CS 381

- Alternative proof without using integrals for upper bound of the sum $\sum_{k=1,n-1} k \cdot \log(k)$
- Closest pair of points
- Farthest pair of points

Alternative proof without using integrals

Exercise:

Bound $\sum_{k=1,n-1} k \cdot \log(k)$ by $\frac{1}{2}\mathrm{n}^2\log(\mathrm{n}) - \mathrm{c}_1 \cdot n^2$

Alternative proof without using integrals

Exercise:

Bound $\sum_{k=1,n-1} k \cdot \log(k)$ by $\frac{1}{2}\mathrm{n}^2\log(\mathrm{n}) - \mathrm{c}_1 \cdot n^2$

Write

$$\sum_{k=1,n-1} k \cdot \log(k) \leq \sum_{k=1,\frac{n}{2}} k \cdot \log\left(\frac{n}{2}\right) + \sum_{k=\frac{n}{2}+1,n-1} k \cdot \log(n)$$

$$\leq \frac{n(n-1)}{2} \cdot \log(n) - \frac{n}{2} \cdot \left(\frac{n}{2} - 1\right) \cdot \log(2)$$

$$\leq \frac{n(n-1)}{2} \cdot \log(n) - \frac{n}{2} \cdot \left(\frac{n}{2} - 1\right) \cdot 0.69$$

Can take $\mathrm{c}_1 = 0.15$.

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems

Algorithm?

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Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

1-D version?

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Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

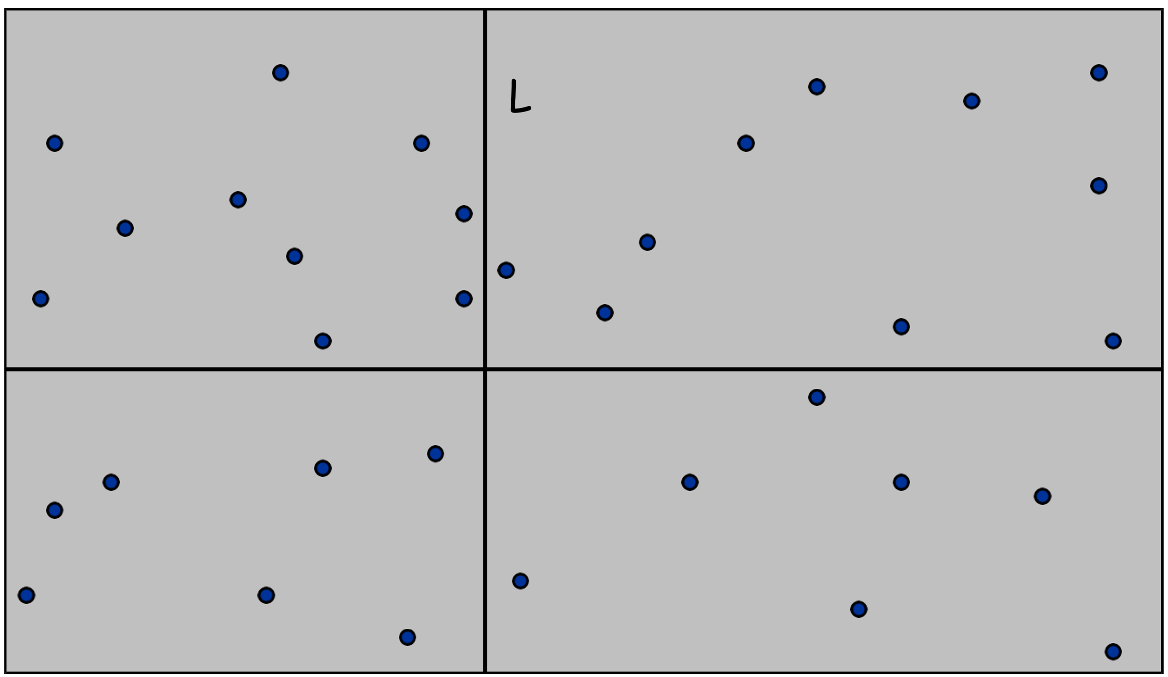
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

