Solutions to Problem Set 7 FYS3140

Problem 7.1

The method of variation of parameters states that if two linearly independent solutions y_1, y_2 to a homogeneous second order differential equation is known, then a particular solution to a non-homogeneous equation is given by

$$y_p(x) = -y_1 \int \frac{y_2(x)R(x)}{W(x)}\, dx + y_2 \int \frac{y_1(x)R(x)}{W(x)}\, dx \tag{1}$$

where $R(x)$ is the right hand side of the DE in canonical form and $W(x)$ is the Wronskian

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \tag{2}$$

a)

The DE is

$$x^2y'' - 2xy' + 2y = x\ln x \tag{3}$$

which in canonical form becomes

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = \frac{\ln x}{x} \tag{4}$$

We are given solutions $y_1 = x$, $y_2 = x^2$ to the homogeneous DE. The Wronskian is

$$W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \tag{5}$$

and the particular solution becomes

$$\begin{aligned} y_p(x) &= -x \int \frac{x^2}{x^2}\frac{\ln x}{x}\, dx + x^2 \int \frac{x}{x^2}\frac{\ln x}{x}\, dx \\ &= -x \int u\ du + x^2 \left[-\frac{\ln x}{x} + \int \frac{1}{x^2}\, dx\right] \\ &= -\frac{x}{2}\ln^2 x + x^2\left[-\frac{\ln x + 1}{x}\right] \\ &= -x\left(\frac{1}{2}\ln^2 x + \ln x + 1\right) \end{aligned} \tag{6}$$

The general solution to the DE is therefore

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x) = \underline{c_1x + c_2x^2 - x\left(\frac{1}{2}\ln^2 x + \ln x + 1\right)} \tag{7}$$

b)

The DE is

$$(x^2+1)y'' - 2xy' + 2y = (x^2+1)^2 \tag{8}$$

which in canonical form becomes

$$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y = (x^2+1) \tag{9}$$

We are given solutions $y_1 = x$, $y_2 = 1 - x^2$. The Wronskian is

$$W(x) = \begin{vmatrix} x & (1-x^2) \\ 1 & -2x \end{vmatrix} = -2x^2 + x^2 - 1 = -x^2 - 1 = -(x^2+1) \tag{10}$$

We find

$$\begin{aligned} y_p(x) &= -x \int -\frac{(1-x^2)(x^2+1)}{(x^2+1)}\ dx + (1-x^2) \int -\frac{x(x^2+1)}{(x^2+1)}\ dx \\ &= x \int 1 - x^2\ dx - (1-x^2) \int x\ dx \\ &= x^2 - \frac{1}{3}x^4 - \frac{1}{2}x^2 + \frac{1}{2}x^4 \\ &= \frac{1}{2}x^2 + \frac{1}{6}x^4 \end{aligned} \tag{11}$$

The general solution is then

$$y(x) = \underline{c_1 x + c_2(1-x^2) + \frac{1}{2}x^2 + \frac{1}{6}x^4} \tag{12}$$

Problem 7.2

12.1.8)

We are to solve the DE

$$(x^2 + 2x)y'' - 2(x+1)y' + 2y = 0 \tag{13}$$

by an elementary method, and then by a series solution. The focus will be on the latter method. For the elementary method, we want to make a guess at one solution $u(x)$. A simplifying choice is one that has $u''(x) = 0$, for instance a linear function $u(x) = Ax + B$. Inserted, we find

$$-2(x+1)A + 2Ax + 2B = 0 \tag{14}$$

which gives

$$A = B \tag{15}$$

That is, one solution is $y_1 = x + 1$. The second solution is found by letting $y_2 = u(x)v(x)$. Inserted into the original DE, this gives us a new DE,

$$\left[(x^2+2x)u'' - 2(x+1)u' + 2u\right] v + \left[(x^2+2x)u\right] v'' + \left[2(x^2+2x)u' - 2(x+1)u\right] v' = 0 \tag{16}$$

