FYS 3120 Classical Mechanics and Electrodynamics
Spring semester 2015

Problem Set 8

Problem 8.1

a) We have below four equations that involve tensors of different ranks. Clearly the consistency rules for covariant equations are not satisfied in all places. Show where there are errors in each equation, and show how the equations can be modified to bring them to correct covariant form.

$$C^\mu = T^\mu{}_\nu \, A^\mu \,, \quad D_\nu = T^\mu{}_\nu \, A_\mu \,, \quad E_{\mu\nu\rho} = T_{\mu\nu} \, S^\nu{}_\rho \,, \quad G = S_{\mu\nu} \, T^\nu{}_\alpha \, A^\alpha \tag{1}$$

b) Assume A^μ and B^μ to be 4-vectors and $T^{\mu\nu}$ to be a rank 2 tensor. Show that by making products of these and by lowering and contracting indices, one can form several new 4-vectors and scalars.

c) We have defined the following four tensor fields as functions of the space-time coordinates $x = (x^0, x^1, x^2, x^3)$,

$$f(x) = x_\mu x^\mu \,, \quad g^\mu(x) = x^\mu \,, \quad b^{\mu\nu}(x) = x^\mu x^\nu \,, \quad h^\mu(x) = \frac{x^\mu}{x_\nu x^\nu} \tag{2}$$

Calculate the following derivatives,

$$\partial_\mu f(x) \,, \quad \partial_\mu g^\mu(x) \,, \quad \partial_\mu b^{\mu\nu}(x) \,, \quad \partial_\mu h^\mu(x) \,; \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \tag{3}$$

(If you are uncertain about the results, a convenient way to check these is by specifying the index values explicitly, for example, in the first case by choosing $\mu = 1$, which gives $\partial_\mu = \partial_x$, and writing $f = -(ct)^2 + x^2 + y^2 + z^2$.)

Problem 8.2

An electron, with charge e, moves in a constant electric field $\mathbf{E}$. The motion is determined by the relativistic Newton's equation

$$\frac{d}{dt}\mathbf{p} = e\mathbf{E} \tag{4}$$

where $\mathbf{p}$ denotes the relativistic momentum $\mathbf{p} = m_e \gamma \mathbf{v}$, with m_e as the electron rest mass, $\mathbf{v}$ as the velocity and $\gamma = 1/\sqrt{1-(v/c)^2}$ as the relativistic gamma factor. We assume the electron to move along the field lines, that is, there is no velocity component orthogonal to $\mathbf{E}$.

a) Show that if $v = 0$ at time $t = 0$, then γ depends on time t as

$$\gamma = \sqrt{1 + \kappa^2 t^2} \tag{5}$$

and determine κ.

b) The proper time τ is related to the coordinate time t by the formula $\frac{dt}{d\tau} = \gamma$. Show that if we write $\gamma = \cosh \kappa\tau$ then τ satisfies the above condition.

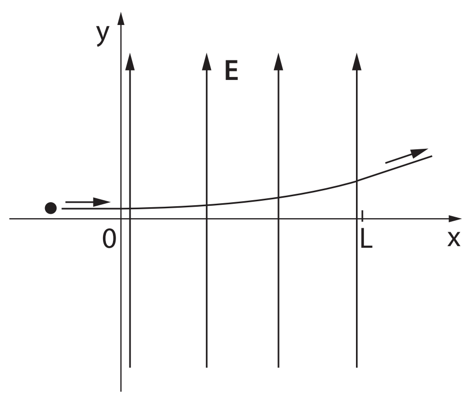
c) For linear motion we have the following relation between the proper acceleration a_0 and the acceleration a measured in a fixed inertial reference frame, $a_0 = \gamma^3 a$. Use this to show that the electron has a constant proper acceleration, given by $\mathbf{a}_0 = e\mathbf{E}/m_e$.

As a reminder we give the following functional relations:

$$\cosh^2 x - \sinh^2 x = 1\,, \quad \frac{d}{dx}\cosh x = \sinh x\,, \quad \frac{d}{dx}\sinh x = \cosh x \tag{6}$$

Problem 8.3 (Exam 2009)

A particle with charge q and mass m moves with relativistic speed through a region $0 < x < L$ where a constant electric field $\mathbf{E}$ is directed along the y-axis, as indicated in the figure. The particle enters the field at $x = 0$ with momentum $\mathbf{p}_0$ in the direction orthogonal to the field. The relativistic energy at this point is denoted $\mathcal{E}_0$. (Note that we write the energy as $\mathcal{E}$ to avoid confusion with the electric field strength E.)



a) Use the equation of motion for a charged particle in an electric field to determine the time dependent momentum $\mathbf{p}(t)$ and relativistic energy $\mathcal{E}(t)$ (without the potential energy) of the particle inside in the electric field. What is the relativistic gamma factor $\gamma(t)$ expressed as a function of coordinate time t?

b) Find the velocity components $v_x(t)$ and $v_y(t)$ and explain the relativistic effect that the velocity in the x-direction decreases with time even if there is no force acting in this direction.

c) Show that the proper time $\Delta\tau$ spent by the particle on the transit through the region $0 < x < L$ is proportional to the length L, $\Delta\tau = \alpha L$, and determine α.

d) What is the transit time Δt through the region when measured in coordinate time?

We remind about the integration formula $\int dx \frac{1}{\sqrt{1+x^2}} = \operatorname{arc\,sinh} x + C$.
We also remind you about the following expressions for relativistic momentum and relativistic energy

$$\mathbf{p} = \gamma m \mathbf{v}\,, \quad \mathcal{E} = \gamma m c^2 \tag{7}$$

and the relation $\mathcal{E}^2 = c^2 p^2 + m^2 c^4$.