to make presentation cleaner

Closest Pair of Points: First Attempt

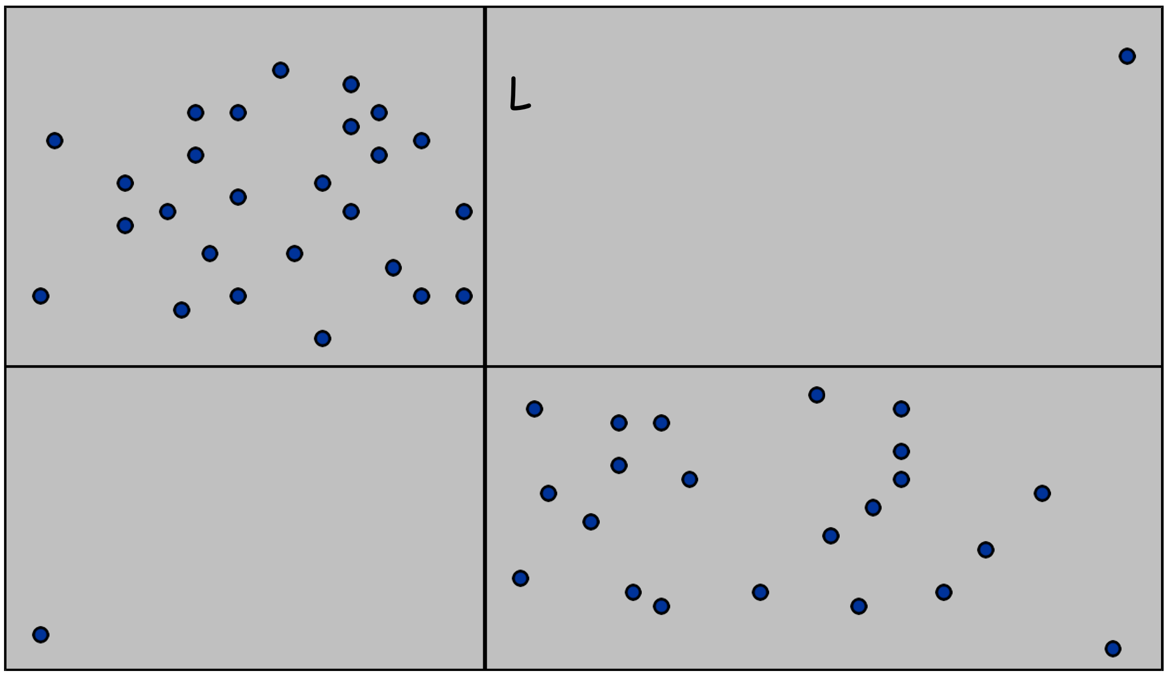
Divide. Sub-divide region into 4 quadrants.



Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.