The part in front of v vanishes because u was a solution to the original DE. Changing variables to $w = v'$ and inserting $u' = 1$ and $u = x + 1$, we find

$$(x^2+2x)(x+1)w' + \left[2(x^2+2x) - 2(x+1)^2\right] w = 0 \tag{17}$$

which is a *first order* DE. Working it out we find

$$(x^2+2x)(2x+2)w' - 4w = 0 \tag{18}$$

which is separable. The solution is

$$w(x) = c_1 \frac{x^2+2x}{(x+1)^2} \tag{19}$$

from which we find

$$y_2 = u(x)v(x) = (x+1)\int w(x)\ dx = c_1(x+1)\frac{x^2+x+1}{(x+1)} = c_1(x^2+x+1) \tag{20}$$

Finally, the general solution is

$$y(x) = \underline{c_1(x+1) + c_2(x^2+x+1) = C_1(x+1) + C_2x^2} \tag{21}$$

Let us now turn to the series expansion solution. We will assume the solution y to be a power series of the Frobenius form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s} \tag{22}$$

Inserting into the DE gives:

$$\begin{aligned} &\sum (n+s)(n+s-1)a_n x^{n+s} + 2\sum (n+s)(n+s-1)a_n x^{n+s-1} \\ &\quad -2\sum (n+s)a_n x^{n+s} - 2\sum (n+s)a_n x^{n+s-1} + 2\sum a_n x^{n+s} = 0 \end{aligned} \tag{23}$$

where all the sums run from $n = 0$. Since all the different powers of x are linearly independent, the coefficients of *all* powers of x must be 0. In the sums where x^{n+s-1}appears, we change the dummy summation indices to n', and then rewrite them in terms of n. As an example, the part

$$\sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s-1} \tag{24}$$

becomes

$$\sum_{n'=0}^{\infty} (n'+s)(n'+s-1)a_{n'} x^{n'+s-1} \tag{25}$$

We want the powers of x to look the same across all the sums, so we require $n' - 1 = n$, that is, $n' = n + 1$. Therefore we get

$$\sum_{n=-1}^{\infty} (n+s+1)(n+s)a_{n+1} x^{n+s} \tag{26}$$

The full DE becomes, after dividing everywhere by 2,

$$\begin{aligned} \sum_{n=0}^{\infty} \left[\frac{1}{2}(n+s)(n+s-1)a_n + (n+s+1)(n+s)a_{n+1} - (n+s)a_n - (n+s+1)a_{n+1} + a_n\right] x^n \\ + \left(s(s-1)a_{-1} - sa_{-1}\right) x^{s-1} = 0 \end{aligned} \tag{27}$$

where we have extracted the terms with $n = -1$ for clarity. The coefficient of x^{s-1} gives us the allowed values for s:

$$s(s-2) = 0 \quad \Rightarrow \quad s = 0 \ \vee \ s = 2 \tag{28}$$

For $s = 0$, the DE then requires that

$$\left[\frac{1}{2}n(n-1) - n + 1\right] a_n = \left[-n(n+1) + (n+1)\right] a_{n+1} \tag{29}$$

which simplifies to

$$a_{n+1} = \frac{(n-1)(\frac{1}{2}n-1)}{(1+n)(1-n)} a_n = \frac{2-n}{2+2n} a_n \tag{30}$$

That is, $a_1 = a_0$, $a_2 = \frac{1}{4}a_0$, $a_3 = 0$, $a_{n>3} = 0$.

For $s = 2$, the DE requires

$$\left[\frac{1}{2}(n+2)(n+1) - (n+2) + 1\right] a_n = \left[-(n+2)(n+3) + (n+3)\right] a_{n+1} \tag{31}$$

which simplifies to

$$a_{n+1} = \frac{-n}{6+2n} a_n \tag{32}$$

That is, $a_1 = 0$, $a_{n>1} = 0$. The full solution is therefore, renaming the two lowest coefficients to c_1, c_2:

$$y(x) = c_1(1 + x + \frac{1}{4}x^2) + c_2 x^2 = \underline{C_1(1+x) + C_2 x^2} \tag{33}$$

like we got with the elementary method.

