SOLUTION TO PROBLEM SET 12 FYS 3140

9.2.3

-problem:

$$\int_{x_1}^{x_2} x\sqrt{1-y'^2}dx$$

-solution:

-independent variable: x

-dependent variable: y

$$F(x:y,y') = x\sqrt{1-y'^2}$$

To minimize the integral or make it stationary we solve the following equation called Euler equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 \tag{1}$$

for this problem is therefore,

$$\frac{\partial F}{\partial y} = 0; \qquad \frac{\partial F}{\partial y'} = \frac{xy'}{\sqrt{1-y'^2}}$$

plugging these values in eqn(1), we obtain

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

$$\frac{xy'}{\sqrt{1-y'^2}} = C$$

where C is some constant. Rearranging the above equation and solving for y we get the following

$$\begin{aligned} y' &= \frac{C}{\sqrt{C^2+x^2}} \\ y &= C\int \frac{dx}{\sqrt{C^2+x^2}} \\ &= \underline{C\ln|x+\sqrt{C^2+x^2}|+C_2} \end{aligned}$$

The last line is using trigonometric substitution or using integral tables, and C_2 is another constant.

9.2.5
-problem:

$$\int_{x_1}^{x_2} (y'^2 + y^2)dx$$

-solution:
-independent variable: x
-dependent variable: y

$$F(x : y, y') = y'^2 + y^2$$
$$\frac{\partial F}{\partial y} = 2y; \qquad \frac{\partial F}{\partial y'} = 2y'$$

Now eqn(1) implies

$$y'' - y = 0$$

Solving the above differential equation where the auxiliary equation is $r^2 - 1 = 0$, we get the following

$$y = Ae^{-x} + Be^{x} \tag{2}$$

9.3.7
-problem:

$$\int_{x_1}^{x_2} (y'^2 + y^2)dx$$

-solution:
-independent variable: x
-dependent variable: y
Now we make the following substitution and see if we simplified the problem further. Substitutions are used when F contains the dependent variable explicitly.

$$y' \to \frac{1}{x'}; \quad dx \to x'dy$$

then the problem is now rewritten as

$$\int_{x_1}^{x_2} (\frac{1}{x'} + y^2x')dy$$

due to the substitution the independent and dependent variables now changes to
-independent variable: y
-dependent variable: x

Therefore,

$$F(y : x, x') = \frac{1}{x'} + y^2 x'$$

$$\frac{\partial F}{\partial x} = 0; \qquad \frac{\partial F}{\partial x'} = -\frac{1}{x'^2} + y^2$$

Eqn(1) now becomes

$$\frac{d}{dy}\left(-\frac{1}{x'^2} + y^2\right) = 0$$

$$\Rightarrow -\frac{1}{x'^2} + y^2 = C$$

Rearranging the above equation and solving for x' we get the following

$$\begin{aligned} x'^2 &= \frac{1}{y^2 - C} \\ x &= \int \frac{dy}{\sqrt{y^2 - C}} \\ &= \ln(y + \sqrt{y^2 - C}) + C_2 \end{aligned}$$

again the last line is using trigonometric substitution or integral table. By rearranging the last line and making y subject, we obtain the following expression

$$y(x) = \frac{e^{(x - C_2)} + Ce^{-(x - C_2)}}{2}$$

which is identical with eqn(2). Note that the substitution method may not guarantee shorter steps.

9.5.11
-problem:
Find Lagrangian equation of motion for freely falling yo-yo of inner radius a and outer radius b
-solution:
based on the given coordinate system

$$U = -mgz$$

and the rotational and transitional kinetic energies will have the form

$$KE = KE_{translation} + KE_{rotation}$$

$$= \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2$$

but we know that $z = a\theta \Rightarrow \dot{z} = a\dot{\theta}$, hence

$$KE = \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{z}^2$$

and the Lagrangian of the problem is

$$L = T - U = \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{z}^2 + mgz$$

Now based on Hamiltonian principle the path followed by the yo-yo is defined by function z such that the Lagrangian stated above satisfies eqn(1). Therefore
-independent variable: t
-dependent variable: z

$$F(x : y, y') = L(t : z, z') = \frac{1}{2}\left(m + \frac{I}{a^2}\right) + mgz$$

$$\frac{\partial L}{\partial z} = mg; \qquad \frac{\partial L}{\partial \dot{z}} = \left(m + \frac{I}{a^2}\right)\dot{z}$$

putting these results in eqn(1) give us

$$\underline{\left(m + \frac{I}{a^2}\right)\ddot{z} - mg = 0}$$

13.4.4
-problem:
A string of length l is disturbed with the form given in Boas. Find the displacement of each point on the string as a function of x and t.
-solution
Starting from the wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ and separation of variable method where the separation constant is negative $-k^2$ we get the following solution requiring that the string is fixed at both ends. i.e.

$$y = \begin{cases} \sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}, \\ \sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}, \end{cases} \tag{3}$$

