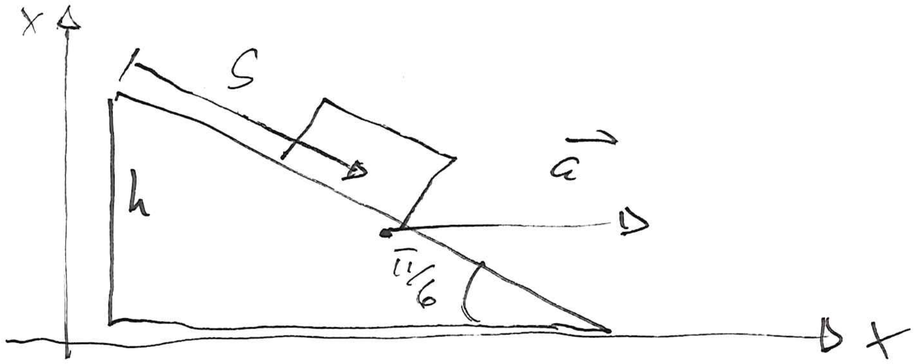
oblig 3 Kristian Tao

Problem 1

a)



$$x = s\cos(\pi/6) = s\frac{\sqrt{3}}{2}$$

$$\dot{x} = \dot{s}\,\sqrt{3}/2$$

$$y = h - s\sin(\pi/6) = h - \frac{s}{2}$$

$$\dot{y} = -\dot{s}/2$$

$$L = \frac{1}{2}m\left(\dot{s}^2\cos^2(\pi/6) + \dot{s}^2\sin^2(\pi/6)\right) + mg\left(h - s/2\right)$$

$$= \frac{1}{2}m\dot{s}^2 - mg\,s/2 + mgh\ \cancel{0}$$

b) $x = s\cos(\pi/6) + \frac{1}{2}at^2$

$\dot{x} = \dot{s}\cos(\pi/6) + at$

$y = h - s\sin(\pi/6)$

$\dot{y} = -\dot{s}\sin(\pi/6)$

$$L = \frac{1}{2}m\left(\dot{s}^2\sin^2(\pi/6) + \dot{s}^2\cos(\pi/6) + 2\dot{s}\cos(\pi/6)at + a^2t^2\right) + mg(h - s\sin(\pi/6))$$

$$= \frac{1}{2}m(\dot{s}^2 + \sqrt{3}\,\dot{s}at + a^2t^2) + mg(h - \tfrac{s}{2})$$

c) $\frac{\partial L}{\partial s} = -\frac{1}{2}mg$

$\frac{\partial L}{\partial \dot{s}} = m\dot{s} + \frac{\sqrt{3}}{2}mat \Rightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{s}}\right] = m\ddot{s} + \frac{\sqrt{3}}{2}ma$

e.o.m $\Rightarrow$ $m\ddot{s} + \frac{\sqrt{3}}{2}ma + \frac{1}{2}mg = 0$

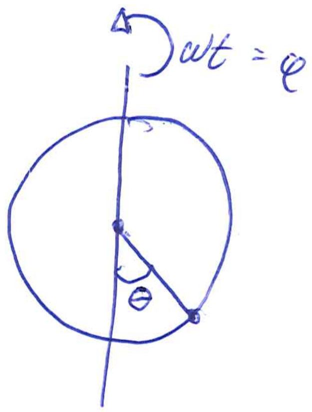
$\Rightarrow \ddot{s} = -\frac{g + \sqrt{3}a}{2}$

$\Rightarrow \dot{s} = \int_0^t \ddot{s}\,dt = -\frac{g + \sqrt{3}a}{2}t$

$$s = \int_0^t \dot{s}\,dt = -\frac{1}{4}(g + \sqrt{3}a)t^2$$

Problem 2

a)



$x = r\sin\theta\cos\varphi \qquad \dot{x} = r(\cos\theta\cos\varphi\,\dot\theta - \omega\sin\theta\sin\varphi$

$y = r\sin\theta\sin\varphi \qquad \dot{y} = r(\dot\theta\cos\theta\sin\varphi + \omega\sin\theta\cos\varphi)$