12.11.2)

Solve

$$x^2 y'' + xy' - 9y = 0 \tag{34}$$

using the Frobenius method. Note however that this in an Euler-Cauchy equation, so we expect a solution of the form $y(x) = C_1 x^{m_1} + C_2 x^{m_2}$ or similar. Inserting power series we find

$$\sum_{n=0}^{\infty} \left[(n+s)(n+s-1)a_n + (n+s)a_n - 9a_n\right] x^{n+s} = 0 \tag{35}$$

The lowest power of x occurs for $n = 0$, yielding

$$s(s-1) + s - 9 = 0 \quad \Rightarrow \quad s = \pm 3 \tag{36}$$

We easily see that all the coefficients $a_{n>0} = 0$ for both values of s. Therefore the general solution is simply

$$y(x) = C_1 x^{0+3} + C_2 x^{0-3} = \underline{C_1 x^3 + C_2 x^{-3}} \tag{37}$$

which seems to agree with our initial estimation.

12.11.6)

Solve

$$3xy'' + 3xy' + y' + y = 0 \tag{38}$$

Inserting a generalised power series gives

$$\sum_{n=0}^{\infty} \left[3(n+s)a_n + a_n\right] x^{n+s} + \sum_{n'=0}^{\infty} \left[3(n'+s)(n'+s-1)a_{n'} + (n'+s)a_{n'}\right] x^{n'+s-1} = 0 \tag{39}$$

equivalent to

$$\sum_{n=0}^{\infty} \left[3(n+s)a_n + a_n\right] x^{n+s} + \sum_{n=-1}^{\infty} \left[3(n+s+1)(n+s)a_{n+1} + (n+s+1)a_{n+1}\right] x^{n+s} = 0 \tag{40}$$

For $n = -1$ we find

$$3s(s-1) + s = 0 \quad \Rightarrow \quad s = 0 \;\vee\; s = \frac{2}{3} \tag{41}$$

For $s = 0$ we find

$$a_{n+1} = -\frac{1}{1+n} a_n \tag{42}$$

which can be written as

$$a_n = \frac{(-1)^n}{n!} a_0 \tag{43}$$

For $s = 2/3$ we find

$$a_{n+1} = -\frac{3}{5+3n} a_n \tag{44}$$

which gives

$$a_1 = -\frac{3}{5} a_0 \quad a_2 = \frac{9}{40} a_0 \quad \ldots \tag{45}$$

The general solution is therefore

$$y(x) = \underline{c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n + c_2 \left(x^{2/3} - \frac{3}{5} x^{5/3} + \frac{9}{40} x^{8/3} + \cdots \right)} \tag{46}$$

The part with c_1 in front is identified as e^{-x}.

12.11.10)

Solve

$$2xy'' - y' + 2y = 0 \tag{47}$$

Inserting a generalised power series gives

$$\sum_{n'=0}^{\infty} \left[2(n'+s)(n'+s-1)a_{n'} - (n'+s)a_{n'} \right] x^{n'+s-1} + \sum_{n=0}^{\infty} 2a_n x^{n+s} = 0 \tag{48}$$

which is equivalent to

$$\sum_{n=-1}^{\infty} \left[2(n+s+1)(n+s)a_{n+1} - (n+s+1)a_{n+1} \right] x^{n+s} + \sum_{n=0}^{\infty} 2a_n x^{n+s} = 0 \tag{49}$$

$n = -1$ gives

$$2s(s-1) - s = 0 \quad \Rightarrow \quad s = 0 \ \vee \ s = \frac{3}{2} \tag{50}$$

For $s = 0$ we find

$$a_{n+1} = -\frac{2}{2n^2 + n - 1} a_n \tag{51}$$

which gives

$$a_1 = 2a_0 \quad a_2 = -2a_0 \quad a_3 = \frac{4}{9} a_0 \quad \ldots \tag{52}$$

For $s = 3/2$ we find

$$a_{n+1} = -\frac{2}{2n^2 + 7n + 5} a_n \tag{53}$$

which gives

$$a_1 = -\frac{2}{5} a_0 \quad a_2 = \frac{4}{5 \cdot 14} a_0 \quad a_3 = -\frac{8}{5 \cdot 14 \cdot 27} a_0 \quad \ldots \tag{54}$$

Therefore the general solution is

$$y(x) = \underline{c_1 \left(1 + 2x - 2x^2 + \frac{4}{9} x^3 + \cdots \right) + c_2 \left(x^{3/2} - \frac{2}{5} x^{5/2} + \frac{4}{5 \cdot 14} x^{7/2} + \cdots \right)} \tag{55}$$