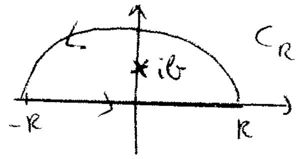
PROBLEM 1

a) $$I = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + b^2}\, dx$$

$a \geq 0$: Contour integral, close in upper half-plane $\Rightarrow$ Jordan's lemma OK (even if $a=0$ since deg(denominator) = deg(num.)+2)

$$\Rightarrow \quad I = \oint_C \frac{e^{iaz}\, dz}{(z+ib)(z-ib)}$$



$$= 2\pi i \,\mathrm{Res}\,(ib) = 2\pi i \frac{e^{-ab}}{2ib} = \frac{\pi}{b} e^{-ab}$$

$a<0$: Close in lower half-plane. NB! Contour has negative orientation

$$I = -2\pi i \,\mathrm{Res}\,(-ib) = -2\pi i \frac{e^{ab}}{(-2ib)} = \frac{\pi}{b} e^{ab} = \frac{\pi}{b} e^{-|a|\cdot b}$$

$$\Rightarrow \quad \underline{\underline{I = \frac{\pi}{b} e^{-|a|\cdot b}}}$$

b) $F(k) = \dfrac{e^{-i\alpha k}}{k^2+\beta^2}$ so $f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} F(k) e^{ikx}\, dk = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \dfrac{e^{ik(x-\alpha)}}{k^2+\beta^2}\, dk$

Identify with a): $(x-\alpha) \leftrightarrow a$ and $\beta \leftrightarrow b$

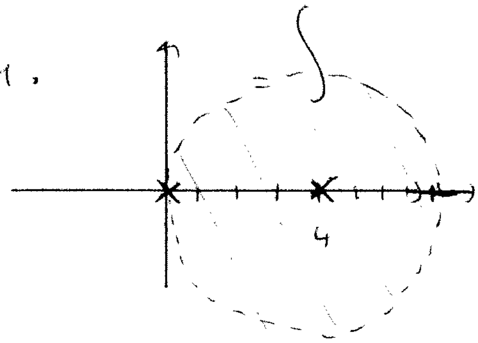
$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\pi}{\beta} e^{-\beta|x-\alpha|} = \underline{\underline{\sqrt{\frac{\pi}{2}} \cdot \frac{1}{\beta} e^{-\beta\cdot|x-\alpha|}}}$$

PROBLEM 2

$$f(z) = \frac{1+z}{z(z-4)^3}$$

$|z-4| < 4$

a) Simple pole $z=0$, 3rd order pole at $z=4$.



b) $|z|<4$: $\underline{\underline{f(z)}} = \left(1+\frac{1}{z}\right)\cdot\frac{1}{(z-4)^3} = \left(1+\frac{1}{z}\right)\left(-\frac{1}{4^3}\right)\cdot\frac{1}{\left(1-\frac{z}{4}\right)^3}$

$$= -\frac{1}{4^3}\left(1+\frac{1}{z}\right)\cdot\left(1+\frac{3}{4}z+\frac{3}{8}z^2+\frac{5}{32}z^3+\dots\right)$$

$$= -\frac{1}{64}\left[\frac{1}{z}+\left(1+\frac{3}{4}\right)+\left(\frac{3}{4}+\frac{3}{8}\right)z+\dots\right]$$

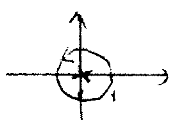
$$= \underline{\underline{-\frac{1}{64}\cdot\frac{1}{z}-\frac{7}{256}-\frac{9}{512}z-\dots}}$$

$$\underline{\underline{\mathrm{Res}(f;0) = -\frac{1}{64}}}$$

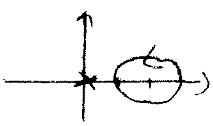
Direct check:

$$\mathrm{Res}(f; z=0) = \lim_{z\to 0}\frac{1+z}{(z-4)^3} = -\frac{1}{4^3} = \underline{\underline{-\frac{1}{64}}} \quad \checkmark$$

c) $\oint_{|z|=1} f(z)\,dz = 2\pi i\,\mathrm{Res}(0) = \underline{\underline{\frac{-\pi i}{32}}}$



$\oint_{|z-2|=1} f(z)\,dx = \underline{\underline{0}}$ (Cauchy)



d) $0 < |z-4| < 4$

* Want powers of $(z-4)$. [Positive powers since $|z-4| < 4$.]

