FYS3120 oblig 2 Kristian Tuv

1a) $\frac{dL}{dt}$ is a total derivative,

meaning if q or $\dot{q}$ depends on t, we need to use the chain-rule and take the derivative of these aswell.

$\frac{\partial L}{\partial t}$ means taking the explicit derivative with respect to t, meaning we only care about explicit dependence of t in L

1b)

$$\frac{d}{dt} L = \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}}{\partial t} + \frac{\partial L}{\partial t}$$

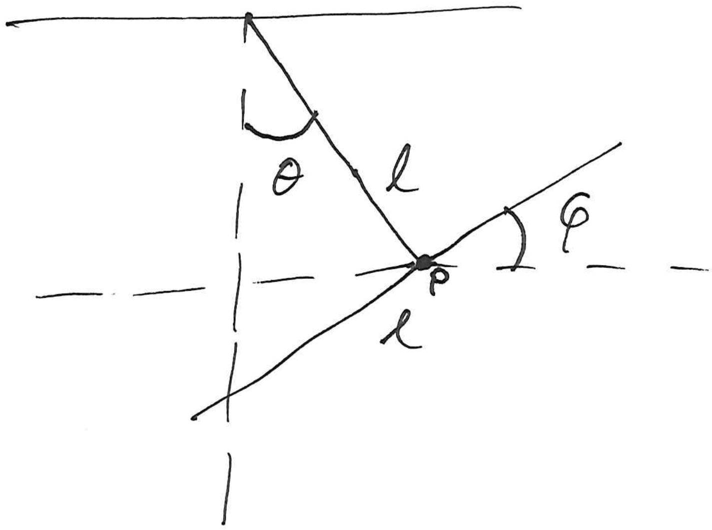
$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) \underbrace{- \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial q_i}\dot{q}_i}_{=0\text{, from Lagrange}} + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} L = \frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) = \frac{\partial L}{\partial t}$$

2)

a)



$K_\theta = \frac{1}{2} I_\theta \dot{\theta}^2 = \frac{1}{2} m \frac{l^2}{3} \dot{\theta}^2$, $K_\varphi = \frac{1}{2} I_\varphi \dot{\varphi}^2 = \frac{1}{2} m \frac{l^2}{12} \dot{\varphi}^2$

$K_P = \frac{1}{2} m l^2 \dot{\theta}^2$

$V = -mg \cdot \left(\frac{1}{2} l\right) \sin\theta - mg \cdot l \cdot \sin\theta = -mg \frac{3}{2} l \sin\theta$

2) a) $L = K - V = \frac{1}{2} m l^2 \left(\frac{4}{3} \dot{\theta}^2 + \frac{\dot{\varphi}^2}{12} \right) + \frac{3}{2} m g l \cos\theta$

b) $\frac{\partial L}{\partial \theta} = -\frac{3}{2} m g l \sin\theta$

$\frac{\partial L}{\partial \varphi} = 0$

$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \frac{4}{3} \dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m l^2 \frac{4}{3} \ddot{\theta}$

$\frac{\partial L}{\partial \dot{\varphi}} = m l^2 \frac{\dot{\varphi}}{12} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = m l^2 \frac{\ddot{\varphi}}{12}$

For θ:

$\frac{4}{3} m l^2 \ddot{\theta} + \frac{3}{2} m g l \sin\theta = 0$, for small θ

$\sin\theta \approx \theta$

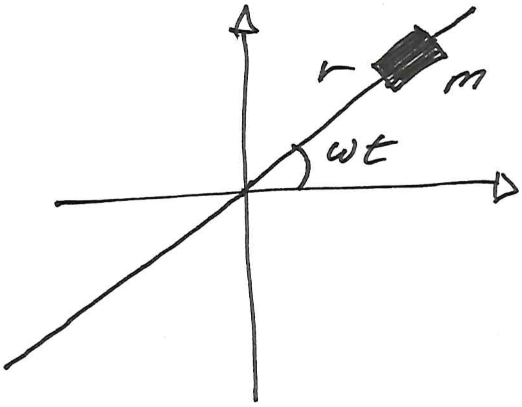
$\Rightarrow \ddot{\theta} \approx -\frac{9}{8} \frac{g}{l} \theta$

A solution is

$\theta = \sin(\omega t) + \cos(\omega t)$

with $\omega = \sqrt{\frac{9}{8} \frac{g}{l}}$

3a)



