Problem 3.2 (Generalized Cauchy integral formula)

you can differentiat with respect to a or z. In both cases you will get same answer. Here I will differentiate eqn(1) with respect to z

$$\frac{d}{dz}f(a) = \frac{1}{2\pi i}\frac{d}{dz}\oint_\Gamma \frac{f(z)}{z-a}dz$$

$$\Rightarrow 0 = \frac{1}{2\pi i}\oint_\Gamma \frac{\partial}{\partial z}\frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\oint_\Gamma \left\{\frac{\frac{\partial}{\partial z}f(z)(z-a) - f(z)}{(z-a)^2}\right\}dz$$

$$= \frac{1}{2\pi i}\oint_\Gamma \left\{\frac{\frac{\partial}{\partial z}f(z)}{z-a} - \frac{f(z)}{(z-a)^2}\right\}dz$$

$$\Rightarrow \frac{1}{2\pi i}\oint_\Gamma \frac{\frac{\partial}{\partial z}f(z)}{z-a}dz = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-a)^2}dz$$

comparing the left side of the above equation with eqn(1), where $f(z)$ in eqn(1) is now $\frac{\partial}{\partial z}f(z) = \frac{df(z)}{dz}$

$$f^{'}(a) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-a)^2}dz \tag{2}$$

doing the same differentiation again on eqn (2)

$$0 = \frac{1}{2\pi i}\oint_\Gamma \frac{\partial}{\partial z}\frac{f(z)}{(z-a)^2}dz$$

$$= \frac{1}{2\pi i}\oint_\Gamma \frac{\frac{\partial}{\partial z}(f(z))(z-a)^2 - f(z)2(z-a)}{(z-a)^4}dz = \frac{1}{2\pi i}\oint_\Gamma \left\{\frac{\frac{\partial}{\partial z}(f(z))}{(z-a)^2} - \frac{2f(z)}{(z-a)^3}\right\}dz$$

again comparing the left part of the integrand with eqn(2), we get

$$f^{''}(a) = \frac{2}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-a)^3}dz \tag{3}$$

If we perform the same operation on eqn(3), we get

$$f^3(a) = \frac{2\times 3}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-a)^4}dz \tag{4}$$

Thus, we can see that n identical operation give us

$$f^n(a) = \frac{2\times 3\times 4\times\cdots\times(n-1)n}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-a)^{n+1}}dz = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-a)^{n+1}}dz \tag{5}$$

Using eqn (5) we can solve

$$\oint_\Gamma \frac{\sin 2z}{(6z-\pi)^3}dz = \oint_\Gamma \frac{1/6^3\sin 2z}{(z-\pi/6)^3}dz$$

where $f(z) = 1/6^3\sin(2z)$ and $f^2(z) = -4/6^3\sin 2z$

$$= \frac{2\pi i}{2!}(f^2(a)) = -\pi i(4/6^3\sin(2(\pi/6))) = \underline{-\frac{i2\sqrt{3}\pi}{6^3}}$$

Problem 3.3 (Laurent Series)

The question is to find Laurent series about the **origin**. Hence, we shall keep the $1/z^2$ part and Taylor expand the remaining expression about zero.

$$f(z) = \frac{z-1}{z^2(z-2)} = \frac{1}{z^2}\left\{1 + \frac{1}{z-2}\right\}$$

good way to find converged Taylor expansion of fraction of the form $\frac{1}{z-2}$ is to compare it with the well know series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n; \qquad \text{disk of convergence} \quad |z| < 1$$

rewriting $\frac{1}{z-2}$ as $-1/2\frac{1}{(1-z/2)}$ and its series becomes

$$-1/2\frac{1}{(1-z/2)} = \frac{-1}{2}\sum_{n=0}^{\infty}(\frac{z}{2})^n \qquad (*)$$

and disk of convergence

$$|\frac{z}{2}| < 1 \Rightarrow |z| < 2$$

This implies in region $|z| < 2$ we have converged series. But outside this region we need to rewrite in $1/z$ form and find another series for convergence reason. i.e.