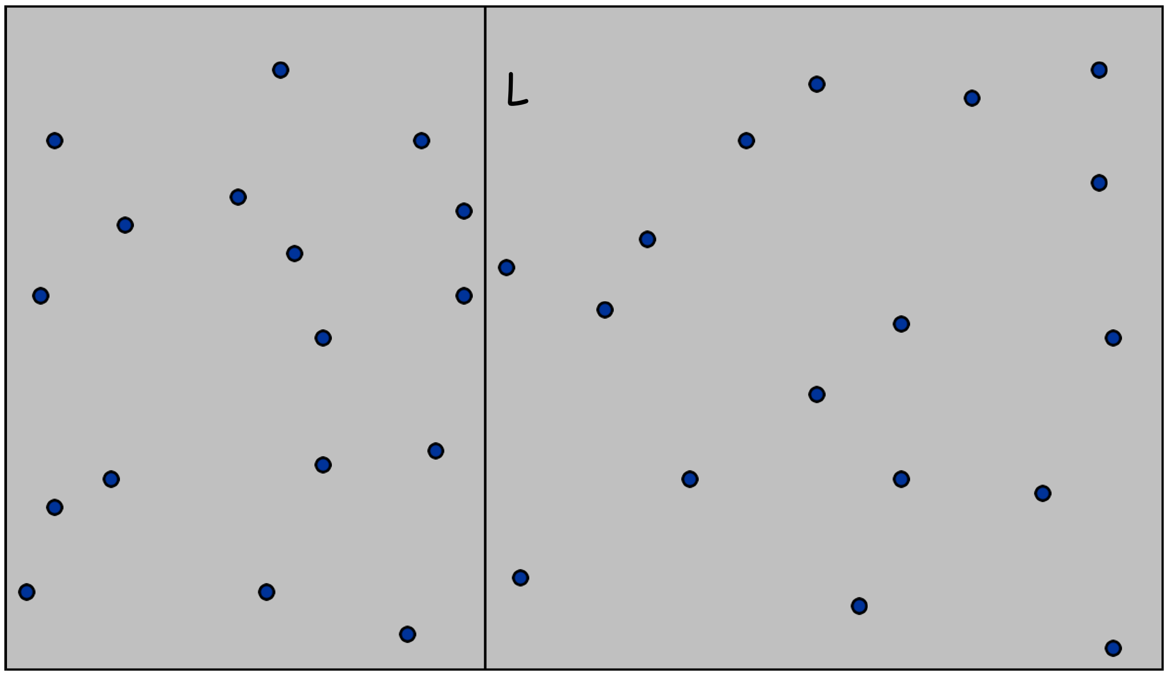
Obstacle. Impossible to ensure n/4 points in each piece.



Closest Pair of Points

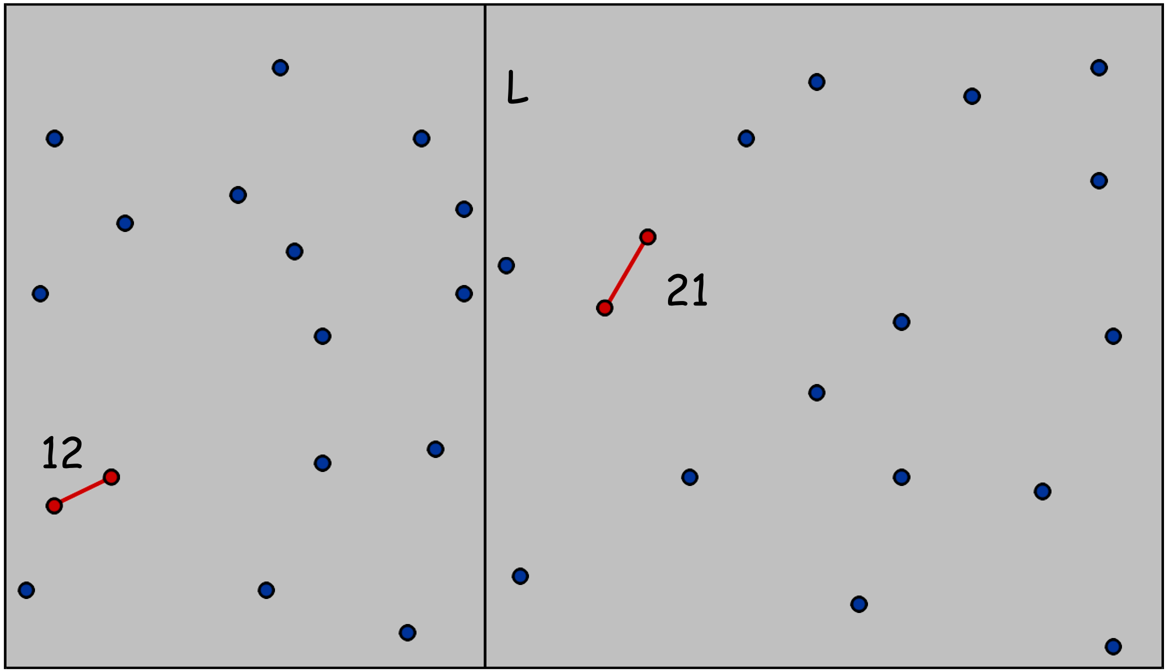
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



