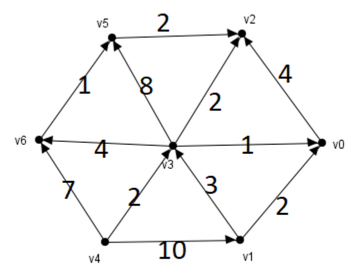
Due Fri Oct 29 at 11:59PM

1. (20 points)

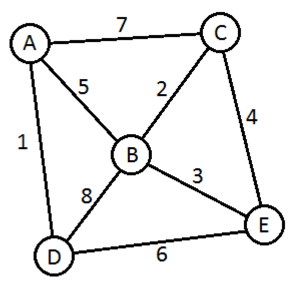
   Consider the recurrence relation $T(n) = T(n/2) + T(n/4) + 2n$ with $n = 2^k$ and $T(1) = T(2) = 1$. Prove by induction that $T(n) = \Theta(n)$.

2. (20 points)

   (a) Run Dijkstra's algorithm on the following directed graph, starting at vertex v_4. What is the order in which vertices are selected, and what is the resulting shortest-path tree?



   (b) Given the graph below, what is the order in which edges are selected by running Kruskal's algorithm?



3. (20 points)

   Country A and B are at war, you are the commander of country B. The map of country B is represented as a square matrix M with side length n. Each element M[i][j] is a city in country B with coordinate row = i and col j (n^2 cities in total) and it can only be 0 or 1. 1 denotes a ruined city, 0 denotes a complete city. A message can be delivered between two adjacent complete cities. city(i, j) is adjacent to city($i-1$, j), city($i+1$, j), city(i, $j-1$) and city(i, $j+1$). Initially, M[i][j] = 0 for all the cities.

   Currently, you get the bombing plan of country A for the next n^2 days. Country A will bomb one different city each day (from day 1 to day n^2). After one city(i, j) is bombed, M[i][j] becomes 1. You are going to calculate when is the last day that a message can be delivered from any northernmost city to any southernmost city. Northernmost means cities with row $i = 0$. Southernmost means cities with row $i = n-1$.

Describe and analyze an algorithm to get the last day that a message can be delivered from any northernmost city to any southernmost city.

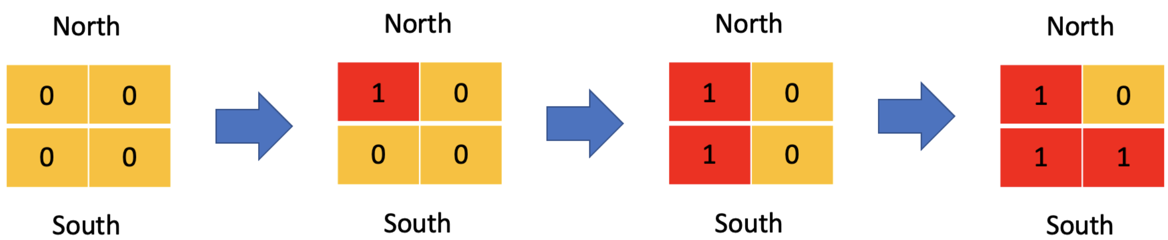


Figure 1: Question 4 example.

Here is a simple example in fig 1. The bombing plan is city(0, 0), city(1, 0), city$(1,1)$ and city(0, 1). For day 2, the message can still be delivered from a northernmost city(0, 1) to a southernmost city(1, 1). But there is no way to deliver at day 3. Thus, the final answer is 2.

4. (20 points)

   There is a un-directed graph $G = (V,E)$, $|V| = n$ and $|E| = m$. For each edge e, its length can only be 0 or 1. Given a start vertex S, **describe and analyze** an algorithm to get the length of shortest path from S to the rest of vertices. (**hint:** double-ended queue is useful in this problem).

5. (20 points)

   Let G be an undirected graph on n nodes. Prove each of the following statements, given the other two.

   1) G is connected.
   2) G does not contain a cycle.
   3) G has $n-1$ edges.

6. (20 points)

   You are consulting for a trucking company that does a large amount of business shipping packages between New York and Boston. The volume is high enough that they have to send several trucks each day between the two locations. Trucks have a fixed limit W on the maximum amount of weight they are allowed to carry. Boxes arrive at the New York station one by one, and each package i has a weight w_i. The trucking station is quite small, so at most one truck can be in the station at any time. Company policy requires that boxes are shipped in the order they arrive; otherwise a customer might get upset. At the moment, the company is using a simple greedy algorithm for packing: they pack boxes in the order they arrive, and whenever the next box does not fit, they send the truck on its way.
   But they wonder if they might be using too many trucks, and they want your opinion on whether the situation can be improved. Here is how they are thinking. Maybe one could decrease the number of trucks needed by sometimes sending off a truck that was less full, and in this way allow the next few trucks to be better packed. Prove that, for a given set of boxes with specified weights, the greedy algorithm currently in use actually minimizes the number of trucks that are needed.