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-2/z} = \frac{1}{z}\sum_{n=0}^{\infty}(\frac{2}{z})^n \qquad (**)$$

and disk of convergence is

$$|\frac{2}{z}| < 1 \Rightarrow |z| > 2$$

Therefore, we would use eqn(*) for problem (a) and eqn(**) for problem (b.

a) in region $|z| < 2$ the valid series using eqn(*) is

$$\frac{1}{z^2}\left\{1 + \frac{-1}{2}\sum_{n=0}^{\infty}(\frac{z}{2})^n\right\} = \frac{1}{z^2} - \sum_{n=0}^{\infty}\frac{z^{n-2}}{2^{n+1}}$$

b) in region $|z| > 2$ the valid series using eqn(**) is

$$\frac{1}{z^2}\left\{1 + \frac{1}{z}\sum_{n=0}^{\infty}(\frac{2}{z})^n\right\} = \frac{1}{z^2} + \sum_{n=0}^{\infty}\frac{2^n}{z^{n+3}}$$

c) residue of f at origin is the coefficient of $1/z$ term in the fist series (the series that converges near zero). Therefore, it is $-1/4$. We can not use series b) to find the residue about 0.

Problem3.4 (singularities)

We solve the following problems by finding the Laurent series about the indicated point and by examining its 'b' series. If the coefficients of the 'b' series is all zero then we call the given point regular(removable); otherwise, its biggest non-zero inverse power would be its pole order. If the biggest non-zero inverse power occurs at infinity, then it is essential singularity.

a)

$$\frac{\sin z}{3z} = \frac{1}{3z}\sum_{n=0}^{\infty}(-1)^n\frac{(z)^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{3z}\left\{z - \frac{(z)^3}{3!} + \frac{(z)^5}{5!} + \cdots,\right\} = \frac{1}{3} - \frac{(z)^2}{3(3!)} + \frac{(z)^4}{3(5!)} + \cdots$$

we can see that all coefficients for the 'b' series are zero. and $z = 0$ is regular point (removable singularity.

b)

$$\frac{\cos z}{z^4} = \frac{1}{z^4}\sum_{n=0}^{\infty}(-1)^n\frac{(z)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(z)^{2n-4}}{(2n)!}$$

or

$$= \frac{1}{z^4}\left\{1 - \frac{(z)^2}{2!} + \frac{(z)^4}{4!} - \cdots,\right\} = \underbrace{-\frac{(1}{z^4} + \frac{(z)^{-2}}{2!}}_{\text{b series terms}} - \underbrace{\frac{(z)^0}{4!} + \frac{(z)^2}{6!} - \cdots}_{\text{a series}}$$

we can see that $z = 0$ is pole of order 4.

(c)

$$\frac{z^3-1}{(z-1)^3} = 1 + \frac{3z}{(z-1)^2} = 1 + \frac{3}{z-1} + \frac{3}{(z-1)^2}$$

hence, $z = 1$ is pole of order 2.

d)

$$\frac{e^z}{z-1} = \frac{ee^{z-1}}{z-1} = \frac{e}{z-1}\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} = e\sum_{n=0}^{\infty}\frac{(z-1)^{n-1}}{n!}$$

hence, $z = 1$ is a simple pole, which you can see the term at $n = 0$.

Extra problems

14.3.20 using Cauchy's integral formula,

$$\oint_C \frac{\cosh z}{2\ln 2 - z}dz = -\oint_C \frac{\cosh z}{z - 2\ln 2}dz$$

a) $= 0$ as $z = 2\ln 2 = \ln 4$ is outside C.
b)

$$= 2\pi i \cdot f(2\ln 2) = 2\pi i(-\cosh(\ln 4)) = -i\pi(e^{\ln 4} + e^{-\ln 4}) = \underline{-i8\pi}$$

14.3.23 using eqn(5)

$$\oint_c \frac{e^{3z}dz}{(z-\ln 2)^4} = \frac{2\pi i}{3!} \cdot f^3(z = \ln 2) = \frac{2\pi i}{6}(27e^{3(\ln 2)}) = \underline{i72\pi}$$

