

## Problem Set 5

### Problem 5.1

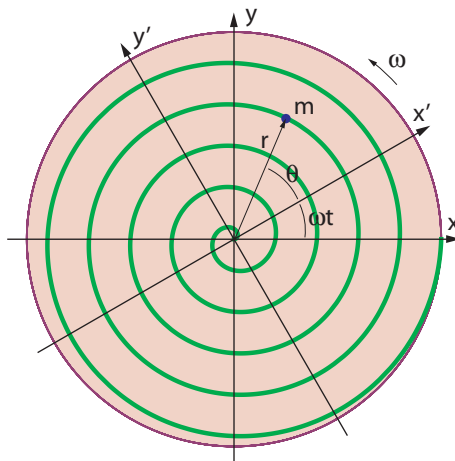
A particle with mass  $m$  moves freely in a horizontal plane. There is no constraint on the motion, but in the following we will consider the free motion described in a rotating reference frame. We refer to the Cartesian coordinates of a fixed frame as  $(x, y)$  and the coordinates of the rotating frame as  $(\xi, \eta)$ . They are related by the standard expressions

$$\begin{aligned}x &= \xi \cos \omega t - \eta \sin \omega t \\y &= \xi \sin \omega t + \eta \cos \omega t\end{aligned}\tag{1}$$

- Find the Lagrangian expressed in terms of the coordinates  $\xi, \eta$  and their time derivatives.
- Find the corresponding equations of motion for the two variables, and identify the Coriolis and centrifugal terms. Compare with the standard expression for Newton's 2. law in a rotating reference frame, as can be found in introductory mechanics text books.

### Problem 5.2

A small body with mass  $m$  is constrained to move (without friction) along a spiral-shaped channel, which is etched in a flat, circular disk (see figure). The disk rotates in the horizontal plane about an axis through the center of the disk, with constant angular velocity  $\omega$ . The points on the spiral are characterized by polar coordinates  $(r, \theta)$ , with  $r = a\theta$ , where  $a$  is a constant. The angle variable  $\theta$  is measured relative to a frame which rotates with the disk. In the expression for  $r$  the angle  $\theta$  is chosen to take positive values and is not restricted to be less than  $2\pi$ . The radius of the disk we refer to as  $R$ .



- Find the Cartesian coordinates of the moving body  $(x, y)$ , measured relative to the non-rotating reference frame, expressed in terms of the (generalized) coordinate  $\theta$ . Use this to determine the Lagrangian  $L$  as a function of  $\dot{\theta}$  and  $\theta$ .

b) Find Lagrange's equation, expressed as a differential equation for  $\theta$ . Assuming the condition  $\theta \gg 1$  to be satisfied, we make in the following the approximation  $(1 + \theta^2)\ddot{\theta} \approx \theta^2\ddot{\theta}$ . Show that the equation then is simplified to

$$\theta\ddot{\theta} + \dot{\theta}^2 - \omega^2 = 0 \quad (2)$$

c) Assume as initial conditions  $r(0) = r_0$  and  $\dot{r}(0) = 0$ , with the above condition satisfied. Show that the (simplified) equation has solution of the form

$$r(t) = \sqrt{At^2 + Bt + C} \quad (3)$$

and determine the constants  $A, B$  and  $C$ .

d) Determine the time  $t$ , which the body takes to reach the edge of the disk, expressed as a function of the initial coordinate  $r_0$ .

