FYS 3120 Classical Mechanics and Electrodynamics
Spring semester 2015

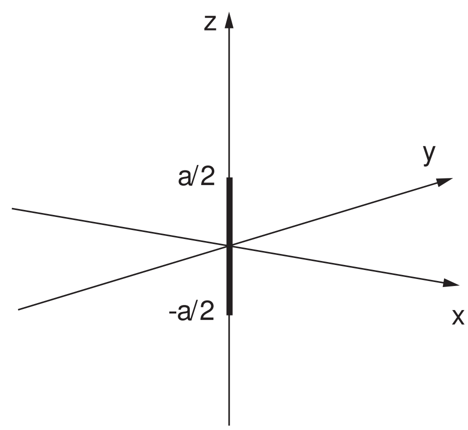
Problem Set 13

Problem 13.1 (Exam 2009)

A so-called *half-wave center-fed* antenna is formed by a thin linear conductor of length a. It is oriented along the z-axis as shown in the figure. An alternating current is running in the antenna, of the form

$$I(z,t) = I_0 \cos\frac{\pi z}{a}\cos\omega t\ , \quad -a/2 < z < a/2 \tag{1}$$

In the following $\lambda(z,t)$ denotes the linear charge density of the antenna (charge per unit length). At time $t=0$ the antenna is charge neutral, so that $\lambda(z,0)=0$.



a) Show that the charge density and current satisfy the relation

$$\frac{\partial\lambda}{\partial t} + \frac{\partial I}{\partial z} = 0 \tag{2}$$

and find λ as a function of z and t.

b) Show that the electric dipole moment of the antenna has the form

$$\mathbf{p}(t) = p_0 \sin\omega t\ \mathbf{k} \tag{3}$$

with $\mathbf{k}$ as the unit vector along the z-axis, and determine the constant p_0.

c) Use the expressions for electric dipole radiation to determine the electric and magnetic fields in a point at a large distance r from the antenna on the x-axis. What is the type of polarization of the radiation from the antenna in this direction?

As a reminder, the general expressions for the radiation fields of an electric dipole positioned at the origin are

$$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0}{4\pi c}\frac{\mathbf{n}}{r}\times\ddot{\mathbf{p}}_{ret}\ , \quad \mathbf{E}(\mathbf{r},t) = c\mathbf{B}(\mathbf{r},t)\times\mathbf{n} \tag{4}$$

with $\mathbf{p}_{ret} = \mathbf{p}(t-r/c)$ and $\mathbf{n} = \mathbf{r}/r$.

Problem 13.2 (Exam 2007)

In a circular loop of radius a an oscillating current of the form $I = I_0 \cos\omega t$ is running. The current loop lies in the x, y plane. We use the notation $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ for the Cartesian unit vectors in the directions x, y and z, in order to reserve the symbol $\mathbf{j}$ for the current density. The current loop is at all times charge neutral.

a) Explain why the electric dipole moment $\mathbf{p}$ of the current loop vanishes, and show that the magnetic dipole moment has the following time dependence, $\mathbf{m}(t) = m_0 \cos\omega t\, \mathbf{e}_z$, with m_0 as a constant. Find m_0 expressed in terms of a and I_0.

As a reminder, the general expressions for the radiation fields of a magnetic dipole are

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c r}\ddot{\mathbf{m}}_{ret} \times \mathbf{n}\,; , \quad \mathbf{B}(\mathbf{r}, t) = -\frac{1}{c}\mathbf{E}(\mathbf{r}, t) \times \mathbf{n} \tag{5}$$

with $\mathbf{m}_{ret} = \mathbf{m}(t - r/c)$ and $\mathbf{n} = \mathbf{r}/r$. In the following we assume that we study the fields far from the current loop (in the radiation zone) where the expressions (5) are valid.

b) Write down the expressions for the radiation fields for points on the x axis far from the current loop and show that that they have the form of electromagnetic waves that propagate in the direction away from the loop. What is the polarization of the waves?

c) Use the general expression for Poynting's vector $\mathbf{S}$ to find the radiated power per unit solid angle $\frac{dP}{d\Omega}$, in the x direction. What is the corresponding radiated power in the direction of the z axis?

Problem 13.3 (Exam 2006)

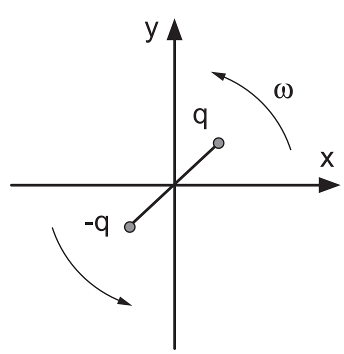


Figure 1:

A thin rigid rod of length ℓ rotates in a horizontal plane (the x,y-plane) as shown in Fig. 2. At the two end points there are fixed charges of opposite sign, $+q$ and $-q$. The rod is rotating with constant angular frequency ω. This gives rise to a time dependent electric dipole moment

$$\mathbf{p}(t) = q\ell(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}) \tag{6}$$

a) Use the general expression for the radiation fields of an electric dipole (see the formula collection of the course) to show that the magnetic field in the present case can be written as

$$\mathbf{B}(\mathbf{r}, t) = B_0(r)(\cos\theta \sin(\omega(t - \frac{r}{c}))\,\mathbf{i} - \cos\theta\cos(\omega(t - \frac{r}{c}))\mathbf{j} - \sin\theta\sin(\omega(t - \frac{r}{c}) - \phi)\,\mathbf{k} \tag{7}$$

with (r, θ, ϕ) as the polar coordinates of $\mathbf{r}$. Find the expression for $B_0(r)$.

What is the general relation between the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ in the radiation zone? (A detailed expression for $\mathbf{E}(\mathbf{r}, t)$ is not needed.)

b) Show that radiation in the x-direction is linearly polarized. What is the polarization of the radiation in the z-direction?

c) Find the time-averaged expression for the energy density of the radiation. In what direction has the radiated energy its maximum?