14.4.8 The question is to find Laurent series about **zero**, for each annular ring between singular points. It means that, for this problem we shall find four Laurent series. First, using partial fraction decomposition,

$$\frac{30}{(1+z)(z-2)(3+z)} = \frac{-5}{1+z} + \frac{2}{2-z} + \frac{3}{3+z}$$

here we have simple poles at $z = -1$; $z = 2$; and $z = -3$. Notice that $z = 0$ is a regular point. Therefore, we find power series for all terms about zero! Because there are three poles, we would have four annular rings in which the Laurent series is computed separately. The regions are thus 1) $|z| < 1$; 2) $1 < |z| < 2$; 3) $2 < |z| < 3$; and 4) $|z| > 3$.
To this end, we find the power series of each of the terms in their regions of convergence. i.e.

$$\frac{-5}{1+z} = \begin{cases} -5\sum_{n=0}^{\infty}(-1)^n z^n = 5\sum_{n=0}^{\infty}(-1)^{n+1}z^n & \text{for } |z| < 1 \\ \frac{1}{z}(\frac{-5}{1+1/z}) = \frac{-5}{z}\sum_{n=0}^{\infty}(-1)^n(\frac{1}{z})^n = 5\sum_{n=0}^{\infty}(-1)^{n+1}(\frac{1}{z})^{n+1} & \text{for } |z| > 1 \end{cases}$$

$$\frac{2}{2-z} = \begin{cases} -\frac{1}{1-z/2} = -\sum_{n=0}^{\infty}(\frac{z}{2})^n & \text{for } |z| < 2 \\ \frac{1}{z}\frac{2}{1-2/z} = \frac{2}{z}\sum_{n=0}^{\infty}(\frac{2}{z})^n = \sum_{n=0}^{\infty}(\frac{2}{z})^{n+1} & \text{for } |z| > 2 \end{cases}$$

$$\frac{3}{3+z} = \begin{cases} \frac{1}{1+z/3} = \sum_{n=0}^{\infty}(-1)^n(\frac{z}{3})^n & \text{for } |z| < 3 \\ \frac{3}{z}\frac{1}{1+3/z} = \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n(\frac{3}{z})^n = \sum_{n=0}^{\infty}(-1)^n(\frac{3}{z})^{n+1} & \text{for } |z| > 3 \end{cases}$$

Laurent series would be constructed by selecting converged series for each term in that region. i.e.
1) $|z| < 1$

$$\frac{30}{(1+z)(z-2)(3+z)} = 5\sum_{n=0}^{\infty}(-1)^{n+1}z^n - \sum_{n=0}^{\infty}(\frac{z}{2})^n + \sum_{n=0}^{\infty}(-1)^n(\frac{z}{3})^n$$

notice that the series for each term converges in $|z| < 1$. The residue is 0. we can not use the remaining series for finding the residue at 0.
2) $1 < |z| < 2$

$$\frac{30}{(1+z)(z-2)(3+z)} = 5\sum_{n=0}^{\infty}(-1)^{n+1}(\frac{1}{z})^{n+1} - \sum_{n=0}^{\infty}(\frac{z}{2})^n + \sum_{n=0}^{\infty}(-1)^n(\frac{z}{3})^n$$

notice that only the first term in the RHS differs from the previous one.
3) $2 < |z| < 3$

$$\frac{30}{(1+z)(z-2)(3+z)} = 5\sum_{n=0}^{\infty}(-1)^{n+1}(\frac{1}{z})^{n+1} + \sum_{n=0}^{\infty}(\frac{2}{z})^{n+1} + \sum_{n=0}^{\infty}(-1)^n(\frac{z}{3})^n$$

4) $|z| > 3$

$$\frac{30}{(1+z)(z-2)(3+z)} = 5\sum_{n=0}^{\infty}(-1)^{n+1}(\frac{1}{z})^{n+1} + \sum_{n=0}^{\infty}(\frac{2}{z})^{n+1} + \sum_{n=0}^{\infty}(-1)^n(\frac{3}{z})^{n+1}$$

14.4.11

a) $\frac{e^z-1-z}{z^2}$ **Ans.** $z = 0$ is regular point.
b) $\frac{\sin z}{z^3}$ **Ans.** $z = 0$ is a pole of order 2.
c) $\frac{z^2-1}{(z-1)^2}$ **Ans.** $z = 1$ is a simple pole.
b) $\frac{\cos z}{(z-\pi/4)^4}$ **Ans.** $z = \pi/2$ is a pole of order 3.