Solutions to Problem Set 10 FYS3140

In this PS, we follow the notation in *Boas*, including the references to the table of Laplace transforms on p. 469-471. The constraints from the third column in the table are omitted here, but do of course apply.

Chapter 8

Problem 8.4

Verify $L12$, namely

$$L[t\cos(at)] = \frac{p^2 - a^2}{(p^2 + a^2)^2} \tag{1}$$

We have $L3$:

$$L[\sin(at) = \frac{a}{p^2 + a^2} \quad \text{Re } p > |\text{Im } a| \tag{2}$$

Differentiation with respect to a variable that is neither of the two involved in the transform (here t and p) can be taken inside the Laplace transform, according to Leibniz' rule of differentiating an integral (see *Boas* p. 236). Differentiating with respect to a gives:

$$\begin{aligned} L[\frac{\partial}{\partial a}\sin(at)] &= \frac{\partial}{\partial a}\frac{a}{p^2 + a^2} \\ L[t\cos(at)] &= \frac{(p^2 + a^2) - 2a^2}{(p^2 + a^2)^2} \\ &= \frac{p^2 - a^2}{(p^2 + a^2)^2} \end{aligned} \tag{3}$$

Problem 8.13

We have

$$F(p) = \frac{6 - p}{p^2 + 4p + 20} = \frac{6 - p}{p^2 + 4p + 4 + 16} = \frac{8}{(p+2)^2 + 16} - \frac{p + 2}{(p+2)^2 + 16} \tag{4}$$

A good advice when doing inverse Laplace transforms is to always take care of the numerators involving p first: after this is done, the eventual constant numerators in other fractions can always be factorized to fit a given entry in the table. In this example, we took care of the factors of p by making the numerator on the right to be $p + 2$. As we will now see, the other fraction with numerator of 8 is no problem. Looking at $L13$ and $L14$, we see that if we rewrite $F(p)$ as

$$F(p) = 2\frac{4}{(p+2)^2 + 4^2} - \frac{p + 2}{(p+2)^2 + 4^2} \tag{5}$$

we have immedeatly

$$f(t) = L^{-1}[F(p)] = 2e^{-2t}\sin(4t) - e^{-2t}\cos(4t) \tag{6}$$

since the Laplace transform is linear.

Problem 8.21

We want to use $L29$ and $L11$ to calculate

$$L[te^{-at}\sin(bt)] \tag{7}$$

According to $L11$ we can write

$$L[t\sin(bt)] = \frac{2bp}{(p^2+b^2)^2} = G(p) \tag{8}$$

with $g(t) = t\sin(bt)$. Then, $L29$ gives

$$L[e^{-at}(t\sin(bt))] = L[e^{-at}g(t)] = G(p+a) = \frac{2b(p+a)}{[(p+a)^2+b^2]^2} \tag{9}$$

Problem 9.10

Solve

$$y'' - 4y' + 4y = 6e^{2t} \tag{10}$$

with $y_0 = y_0' = 0$. We take the Laplace transform of the DE:

$$p^2Y - py_0 - y_0' - 4(pY - y_0) + 4Y = 6L[e^{-(-2)t}] \tag{11}$$

which according to $L2$ is

$$Y(p^2 - 4p + 4) = 6\frac{1}{p-2} \tag{12}$$

for $p > 2$. Factorizing gives:

$$\begin{aligned} Y(p-2)^2 &= 6\frac{1}{p-2} \\ Y &= 6\frac{1}{(p-2)^3} \end{aligned} \tag{13}$$

Comparing with $L6$ we see that

$$y(t) = L^{-1}[Y] = L^{-1}\left[3\frac{2!}{(p-2)^{2+1}}\right] = 3t^2e^{2t} \tag{14}$$

Problem 9.26

Solve

$$y'' + 2y' + 10y = -6e^{-t}\sin(3t) \tag{15}$$

with $y_0 = 0$, $y_0' = 1$. Again taking the transform, we find

$$p^2Y - 1 + 2pY + 10Y = -6L\left[e^{-1t}\sin(3t)\right] = -6\frac{3}{(p+1)^2+3^2} \tag{16}$$

where $L13$ has been used. We find

$$\begin{aligned} Y(p^2+2p+10) &= 1 - \frac{18}{(p+1)^2+3^2} \\ Y(p^2+2p+1+9) &= \frac{(p+1)^2+3^2}{(p+1)^2+3^2} - \frac{18}{(p+1)^2+3^2} \\ Y\left[(p+1)^2+3^2\right] &= \frac{(p+1)^2+3^2}{(p+1)^2+3^2} - \frac{18}{(p+1)^2+3^2} \\ Y &= \frac{1}{(p+1)^2+3^2} - \frac{18}{\left[(p+1)^2+3^2\right]^2} \end{aligned} \tag{17}$$

