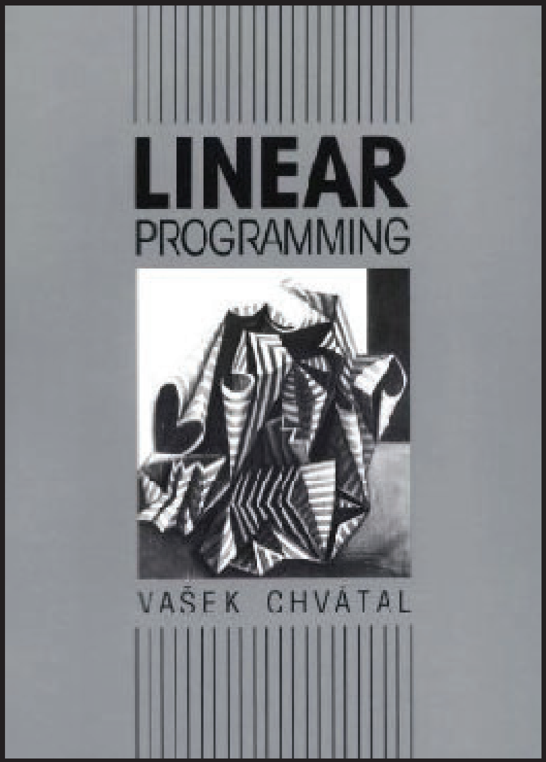

Lecture slides by Kevin Wayne

LINEAR PROGRAMMING I

- *a refreshing example*
- *standard form*
- *fundamental questions*
- *geometry*
- *linear algebra*
- *simplex algorithm*

Last updated on 7/25/17 11:09 AM

Linear programming

Linear programming. Optimize a linear function subject to linear inequalities.

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\begin{array}{llrcll}
\text{(P)} & \max & \sum_{j=1}^{n} c_j x_j & & & \\
 & \text{s. t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \le i \le m \\
 & & x_j & \ge & 0 & 1 \le j \le n
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Linear programming

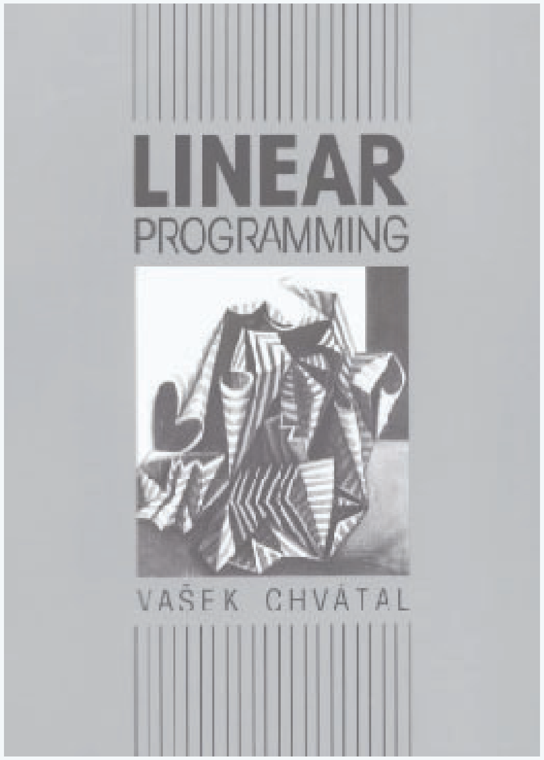
Linear programming. Optimize a linear function subject to linear inequalities.

Generalizes: $Ax = b$, 2-person zero-sum games, shortest path, max flow, assignment problem, matching, multicommodity flow, MST, min weighted arborescence, …

Why significant?

- Design poly-time algorithms.
- Design approximation algorithms.
- Solve NP-hard problems using branch-and-cut.

Ranked among most important scientific advances of 20th century.



Linear Programming I

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Reference: *The Allocation of Resources by Linear Programming,* Scientific American, by Bob Bland

Brewery problem

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

How can brewer maximize profits?

- Devote all resources to ale: 34 barrels of ale $\Rightarrow$ \$442
- Devote all resources to beer: 32 barrels of beer $\Rightarrow$ \$736
- 7.5 barrels of ale, 29.5 barrels of beer $\Rightarrow$ \$776
- 12 barrels of ale, 28 barrels of beer $\Rightarrow$ \$800