$$f(z) = \frac{1}{(z-4)^3} + \frac{1}{(z-4)^3} \cdot \frac{1}{z} = \frac{1}{(z-4)^3} + \frac{1}{(z-4)^3} \cdot \frac{1}{4 + z - 4}$$

$$= \frac{1}{(z-4)^3} + \frac{1}{(z-4)^3} \cdot \frac{1}{4} \frac{1}{1 - \left(-(z-4)/4\right)}$$

$$= \frac{1}{(z-4)^3} + \frac{1}{(z-4)^3} \cdot \frac{1}{4} \sum_{n=0}^{\infty} \left[\frac{-(z-4)}{4}\right]^n$$

$$= \frac{1}{(z-4)^3} + \frac{1}{(z-4)^3} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} (z-4)^n$$

$$= \left(1 + \frac{1}{4}\right) \cdot \frac{1}{(z-4)^3} + \cancel{\frac{1}{(z-4)^3}} \sum_{n=1}^{\infty} \frac{(-1)^n}{4^{n+1}} (z-4)^{n-3} \qquad \boxed{s = n-3}$$

$$= \frac{5}{4} \cdot \frac{1}{(z-4)^3} + \sum_{s=-2}^{\infty} \frac{(-1)^{s+1}}{4^{s+4}} (z-4)^s$$

PROBLEM 3

$y'' + P(x)y' + Q(x)y = R(x)$; $y(a) = y(b) = 0$;

$y_h(x) = C_1 y_1 + C_2 y_2$ with $y_1(a) = y_2(b) = 0$

a) DE for $G(x,z)$: $[\partial_x^2 + P\partial_x + Q]\, G(x,z) = \delta(x-z)$

$\Rightarrow \quad G(x,z) = A(z)\, y_1(x) + B(z)\, y_2(x) \quad ; \quad a < x < z$

$\wedge \quad G(x,z) = C(z)\, y_1(x) + D(z)\, y_2(x) \quad ; \quad z < x < b$

Boundary conditions:

(i) $x=a$: $G(a,z) = A(z)\, \underbrace{y_1(a)}_{0} + B(z) y_2(a) = B(z) y_2(a) = 0$

$\Rightarrow \underline{B(z) = 0}$

(ii) $x=b$: $G(b,z) = C(z)\, y_1(b) + D(z) \underbrace{y_2(b)}_{0} = 0 = C(z)\, y_1(b) = 0$

$\Rightarrow \quad C(z) = 0$

so

$$G(x,z) = \begin{cases} A(z)\, y_1(x) \; ; & a < x < z \\ D(z)\, y_2(x) \; ; & z < x < b \end{cases}$$

Continuity & derivatives at $x=z$:

$$\left.\begin{aligned} A(z)\, y_1(z) - D(z)\, y_2(z) &= 0 \\ \wedge \quad D(z)\, y_2'(z) - A(z)\, y_1'(z) &= 1 \end{aligned}\right\} \text{ Solve for A and D}$$

$$\left.\begin{array}{ll}(i) & Ay_1y_1' - Dy_2y_1' = 0 \\ \wedge & -Ay_1'y_1 + Dy_2'y_1 = y_1\end{array}\right\} \Rightarrow D\underbrace{(y_1y_2' - y_2y_1')}_{\text{Wronskian}} = y_1$$

$$\Rightarrow \underline{D(z) = \frac{y_1(z)}{W(z)}}$$

$$\left.\begin{array}{ll}(ii) & Ay_1y_2' - Dy_2y_2' = 0 \\ \wedge & -Ay_1'y_2 + Dy_2'y_2 = y_2\end{array}\right\} \Rightarrow A\underbrace{(y_1y_2' - y_2y_1')}_{W(z)} = y_2$$

$$\Rightarrow \underline{A(z) = \frac{y_2(z)}{W(z)}}$$

Thus,

$$G(x,z) = \begin{cases} y_1(x)\,y_2(z)/W(z), & x < z \\ y_2(x)\cdot y_1(z)/W(z), & x > z \end{cases}$$

b) The full solution is given by

$$y(x) = \int_a^b G(x,z)\,R(z)\,dz$$

$$= \underset{(z<x)}{\int_a^x y_2(x)\,y_1(z)\,R(z)/_{W(z)}\,dz} + \underset{(z>x)}{\int_x^b y_1(x)\,y_2(z)\,R(z)/_{W(z)}\,dz}$$

$$= y_2(x)\int_a^x \frac{y_1(z)\,R(z)}{W(z)}\,dz - y_1(x)\int_b^x \frac{y_2(z)\,R(z)}{W(z)}\,dz$$

switched limits

c) The argument is given in Boas. Note that

$$\int_\alpha^x f(x')\,dx' = \underbrace{F(x)}_{\text{some function of } x} - \text{constant}$$

