Solution to PROBLEM SET 2 FYS3140

Problem 2.1
Use $z = x + iy$ and write it in $f(z) = u(x,y) + iv(x,y)$ form.

a) $\frac{-i+2z}{2+iz} = \frac{-i+2x+i2y}{2+ix-y} = \frac{(2x+i(2y-1))((2-y)-ix)}{((2-y)+ix)((2-y)-ix)} = \frac{3x+i(-2y^2+5y-2x^2-2)}{(2-y)^2+x^2}$
Ans. $u(x,y) = \frac{3x}{(2-y)^2+x^2}$; $\quad v(x,y) = \frac{-2y^2+5y-2x^2-2}{(2-y)^2+x^2}$

b) $e^{iz} = e^{i(x+iy)} = e^{-y}(\cos x + i \sin x)$
Ans. $u(x,y) = e^{-y}\cos x$; $\quad v(x,y) = e^{-y}\sin x$

Problem 2.2 (Derivatives)
Given definition for $\frac{df}{dz}$; we now show product rule of derivatives for complex variables follows same rule as real variables. Assume that higher order derivatives of single functions f and g exit, we taylor expand f and g as $f(z+\triangle z) = f(z) + \triangle z f'(z) + R_f$ and $g(z+\triangle z) = g(z) + \triangle z g'(z) + R_g$. R_f and R_g contains higher powers of $\triangle z$

$$\frac{d}{dz}[f(z)g(z)] = \lim_{\triangle z \to 0} \frac{\triangle[f(z)g(z)]}{\triangle z} = \lim_{\triangle z \to 0} \frac{f(z+\triangle z)g(z+\triangle z) - f(z)g(z)}{\triangle z}$$
$$= \lim_{\triangle z \to 0} \frac{(f(z) + \triangle z f'(z) + R_z)(g(z) + \triangle z g'(z) + R_g) - f(z)g(z)}{\triangle z}$$

expanding and rearranging gives

$$= \lim_{\triangle z \to 0} \frac{\triangle z g f' + g R_f + \triangle z f g' + f R_g}{\triangle z} = \underline{g f' + f g'}$$

terms containing R_f and R_g are zero, since they contain $\triangle z$ and powers of $\triangle z$ after cancellations with the denominator $\triangle z$.

Problem 2.3 (Cauchy-Riemann conditions)
$f(x,y) = u(x,y) + iv(x,y)$ or
$f(r,\theta) = u(r,\theta) + iv(r,\theta)$ and $z(r,\theta) = re^{i\theta}$

$$\frac{\partial z}{\partial r} = e^{i\theta}; \quad \frac{\partial z}{\partial \theta} = ire^{i\theta}$$
$$\frac{\partial f}{\partial r} = \frac{df}{dz}\frac{\partial z}{\partial r} = \frac{df}{dz}e^{i\theta}; \quad \frac{\partial f}{\partial \theta} = \frac{df}{dz}\frac{\partial z}{\partial \theta} = \frac{df}{dz}ire^{i\theta}$$

we can also have

$$\frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{df}{dz}e^{i\theta} \qquad (*)$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial f}{\partial \theta} = \frac{df}{dz}ire^{i\theta} \qquad (**)$$

combining (*) and (**) based on the condition that the derivative has to be unique; we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}]$$

by equating the complex and the real values of both sides we get the Cauchy-Riemann conditions in polar coordinates

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}; \quad \text{and} \quad \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}$$

Problem 2.4 (Harmonic functions)

Here we use quotient rule $\frac{d}{dx}(f/g) = \frac{f^{'}g - fg^{'}}{g^2}$, $[(1-x)^2+y^2]^2 = (1-x)^4 + y^4 + 2(1-x)^2y^2$; and let $[(1-x)^2+y^2]^4 = \Omega$

a)

$$\frac{\partial}{\partial x}(u) = \frac{2y(1-x)}{(1-x)^4+y^4+2(1-x)^2y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[\frac{2y(1-x)}{(1-x)^4+y^4+2(1-x)^2y^2}]$$

$$= \frac{6y(1-x)^2+4y^3(1-x)^2-2y^5}{\Omega} \tag{1}$$

$$\frac{\partial}{\partial y}(u) = \frac{(1-x)^2-y^2}{(1-x)^4+y^4+2(1-x)^2y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}[\frac{(1-x)^2-y^2}{(1-x)^4+y^4+2(1-x)^2y^2}]$$