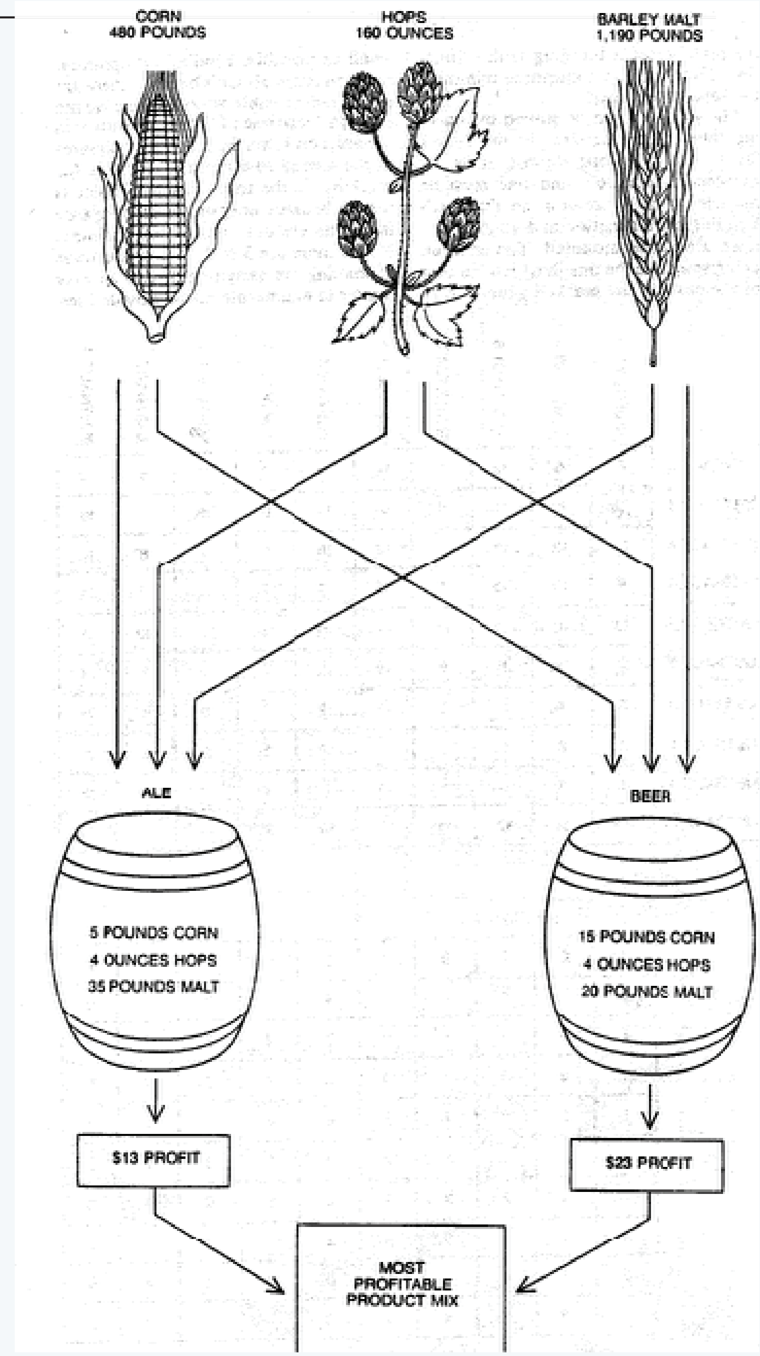
Brewery problem

objective function

$$
\begin{array}{lrcrcr l}
 & \text{Ale} & & \text{Beer} & & & \\
\max & 13A & + & 23B & & & \text{Profit} \\
\text{s. t.} & 5A & + & 15B & \leq & 480 & \text{Corn} \\
 & 4A & + & 4B & \leq & 160 & \text{Hops} \\
 & 35A & + & 20B & \leq & 1190 & \text{Malt} \\
 & A & , & B & \geq & 0 &
\end{array}
$$

constraint

decision variable



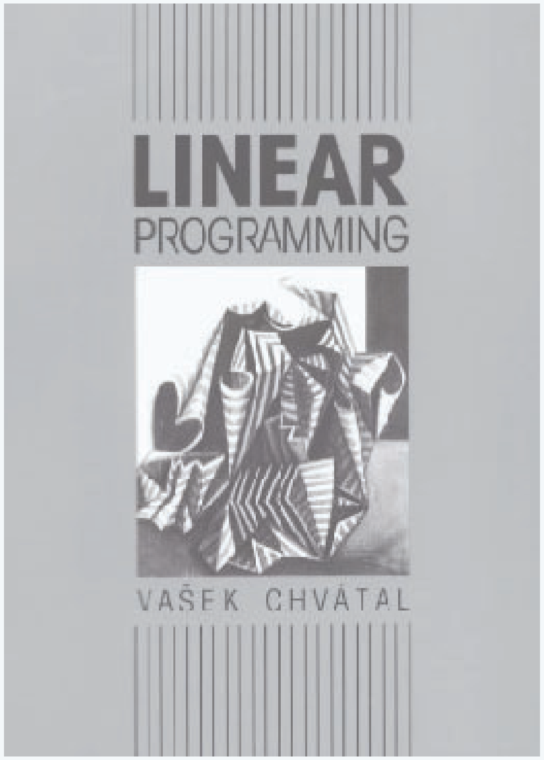
Reference

SCIENTIFIC AMERICAN JUNE 1981

The Allocation of Resources by Linear Programming

Abstract, crystal-like structures in many geometrical dimensions can help to solve problems in planning and management. A new algorithm has set upper limits on the complexity of such problems

By Robert G. Bland



LINEAR PROGRAMMING I

- a refreshing example
- standard form
- fundamental questions
- geometry
- linear algebra
- simplex algorithm

Standard form of a linear program

"Standard form" of an LP.

