FYS 3120 Classical Mechanics and Electrodynamics
Spring semester 2015

Problem Set 7

Problem 7.1

This problem is an exercise in using space-time diagrams (Minkowski diagrams) when studying how elementary relativistic effects are represented in different inertial reference frames.

A railway carriage is moving in a straight line with constant velocity v relative to the earth. The earth is considered as an inertial reference frame S, and in this reference frame the moving carriage has the length L. The situation is shown in Figure 1, where A and B indicate points on the rear wall and front wall of the carriage, respectively. C is a point in the middle of the carriage.

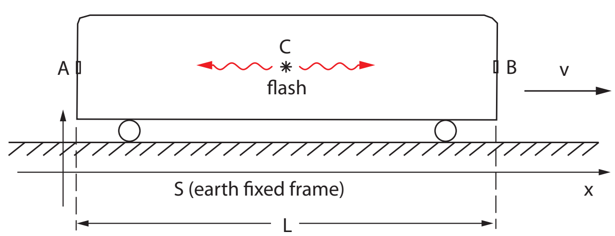


Figure 1:

a) In Figure 2 we have drawn the world line (space-time trajectory) for the midpoint C in a two-dimensional Minkowski diagram of reference frame S. Draw the world lines for the points A and B in the same diagram and show that the angle α between these lines and the time axis is given by $\tan\alpha = v/c$. (Choose the origin of the coordinate system in S so that A has coordinate $x = 0$ at time $t = 0$.)

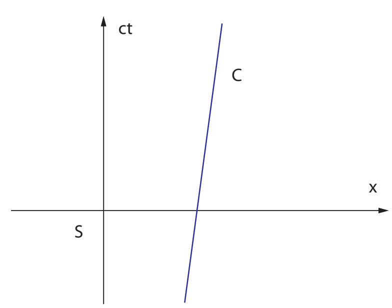


Figure 2:

At a given time t_0 a flash tube is discharged at point C. We will call this event (space-time point) E_0. Some of the light will propagate backwards in the compartment and some will propagate forwards. Let E_1 and E_2 be the events where the light signals hit the rear wall and front wall, respectively. Let us assume that the light is reflected from A and B, and that the two reflected light signals meet at a space-time point E_3.

b) Draw the world lines of the light signals as well as the four events E_0, E_1, E_2 and E_3 in the Minkowski diagram of reference frame S.

c) We introduce the co-moving reference frame S' of the carriage. Explain why E_1 and E_2 are simultaneous in this reference frame and why E_0 and E_3 are at the same point in space in S'. Is this consistent with the drawing of point b)?

d) Draw the straight line from from E_1 to E_2 in the Minkowski diagram of S and show that the angle between the x-axis and this line is α.

e) Show that if a signal should connect the two space-time points E_1 and E_2 it must have the velocity c^2/v (which is greater than c).

f) In the Minkowski diagram of S, the coordinate axes of x and t appear as orthogonal lines. Draw in the same diagram the coordinate axes of S', corresponding to $t' = 0$ and $x' = 0$. Choose the coordinates so that $x' = 0$, $t' = 0$ corresponds to $x = 0$, $t = 0$.

g) The space-time position for any event E can in the diagram be read out either as x and ct in the orthogonal coordinate system of S or as x' and ct' in the non-orthognal coordinate system of S'. Show how this is done by plotting in the space-time position of an arbitrarily chosen event E, and indicate by lines to the coordinate axes of S and S' how the coordinates of E can be read out of the diagram.

Problem 7.2

Consider an inertial reference frame S in four-dimensional space-time, with coordinate axes defined by a set of unit vectors $\underline{\mathbf{e}}_\mu$, $\mu = 0, 1, 2, 3$. They satisfy the generalized orthonormalization condition

$$\underline{\mathbf{e}}_\mu \cdot \underline{\mathbf{e}}_\nu = g_{\mu\nu} \tag{1}$$

with $g_{\mu\nu}$ as the standard metric tensor (see the lecture notes). A second inertial frame S' has coordinate axes with unit vectors that mix those of S in the following way,