Closest Pair of Points

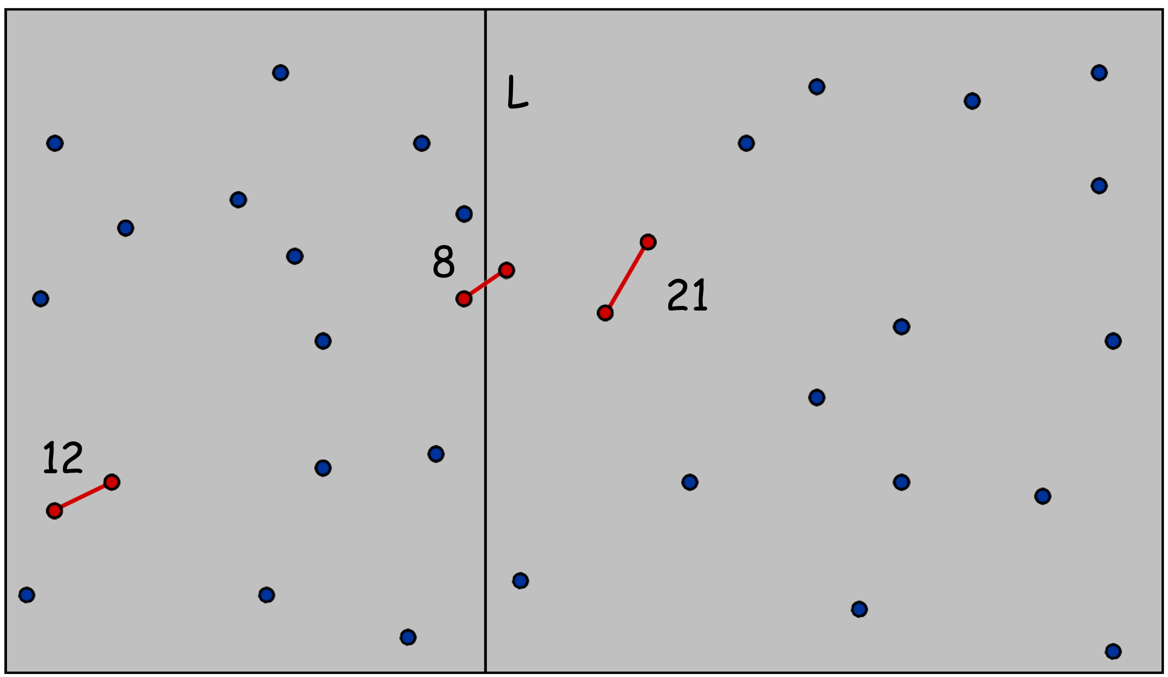
Algorithm.
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

Algorithm.

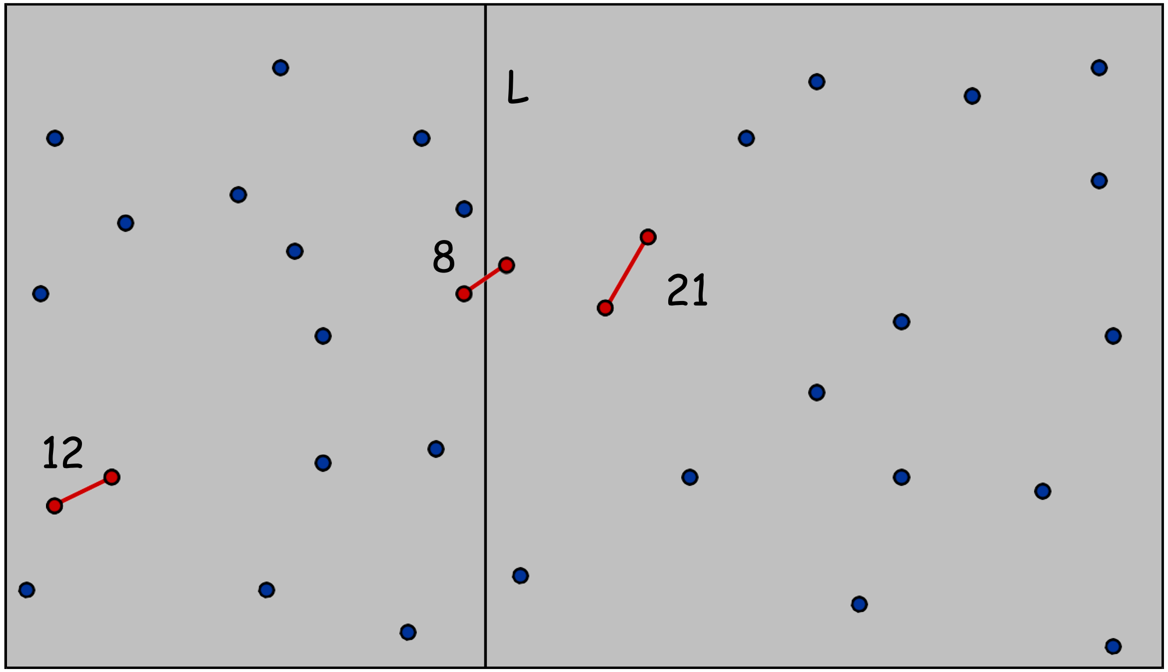
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← ***How much work is this?***
- Return best of 3 solutions.



