FYS3120
Problem Set 8
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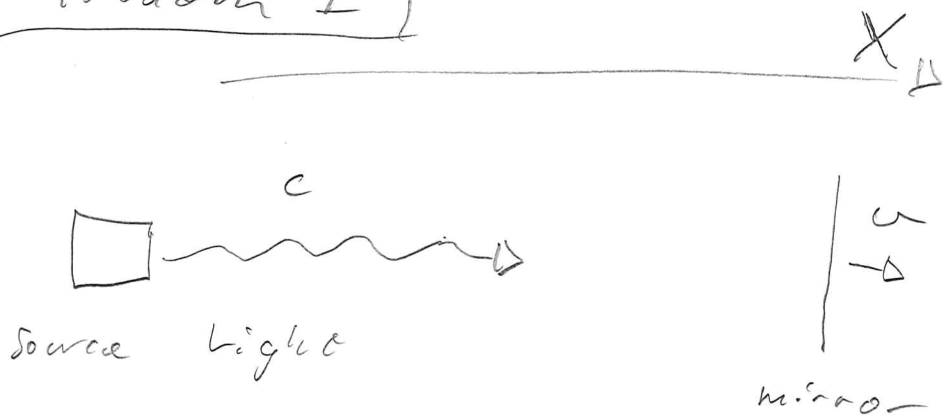
Problem I



$p^\mu = (E/c, \vec{p})$, $E = h\nu$, $\vec{p} = (\frac{h\nu}{c}, 0, 0)$

$\nu_{emitted} = \nu_0$

So

$p^\mu = (\frac{E}{c}, \vec{p}) = (\frac{h\nu_0}{c}, \frac{h\nu_0}{c}, 0, 0)$ in frame S, (the laboratory frame)

We use Lorentz transformation to find the four-momentum in frame S' (the rest frame of the mirror)

$p'^\mu = L^\mu_\rho p^\rho$, when $L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

$p'^0 = L^0_0 p^0 + L^0_1 p^1 = \frac{h\nu_0}{c}\gamma(1-\beta) = \frac{h}{c}\nu_0'$

We see the frequency in the new frame is now $\nu_0' = \nu_0\gamma(1-\beta)$. When reflected, the light is moving in the opposite direction, so $-\beta \to +\beta$ and we have
$\nu_r' = \nu_0(1+\beta)\gamma$

I b) $S' \to S$

$$p'^{\mu}_{r} = \left(\frac{h\nu_r'}{c}, \frac{h\nu_r'}{c}, 0, 0\right)$$

$$p'^{\mu} = L^{\mu}_{\rho} p^{\rho} \Rightarrow p^{\rho} = \left(L^{\mu}_{\rho}\right)^{-1} p'^{\mu} = L^{\rho}_{\mu} p'^{\mu}$$

where $L^{-1} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

so

$$p^{\rho} = \gamma \frac{h\nu_r'}{c}(1+\beta) = \gamma^2 \frac{h}{c}\nu_0 (1+\beta)^2 = \frac{h}{c}\nu_0 \frac{(1+\beta)^2}{1-\beta^2} = \frac{h}{c}\nu_0 \frac{1+\beta}{1-\beta}$$

and we see that

$$\nu_r = \nu_0 \frac{1+\beta}{1-\beta}$$

Problem 2

a)

$$K = (\gamma - 1)mc^2 \Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 = \frac{K}{mc^2} = \alpha$$

$$\Rightarrow 1 = (\alpha - 1)\sqrt{1-\frac{v^2}{c^2}}$$

$$\Rightarrow v = c\sqrt{1-(\alpha+1)^{-2}}$$

b) Particle 1: $P_2 = \gamma m v$, $E_1 = \cancel{mc^2 + \gamma mc^2}$ γmc^2

Particle 2: $P_2 = 0$, $E_2 = mc^2$

Composed: $p = P_1 + P_2 = \gamma m v$

$$E = E_1 + E_2 = mc^2 + \gamma mc^2 = (1+\gamma)mc^2$$

b) $E^2 = M^2c^4 + p^2c^2 \Rightarrow M^2c^4 = (1+\gamma)^2 m^2c^4 - \gamma^2 m^2 v^2 c^2$

$$\Rightarrow M^2 = (1+\gamma)^2 m^2 - \gamma^2 m^2 \frac{v^2}{c^2} = \left(\gamma^2 + 2\gamma + 1 - \gamma^2 \frac{v^2}{c^2}\right) m^2$$

$$= m^2\Big(2\gamma + 1 + \underbrace{\gamma^2\left(1 - \frac{v^2}{c^2}\right)}_{=\,\gamma^2 \cdot \frac{1}{\gamma^2} = 1}\Big) = m^2(2\gamma + 1)$$

$$\Rightarrow M = m\sqrt{2\gamma + 1}$$

$$V = c\sqrt{1 - \left(\frac{K}{Mc^2} + 1\right)^{-2}} = c\sqrt{1 - \left(\alpha\frac{1}{\sqrt{2\gamma+1}} + 1\right)^{-2}}$$