$$\begin{aligned} \underline{\mathbf{e}}'_0 &= \cosh\chi\,\underline{\mathbf{e}}_0 + \sinh\chi\,\underline{\mathbf{e}}_1 \\ \underline{\mathbf{e}}'_1 &= \sinh\chi\,\underline{\mathbf{e}}_0 + \cosh\chi\,\underline{\mathbf{e}}_1 \\ \underline{\mathbf{e}}'_2 &= \underline{\mathbf{e}}_2 \\ \underline{\mathbf{e}}'_3 &= \underline{\mathbf{e}}_3 \end{aligned} \tag{2}$$

with χ as a velocity parameter called the *rapidity*.

a) Assume S' has velocity v relative to S. Show that

$$\cosh\chi = \gamma\,, \quad \sinh\chi = \beta\gamma \tag{3}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, by relating (3) to the standard Lorentz transformation formula for the coordinates of the two reference frames.

b) Assume the two sets of basis vectors $(\underline{\mathbf{e}}_0, \underline{\mathbf{e}}_1)$ and $(\underline{\mathbf{e}}'_0, \underline{\mathbf{e}}'_1)$ of S and S', are plotted in a common two-dimensional Minkowski diagram, with $\underline{\mathbf{e}}_0$ and $\underline{\mathbf{e}}_1$ defining the orthogonal, horizontal and vertical directions of the diagram. With ϕ as the angle between the two time axes $\underline{\mathbf{e}}_0$ and $\underline{\mathbf{e}}'_0$ in the diagram, show that

$$\tan\phi = \tanh\chi \tag{4}$$

For the cases $\phi = 15°$ and $\phi = 30°$ determine the corresponding values of $\beta = v/c$.

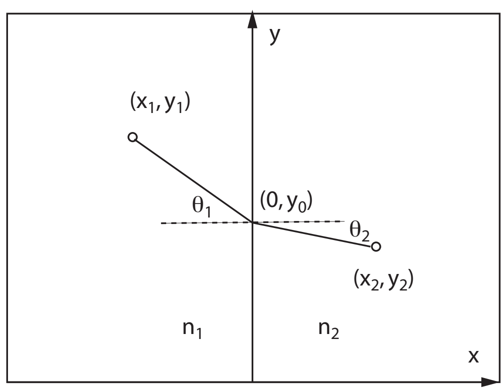
c) Show that the vector $\underline{\mathbf{e}}'_0$ will trace out an hypebola when the velocity v changes, and that $\underline{\mathbf{e}}'_1$ will trace out another hyperbola.

d) Draw a two-dimensional Minkowski diagram, as referred to in b), where the basis vectors of S', for both angles $\phi = 15°$ and $\phi = 30°$, are shown. Also plot in the diagram the two hyperbolas referred to in c), as well as the lines of the light cone.

Problem 7.3 (Midterm Exam 2014)

According to Fermat's Principle, a light ray will follow the path between two points which makes the *optical path length* extremal. For simplicity we consider here paths constrained to a two dimensional plane (the x, y plane), in an optical medium with a position dependent index of refraction $n(x, y)$. The optical path length between two points (x_1, y_1) and (x_2, y_2) along $y(x)$ can be written as the integral

$$S[y(x)] = \int_{x_1}^{x_2} n(x,y)\sqrt{1+y'^2}dx\ , \qquad y' = \frac{dy}{dx} \tag{5}$$



a) Find Lagrange's equation for the variational problem $\delta S = 0$, and show that if the index of refraction is constant the equation has the straight line between the two points as solution.

b) Assume the medium to have two different, constant indices of refraction, $n = n_1$ for $x < 0$ og $n = n_2$ for $x > 0$ (see the figure). Explain why the variational problem can now be simplified to the problem of finding the coordinate $y = y_0$ for the point where the light ray crosses the boundary between the two media at $x = 0$. Find the equation for y_0 that gives the shortest optical path length. (Solving the equation is not needed.)

c) Show that the equation for y_0 at point b) implies that the path of the light ray satisfies Snell's law of refraction,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{6}$$

with θ_1 and θ_2 as the angle of the light ray relative to the normal on the two sides of the the boundary.