$z = r\cos\theta \qquad \dot{z} = -r\sin\theta$

$$\dot{\vec{r}}^2 = r^2(\dot\theta^2\cos^2\theta\cos^2\varphi - 2\cos\theta\cos\varphi\,\dot\theta\omega\sin\varphi\sin\theta$$
$$+ \omega^2\sin^2\theta\sin^2\varphi$$
$$+ \dot\theta^2\cos^2\theta\sin^2\varphi + 2\dot\theta\omega\cos\theta\sin\varphi\sin\theta\cos\varphi$$
$$+ \dot\theta^2\sin^2\theta$$

$$= r^2(\dot\theta^2\cos^2\theta + \omega^2\sin^2\theta + \dot\theta^2\sin^2\theta)$$
$$= r^2(\dot\theta^2 + \omega^2\sin^2\theta)$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}mr^2(\dot\theta^2 + \omega^2\sin^2\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = mr^2\omega^2 \sin\theta\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = mr^2\dot\theta \quad \Bigg| \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = mr^2\ddot\theta$$

$$\Rightarrow \quad mr^2\ddot\theta - mr^2\omega^2 \sin\theta\cos\theta = 0$$

b) For a particle moving in a circle with radius r and angle θ the kinetic energy is:

$$\cancel{K = \frac{1}{2}m\dot{r}^2}$$

$$K = \frac{1}{2}mr^2\dot\theta^2$$

$$\Rightarrow \quad \mathcal{L} = K - V = \frac{1}{2}mr^2\dot\theta^2 - V$$

And we se from our lagrangian that this makes our potential

$V = \frac{1}{2}mr^2\omega^2\sin^2\theta$, wich is explicitly

time-independant and periodic with two stable $(\pi/2, \frac{3\pi}{2})$ and two unstable $(0, \pi)$ equilibriumpoints within the interval $[0, 2\pi)$

c) $\theta_0 = \pi/2$

$$\theta = \varphi + \theta_0 = \varphi + \pi/2 \quad \Big| \quad \frac{d^2}{dt^2}\left(\varphi + \pi/2\right) = \ddot{\varphi}$$

$$\sin(\varphi + \pi/2) = \cos(\varphi)$$
$$\cos(\varphi + \pi/2) = -\sin(\varphi)$$

E.o.m:

$$\ddot{\theta} = \omega^2 \sin\theta \cos\theta$$

$$\Rightarrow \ddot{\varphi} = \omega^2 \sin(\varphi + \theta_0)\cos(\varphi + \theta_0)$$

$$= -\omega^2 \cos\varphi \sin\varphi$$

For small φ:

$$\cos\varphi \approx 1$$

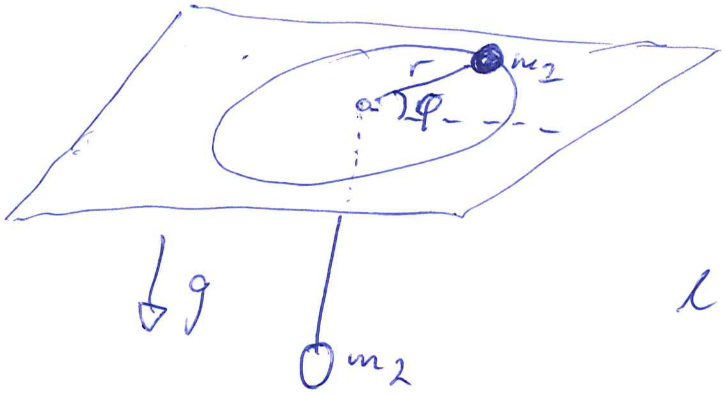
$$\sin\varphi \approx \varphi$$

$$\Rightarrow \ddot{\varphi} = -\omega^2 \varphi \longleftarrow \text{Harmonic oscillator!!}$$

The angular frequency is ω

Problem 3

a)



