Solutions to Problem Set 9 FYS3140

In this PS, we will use the *symmetrized* definitions of the Fourier transforms, both exponential and trigonometric:

$$\text{Fourier transform: } g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\ dx$$
$$\text{Inverse transform: } f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx}\ dk$$
$$\text{Sine transform: } g_s(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \sin(kx)\ dx$$
$$\text{Inverse: } f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(k) \sin(kx)\ dk$$
$$\text{Cosine transform: } g_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(x) \cos(kx)\ dx$$
$$\text{Inverse: } f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(k) \cos(kx)\ dk$$

Chapter 7

Problem 12.6

Find the exponential Fourier transform of

$$f(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \tag{1}$$

We find

$$\begin{aligned} g(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\ dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x e^{-ikx}\ dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{i}{k} x e^{-ikx} + \frac{1}{k^2} e^{-ikx} \right]_{x=-1}^{x=1} \\ &= \frac{1}{\sqrt{2\pi}} \frac{ki(e^{-ik} + e^{ik}) + (e^{-ik} - e^{ik})}{k^2} \end{aligned} \tag{2}$$

Therefore $f(x)$ can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx}\ dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{ki(e^{ik(x+1)} + e^{ik(x-1)}) - (e^{ik(x+1)} - e^{ik(x-1)})}{k^2}\ dk \tag{3}$$

Problem 12.18

Since the function in the previous problem is an odd function, it can be written as an inverse of the sine transform $g_s(k)$. We get in this case

$$\begin{aligned}
g_s(k) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(kx)\;dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^1 x\sin(kx)\;dx \\
&= \sqrt{\frac{2}{\pi}} \left[-\frac{1}{k}x\cos(kx) + \frac{1}{k^2}\sin(kx)\right]_{x=0}^{x=1} \\
&= \sqrt{\frac{2}{\pi}} \frac{-k\cos(k)+\sin(k)}{k^2}
\end{aligned} \tag{4}$$

Therefore $f(x)$ can be written as

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{-k\cos(k)+\sin(k)}{k^2}\sin(kx)\;dk \tag{5}$$

We replace the trigonometric functions with their complex exponential forms and find

$$\begin{aligned}
f(x) &= \frac{2}{\pi}\int_0^\infty \frac{-k(e^{ik}+e^{-ik}) - i(e^{ik}-e^{-ik})}{2k^2}\frac{1}{2i}(e^{ikx}-e^{-ikx})\;dk \\
&= \frac{1}{2\pi}\int_0^\infty \frac{ki(e^{ik}+e^{-ik}) - (e^{ik}-e^{-ik})}{k^2}(e^{ikx}-e^{-ikx})\;dk \\
&= \frac{1}{2\pi}\int_0^\infty \frac{ki(e^{ik(x+1)}+e^{ik(x-1)}) - (e^{ik(x+1)}-e^{ik(x-1)})}{k^2}\;dk \\
&\quad - \frac{1}{2\pi}\int_0^\infty \frac{ki(e^{-ik(x-1)}+e^{-ik(x+1)}) - (e^{-ik(x-1)}-e^{-ik(x+1)})}{k^2}\;dk
\end{aligned} \tag{6}$$

Let the integration variable in the latter integral go from k to $-k$, and change the upper and lower limit. Then the latter integral becomes

$$\begin{aligned}
I_2 &= -\frac{1}{2\pi}\int_{-\infty}^0 \frac{-ki(e^{ik(x-1)}+e^{ik(x+1)}) - (e^{ik(x-1)}-e^{ik(x+1)})}{k^2}\;dk \\
&= \frac{1}{2\pi}\int_{-\infty}^0 \frac{ki(e^{ik(x+1)}+e^{ik(x-1)}) - (e^{ik(x+1)}-e^{ik(x-1)})}{k^2}\;dk
\end{aligned} \tag{7}$$

such that the two parts combine into our result from Problem 12.6.

Problem 12.22

The correct definition of the function is

$$j_1(k) = \frac{-k\cos k + \sin k}{k^2} \tag{8}$$

where I have used k instead of α. We recognize that this is proportional to $g_s(k)$ from the previous problem. That is:

$$j_1(k) = \sqrt{\frac{\pi}{2}}g_s(k) \tag{9}$$

Therefore,

$$\int_0^\infty j_1(k)\sin(kx)\ dk = \sqrt{\frac{\pi}{2}}\int_0^\infty g_s(k)\sin(kx)\ dk = \frac{\pi}{2}f(x) = \begin{cases}\pi x/2 & |x|<1\\ 0 & |x|>1\end{cases} \tag{10}$$

Chapter 8

Problem 11.14

The first part is straight-forward:

$$\int_{-\infty}^{\infty}\phi(x)\delta''(x-a)\ dx = \phi(x)\delta'(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\phi'(x)\delta'(x-a)\ dx = -\left(-\phi'(a)\right)' = \phi''(a) \tag{11}$$

where we have used the result (11.14) in the book with ϕ replaced with $-\phi'$ in the last integral. The general case is proved by induction. Assume

$$\int_{-\infty}^{\infty}\phi(x)\delta^{(t)}(x-a)\ dx = (-1)^t\phi^{(t)}(a) \tag{12}$$

for some $t>0$, as we already know the formula holds for $t=0$. Then

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(x)\delta^{(t+1)}(x-a)\ dx &= \phi(x)\delta^{(t)}(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\phi'(x)\delta^{(t)}(x-a)\ dx\\ &= 0-(-1)^t(\phi')^{(t)}(a)\\ &= (-1)^{t+1}\phi^{(t+1)}(a)\end{aligned} \tag{13}$$

QED.

