Solutions to Problem Set 4 FYS3140

All problems this week are taken from Chapter 14 in Boas.

Problem 6.9

Find the Laurent series of

$$f(z) = \frac{1}{z^2 - 5z + 6} \tag{1}$$

about the point $z_0 = 2$. We introduce $w = z - z_0 = z - 2$ which gives $z = w + 2$. Inserted into f we find

$$f(w) = \frac{1}{(w+2)^2 - 5(w+2) + 6} = \frac{1}{w^2 - w} = \frac{1}{w}\frac{1}{w-1} \tag{2}$$

Expanding $f(z)$ around z_0 is the same as expanding $f(w)$ around the origin. Therefore we keep the factor $1/w$ and expand the other factor in a geometric series:

$$f(w) = -\frac{1}{w}\sum_{n=0}^{\infty} w^n = -\sum_{n=0}^{\infty} w^{n-1} \tag{3}$$

which converges for $|w| < 1$, which is a disk around $z_0 = 2$ with radius 1. From this we see that the Laurent series for $f(z)$ around z_0 is:

$$f(z) = -\sum_{n=0}^{\infty}(z-2)^{n-1} \tag{4}$$

which has $b_i = 0 \ \forall \ i > 1$ and residue $b_1 = -1$.

Problem 6.19

Find the residue of the function

$$f(z) = \frac{\sin^2 z}{2z - \pi} = \frac{\frac{1}{2}\sin^2 z}{z - \pi/2} \tag{5}$$

at $z_0 = \pi/2$. We can easily calculate the quantity (Eq. 6.1 in Boas):

$$R(z_0) = \lim_{z\to z_0}(z - z_0)f(z) \tag{6}$$

If this limit exists (is not infinite) then the value is equal to the residue at the point z_0. If the value is 0, the function f is analytic and the singularity was removable; if the value is some finite number, then the function had a simple pole at z_0. We find:

$$R(z_0) = \lim_{z\to z_0}(z - z_0)f(z) = \lim_{z\to\pi/2}(z - \pi/2)\frac{\frac{1}{2}\sin^2 z}{(z - \pi/2)} = \frac{1}{2}\sin^2(\pi/2) = \frac{1}{2} \tag{7}$$

so it was a simple pole and the residue is 1/2.

Problem 6.28

Find the residue of the function

$$f(z) = \frac{z+2}{(z^2+9)(z^2+1)} \tag{8}$$

at the point $z_0 = 3i$. We rewrite:

$$f(z) = \frac{z+2}{(z^2+9)(z^2+1)} = \frac{z+2}{(z+3i)(z-3i)(z+i)(z-i)} \tag{9}$$

Again we calculate

$$\begin{aligned} R(z_0) &= \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to 3i}(z-3i)\frac{z+2}{(z+3i)(z-3i)(z+i)(z-i)} \\ &= \frac{3i+2}{(3i+3i)(3i+i)(3i-i)} = \frac{3i+2}{6i\cdot 4i\cdot 2i} = \frac{3i+2}{-48i} = -\frac{1}{16}+\frac{1}{24}i \end{aligned} \tag{10}$$

and since the limit exists, this is the residue and the pole was simple.

Problem 7.7

Calculate

$$I = \int_0^{2\pi} \frac{\cos 2\theta d\theta}{5+4\cos\theta} \tag{11}$$

We view this as a complex contour integral around the unit circle. We have

$$z = e^{i\theta} \quad \Rightarrow \quad \frac{dz}{d\theta} = ie^{i\theta} = iz \quad \Rightarrow \quad d\theta = \frac{dz}{iz} \tag{12}$$

and

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} \quad \Rightarrow \quad \cos 2\theta = \frac{z^2+z^{-2}}{2} \tag{13}$$

This gives us

$$\begin{aligned} I &= \oint \frac{(z^2+z^{-2})/2}{5+4(z+z^{-1})/2}\frac{dz}{iz} = \oint \frac{z^2+z^{-2}}{10+4z+4z^{-1}}\frac{dz}{iz} = \frac{1}{i}\oint \frac{z^2+z^{-2}}{10z+4z^2+4}dz \\ &= \frac{1}{i}\oint \frac{z^2+z^{-2}}{4(z+1/2)(z+2)}dz = \frac{1}{4i}\oint \frac{z^4+1}{z^2(z+1/2)(z+2)}dz = \frac{1}{4i}\Omega \end{aligned} \tag{14}$$