$K_{Before} = \gamma mc^2 - mc^2$ (total energy - rest energy)

$K_{after} = E - Mc^2 = (1+\gamma)mc^2 - mc^2\sqrt{2\gamma + 1}$

$$K_{After} - K_{Before} = \gamma mc^2 - mc^2 - (1+\gamma)mc^2 + mc^2\sqrt{2\gamma + 1}$$

$$= mc^2(\gamma - 1 - 1 - \gamma + \sqrt{2\gamma + 1})$$

$$= mc^2(-2 + \sqrt{2\gamma + 1})$$

c)

$$\vec{P}_1 = P\cos\theta\,\hat{i} + P\sin\theta\,\hat{j}$$

$$\vec{P}_2 = P\cos\theta\,\hat{i} - P\sin\theta\,\hat{j}$$

$|\vec{P}_1| = |\vec{P}_2|$ obviously

$$E_1^2 = P_1^2 c^2 + m_1^2 c^4 \quad , \quad E_2^2 = m_2^2 c^4 + P_2^2 c^2$$

The masses are the same, and we showed that $P_1 = P_2$

so $E_1 = E_2$

d) $E_{total} = E_1 + E_2 = 2E \Rightarrow E = \frac{1}{2} E_{total} = \frac{1}{2}\left[mc^2 + \gamma mc^2\right]$

$$= \frac{1}{2} mc^2 \left[1 + \gamma\right]$$

e) $v = \sqrt{1 - (\alpha + 1)^{-2}}\, c$

$P = \frac{P_x}{\cos\theta} = \frac{\frac{1}{2}\gamma m v}{\cos\theta}$ ⟵ looking at one particle = half the total momentum

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\Rightarrow \frac{1}{4} \cancel{m^2} c^4 (1+\gamma)^2 = \frac{1}{4} \frac{\gamma^2 \cancel{m^2} v^2}{\cos^2\theta} \cdot c^2 + \cancel{m^2} c^4$$

$$\Rightarrow (1+\gamma)^2 = \gamma^2 \frac{v^2}{c^2} \cos^{-2}\theta + 4 \Rightarrow \cos^2\theta = \frac{\gamma^2 v^2}{c^2((\gamma+1)^2 - 4)}$$

e) $\gamma - 1 = \alpha \Rightarrow \gamma = \alpha + 1$

$$\Rightarrow \cos^2\theta = \frac{(\alpha+1)^2 v^2}{c^2((\alpha+2)^2-4)} = \frac{(\alpha+1)^2(1-(\alpha+1)^{-2})}{c^2((\alpha+2)^2-4)}$$

$$= \frac{(\alpha+1)^2 - 1}{c^2(\alpha^2+4\alpha+4-4)} = \frac{\alpha^2+2\alpha+1-1}{c^2(\alpha^2+4\alpha)}$$

$$= \frac{1}{c^2}\frac{\alpha+2}{\alpha+4}$$

$$\Rightarrow \theta = a\cos\left(\frac{1}{c}\sqrt{\frac{\alpha+2}{\alpha+4}}\right)$$

$\alpha \to 0$ gives

$\theta = a\cos\left(\frac{1}{c}\frac{1}{\sqrt{2}}\right) = \pi/4$ if we for some reason set $c = 1$

$\alpha \to \infty$

L'Hopital gives

$\theta = a\cos\left(\frac{1}{c}(\sqrt{1})\right) = 0$. I think it makes sense that

$\theta < \pi/4$, because $2\theta = \pi/2 = 90°$. And two particles or biljardballs or whatever can not have a bigger angle between them after impact than $90°$

3a) $U^\mu = \frac{dx^\mu}{d\tau}$

$X = (ct, x, 0, 0)$ (only moving in x-direction)

$U = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, 0, 0\right)$

$x = \frac{c^2}{a}\cosh\left(\frac{a}{c}(\tau - \tau_I)\right) - x_I$

$t = \frac{c}{a}\sinh\left(\frac{a}{c}(\tau - \tau_I)\right) - t_I$

$\frac{dx}{d\tau} = \frac{c^2}{a}\cdot\frac{a}{c}\sinh\left(\frac{a}{c}\tau - \tau_I\right) = c\sinh\left(\frac{a}{c}\tau - \tau_I\right)$