$x = r\cos(\omega t)$

$y = r\sin(\omega t)$

$\vec{r} = x\hat{i} + y\hat{j} = r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}$

$\dot{x} = \dot{r}\cos(\omega t) - r\sin(\omega t)\omega$

$\dot{y} = \dot{r}\sin(\omega t) + r\cos(\omega t)\omega$

$\dot{x}^2 = \dot{r}^2\cos^2(\omega t) - \underline{\dot{r}r\cos(\omega t)\sin(\omega t)} + r^2\sin^2(\omega t)\omega^2$

$\dot{y}^2 = \dot{r}^2\sin^2(\omega t) + \underline{\dot{r}r\cos(\omega t)\sin(\omega t)} + r^2\cos^2(\omega t)\omega^2$

$\cancel{\dot{\vec{r}}^2 = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{r}^2\cos^2(\omega t) + r^2\sin}}$

$\dot{\vec{r}}^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2\cos^2(\omega t) + r^2\sin^2(\omega t)\omega^2$

$+ \dot{r}^2\sin^2(\omega t) + r^2\cos^2(\omega t)\omega^2$

$= \dot{r}^2 + r^2\omega^2$

$L = K = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$

3a) $p = \frac{\partial L}{\partial \dot r} = m\dot r \Rightarrow \frac{d}{dt} p = m\ddot r$

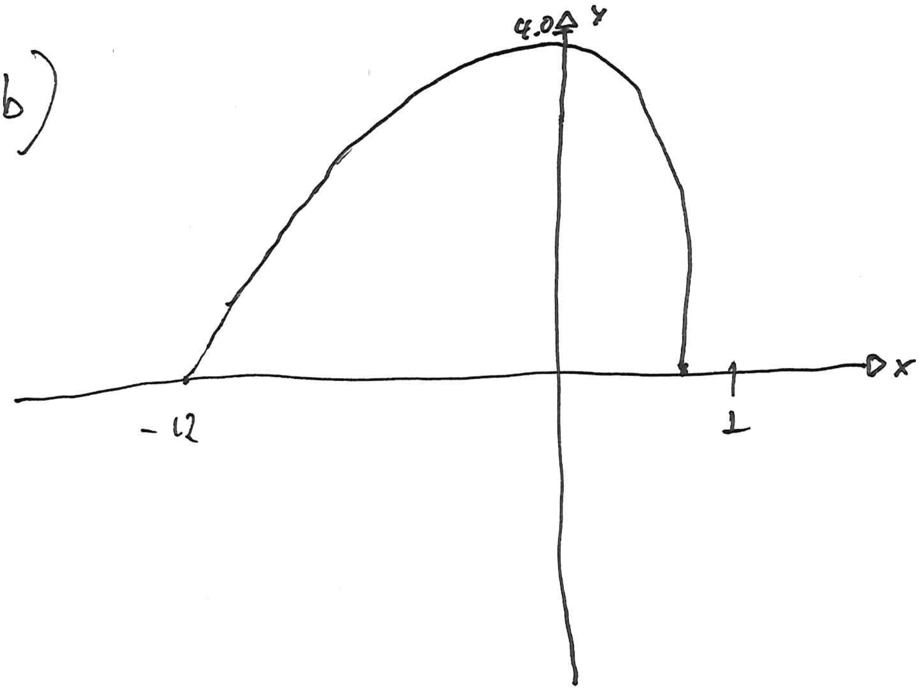
$$\frac{\partial L}{\partial r} = m r \omega^2$$

$$\Rightarrow \quad m\ddot r + m r \omega^2 = 0 \Rightarrow \ddot r = -r\omega^2$$

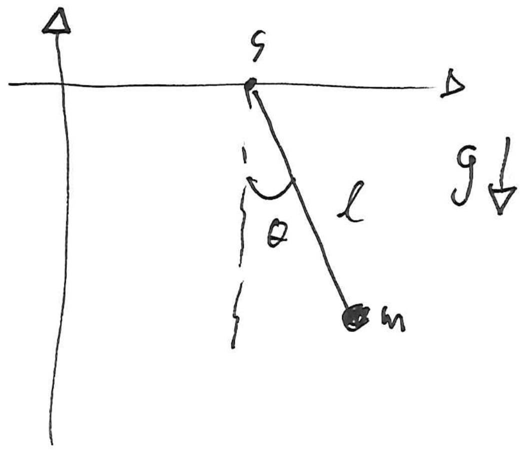
$$r = r_0 \cosh(\omega t)$$
$$\dot r = -r_0 \sinh(\omega t)\omega$$
$$\ddot r = -r_0\omega^2\cosh(\omega t) = -\omega^2 r$$

