Linear programming

CS 381

An Example

[Diet Problem]. An athlete wants to maximize her daily protein consumption according to the next constraints:

- At most 5 units of fat per day
- Spending at most $6 per day.
- ***Ingredients:***
    - Steak: cost $4 per pound; 2 units of protein and 1 unit of fat per pound.
    - Peanut butter: cost $1 per pound; 1 unit of protein and 2 units of fat per pound

Exercise: Write as a linear program (LP).

An Example

Let $\mathbf{x_1}$ = number of pounds of steak; $\mathbf{x_2}$ = number of pounds of peanut butter per day.

Linear Program (LP1):

$$\begin{aligned} &\max\ 2x_1 + x_2 \\ \text{subject to} \quad & 4x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

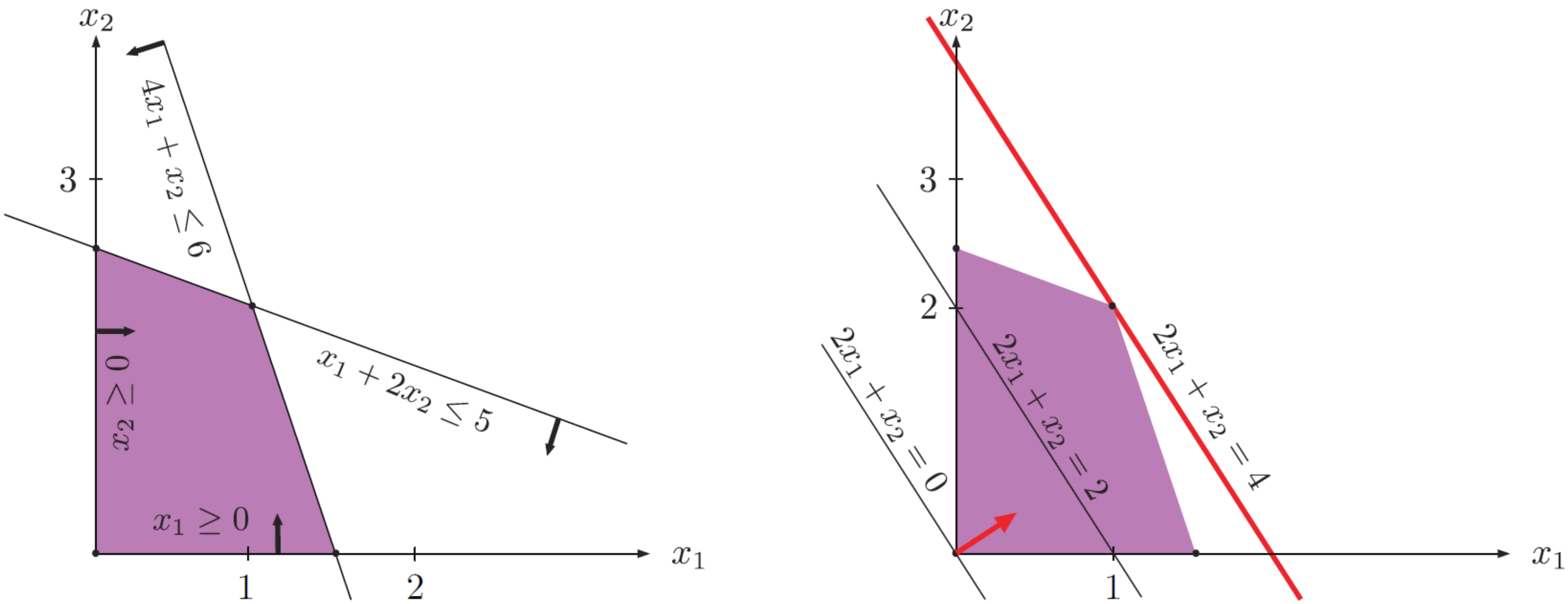
Objective function: $2x_1 + x_2$

Feasible set: the set of feasible vectors that satisfy the constraints. In this example, this is the set of vectors (x_1, x_2) such that $4x_1 + x_2 \leq 6$ and $x_1 + 2x_2 \leq 5$.