$$= \frac{-6y(1-x)^4-4y^3(1-x)^2+2y^5}{\Omega} \tag{2}$$

From eqn(1) and eqn(2) we see that the function satisfies Laplace's equation $\bigtriangledown^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

b) From Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{2y(1-x)}{[(1-x)^2+y^2]^2}$$

and integrating with respect to y gives

$$v(x,y) = \frac{-(1-x)}{(1-x)^2+y^2} + g(x) \tag{3}$$

$g(x)$ is a function only of x that needs to be determined by the second Cauchy-Riemann equation:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4}$$

giving

$$\frac{(1-x)^2 - y^2}{[(1-x)^2 + y^2]^2} = -\frac{y^2 - (1-x)^2}{[(1-x)^2 + y^2]^2} - \frac{\partial g(x)}{\partial x} \tag{5}$$

that is

$$g'(x) = 0 \quad \Rightarrow g(x) = \text{constant} = K \tag{6}$$

In total, this gives us

$$\begin{aligned} f = u + iv &= \frac{y}{(1-x)^2 + y^2} + i\left(\frac{x-1}{(1-x)^2 + y^2} + K\right) \\ &= \frac{i(x - iy - 1)}{|(x-1) + iy|^2} + iK \\ &= \frac{i(\bar{z} - 1)}{(z-1)(\bar{z} - 1)} + iK \\ &= \frac{i}{z-1} + iK \end{aligned} \tag{7}$$

c)

$$\frac{\partial^2 v}{\partial x^2} = \frac{-2(1-x)\left((1-x)^2 - 3y^2\right)}{\left((1-x)^2 + y^2\right)^3} = -\frac{\partial^2 v}{\partial y^2} \tag{8}$$

Problem 2.5

Cauchy's formula:

$$\oint_\Gamma \frac{f(z)}{z-a} dz = 2\pi i f(a) \tag{9}$$

if a is enclosed inside the curve Γ. If a is outside then the integral is 0.

a)

$$\oint_\Gamma \frac{\sin z}{2z - \pi} dz = \oint_\Gamma \frac{\frac{1}{2}\sin z}{z - \pi/2} dz \tag{10}$$

The simple pole is located at $a = \pi/2$, which is inside Γ. The analytic function is $f(z) = \frac{1}{2}\sin z$, and the integral formula gives

$$\oint_\Gamma \frac{\sin z}{2z - \pi} dz = 2\pi i f(\pi/2) = 2\pi i \cdot \frac{1}{2}\sin(\pi/2) = \pi i \tag{11}$$

b)

The pole $a = \pi/2 \approx 1.57$ is outside Γ, hence Cauchy's theorem gives

$$\oint_\Gamma \frac{\sin z}{2z - \pi} dz = 0 \tag{12}$$

in this case.

c)

$$\oint_\Gamma \frac{\sin(2z)}{6z-\pi}dz = \oint_\Gamma \frac{\frac{1}{6}\sin(2z)}{z-\pi/6}dz \tag{13}$$

The pole $a = \pi/6 \approx 0.52$ is inside Γ, and the analytic function is $f(z) = \frac{1}{6}\sin(2z)$. Therefore

$$\oint_\Gamma \frac{\sin(2z)}{6z-\pi}dz = 2\pi i f(\pi/6) = 2\pi i \cdot \frac{1}{6}\sin(\frac{2\pi}{6}) = \frac{\sqrt{3}}{6}\pi i \tag{14}$$

d)

$$\oint_\Gamma \frac{e^{2z}}{z-\ln 2}dz \tag{15}$$

has a pole at $a = \ln 2 \approx 0.69$ which is inside Γ. The analytic function is simply $f(z) = e^{2z}$, so

$$\oint_\Gamma \frac{e^{2z}}{z-\ln 2}dz = 2\pi i e^{2\ln 2} = 2\pi i \cdot 2^2 = 8\pi i \tag{16}$$

