FYS3140 Mathematical Methods, Home exam 2014

Due Monday April 7 at 14:30h (strict deadline!)

28. mars 2014

Important – please read:

- **Mark your paper with your *candidate number*, not your name!**
- **Hand in at the front office, same place as for homeworks – do not use email.**
- **Please sign your name on the list at the counter when handing in!**
- **Keep a copy of your paper!**

Good luck! :)

Problem 1: Contour integrals and Fourier transforms

a) Use contour integration to calculate the following integral, briefly explaining your reasoning at the most important steps:

$$I = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + b^2}\, dx, \tag{1}$$

where b is a positive real constant, and a is a real constant. Make sure your calculation covers both possibilities, $a \geq 0$ and $a < 0$.

b) Use your result from **a)** to calculate the (inverse) Fourier transform $f(x)$ of

$$F(k) = \frac{e^{-i\alpha k}}{k^2 + \beta^2} \tag{2}$$

where α and β are positive real constants. Preferably use the same conventions (signs and overall constants) for Fourier transforms as in the lectures (if you do not attend the lectures, you can find the definitions in the slide presentations file published on the web page).

Problem 2: Laurent series

Consider the complex function

$$f(z) = \frac{1 + z}{z(z-4)^3}. \tag{3}$$

a) Identify the singularities (type and order) of $f(z)$. Draw a figure where you indicate the singularities, as well as the disk $|z - 4| < 4$.

b) Find the first few terms of the Laurent series of $f(z)$ around the origin, valid for $|z| < 4$. From this, identify Res($f, z = 0$). You can check your result by computing the residue directly. *Hint:* Relevant series expansion formulas can be looked up in Rottmann p.115.

c) Use your result from **b)** to compute the integrals

$$\oint_{|z|=1} f(z) dz \tag{4}$$

and

$$\oint_{|z-2|=1} f(z) dz. \tag{5}$$

d) Finally, compute the Laurent expansion of $f(z)$ for the disk $|z - 4| < 4$.

Answer:

$$f(z) = \frac{5}{4} \frac{1}{(z-4)^3} + \sum_{s=-2}^{\infty} \frac{(-1)^{s+1}}{4^{s+4}} (z-4)^s. \tag{6}$$

Problem 3: Green's functions

In this problem we will examine the Green's function method for ordinary DE's in a somewhat more general way than presented in the lectures. Consider the differential equation

$$y'' + P(x)y' + Q(x)y = R(x) \tag{7}$$

with specified boundary conditions

$$y(a) = y(b) = 0. \tag{8}$$

We assume that two linearly independent solutions, $y_1(x)$ and $y_2(x)$, of the homogeneous equation are known, and that $y_1(a) = y_2(b) = 0$ [see problem **d)** for a discussion of this point].

a) Show that the Green's function is

$$G(x,z) = \begin{cases} y_1(x)y_2(z)/W(z) & \text{if } a < x < z \\ y_2(x)y_1(z)/W(z) & \text{if } z < x < b \end{cases} \tag{9}$$

where $W(z) = y_1(z)y_2'(z) - y_2(z)y_1'(z)$ is the Wronskian. (Make sure to keep track of which variables are x's and which are z's...)

b) Proceed by showing that the full solution of the DE can be written as

$$y(x) = y_2(x)\int_a^x \frac{y_1(z)R(z)}{W(z)}dz - y_1(x)\int_b^x \frac{y_2(z)R(z)}{W(z)}dz. \tag{10}$$

c) Show that this result can also be used to find a *particular* solution once the homogeneous solution is known. In other words, that this reproduces the result found from variation of parameters, Eq.(61) in the lecture note on DE's,

$$y_p(x) = y_2(x)\int \frac{y_1(x)R(x)}{W(x)}dx - y_1(x)\int \frac{y_2(x)R(x)}{W(x)}dx. \tag{11}$$

now with *indefinite* integrals. *Hint:* See section 8.12 in Boas.

d) Finally, let us return to the starting point of this problem where we demanded that $y_1(a) = y_2(b) = 0$, which at first sight may look like a rather strong restriction. Assume we have found two linearly independent solutions $y_1(x)$ and $y_2(x)$ to the homogeneous equation that are *not* zero at these points,

$$y_1(a) = c_1;\ y_1(b) = c_2;\ y_2(a) = d_1;\ y_2(b) = d_2, \tag{12}$$

all nonzero. Let's examine the possibility to construct a new set of linearly independent solutions as linear combinations of the old ones,

$$\tilde{y}_1(x) = \alpha y_1(x) + \beta y_2(x) \tag{13}$$

$$\tilde{y}_2(x) = \gamma y_1(x) + \delta y_2(x) \tag{14}$$

such that $\tilde{y}_1(a) = \tilde{y}_2(b) = 0$. Find the conditions on $\alpha, \beta, \gamma, \delta$ that follow from demanding linear independence and, consequentially, show that

$$\begin{vmatrix} y_1(a) & y_1(b) \\ y_2(a) & y_2(b) \end{vmatrix} \neq 0. \tag{15}$$