Matrix Multiplication

Problem. Given two n-by-n matrices A and B, compute matrix $C = A \cdot B$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Standard multiplication computes each c_{ij} as:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Complexity. $\Theta(n^3)$ operations (scalar multiplications)

Slide credit: Shikha Singh

Block Matrix Multiplication

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21}$$

C_{11} A_{11} A_{12} B_{11}

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

B_{21}

Block Matrix Multiplication

To multiply two n-by-n matrices A and B:

- **Divide**: partition A and B into $\frac{n}{2}$ by $\frac{n}{2}$ matrices
- **Conquer**: multiply 8 pairs of $\frac{n}{2}$ by $\frac{n}{2}$ matrices recursively
- **Combine**: Add products using 4 matrix additions

n-by-n matrices

$$C = A \times B$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

½n-by-½n matrices

8 matrix multiplications (of ½n-by-½n matrices)

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

4 matrix additions (of ½n-by-½n matrices)

Block MM: Running Time

- $T(n) = 8T(n/2) + \Theta(n^2)$

n-by-n matrices

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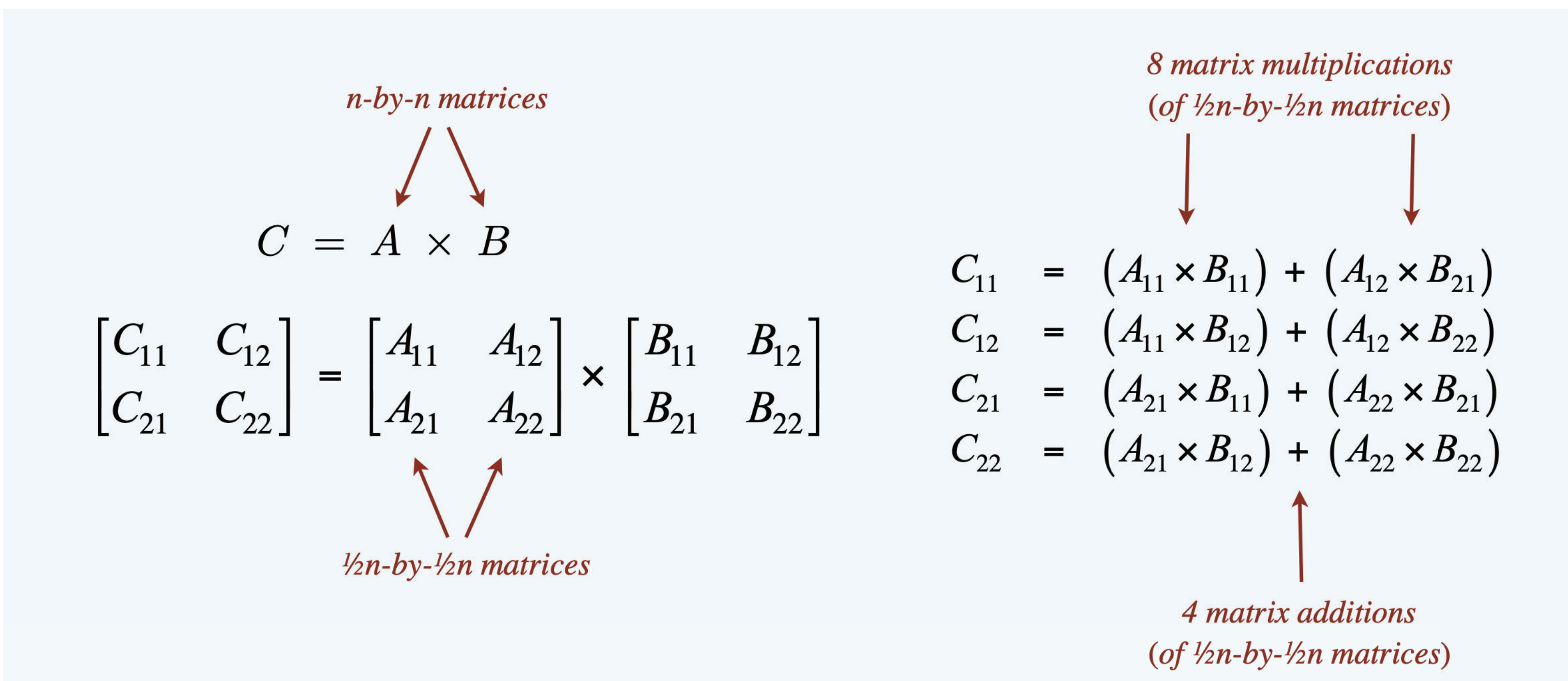
4 matrix additions (of ½n-by-½n matrices)

Block MM: Running Time

- $T(n) = 8T(n/2) + \Theta(n^2)$
- $T(n) = \Theta(n^3)$
- Nice idea but it didn't improve the run time, oh well!