$\frac{dt}{d\tau} = \cosh\left(\frac{a}{c}(\tau - \tau_I)\right)$

$\Rightarrow U = \left(c\cosh\left(\frac{a}{c}(\tau - \tau_I)\right), c\sinh\left(\frac{a}{c}(\tau - \tau_I)\right), 0, 0\right)$

$A = \frac{dU}{d\tau} = a\left(\sinh\left(\frac{a}{c}(\tau - \tau_I)\right), \cosh\left(\frac{a}{c}(\tau - \tau_I)\right), 0, 0\right)$

$a_0^2 = A^2 = \left(\cosh^2\left(\frac{a}{c}(\tau - \tau_I)\right) - \sinh^2\left(\frac{a}{c}(\tau - \tau_I)\right)\right)a^2 = a^2$

$\Rightarrow a_0 = a$. The space-time path is hyperbolic because in a minkowski-diagram, the world line of the space-ship is shaped as a hyperbola when a_0 is constant and approaching the speed is approaching c

3b) when leaving Earth $t=0$ & $v=0$, $x=0$, $\tau=0$, $a=g$

$$U=\left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, 0, 0\right) = \left(c\gamma, \frac{dx}{dt}\frac{dt}{d\tau}, 0, 0\right) = (\gamma c, \gamma v, 0, 0)$$

$$= \left(c\cosh(-), c\sinh(-), 0, 0\right)$$

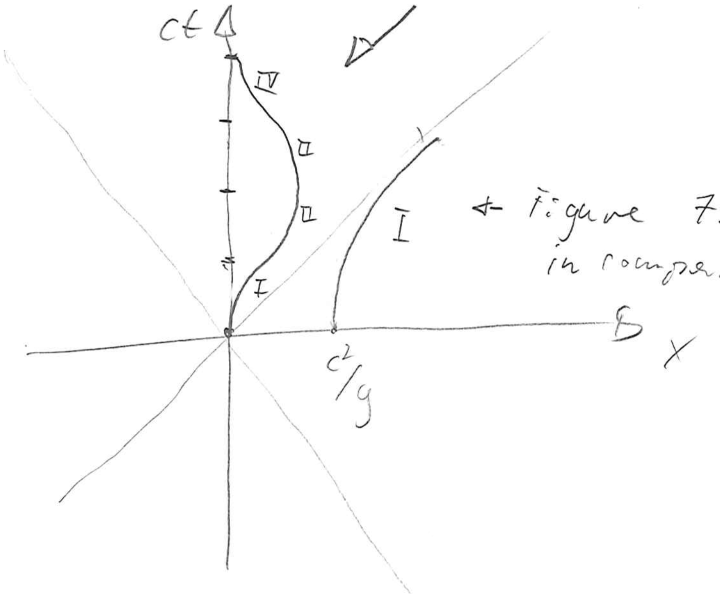
$\Rightarrow \gamma v = c\sinh\left(\frac{g}{c}(-\tau_I)\right) = 0$, so $\sinh(-)=0$

$\Rightarrow \tau_I = 0$

$$x_I = -\frac{c^2}{g}\cosh\left(\frac{g}{c}(0-0)\right) = -\frac{c^2}{g} \quad \Big| \quad x+\frac{c^2}{g} = \frac{c^2}{g}\cosh\left(\frac{g}{c}(\tau)\right)$$

$$t_I = -\frac{c}{g}\sinh\left(\frac{a}{c}(0-0)\right) = 0 \quad \Big| \quad t = \frac{c}{g}\sinh\left(\frac{g}{c}\tau\right)$$

How i want it to be



← figure 7.7 in compendium] (I dont understand why we are on this side of the light, and I dont understand how part II, III & IV will look like)

3c) $\tau_0 = \frac{c}{a_0} = \frac{c}{g} \approx 1$ year

3d) Maximum speed is when the ship is halfway to Proxima.

So

$$x = \frac{c^2}{g}\left(\cosh\left(\frac{g}{c}\tau\right) - 1\right) = 2.2\ l.y = 2.2\ \tau_0 c = 2.2\frac{c^2}{g}$$

$$\Rightarrow \cosh\left(\frac{g}{c}\tau\right) - 1 = 2.2$$

$$\Rightarrow \tau \approx 1.86\ \tau_0$$

$$v = \frac{x}{t} = \frac{\frac{c^2}{g}\left(\cosh\left(\frac{g}{c}\tau\right) - 1\right)}{\frac{c}{g}\sinh\left(\frac{g}{c}\tau\right)} = c \cdot \frac{2.2}{\sinh(1.86)}$$

$$= 0.66\ c$$