So the result of b) can be schematically written as

$$y(x) = y_2(x)\cdot\left[F_1(x) - \alpha\right] - y_1(x)\left[F_2(x) - \beta\right]$$

$$= \underbrace{C_1\,y_1(x) + C_2\,y_2(x)}_{} + \underbrace{y_2(x)\,F_1(x) - y_1(x)\,F_2(x)}_{\text{PARTICULAR SOLUTION}}$$

C_1, C_2: SOME constants

The constant limits give the homogeneous solution → SKIP!

$$\Rightarrow \boxed{y_p = y_2(x)\int \frac{y_1(x)\,R(x)}{W(x)}\,dx - y_1(x)\int \frac{y_2(x)\,R(x)}{W(x)}\,dx}$$

with INDEFINITE integrals

d) $\tilde{y}_1 = \alpha y_1 + \beta y_2 \,; \quad \tilde{y}_2 = \gamma y_1 + \delta y_2$

$$\tilde{y}_1(a) = \alpha c_1 + \beta d_1 \overset{!}{=} 0 \quad \Rightarrow \quad \boxed{\alpha = -\beta \frac{d_1}{c_1}} \qquad (1)$$

$$\tilde{y}_2(b) = \gamma c_2 + \delta d_2 \overset{!}{=} 0 \quad \Rightarrow \quad \boxed{\gamma = -\delta \frac{d_2}{c_2}} \qquad (2)$$

<u>LINEAR INDEPENDENCE - TWO WAYS:</u>

(i) Use the Wronskian, demand $\tilde{W} \neq 0$:

$$\tilde{W} = \tilde{y}_1 \tilde{y}_2' - \tilde{y}_1' \tilde{y}_2$$

$$= (\alpha y_1 + \beta y_2)(\gamma y_1' + \delta y_2') - (\alpha y_1' + \beta y_2')(\gamma y_1 + \delta y_2)$$

$$= \alpha\gamma \cancel{y_1 y_1'} + \alpha\delta y_1 y_2' + \beta\gamma y_1' y_2 \underbrace{}_{+\beta\delta \cancel{y_2 y_2'}} - \left[\alpha\gamma \cancel{y_1 y_1'} + \alpha\delta y_1' y_2 + \beta\gamma y_1 y_2' + \beta\delta \cancel{y_2 y_2'}\right]$$

$$= \alpha\delta(y_1 y_2' - y_1' y_2) - \beta\gamma(y_1 y_2' - y_2 y_1') = (\alpha\delta - \beta\gamma) \cdot W$$

Know that the original Wronskian, $W \neq 0$.
So for $\tilde{W} \neq 0$ we must have

$$\underline{\underline{\alpha\delta - \beta\gamma \neq 0}} \quad \text{or} \quad \underline{\underline{\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0}} \qquad (3)$$

(ii) By inspection, noting that we can scale away overall factors,

$\tilde{y}_1 = y_1 + \frac{\beta}{\alpha} y_2$ (divided by α)

$\tilde{y}_2 = y_1 + \frac{\delta}{\gamma} y_2$ (— " — γ)

For linear independence, must have $\underline{\underline{\frac{\beta}{\alpha} \neq \frac{\delta}{\gamma}}}$, same as (3).

Finally, combine (3) with (1) and (2),

$$\alpha\delta - \beta\gamma = -\beta\delta\frac{d_1}{c_1} - \beta\left(-\delta\frac{d_2}{c_2}\right)$$

$$= \beta\delta\left(\frac{d_2}{c_2} - \frac{d_1}{c_1}\right) = \frac{\beta\delta}{c_1 c_2}\left(c_1 d_2 - c_2 d_1\right)$$

For this to be non-zero we must have

$$c_1 d_2 - c_2 d_1 = y_1(a)\, y_2(b) - y_2(a)\, y_1(b) \neq 0, \quad \text{i.e.}$$

$$\underline{\underline{\begin{vmatrix} y_1(a) & y_2(a) \\ y_1(b) & y_2(b) \end{vmatrix} \neq 0}}$$