Closest Pair of Points

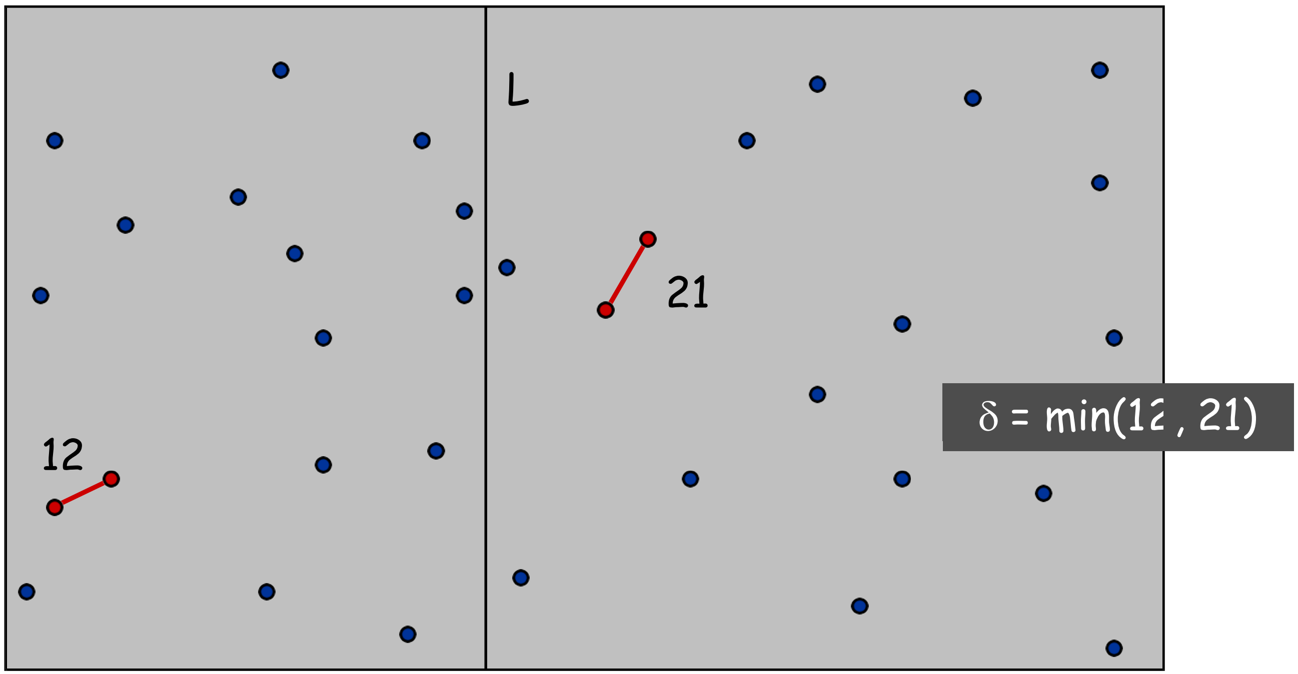
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.