An Example

Let x_1 = number of pounds of steak; x_2 = number of pounds of peanut butter per day.

In the following graphs, the purple region shows the feasible set for LP1.

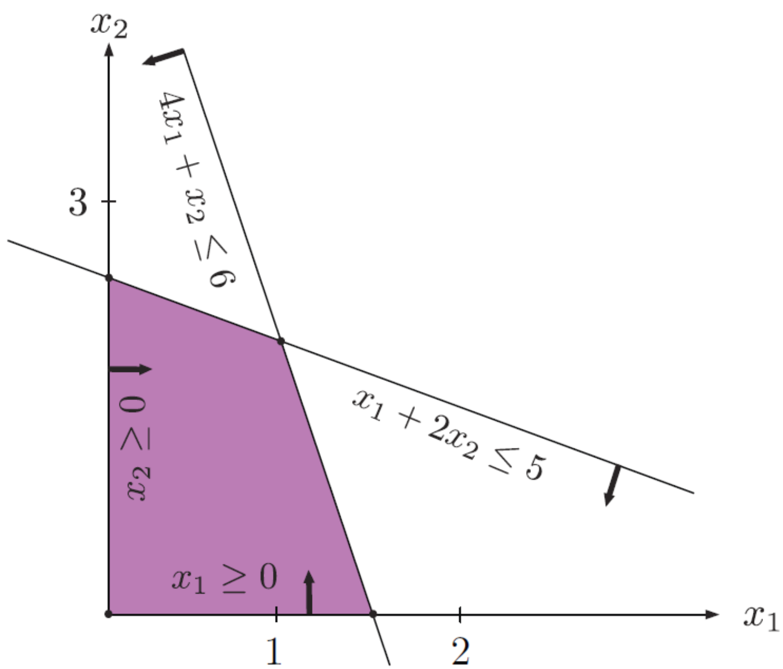


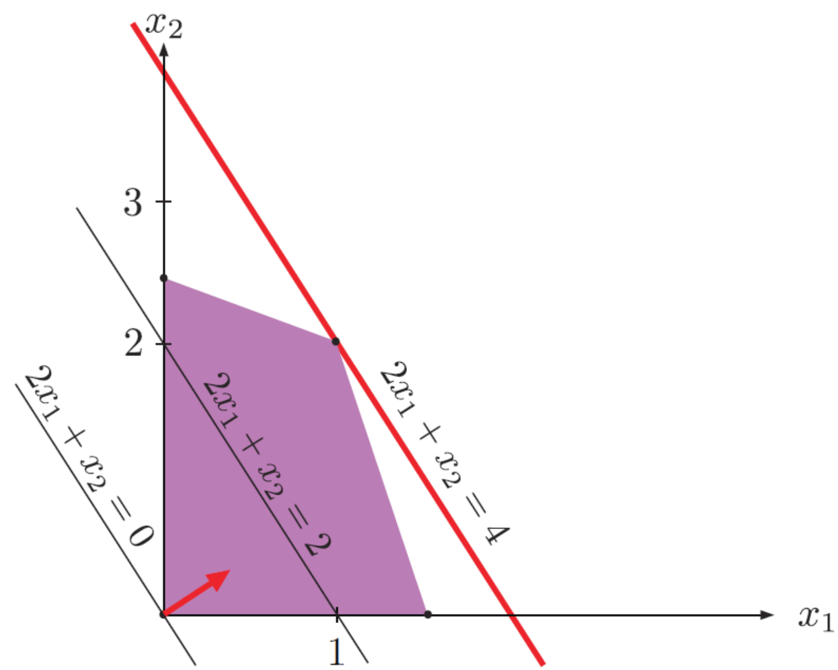
The largest c for which the line $2 \cdot x_1 + x_2 = c$ intersects the feasible set is $c = 4$. The red arrow from the origin to the right is perpendicular to the lines $2x_1 + x_2 = k$ for all k

An Example

Let x_1 = number of pounds of steak; x_2 = number of pounds of peanut butter per day.