where we have used the ABC equation to write the denominator az^2+bz+c as $a(z-z_1)(z-z_2)$ where z_1, z_2 are the two roots. We use the residue theorem to evaluate the integral Ω:

$$\oint f(z)dz = 2\pi i \sum_i R(z_i) \tag{15}$$

where the sum runs over all the residues of the function f inside the integral contour. We have two residues: one at the origin $z_1 = 0$ and one at the point $z_2 = -1/2$; the point $z_3 = -2$ is outside the unit circle. The residue at the origin is found by the procedure detailed on page 685 in Boas;

$$F(z) = (z-z_1)^2 f(z) = z^2\frac{z^4+1}{z^2(z+1/2)(z+2)} = \frac{z^4+1}{(z+1/2)(z+2)} \tag{16}$$

$$\frac{dF}{dz} = \frac{4z^3(z+1/2)(z+2) - (z^4+1)(2z+5/2)}{(z+1/2)^2(z+2)^2} \tag{17}$$

$$R(z_1) = \frac{1}{(2-1)!}\frac{dF}{dz}(z_1) = \frac{0-(0+1)(0+5/2)}{(0+1/2)^2(0+2)^2} = \frac{-5/2}{(1/4)\cdot 4} = -5/2 \tag{18}$$

The residue at z_2 is much easier:

$$R(z_2) = \lim_{z\to z_2}(z-z_2)f(z) = \lim_{z\to -1/2}(z+1/2)\frac{z^4+1}{z^2(z+1/2)(z+2)} = \frac{1/16+1}{1/4\cdot(-1/2+2)} = \frac{17}{6} \tag{19}$$

Finally, the integral becomes

$$I = \frac{1}{4i}\Omega = \frac{1}{4i}2\pi i\left(R(z_1)+R(z_2)\right) = \frac{\pi}{2}\left(\frac{17}{6}-\frac{5}{2}\right) = \frac{\pi}{6} \tag{20}$$

which we can verify using a computer, i.e. on WolframAlpha, or simply on a calculator.

Problem 7.9

Calculate

$$\int_0^{2\pi} \frac{d\theta}{1+\sin\theta\cos\alpha} \tag{21}$$

where α is come constant, which we will assume to be real. I will think of this integral as a function of α:

$$I(\alpha) = \int_0^{2\pi} \frac{d\theta}{1+\sin\theta\cos\alpha} \tag{22}$$

Let us deal with some special cases straight away. If $\alpha = \pi/2 + k\pi$ where $k \in \mathbb{Z}$, then $\cos\alpha = 0$. It is then obvious that

$$I(\pi/2+k\pi) = \int_0^{2\pi} d\theta = 2\pi \tag{23}$$

If $\alpha = k\pi$ we have $\cos\alpha = 1$, and the integral looks like it will diverge. In the following we will therefore assume that α does not take any of these values.

We make the same change of coordinates as in 7.7 and obtain:

$$I(\alpha) = \oint \frac{1}{1+\cos\alpha(z-z^{-1})/2i}\frac{dz}{iz} = \oint \frac{2i}{-2z+i\cos\alpha(z^2-1)}dz = 2i\oint\frac{1}{az^2-2z-a}dz \tag{24}$$

where $a = i\cos\alpha$. We factorize the denominator: The roots of the denominator are

$$z = a^{-1}\left(1\pm\sqrt{1+a^2}\right) \tag{25}$$

We also notice that since $\cos\alpha \in (-1,1)\backslash\{0\}\ \forall\ \alpha$, a will always lie between i and $-i$ on the imaginary axis. That is:

$$z = \frac{1}{i\cos\alpha}\left(1\pm\sqrt{1+i^2\cos^2\alpha}\right) = -i\left(\frac{1}{\cos\alpha}\pm\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}\right) = -i\left(\frac{1\pm|\sin\alpha|}{\cos\alpha}\right) \tag{26}$$

As long as $\cos\alpha \neq 0$, the root with addition in the numerator lies outside the unit circle. In this case, the only singularity comes from the point

$$z_0 = -i\left(\frac{1-|\sin\alpha|}{\cos\alpha}\right) \tag{27}$$

The residue is

$$\begin{aligned}
R(z_0) &= \lim_{z\to z_0}(z-z_0)f(z) \\
&= \lim_{z\to z_0}(z-z_0)\frac{1}{a(z-z_0)(z+i\,(1+|\sin\alpha|)\,/\cos\alpha)} \\
&= \frac{1}{a}\left(-i\left(\frac{1-|\sin\alpha|}{\cos\alpha}\right)+i\left(\frac{1+|\sin\alpha|}{\cos\alpha}\right)\right)^{-1} \\
&= \frac{1}{a}\left(i\frac{2|\sin\alpha|}{\cos\alpha}\right)^{-1} \\
&= \frac{1}{i\cos\alpha}\cdot\frac{\cos\alpha}{2i|\sin\alpha|} \\
&= \frac{-1}{2|\sin\alpha|}
\end{aligned} \tag{28}$$