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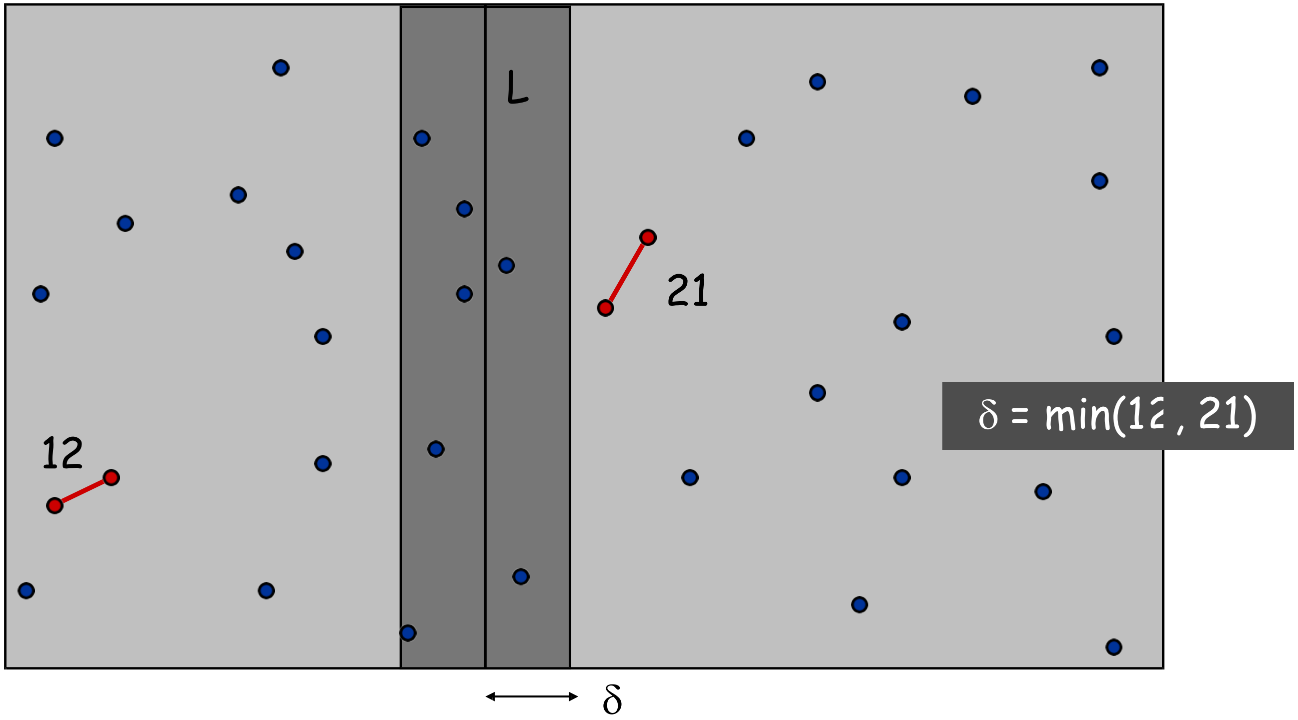