$m_1 = m_2 = m$

l = length of rope

not dot

$$m_1: \quad x = r\cos\varphi \quad \Big| \quad \dot{x} = \dot{r}\cos\varphi - \dot{r}\dot{\varphi}\sin\varphi$$
$$\qquad y = r\sin\varphi \quad \Big| \quad \dot{y} = \dot{r}\sin\varphi + \dot{r}\dot{\varphi}\cos\varphi$$

$$m_2: \quad z = l - r \quad \Big| \quad \dot{z} = -\dot{r}$$

$$\dot{\vec{r}}^2 = \dot{r}^2\cos^2\varphi - 2\dot{r}r\dot{\varphi}\sin\varphi\cos\varphi + r^2\dot{\varphi}^2\sin^2\varphi$$
$$+ \dot{r}^2\sin^2\varphi + 2r\dot{r}\dot{\varphi}\sin\varphi\cos\varphi + r^2\dot{\varphi}^2\cos^2\varphi$$
$$+ \dot{r}^2$$
$$= 2\dot{r}^2 + r^2\dot{\varphi}^2$$

$$T = \frac{1}{2}m\left(2\dot{r}^2 + r^2\dot{\varphi}^2\right)$$

$$V = -mg(l - r)$$

$$\mathcal{L} = \frac{1}{2}m\left(2\dot{r}^2 + r^2\dot{\varphi}^2\right) + mg(l - r)$$

$$\frac{\partial \mathcal{L}}{\partial r} = m r \dot{\varphi}^2 - mg \quad , \quad \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = 2m\dot{r} \quad , \quad \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] = 2m\ddot{r}$$

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = \frac{d}{dt}\Big(\underbrace{m r^2 \dot{\varphi}}_{\substack{\text{constant} \\ = k}}\Big)$$

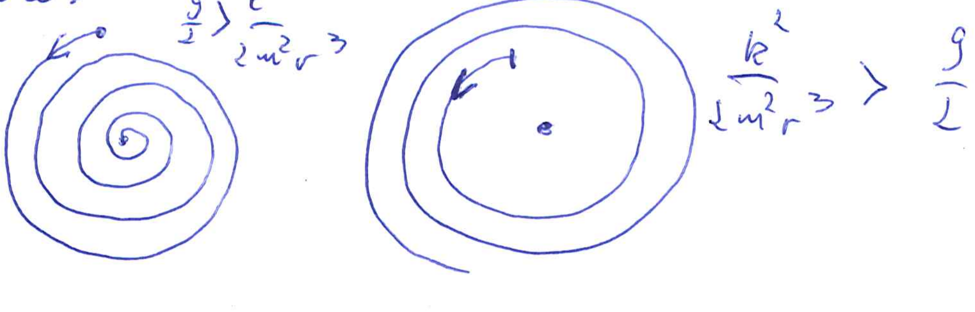
$$r: \; 2m\ddot{r} + mg - m r \dot{\varphi}^2 = 0$$

$$\varphi: \; \dot{\varphi} = \cancel{mr^2} \; \frac{k}{mr^2}$$

b) $$2m\ddot{r} - mr\frac{k^2}{m^2 r^4} + mg = 0$$

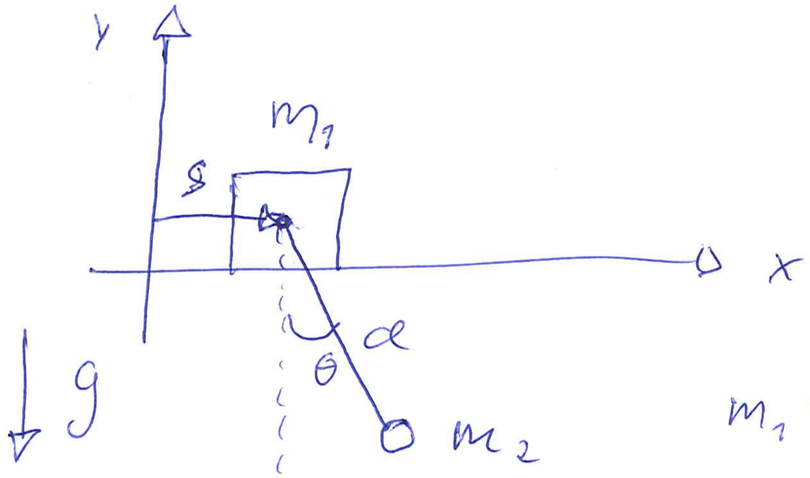
$$\Rightarrow \ddot{r} = \left(-mg + \frac{k^2}{mr^3}\right) \cdot \frac{1}{2m} = \frac{k^2}{2m^2 r^3} - \frac{g}{2}$$

Gravity is trying to pull mass m_2 towards the hole, but the centrifugal force is trying to pull the mass away from the hole.