$r = r_0 \cosh(\omega t)$ is a solution

b)



4a)



$$L = K - V = \frac{1}{2} m l^2 \dot{\theta}^2 + m \vec{g} l \cos\theta$$

$$\Rightarrow \frac{d}{dt}\left[m l^2 \dot{\theta}\right] - m \vec{g} l \sin\theta = 0$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\theta$$

4b)

$x = s + l\sin\theta \quad\Big|\quad \dot{x} = \dot{s} + \dot{\theta} l\cos\theta$

$y = -l\cos\theta \quad\Big|\quad \dot{y} = +l\sin\theta\,\dot{\theta}$

$\vec{r} = x\hat{i} + y\hat{j} = s + l\sin\theta\,\hat{i} + l\cos\theta\,\hat{j}$

$\dot{\vec{r}} = \dot{s} + \dot{\theta} l\cos\theta\,\hat{i} + l\sin\theta\,\dot{\theta}\,\hat{j}$

$\dot{\vec{r}}^2 = \dot{s}^2 + 2\dot{s}\dot{\theta} l\cos\theta + \dot{\theta}^2 l^2\cos^2\theta + l^2\sin^2\theta\,\dot{\theta}^2$

$K = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m\left[\dot{s}^2 + 2\dot{s}\dot{\theta} l\cos\theta + \dot{\theta}^2 l^2\cos^2\theta + l^2\sin^2\theta\,\dot{\theta}^2\right]$

$V = -mgl\cos\theta$

$L = K - V = \frac{1}{2}m\left[\dot{s}^2 + 2\dot{\theta}\dot{s}\,l\cos\theta + \dot{\theta}^2 l^2\right] + mgl\cos\theta$

$\frac{\partial L}{\partial s} = 0 \quad , \quad \frac{\partial L}{\partial \theta} = \frac{1}{2}m\left[-2\dot{\theta}\dot{s}\,l\sin\theta\right] + (-mgl\sin\theta)$

$= ml\left[-\dot{\theta}\dot{s}\sin\theta - g\sin\theta\right]$

$\frac{\partial L}{\partial \dot{s}} = m\left[\dot{s} + \dot{\theta} l\cos\theta\right] \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = m\left[\ddot{s} + \ddot{\theta} l\cos\theta + \dot{\theta}(-l)\sin\theta\right]$

$\frac{\partial L}{\partial \dot{\theta}} = m\left[\dot{s}\,l\cos\theta + \dot{\theta} l^2\right] \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m\left[\ddot{s}\,l\cos\theta + \dot{s}\,l(-\sin\theta)\dot{\theta} + \ddot{\theta} l^2\right]$

$s\text{:}\quad \ddot{s} + \ddot{\theta} l\cos\theta - \dot{\theta} l\sin\theta = 0$

$\Rightarrow \ddot{s} = \dot{\theta} l\sin\theta - \ddot{\theta} l\cos\theta$

$\theta\text{:}\quad m\left[\ddot{s}\,l\cos\theta + \dot{\theta}\dot{s}\,l(-\sin\theta) + \ddot{\theta} l^2\right] - ml\left[-\dot{\theta}\dot{s}\sin\theta - g\sin\theta\right] = 0$

$\Rightarrow \ddot{s}\,l\cos\theta + \ddot{\theta} l^2 + lg\sin\theta = 0$

$\Rightarrow \ddot{s}\cos\theta + \ddot{\theta} l + g\sin\theta = 0$

4c) $\ddot{s} = \dot{\theta} l \sin\theta - \ddot{\theta} l \cos\theta$

$\ddot{s}\cos\theta + \ddot{\theta} l + g\sin\theta = 0$

$\Rightarrow l\dot{\theta}\sin\theta\cos\theta - \ddot{\theta} l \cos^2\theta + \ddot{\theta} l + g\sin\theta = 0$

$\Rightarrow l\dot{\theta}\sin\theta\cos\theta + \ddot{\theta} l \sin^2\theta + g\sin\theta = 0$

~~$\Rightarrow \frac{1}{2}\dot{\theta}\sin(2\theta) + \ddot{\theta} l \sin^2\theta + g\sin\theta = 0$~~

$\Rightarrow l\dot{\theta}\cos\theta + \ddot{\theta} l \sin\theta = -g$

$$\boxed{\ddot{y} = -l[\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}] = g}$$