Extra problems

Problem 2.17.25

a)

$$\overline{\cos z} = \frac{1}{2}\overline{(e^{iz}+e^{-iz})} = \frac{1}{2}\left(e^{-i\bar{z}}+e^{i\bar{z}}\right) = \cos\bar{z} \tag{17}$$

b)

$$\overline{\sin z} = -\frac{1}{2i}\overline{(e^{iz}-e^{-iz})} = -\frac{1}{2i}\left(e^{-i\bar{z}}-e^{i\bar{z}}\right) = \frac{1}{2i}\left(e^{i\bar{z}}-e^{-i\bar{z}}\right) = \sin\bar{z} \tag{18}$$

c)

$$f(z) = 1+iz \quad \Rightarrow \quad \overline{f(z)} = 1-i\bar{z} \neq 1+i\bar{z} = f(\bar{z}) \tag{19}$$

d) Assume

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{20}$$

and further assume that $c_n \in \mathbb{R}\ \forall\ n$. Then

$$\overline{f(z)} = \sum_{n=0}^{\infty}\overline{c_n z^n} = \sum_{n=0}^{\infty} c_n \bar{z}^n = f(\bar{z}) \tag{21}$$

See that this holds only when $\bar{c}_n = c_n$, that is when they are all real. For instance, $f(z) = 1+iz$ has $c_0 = 1$, $c_1 = i$ and $c_n = 0\ \forall\ n > 1$, so c_1 is not real and therefore $\overline{f(z)} \neq f(\bar{z})$.

e) We have

$$\overline{\sinh z} = \frac{1}{2}\left(e^{\bar{z}}-e^{-\bar{z}}\right) = \sinh\bar{z} \tag{22}$$

Therefore

$$\overline{i[\sinh(1+i)-\sinh(1-i)]} = -i[\sinh(1-i)-\sin(1+i)] = i[\sinh(1+i)-\sinh(1-i)] \tag{23}$$

and the expression is therefore real.

Problem 2.17.28

We are to simplify the expression

$$|Z|^2 = \left| \frac{(a+ib)^2 e^b - (a-ib)^2 e^{-b}}{4abie^{-ia}} \right|^2 \tag{24}$$

and write it in terms of a hyperbolic function, assuming $a, b \in \mathbb{R}$. Let $z = a + ib = re^{i\theta}$.

$$\begin{aligned}
|Z|^2 = Z\bar{Z} &= \left(\frac{(a+ib)^2 e^b - (a-ib)^2 e^{-b}}{4abie^{-ia}} \right) \left(\frac{(a-ib)^2 e^b - (a+ib)^2 e^{-b}}{-4abie^{ia}} \right) \\
&= \left(\frac{1}{4ab} \right)^2 \left([(a+ib)(a-ib)]^2 e^{2b} - (a+ib)^4 - (a-ib)^4 + [(a+ib)(a-ib)]^2 e^{-2b} \right) \\
&= \left(\frac{1}{4ab} \right)^2 \left(r^4 e^{2b} - 2a^4 + 12a^2b^2 - 2b^4 + r^4 e^{-2b} \right) \\
&= \left(\frac{r^2}{2ab} \right)^2 \frac{1}{2} \left(\frac{1}{2} \left(e^{2b} + e^{-2b} \right) - \frac{a^4 - 6a^2b^2 + b^4}{r^4} \right) \\
&= \left(\frac{r^2}{2ab} \right)^2 \frac{1}{2} \left(\cosh(2b) - \frac{a^4 + 2a^2b^2 + b^4 - 8a^2b^2}{a^4 + 2a^2b^2 + b^4} \right) \\
&= \left(\frac{r^2}{2ab} \right)^2 \frac{1}{2} \left(\cosh(2b) - 1 + 2\frac{4a^2b^2}{r^4} \right) \\
&= \left(\frac{r^2}{2ab} \right)^2 \frac{1}{2} \left(\cosh(2b) - 1 \right) + 1 \\
&= \left(\frac{r^2}{2ab} \right)^2 \frac{1}{2} \left(1 + 2\sinh^2 b - 1 \right) + 1 \\
&= 1 + \left(\frac{a^2 + b^2}{2ab} \right)^2 \sinh^2 b
\end{aligned} \tag{25}$$

where we have used the fact that $\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x$.

Problem 2.17.32

We are going to show that

$$\sum_{n=0}^{\infty} \frac{(1+i\pi)^n}{n!} = -e \tag{26}$$

We write

$$-e = e^{i\pi} e^1 = e^{1+i\pi} \tag{27}$$

and use the series expansion for the exponential function:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{28}$$

giving

$$-e = e^{1+i\pi} = \sum_{n=0}^{\infty} \frac{(1+i\pi)^n}{n!} \tag{29}$$