Since the string has zero initial velocity the solution to the problem is given using the superposition of the following terms,

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}$$

where the coefficients b_n are obtained by equating the above function with the initial shape of the string. At $t = 0$ the shape of string can be represented by the following function.

$$y_0 = f(x) = \begin{cases} \frac{4h}{l}x, & 0 < x < \frac{l}{4} \\ -\frac{4h}{l}x + 2h, & \frac{l}{4} < x < \frac{3l}{4} \\ \frac{4h}{l}x - 4h, & \frac{3l}{4} < x < l \end{cases}$$

at $t = 0$ we require

$$y_0 = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is a Fourier sine series for function $f(x)$ of period $2l$. Value of each coefficient is then obtained using

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l y_0 \sin \frac{n\pi x}{l} dx \\ &= \frac{2}{l}\left\{\int_0^{l/4} \frac{4h}{l} x \sin \frac{n\pi x}{l} dx - \int_{l/4}^{3l/4} \frac{4h}{l} x \sin \frac{n\pi x}{l} dx + \int_{l/4}^{3l/4} 2h \sin \frac{n\pi x}{l} dx \right. \\ &+ \left. \int_{3l/4}^{l} \frac{4h}{l} x \sin \frac{n\pi x}{l} dx - \int_{3l/4}^{l} 4h \sin \frac{n\pi x}{l} dx \right\} \\ &= \frac{16h}{(n\pi)^2}\left\{\sin \frac{n\pi}{4} + \sin \frac{3n\pi}{4}\right\} \end{aligned}$$

the solution to the problem is thus

$$y = \underline{\frac{16h}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\left\{\sin \frac{n\pi}{4} + \sin \frac{3n\pi}{4}\right\} \sin \frac{n\pi x}{l} \cos \frac{n\pi v t}{l}}$$

13.4.5

-problem:

A string of length l is disturbed with initial velocity of form given in Boas. Find the displacement of each point on the string as a function of x and t.

-solution

The shape of velocity at $t = 0$ can be represented by the following function

$$v_0(x) = \begin{cases} \frac{2h}{l}x, & 0 < x < \frac{l}{2} \\ -\frac{2h}{l}x + 2h, & \frac{l}{2} < x < l \end{cases}$$

Requiring the initial velocity is none-zero, From eqn(3) we choose to have basis function which contains products of two sine functions. The solution to the problem is thus

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}$$

where the coefficients b_n are determined by fitting the first derivative of the above equation with the initial velocity function stated above. i.e.

$$v_0(x) = \frac{\partial y}{\partial t}\bigg\|_{t=0} = \sum_{n=1}^{\infty} b_n \frac{n\pi v}{l}\sin\frac{n\pi x}{l}$$

and

$$\frac{l}{n\pi v}v_0(x) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

which is another Fourier sine series where the coefficients are determined using

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l \frac{l}{n\pi v}v_0(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\frac{l}{n\pi v}\left\{\int_0^{l/2}\frac{2h}{l}x\sin\frac{n\pi x}{l}dx - \int_{l/2}^{l}\frac{2h}{l}x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} 2h\sin\frac{n\pi x}{l}dx\right\} \\
&= \frac{l}{n\pi v}\frac{8h}{(n\pi)^2}\sin\frac{n\pi}{2}
\end{aligned}$$

The solution to the problem is thus

$$y = \underline{\frac{8h}{\pi^3 v}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}}$$

Extra problem 13.4.8

-problem:

Solve problem 13.4.5 with inital velocity function given by

$$v_0(x) = \begin{cases} \sin\frac{2\pi x}{l}, & 0 < x < \frac{l}{2} \\ 0, & \frac{l}{2} < x < l \end{cases}$$

-solution

Following same procedure as the previous problem, solution to the problem can be written as

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}$$

where the coefficients b_n are determined by fitting the first derivative of the above equation with the initial velocity function stated above. i.e.

$$v_0(x) = \left.\frac{\partial y}{\partial t}\right\|_{t=0} = \sum_{n=1}^{\infty} b_n \frac{n\pi v}{l}\sin\frac{n\pi x}{l}$$

and

$$\frac{l}{n\pi v}v_0(x) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

which is a Fourier sine series where the coefficients b_n are determined using

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l \frac{l}{n\pi v}v_0(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\frac{l}{n\pi v}\int_0^{l/2}\sin\frac{2\pi x}{l}\sin\frac{n\pi x}{l}dx \\
&= \frac{2}{n\pi v}\frac{n\sin l\pi\cos\frac{nl\pi}{2} - 2\sin\frac{nl\pi}{2}\cos l\pi}{\pi(4-n^2)}
\end{aligned}$$

The solution to the problem is thus

$$\underline{y = \frac{2}{\pi^2 v}\sum_{n=1}^{\infty}\frac{n\sin l\pi\cos\frac{nl\pi}{2} - 2\sin\frac{nl\pi}{2}\cos l\pi}{n(4-n^2)}\sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}}$$