Problem 4

a)



$m_1 = m_2 = m$

m_1: $x_1 = s$ | $\dot{x}_1 = \dot{s}$

m_2: $x_2 = s + d\sin\theta$ | $\dot{x}_2 = \dot{s} + d\cos\theta\,\dot{\theta}$

$y = -d\cos\theta$ | $\dot{y} = d\sin\theta\,\dot{\theta}$

$\cancel{T} =$

$$\dot{r}^2 = \dot{s}^2 + \dot{s}^2 + 2\dot{s}d\cos\theta\,\dot{\theta} + d^2\cos^2\theta\,\dot{\theta}^2 + d^2\sin^2\theta\,\dot{\theta}^2$$

$$= 2\dot{s}^2 + d^2\dot{\theta}^2 + 2\dot{s}d\cos\theta\,\dot{\theta}$$

$T = \frac{1}{2}m(\dot{r})^2$; $V = -mgh = -mg(-d\cos\theta) = mgd\cos\theta$

~~$V = -mgh = -mg(-d\cos\theta) = -mgd\cos\theta$~~

$$\mathcal{L} = \frac{1}{2}m(2\dot{s}^2 + d^2\dot{\theta}^2 + 2\dot{s}d\cos\theta\,\dot{\theta}) + mgd\cos\theta$$

b) We have a cyclic coordinate when

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Then we have a conserved quantity, a "constant of motion" because

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$

which means

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{Constant},$$

and we can use this to eliminate $\dot{q}_i$ from the other e.o.m/Lagrangian.

$$\cancel{x}\quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) = \frac{d}{dt}\underbrace{\left(2m\dot{s} + m\frac{2}{2}d\dot{\theta}\cos\theta\right)}_{k} = 0$$

$$\Rightarrow \dot{s} = \frac{k - d\dot{\theta}\cos\theta m}{2 \cdot m} = \frac{1}{2}\left(\frac{k}{m} - d\dot{\theta}\cos\theta\right)$$

$$\mathcal{L} = \frac{1}{2}m\left(\frac{2}{4}\left[\frac{k}{m} - d\dot{\theta}\cos\theta\right]^2 + \frac{2}{2}\left[\frac{k}{m} - d\dot{\theta}\cos\theta\right]d\cos\theta\,\dot{\theta} + d^2\dot{\theta}^2\right)$$

$$+ mgd\cos\theta =$$

$$= \frac{k^2}{4m} + \left(1 - \frac{1}{2}\cos^2\theta\right)\frac{md^2\dot{\theta}^2}{2} + mg\cos\theta$$

c) $\mathcal{L} = \frac{k^2}{4m} + \left(1 - \frac{1}{2}\cos^2\theta\right)\frac{md^2\dot{\theta}^2}{2} + mg\cos\theta$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2} md^2\dot{\theta}^2 \cos\theta \sin\theta - mg\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = md^2\dot{\theta}\left(1 - \frac{1}{2}\cos^2\theta\right)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = md^2\ddot{\theta} - \left[\frac{md^2}{2}\left(\ddot{\theta}\cos^2\theta - \dot{\theta}\cdot 2\cos\theta\cdot\sin\theta\cdot\dot{\theta}\right)\right]$$

$$= md^2\ddot{\theta} - \frac{md^2}{2}\ddot{\theta}\cos^2\theta + \dot{\theta}^2 md^2 \cos\theta\sin\theta$$

$$= md^2\ddot{\theta}\left(1 - \cos^2\theta\right) + \dot{\theta}^2 md^2\cos\theta\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\Rightarrow \ddot{\theta}\left(1 - \cos^2\theta\right) + \frac{1}{2}\dot{\theta}^2\cos\theta\sin\theta + \frac{\cancel{m}g}{d}\sin\theta = 0$$

d)

$$\cos^2\theta \approx 1$$

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1$$

$$\Rightarrow \left(1-\frac{1}{2}\right)\ddot{\theta} + \frac{1}{2}\theta\dot{\theta}^2 + \frac{g}{d}\theta = 0$$

we do not like this, so we set $\dot{\theta}^2 \approx 0$

$$\Rightarrow \ddot{\theta} + \frac{2g}{d}\theta = 0$$

$$\omega = \sqrt{\frac{2g}{d}}$$