We know that $p \to p+1$ simply correspond to a factor e^{-t} in the inverse. Overlooking this, the two entries $L13$ and $L17$ stand out as candidates for the inverse transform. We find:

$$\begin{aligned} y(t) &= L^{-1}[Y] \\ &= L^{-1}\left[\frac{1}{3}\frac{3}{(p+1)^2+3^2} - \frac{1}{3}\frac{2\cdot 3^3}{[(p+1)^2+3^2]}\right] \\ &= \frac{1}{3}\left[e^{-t}\sin(3t) - e^{-t}\left(\sin(3t) - 3t\cos(3t)\right)\right] \\ &= te^{-t}\cos(3t) \end{aligned} \tag{18}$$

Problem 10.5

Find the inverse transform of

$$Y(p) = \frac{p}{(p+a)(p+b)^2} \tag{19}$$

There are many possibilities here, but we choose to write:

$$Y = \frac{p}{(p+a)(p+b)} \cdot \frac{1}{p+b} \tag{20}$$

with the intention of using $L8$ and $L2$. Let

$$G(p) = \frac{p}{(p+a)(p+b)}, \qquad H(p) = \frac{1}{p+b} \tag{21}$$

such that

$$g(t) = \frac{ae^{-at} - be^{-bt}}{a-b}, \qquad h(t) = e^{-bt} \tag{22}$$

The convolution theorem then gives

$$\begin{aligned} y(t) &= \int_0^t g(\tau)h(t-\tau)\, d\tau \\ &= \frac{1}{a-b}\int_0^t \left(ae^{-a\tau} - be^{-b\tau}\right)e^{-b(t-\tau)}\, d\tau \\ &= \frac{1}{a-b}\left[ae^{-bt}\int_0^t e^{-(a-b)\tau}\, d\tau - be^{-bt}\int_0^t d\tau\right] \\ &= \frac{e^{-bt}}{a-b}\left[\frac{-a}{a-b}\left(e^{-(a-b)t} - 1\right) - bt\right] \\ &= \frac{e^{-bt}}{b-a}\left[bt + \frac{a}{a-b}\left(e^{(b-a)t} - 1\right)\right] \end{aligned} \tag{23}$$

Problem 10.14

Solve

$$y'' + 5y' + 6y = e^{-2} \tag{24}$$

with $y_0 = y_0' = 0$. Taking the transform and using $L2$:

$$\begin{aligned}
Y\left(p^2 + 5p + 6\right) &= \frac{1}{p+2} \\
Y(p+2)(p+3) &= \frac{1}{p+2} \\
Y &= \frac{1}{(p+3)(p+2)^2} \\
&= \frac{1}{p+3} \cdot \frac{1!}{(p+2)^{1+1}}
\end{aligned} \tag{25}$$

Using $L2$, $L5$ and convolution, with $G(p)$ and $H(p)$ corresponding to the first and second fraction respectively,

$$\begin{aligned}
y(t) &= \int_0^t g(t-\tau)h(\tau)\ d\tau \\
&= \int_0^t e^{-3(t-\tau)}\tau e^{-2\tau}\ d\tau \\
&= e^{-3t}\int_0^t \tau e^{\tau}\ d\tau \\
&= e^{-3t}\left[te^t - e^t + 1\right] \\
&= (t-1)e^{-2t} + e^{-3t}
\end{aligned} \tag{26}$$

Problem 11.8

Find the response of a unit pulse to a system described by

$$y'' + 4y' + 5y = f(t) \tag{27}$$

A unit pulse at time t_0 is equivalent to replacing $f(t)$ with $\delta(t - t_0)$, and letting the system start in the state $y_0 = y_0' = 0$. Therefore we can take the transform of the equation:

$$\begin{aligned}
Y\left(p^2 + 4p + 5\right) &= \begin{cases} e^{-pt_0} & t > t_0 \\ 0 & \text{otherwise} \end{cases} \\
Y &= e^{-pt_0}\frac{1}{(p+2)^2 + 1^2}
\end{aligned} \tag{28}$$

From now on we assume $t > t_0$. Let

$$G(p+2) = \frac{1}{(p+2)^2 + 1^2} \quad \Rightarrow \quad g(t) = \sin(1t) \tag{29}$$

Then according to $L28$ and $L29$,

$$y(t) = L^{-1}[Y] = \begin{cases} e^{-2(t-t_0)}\sin(t - t_0) & t > t_0 \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

The exponential comes from $L29$ and the split definition along with the replacement $t \to t - t_0$ comes from $L28$.