SOLUTION TO PROBLEM SET 6 FYS3140

8.6.6

- *Problem to solve:* $y^{''} + 6y^{'} + 9y = 12e^{-x}$
- *Auxiliary equation and its roots:* $r^2 + 6r + 9 = 0 \Rightarrow r = -3$.
- The two roots are identical. the problem can now be rewritten as

$$(D+3)(D+3)y = 12e^{-x} \qquad (6.6a)$$

$i)$ **Method of successive integration of two first-order equations:** This method can always be used to solve second order linear equations with constant coefficients. It involves solving successive two first order equations. Let $u = (D+3)y$, then eqn(6.6a) can be solved as

$$\begin{aligned}
(D+3)u = 12e^{-x} \quad &\Rightarrow \quad u^{'} + 3u = 12e^{-x} \\
P(x) = 3; \quad & \quad Q(x) = 12e^{-x} \\
e^{I} \quad &= \quad e^{\int 3dx} = e^{3x} \\
u \quad &= \quad e^{-I}\left\{\int e^{I}Q(x)dx + C\right\} \\
&= \quad e^{-3x}\left\{\int e^{3x}(12e^{-x})dx + C\right\} \\
&= \quad e^{-3x}\left\{6e^{2x} + C\right\} = 6e^{-x} + Ce^{-3x} \qquad (6.6b)
\end{aligned}$$

Since $u = (D+3)y$, we perform similar integration once more to get the final general solution. Now $Q(x) = u$ as in eqn(6.6b); and again $P(x) = 3 \Rightarrow e^{I} = e^{3x}$, then

$$\begin{aligned}
y \quad &= \quad e^{-I}\left\{\int e^{I}Q(x)dx + D\right\} \\
&= \quad e^{-3x}\left\{\int e^{3x}\{6e^{-x} + Ce^{-3x}\}dx + D\right\} \\
&= \quad \underline{3e^{-x} + Cxe^{-3x} + De^{-3x}} \qquad (6.6c)
\end{aligned}$$

eqn(6.6c) is general solution. It contains both the characteristic function and particular solution. We can see from eqn(6.6c) that $y_c = Cxe^{-3x} + De^{-3x}$ and $y_p = 3e^{-x}$; where C and D are arbitrary constants.

$ii)$ **Method of undetermined coefficients**

We shall use this method afterwards. It is a method of choice for many practical problems.

- *Characteristic function:*$y_c = Cxe^{-3x} + De^{-3x}$.
- *Particular solution:* Assume $y_p = Ae^{-x}$.

Evaluating eqn(6.6a) for constant A we get,

$$\begin{array}{rcll} y_p = Ae^{-x} & & y_p^{'} = -Ae^{-x} & y_p^{''} = Ae^{-x} \\ \text{eqn(6.6a):} \quad Ae^{-x} - 6Ae^{-x} + 9Ae^{-x} & = & 12e^{-x} & \\ \Rightarrow 4Ae^{-x} & = & 12e^{-x} & \\ \Rightarrow A & = & 3 & \\ \therefore y_p & = & 3e^{-x} & \end{array}$$

General solution is, therefore, $y = \underline{y_c + y_p = Cxe^{-3x} + De^{-3x} + 3e^{-x}}$, which is identical with eqn(6.6c).

6.6.12

- *Problem to solve:* $(D^2 + 4D + 12)y = 80 \sin 2x \qquad (8.6.12a)$
- *We first solve:* $(D^2 + 4D + 12)y = 80e^{i2x} \quad (8.6.12b)$ and take the complex part of the particular solution as solution to the problem.
- *Auxiliary equation:* $r^2 + 4r + 12 = 0 \Rightarrow r = -2 \pm i2\sqrt{2}$.
- *Characteristic function:* $y_c = e^{-2x}\left(Ae^{i2\sqrt{2}x} + Be^{-i2\sqrt{2}x}\right)$
- *particular solution:* assume $y_{ep} = Ce^{i2x}$ and evaluate the constant C. i.e.

$$\begin{array}{rcll} y_{ep} = Ce^{i2x} \quad y_{ep}^{'} = i2Ce^{i2x} & & y_{ep}^{''} = -4Ce^{i2x} \\ \text{eqn(8.6.12b):} \quad -4Ce^{i2x} + i8Ce^{i2x} + 12Ce^{i2x} & = & 80e^{i2x} \\ \Rightarrow 8C(1+i) & = & 80 \\ C & = & \dfrac{10}{1+i} = 5(1-i) \\ y_{ep} & = & 5(1-i)e^{i2x} \end{array}$$

The particular solution for eqn(8.6.12a) is thus the imaginary value of y_{ep}

$$\begin{array}{rcl} y_p = Im(y_{ep}) & = & \dfrac{y_{ep} - y_{ep}^{*}}{2i} \\ & = & \dfrac{5(1-i)e^{i2x} - 5(1+i)e^{-i2x}}{2i} = 5\left(\sin 2x - \cos 2x\right) \\ \therefore y = y_p + y_c & = & \underline{5\left(\sin 2x - \cos 2x\right) + e^{-2x}\left(Ae^{i2\sqrt{2}x} + Be^{-i2\sqrt{2}x}\right)} \end{array}$$

where A and B are arbitrary constants.