And the integral is

$$I(\alpha) = 2i\cdot 2\pi i\cdot\frac{-1}{2|\sin\alpha|} = \frac{2\pi}{|\sin\alpha|} \tag{29}$$

We see that the limit where $\alpha\to\pi/2+k\pi$ reproduces the first result nicely, and also that the integral diverges as $\alpha\to k\pi$ as we expected.

Problem 7.11

Calculate the integral

$$I = \int_0^\infty \frac{dx}{(4x^2+1)^3} \tag{30}$$

We make an integration path along the real axis from $z_1=-r$ to $z_2=r$ and connect it by a semicircle in the upper half of the complex plane with radius r: see page 688 in Boas for a figure. We then look at the integral

$$I^* = \oint \frac{dz}{(4z^2+1)^3} = \int_{-r}^{r}\frac{dx}{(4x^2+1)^3} + \int_0^\pi \frac{rie^{i\theta}d\theta}{(4r^2e^{2i\theta}+1)^3} \tag{31}$$

If we take the limit $r\to\infty$, then the complex integral will contain the singularity $z_0=i/2$, and we can find its value by the residue theorem. We have

$$\oint\frac{dz}{(4z^2+1)^3} = \frac{1}{4^3}\oint\frac{dz}{(z^2+1/4)^3} = \frac{1}{4^3}\oint\frac{dz}{(z+i/2)^3(z-i/2)^3} \tag{32}$$

Again following page 685 in Boas:

$$F = (z-z_0)^3 f(z) = \frac{1}{(z+i/2)^3} = (z+i/2)^{-3} \tag{33}$$

$$\frac{d^2F}{dz^2} = \frac{d}{dz}\left(-3(z+i/2)^{-4}\right) = 12(z+i/2)^{-5} \tag{34}$$

$$R(z_0) = \frac{1}{2!}\frac{d^2F}{dz^2}(z_0) = 6(i/2+i/2)^{-5} = \frac{6}{i} \tag{35}$$

As $r\to\infty$ the integral over θ vanishes because the integrand goes to zero along the whole integration path as r/r^6. We can conclude:

$$\oint\frac{dz}{(4z^2+1)^3} = \frac{1}{4^3}\cdot 2\pi i\cdot\frac{6}{i} = \int_{-\infty}^{\infty}\frac{dx}{(4x^2+1)^3}+0 \tag{36}$$

Meaning

$$\int_{-\infty}^{\infty} \frac{dx}{(4x^2+1)^3} = \frac{12\pi}{64} = \frac{3\pi}{16} \tag{37}$$

The integrand is an even function, since $(4(-x)^2+1) = (4x^2+1)$, so our final result is

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(4x^2+1)^3} = \frac{3\pi}{32} \tag{38}$$

Problem 7.13

Calculate the integral

$$I = \int_{0}^{\infty} \frac{x^2 dx}{(x^2+4)(x^2+9)} \tag{39}$$

The steps are very similar to the previous problem, so we will be breifer. The function is even, so

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+4)(x^2+9)} \tag{40}$$

We consider

$$\oint \frac{z^2 dz}{(z^2+4)(z^2+9)} = \oint \frac{z^2}{(z+2i)(z-2i)(z+3i)(z-3i)} dz \tag{41}$$

along the same kind of curve as in Problem 7.11. The singularities that end up inside the curve are $z_1 = 2i$ and $z_2 = 3i$, while the other two are outside. The poles are all simple, so

$$R(z_1) = \frac{-4}{4i \cdot 5i \cdot (-i)} = \frac{-1}{5i} = \frac{i}{5} \tag{42}$$

and

$$R(z_2) = \frac{-9}{5i \cdot i \cdot 6i} = \frac{9}{30i} = \frac{-3i}{10} \tag{43}$$

We are left with

$$I = \frac{1}{2} \cdot 2\pi i \cdot (\frac{i}{5} - \frac{3i}{10}) = \frac{\pi}{10} \tag{44}$$