### Problem 5.3 (Midterm Exam 2008)

*The brachistochrone challenge.*

This is a classical problem in analytical mechanics. It was discussed by Galileo Galilei, who suggested a solution (but not the correct one), and studied the problem experimentally. In 1696 the problem was formulated as a challenge to the mathematicians at the time by Johann Bernoulli. He wrote in the journal *Acta Eruditorum*:

*I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.*

The problem he formulated was the following:

*Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.*

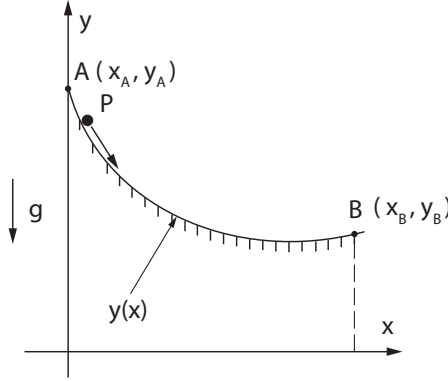
Five solutions were obtained from scientist and mathematicians we are all acquainted with, Newton, Jacob Bernoulli (the older brother of Johann), Leibniz and de L'Hôpital, in addition to Johann himself. Johann Bernoulli gave a formulation of the problem where he could use an analogy to Snell's law of refraction in optics to solve the problem.

Let us now rephrase the problem with a few more words:

Assume a small body ( $P$  in the figure) moves in a vertical plane under the influence of gravity. It leaves a point  $A$  with zero velocity and follows (without friction) a given path in the plane which passes through a second point  $B$ , as shown in the figure. Assume the path between the two points  $A$  and  $B$  can be changed, while the points themselves stay fixed. For which path between the two points does the body spend the least time on the transit from point  $A$  to point  $B$ ?

The challenge for you is the following:

Find the solution to the brachistochrone problem by using the correspondence between the variational problem (finding the "path of shortest time") and the Lagrange equation, in the way discussed in the lectures.



The body  $P$  is to be treated as a point particle of mass  $m$  and the path is represented by a function  $y(x)$  with  $x$  as the horizontal and  $y$  as the vertical coordinate. The boundary conditions, which fix the positions of point  $A$  and  $B$ , are specified as,  $y(x_A) = y_A$ ,  $y(x_B) = y_B$ . To simplify the equations assume in the following the initial coordinates are  $x_A = y_A = 0$ .

a) Show that the time  $T$  spent by the body on the way between  $A$  and  $B$  can be expressed as an integral of the form

$$T[y(x)] = \int_{x_A}^{x_B} L(y, y') dx \quad (4)$$

with  $y' = \frac{dy}{dx}$  and with

$$L(y, y') = \sqrt{\frac{1 + y'^2}{-2gy}} \quad (5)$$

For the derivation, make use of energy conservation in the form  $\frac{1}{2}mv^2 + mgy = 0$ , with  $y < 0$ .

b) With  $L(y, y')$  interpreted as a Lagrangian ( $x$  then plays the role of  $t$  in the usual formulation) the canonically conjugate momentum is  $p = \frac{\partial L}{\partial y'}$  and the Hamiltonian is  $H = py' - L$ . Explain why  $H$  is a constant of motion and use this fact to show that  $y(x)$  satisfies a differential equation of the form

$$(1 + y'^2)y = -k^2 \quad (6)$$

with  $k$  as a constant.

c) The equation has a solution which can be written on *parametric form* as

$$\begin{aligned} x &= \frac{1}{2}k^2(\theta - \sin \theta) \\ y &= \frac{1}{2}k^2(\cos \theta - 1) \end{aligned} \quad (7)$$

where  $\theta$  has been introduced as a curve parameter. Show that (7) is a solution of the differential equation (6) by changing from  $x$  to  $\theta$  as a variable in the equation and by using the above expression for  $y(\theta)$ . In what way are the boundary conditions taken care of by this solution?

d) The curve  $y(x)$  defined by the solution of the brachistochrone problem is a section of a *cycloid*, known for example as the curve traced out by a point on the periphery of a rolling wheel. Make a plot which shows the form of the curve.

e) Assume that point  $B$  lies at the lowest point of the cycloid. Show that in this case the following relation has to be satisfied,  $y_B = -\frac{2}{\pi} x_B$ . Calculate for that situation the time used by the body to reach point  $B$  from  $A$  and compare with the time used when the body instead follows a straight line between the two points.