8.6.23

- *Problem to solve:* $y^{''} + y = 2xe^{x} \qquad (8.6.23a)$
- *Auxiliary equation:* $r^2 + 1 = 0 \Rightarrow r = \pm i$
- *Characteristic function:* $y_c = A \cos x + B \sin x$

- *Particular solution:* Assume $y_p = (Ax+B)e^x$ and find A and B using eqn(8.6.23a),

$$y_p = (Ax+B)e^x; \quad y_p^{'} = (Ax+B)e^x + Ae^x; \quad y_p^{''} = (Ax+B)e^x + 2Ae^x$$

$$\begin{aligned} \text{eqn(8.6.23a):} \quad (Ax+B)e^x + 2Ae^x + (Ax+B)e^x &= 2xe^x \\ \Rightarrow A+B=0 \quad \text{and} \quad & 2A = 2 \\ \Rightarrow A=1 \quad \text{and} \quad & B=-1 \\ \therefore y_p &= (x-1)e^x \\ \Rightarrow y = y_p + y_c &= \underline{(x-1)e^x + A\cos x + B\sin x} \end{aligned}$$

where A and B are arbitrary constants.

8.7.17

Problem to solve: $x^2y^{''} + xy^{'} - 16y = 8x^4$
Change of variable $x = e^z$ gives: $y^{''} - 16y = 8e^{4z}$ $\quad (7.17a)$
Auxiliary equation: $r^2 - 16 = 0 \Rightarrow r = \pm 4$
Characteristic function: $y_{zc} = Ae^{4z} + Be^{-4z}$
Particular solution: Assume $y_{zp} = Cze^{4z}$

Solving for constant C in eqn(7.17a).

$$y_{zp} = Cze^{4z} \quad y_{zp}^{'} = 4Cze^{4z} + Ce^{4z} \quad y_{zp}^{'} = 16Cze^{4z} + 8Ce^{4z}$$

$$\begin{aligned} \text{eqn(7.17a):} \quad 16Cze^{4z} + 8Ce^{4z} - 16Cze^{4z} &= 8e^{4z} \\ \Rightarrow C &= 1 \\ \therefore y_{zp} &= ze^{4z} \\ y_z = y_{zp} + y_{cp} &= ze^{4z} + Ae^{4z} + Be^{-4z} \end{aligned}$$

change of variable $x = e^z$ gives $y(x) = y_z(\ln|x|) = \underline{(\ln|x|)x^4 + Ax^4 + Bx^{-4}}$; where A and B are arbitrary constants.

8.7.19

Problem to solve: $x^2y^{''} - 5xy^{'} + 9y = 2x^3$
Change of variable $x = e^z$ *gives:* $y^{''} - 6y + 9y = 2e^{3z}$ $\quad (7.19a)$
Auxiliary equation: $r^2 - 6r + 9 = 0 \Rightarrow r = 3$
Characteristic function: $y_{zc} = Aze^{3z} + Be^{3z}$
particular solution: Assume $y_{zp} = Cz^2e^z$

Solving for constant C in eqn(7.19a).

$$
\begin{aligned}
y_{zp} = Cz^2e^{3z}\ y'_{zp} = 3Cz^2e^{3z} + 2Cze^{3z} & \qquad y''_{zp} = 9Cz^2e^{3z} + 12Cze^{3z} + 2Ce^{3z} \\
\text{eqn(7.19a)}:\ 9Cz^2e^{3z} + 12Cze^{3z} + 2Ce^{3z} &- 6\left(3Cz^2e^{3z} + 2Cze^{3z}\right) + 9Cz^2e^{3z} = 2e^{3z} \\
\Rightarrow C &= 1 \\
\therefore y_{zp} &= z^2e^{3z} \\
y_z = y_{zp} + y_{cp} &= z^2e^{3z} + Aze^{3z} + Be^{3z}
\end{aligned}
$$

change of variable $x = e^z$ gives $y(x) = y_z(\ln|x|) = \underline{(\ln|x|)^2x^3 + A(\ln|x|)x^3 + Bx^3}$; where A and B are arbitrary constants.

8.7.22

Problem to solve: $x^2y'' + xy' + y = 2x$
Change of variable $x = e^z$ *gives:* $y'' + y = 2e^z$ (7.22*a*)
Auxiliary equation: $r^2 + 1 = 0 \Rightarrow r = \pm i$
Characteristic function: $y_{zc} = Ae^{iz} + B^{-iz} = D\cos z + E\sin z$
Starting guess for particular solution: $y_{zp} = Ce^z$

Solving for constant C in eqn(7.22a).

$$
\begin{aligned}
y_{zp} = Ce^z \quad y'_{zp} = Ce^z & \qquad y''_{zp} = Ce^z \\
\text{eqn(7.22a)}:\ Ce^z + Ce^z &= 2e^{4z} \\
\Rightarrow C &= 1 \\
\therefore y_{zp} &= e^z \\
y_z = y_{zp} + y_{cp} &= e^z + D\cos z + E\sin z
\end{aligned}
$$

change of variable $x = e^z$ gives $y(x) = y_z(\ln|x|) = \underline{x + D\cos(\ln|x|) + E\sin(\ln|x|)}$; where D, and E are arbitrary constants.