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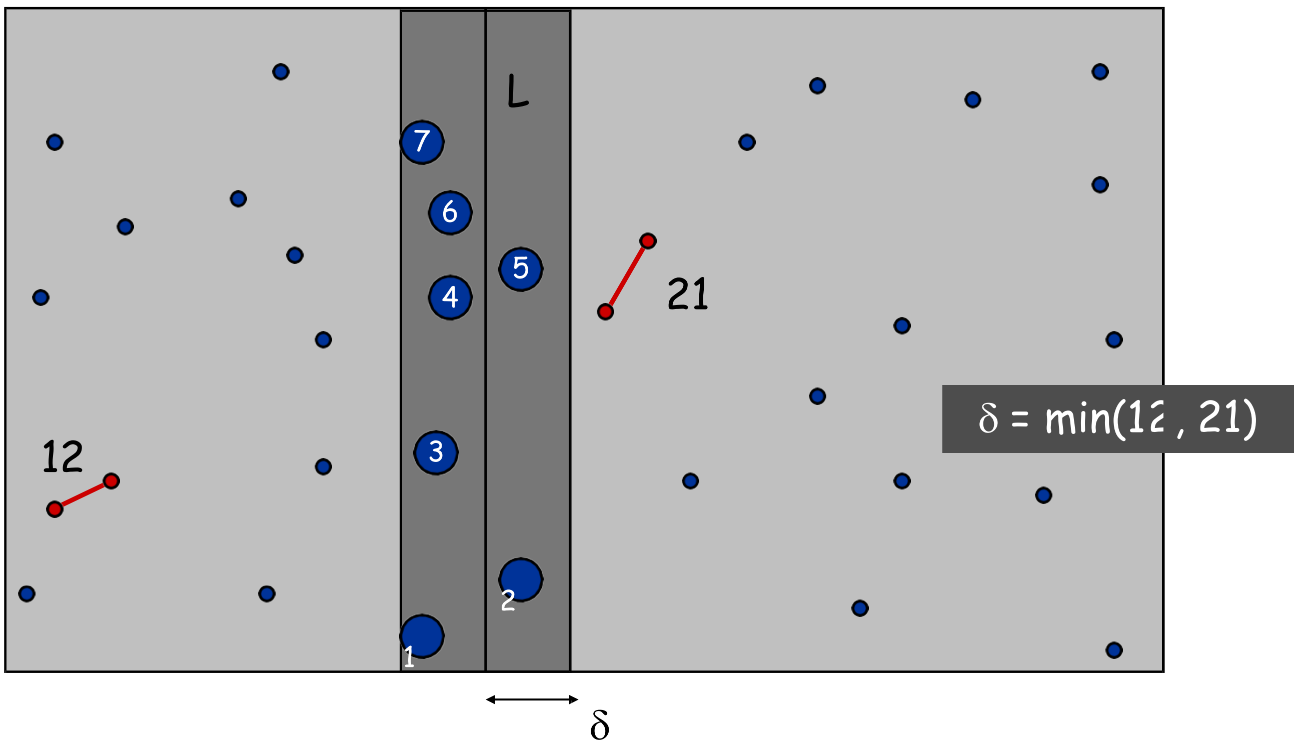
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