- Input: real numbers a_{ij}, c_j, b_i.
- Output: real numbers x_j.
- n = # decision variables, m = # constraints.
- Maximize linear objective function subject to linear inequalities.

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Recall transpose:

If the vector is $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ => the transpose of x is $\begin{bmatrix} x_1 \ x_2 \ \ldots \ x_m \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$

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Linear. No x^2, $x\,y$, $\arccos(x)$, etc.

Programming. Planning (term predates computer programming).

The Diet Problem:

For a moderately active man weighing 154 pounds, how much of each of 77 foods should be eaten on a daily basis so that the man's intake of nine nutrients will be at least equal to the recommended dietary allowances (RDAs) suggested by the National Research Council in 1943, with the cost of the diet being minimal?

The nutrient RDAs required to be met in Stigler's experiment were calories, protein, calcium, iron, as well as vitamins A, B1, B2, B3, and C.

Table of nutrients considered in Stigler's diet

Nutrient	Daily Recommended Intake
Calories	3,000 Calories
Protein	70 grams
Calcium	.8 grams
Iron	12 milligrams
Vitamin A	5,000 IU
Thiamine (Vitamin B_1)	1.8 milligrams
Riboflavin (Vitamin B_2)	2.7 milligrams
Niacin	18 milligrams
Ascorbic Acid (Vitamin C)	75 milligrams

Stigler's 1939 Diet

Food	Annual Quantities	Annual Cost
Wheat Flour	370 lb.	$13.33
Evaporated Milk	57 cans	$3.84
Cabbage	111 lb.	$4.11
Spinach	23 lb.	$1.85
Dried Navy Beans	285 lb.	$16.80
Total Annual Cost		$39.93

The Diet Problem: a simple example

Design a diet of minimal cost that meets the dietary requirements. Write it as an optimization problem.

Vitamin	Milk	Eggs	Daily Requirement
F	3	7	45
G	4	2	60
Intake	x_1	x_2	
Unit cost	2	5	

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- Total cost: $2x_1 + 5x_2$
- Dietary Constraints
  * Vitamin F: $3x_1 + 7x_2 \geq 45$
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  * Physical constraints: $x_1 \geq 0, x_2 \geq 0$

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- (LP) in matrix form

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\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \geq b \\
& x_1 \geq 0, x_2 \geq 0
\end{array}
$$

$$
A = \begin{bmatrix} 3 & 7 \\ 4 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 45 \\ 60 \end{bmatrix}, \quad c = \begin{bmatrix} 2 \\ 5 \end{bmatrix}
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But this is not the end of the diet tale. In the early 1950s, I moved to Santa Monica to work for the RAND Corporation. My doctor advised me to go on a diet to lose weight. I decided I would model my diet problem as a linear program and let the computer decide my diet. Some revisions of the earlier model, of course, would be necessary in order to give me a greater variety of foods to choose from; the calorie intake had to be reduced to under 1,500 calories per day; and the objective function had to be changed (I wasn't interested in saving money). I said to myself: "The trouble with a diet is that one's always hungry. What I need to do is maximize the feeling of feeling full." After giving much thought to what the coefficients in the objective form should be, I used the weight (per unit amount) of a food minus the weight of its water content. Input data for over 500 different foods was punched into cards and fed into Rand's IBM 701 computer.

My colleague Ray Fulkerson (famous for his contributions to network flow and matroid theory) was skeptical. "You crazy or something? We solve models to obtain optimal schedules of activities for others to follow, not for ourselves." Nevertheless I was determined to do just that.

One day I said to Anne, my wife, "Today is Der Tag, whatever the 701 says that's what I want you to feed me each day starting with supper tonight."

Around 5:00 PM, Anne called, "Nu, it's five and you haven't called. What should I be cooking?" I replied that she didn't really want to know. I then read off the amounts of foods in the optimal diet. Her reaction: "The diet is a bit weird but conceivable. Is that it?"

"Not exactly," I replied, "AND 500 gallons of vinegar." She thought it funny and laughed.

I figured there had to be a mistake somewhere. It turned out that our data source listed vinegar as a very weak acid with water content = zero. Therefore, according to the way the model was formulated the more vinegar you drank the greater would be your feeling of feeling full. I decided that vinegar wasn't a food.

The next day the above scene was repeated, except this time the diet called, among other things, for two pounds of bran per day. Anne said, "If you consume that much bran, I doubt you'll make it to the hospital."

The model was revised with an upper bound put on the amount of bran. The next day the proposed menu was almost exactly the same except this time it was two pounds of blackstrap molasses which substituted for the bran; apparently their nutritional contents were quite similar.

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