Problem 11.15

a)

$$\int_0^\pi \sin x\delta(x-\frac{\pi}{2})\ dx = \sin\frac{\pi}{2} = 1 \tag{14}$$

b)

$$\int_0^\pi \sin x\delta(x+\frac{\pi}{2})\ dx = 0 \tag{15}$$

because $x=-\pi/2$ is outside the range of integration.

c)

$$\int_{-1}^{1} e^{3x}\delta'(x)\ dx = -3e^{3\cdot 0} = -3 \tag{16}$$

d)

$$\int_0^\pi \cosh x \delta''(x-1)\ dx \tag{17}$$

We find

$$\frac{d}{dx}\cosh x = \sinh x \quad \Rightarrow \quad \frac{d^2}{dx^2}\cosh x = \cosh x \tag{18}$$

Therefore

$$\int_0^\pi \cosh x \delta''(x-1)\ dx = (-1)^2 \cosh 1 = \frac{e+e^{-1}}{2} \tag{19}$$

Extra problems

Problem 7.12.12

Find the exponential Fourier transform of

$$f(x) = \begin{cases} \sin x, & |x| < \pi/2 \\ 0, & |x| > \pi/2 \end{cases} \tag{20}$$

We find

$$\begin{aligned} g(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} \sin x e^{-ikx}\ dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \int_{-\pi/2}^{\pi/2} e^{ix(1-k)} - e^{-ix(1+k)}\ dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \left[\frac{-i}{1-k} e^{ix(1-k)} - \frac{i}{1+k} e^{-ix(1+k)} \right]_{-\pi/2}^{\pi/2} \\ &= -\frac{1}{\sqrt{2\pi}} \frac{1}{2} \left[\frac{1}{1-k} \left(e^{i\pi/2} e^{-ik\pi/2} - e^{-i\pi/2} e^{ik\pi/2} \right) + \frac{1}{1+k} \left(e^{-i\pi/2} e^{-ik\pi/2} - e^{i\pi/2} e^{ik\pi/2} \right) \right] \\ &= -\frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-k} i \cos(\pi k/2) - \frac{1}{1+k} i \cos(\pi k/2) \right] \\ &= \frac{1}{\sqrt{2\pi}} \frac{2ik \cos(\pi k/2)}{k^2 - 1} \end{aligned} \tag{21}$$

Problem 7.12.25

Find the exponential Fourier transform of

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

We find

$$
\begin{aligned}
g(k) &= \frac{1}{\sqrt{2\pi}} \int_0^{\pi} \sin x e^{-ikx} \, dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \left[\frac{-i}{1-k} e^{ix(1-k)} - \frac{i}{1+k} e^{-ix(1+k)} \right]_0^{\pi} \\
&= -\frac{1}{\sqrt{2\pi}} \frac{1}{2} \left[\frac{1}{1-k} \left(-e^{-ik\pi} - 1 \right) + \frac{1}{1+k} \left(-e^{-ik\pi} - 1 \right) \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{1 + e^{-ik\pi}}{1-k^2}
\end{aligned} \tag{23}
$$

From this we can find an expression for f:

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1 + e^{-ik\pi}}{1-k^2} e^{ikx} \, dk \\
&= \frac{1}{2\pi} \int_0^{\infty} \frac{1 + e^{-ik\pi}}{1-k^2} e^{ikx} \, dk + \frac{1}{2\pi} \int_{-\infty}^{0} \frac{1 + e^{-ik\pi}}{1-k^2} e^{ikx} \, dk \\
&= \frac{1}{2\pi} \int_0^{\infty} \frac{1 + e^{-ik\pi}}{1-k^2} e^{ikx} \, dk + \frac{1}{2\pi} \int_0^{\infty} \frac{1 + e^{ik\pi}}{1-k^2} e^{-ikx} \, dk \\
&= \frac{1}{2\pi} \int_0^{\infty} \frac{1}{1-k^2} \left[\left(1 + e^{-ik\pi} \right) e^{ikx} + \left(1 + e^{ik\pi} \right) e^{-ikx} \right] \, dk \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{1}{1-k^2} \left[\cos(kx) + \cos k(x-\pi) \right] \, dk
\end{aligned} \tag{24}
$$

QED.