Block MM: Strassen's Trick

Key idea. Can multiply two 2-by-2 matrices via 7 scalar multiplications (plus 11 additions and 7 subtractions).

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_1 + P_5 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &\leftarrow A_{11} \times (B_{12} - B_{22}) \\ P_2 &\leftarrow (A_{11} + A_{12}) \times B_{22} \\ P_3 &\leftarrow (A_{21} + A_{22}) \times B_{11} \\ P_4 &\leftarrow A_{22} \times (B_{21} - B_{11}) \\ P_5 &\leftarrow (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &\leftarrow (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &\leftarrow (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

Block MM: Strassen's Trick

Key idea. Can multiply two $\frac{n}{2}$ by $\frac{n}{2}$ matrices via 7 $\frac{n}{2}$ by $\frac{n}{2}$ matrix multiplications (using additions and subtractions).

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

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Strassen's MM Algorithm

assume n is a power of 2

STRASSEN(n, A, B)

IF $(n = 1)$ RETURN $A \times B$.

Partition A and B into ½n-by-½n blocks.

$P_1 \leftarrow$ STRASSEN$(n / 2, A_{11}, (B_{12} - B_{22}))$.

$P_2 \leftarrow$ STRASSEN$(n / 2, (A_{11} + A_{12}), B_{22})$.

$P_3 \leftarrow$ STRASSEN$(n / 2, (A_{21} + A_{22}), B_{11})$.

$P_4 \leftarrow$ STRASSEN$(n / 2, A_{22}, (B_{21} - B_{11}))$.

$P_5 \leftarrow$ STRASSEN$(n / 2, (A_{11} + A_{22}), (B_{11} + B_{22}))$.

$P_6 \leftarrow$ STRASSEN$(n / 2, (A_{12} - A_{22}), (B_{21} + B_{22}))$.

$P_7 \leftarrow$ STRASSEN$(n / 2, (A_{11} - A_{21}), (B_{11} + B_{12}))$.

$7\,T(n / 2) + \Theta(n^2)$

$C_{11} = P_5 + P_4 - P_2 + P_6$.

$C_{12} = P_1 + P_2$.

$C_{21} = P_3 + P_4$.

$C_{22} = P_1 + P_5 - P_3 - P_7$.

$\Theta(n^2)$

RETURN C.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Strassen’s MM Algorithm Analysis

- We get the following recurrence

$$T(n) = 7T(n/2) + \Theta(n^2)$$

- What is the running time solve to?

Strassen's MM Algorithm Analysis

- We get the following recurrence

$$T(n) = 7T(n/2) + \Theta(n^2)$$

- What is the running time solve to?
    - We have a increasing geometric series and the cost is dominated by the leaves
    - $T(n) = \Theta(r^L) = \Theta(7^{\log_2 n}) = \Theta(n^{\log_2 r}) \approx \Theta(n^{2.81})$
    - Improvement in the exponent means this is a much faster algorithm!

History of Matrix Multiplication

year	algorithm	arithmetic operations
1858	**“grade school”**	$O(n^3)$
1969	**Strassen**	$O(n^{2.808})$
1978	**Pan**	$O(n^{2.796})$
1979	**Bini**	$O(n^{2.780})$
1981	**Schönhage**	$O(n^{2.522})$
1982	**Romani**	$O(n^{2.517})$
1982	**Coppersmith-Winograd**	$O(n^{2.496})$
1986	**Strassen**	$O(n^{2.479})$
1989	**Coppersmith-Winograd**	$O(n^{2.3755})$
2010	**Strother**	$O(n^{2.3737})$
2011	**Williams**	$O(n^{2.372873})$
2014	**Le Gall**	$O(n^{2.372864})$
	???	$O(n^{2+\varepsilon})$

galactic algorithms

number of arithmetic operations to multiply two n-by-n matrices

Tons of Applications

- Lots of problem reduce to matrix multiplication complexity

linear algebra problem	expression	arithmetic complexity
matrix multiplication	$A \times B$	$MM(n)$
matrix squaring	A^2	$\Theta(MM(n))$
matrix inversion	A^{-1}	$\Theta(MM(n))$
determinant	$\lvert A \rvert$	$\Theta(MM(n))$
rank	$rank(A)$	$\Theta(MM(n))$
system of linear equations	$Ax = b$	$\Theta(MM(n))$
LU decomposition	$A = LU$	$\Theta(MM(n))$
least squares	$\min \lVert Ax - b \rVert_2$	$\Theta(MM(n))$

numerical linear algebra problems with the same arithmetic complexity MM(n) as matrix multiplication