In the following graphs, the purple region shows the feasible set for LP1.





Which point in the feasible set maximizes the objective $2x_1 + x_2$?

For LP1, this point is $(x_1, x_2) = (1,2)$; the value of the objective is $2x_1 + x_2 = 4$.

Linear Programming Duality

Linear Program (LP1):

$$
\begin{aligned}
& \max \; 2x_1 + x_2 \\
\text{subject to} \quad & 4x_1 + x_2 \leq 6 \\
& x_1 + 2x_2 \leq 5 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

- The first constraint of LP1 ensures the objective is upper bounded by 6.
- The second constraint (multiplied by 2) gives an upper bound of 10.

Multiply the first constraint by $y_1 \geq 0$ and the second constraint by $y_2 \geq 0$. We get:

$y_1(4\,x_1 + x_2) + y_2(x_1 + 2x_2) \leq 6y_1 + 5y_2$ **(*)**

The LHS of (*) is greater than the objective of LP1 whenever:

$$
\begin{aligned}
4y_1 + y_2 &\geq 2 \\
y_1 + 2y_2 &\geq 1 \\
y_1, y_2 &\geq 0\,.
\end{aligned}
$$

Linear Programming Duality

Linear Program (LP1):

$$\begin{aligned} & \max \; 2x_1 + x_2 \\ \text{subject to} \quad & 4x_1 + x_2 \le 6 \\ & x_1 + 2x_2 \le 5 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Constraints on y (**)

$$\begin{aligned} 4y_1 + y_2 &\ge 2 \\ y_1 + 2y_2 &\ge 1 \\ y_1, y_2 &\ge 0\,. \end{aligned}$$

Then for any (y_1, y_2) that is feasible for (**), we get: $2x_1 + x_2 \le 6y_1 + 5y_2$ for all feasible (x_1, x_2).

The best upper bound we can obtain this way on the optimal value of LP1 is given by the solution to the linear program:

Linear Program (LP2):

$$\begin{aligned} & \min \; 6y_1 + 5y_2 \\ \text{subject to} \quad & 4y_1 + y_2 \ge 2 \\ & y_1 + 2y_2 \ge 1 \\ & y_1, y_2 \ge 0\,. \end{aligned}$$

Linear Programming Duality

In general, a maximization LP in standard form is written as:

$$\left.\begin{aligned} \max\ \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad \\ A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{aligned}\right\}, \qquad (P)$$

where $A \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$

(P) is known as the primal LP. This linear program is feasible if the feasible set is non-empty:

$$\mathcal{F}(P) := \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0\}$$

Linear Programming Duality

As discussed before, if a vector $\mathrm{y} \geq 0 \in R^m$ satisfies the inequality

$\mathrm{y}^{\mathrm{T}}\mathrm{A} \geq \mathrm{c}^{\mathrm{T}}$, then:

$$\forall \mathbf{x} \in \mathcal{F}(P),\ \mathbf{y}^T\mathbf{b} \geq \mathbf{y}^T A\mathbf{x} \geq \mathbf{c}^T\mathbf{x}.$$

This motivates introducing the following linear program, known as the dual LP:

$$\left.\begin{array}{r} \min\ \mathbf{b}^T\mathbf{y} \\ \text{subject to} \\ \mathbf{y}^T A \geq \mathbf{c}^T \\ \mathbf{y} \geq 0 \end{array}\right\} \qquad (D)$$

where $\mathrm{y} \geq 0 \in R^m$.

Exercise: Check that the dual of the dual LP is the primal LP.