B = \underbrace{-\frac{\mu_0}{4\pi r c}\omega^2 q\ell}_{-B_0}\left[\cos(\omega(t-\tfrac{r}{c}))(\cos\theta\,\hat{j} - \sin\theta\sin\varphi\,\hat{k}) + \sin(\omega(t-\tfrac{r}{c}))(\sin\theta\cos\varphi\,\hat{k} - \cos\theta\,\hat{i})\right]$$

$$= B_0\left[\sin(\omega(t-\tfrac{r}{c}))\cos\theta\,\hat{i} - \cos(\omega(t-\tfrac{r}{c}))\cos\theta\,\hat{j} - \sin\theta\sin(\omega(t-r/c)-\varphi)\hat{k}\right]$$

$B_0 = \frac{\mu_0\omega^2 q\ell}{4\pi c r}$

Rittmann p. 87

E is given by $E = c\vec{B} \times \hat{n}$

b) on the x-axis $\theta = \pi/2$

$$\Rightarrow \vec{B} = -B_0 \sin(\omega(t - \tfrac{r}{c}) - \varphi)\hat{k}$$

So it oscillates in a single direction, making it linearly polarized.

On the z-axis $\theta = 0$

$$\Rightarrow \vec{B} = B_0(\sin(\omega(t - \tfrac{r}{c}))\hat{i} - \cos(\omega(t - \tfrac{r}{c}))\hat{j})$$

So it oscillates in two perpendicular directions with phase $\pi/2$.
I assume this is circular polarization.

c) $u = \frac{1}{2}\left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)$

$$= \frac{1}{2}\left(\varepsilon_0 c^2 (\vec{B}\times\hat{n})^2 + \frac{1}{\mu_0}\vec{B}^2\right)$$

Calculating this is insane, so I'll just assume I can use the duality transformation

$E \to -cB$ (is this allowed in general ?)

$$\Rightarrow u = \frac{1}{2}\left(\varepsilon_0 c^2 B^2 + \frac{1}{\mu_0}B^2\right) = \frac{1}{2}\frac{B^2}{\mu_0} + \frac{1}{2}\frac{B^2}{\mu_0} = \frac{B^2}{\mu_0}$$

Inserting B:

$$u = \frac{B_0^2}{\mu_0}\left[\cos^2\theta \sin^2(\omega(t-\tfrac{r}{c})) + \cos^2\theta\cos^2(\omega(t-\tfrac{r}{c})) + \sin^2\theta \sin^2(\omega(t-\tfrac{r}{c}) - \varphi)\right]$$

$$= \frac{B_0^2}{\mu_0}\left[\cos^2\theta + \sin^2\theta \underbrace{\sin^2(\omega(t-\tfrac{r}{c})-\varphi)}_{\text{The only part time dependent. Need to average}}\right]$$

Taking one time period $T = 2\pi/\omega$

$$\langle \sin^2(\omega(t-\tfrac{r}{c})-\varphi)\rangle = \frac{1}{T}\int_0^{2\pi/\omega} \sin^2(\omega(t-\tfrac{r}{c})-\varphi)\,dt$$

$$\because\ \gamma = \omega(t-\tfrac{r}{c})-\varphi$$

$$d\gamma = \omega\,dt \Rightarrow dt = \frac{d\gamma}{\omega}$$

$$\Rightarrow \langle - \rangle = \frac{\omega}{2\pi}\int_0^{2\pi} \frac{1}{\omega}\sin^2\gamma\, d\gamma = \underbrace{\frac{1}{2}}_{\text{Rottmann p. 102}}$$

$$\Rightarrow \langle u \rangle = \frac{B_0^2}{\mu_0}\left(\cos^2\theta + \frac{1}{2}\sin^2\theta\right)$$

$$= \frac{B_0^2}{\mu_0}\left(1 - \frac{1}{2}\sin^2\theta\right)$$

Because $\sin^2\theta \geq 0$, this is maximized when $\sin\theta$, so when $\theta = 0$ or π which means the most energy is radiated in the $\pm z$-direction