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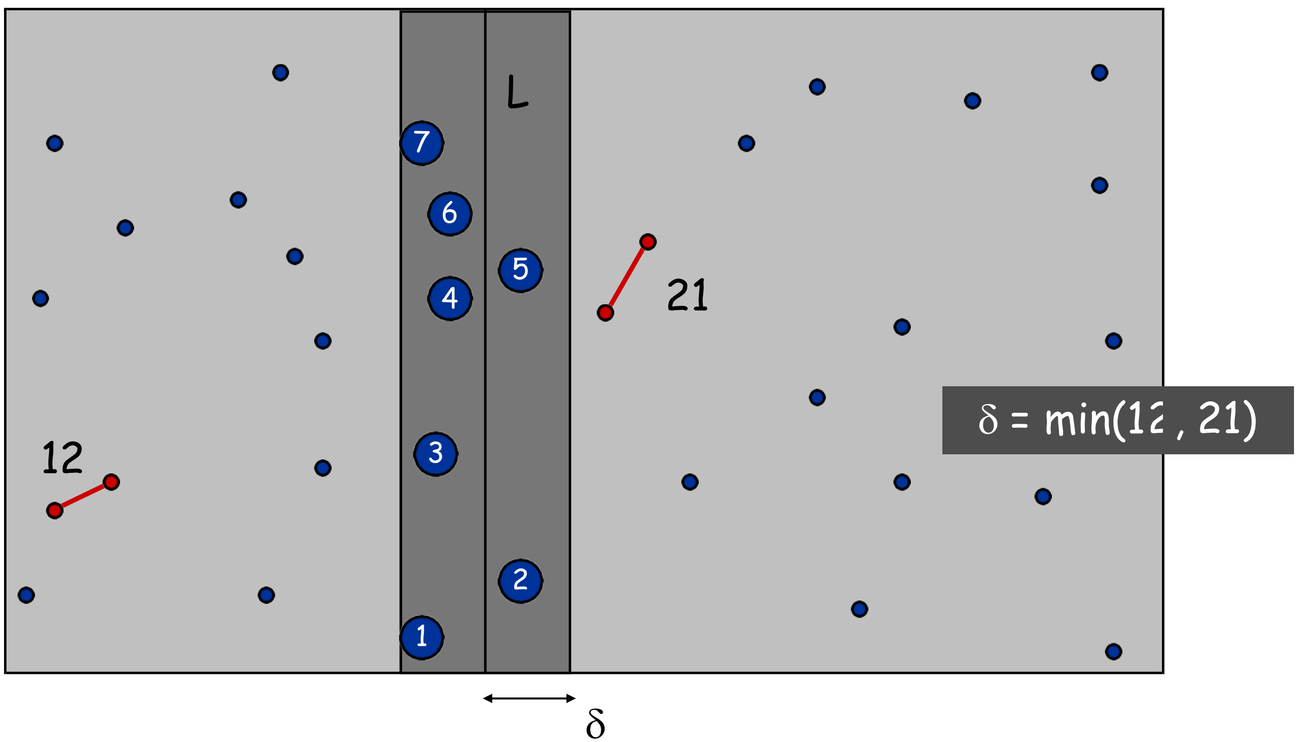
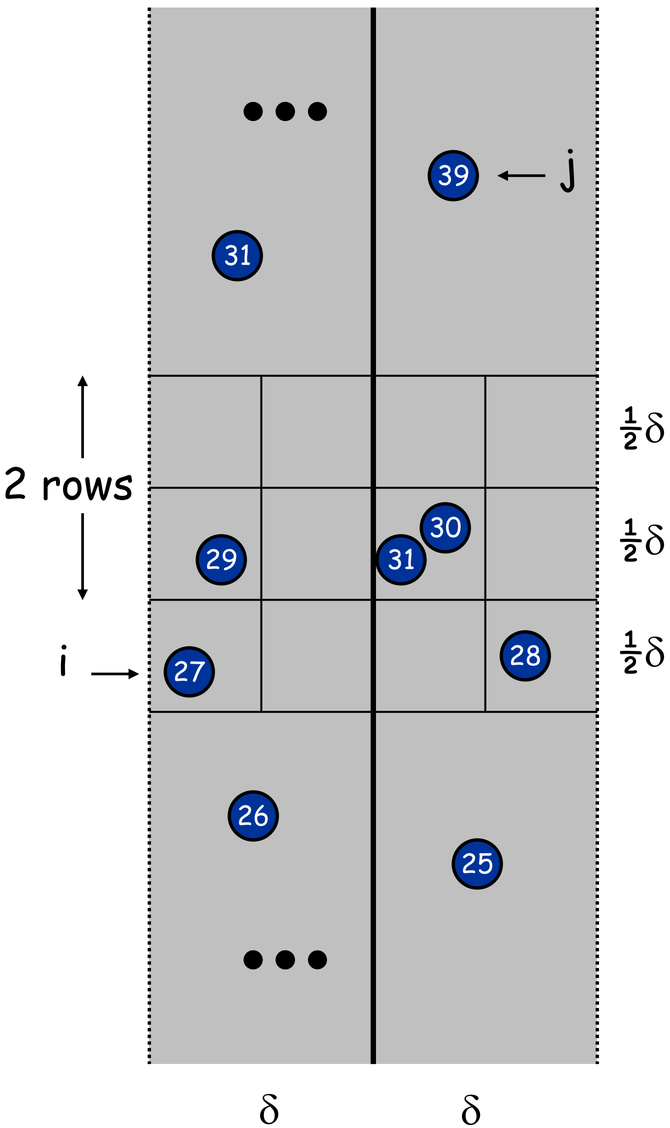
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



Closest Pair of Points

Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.



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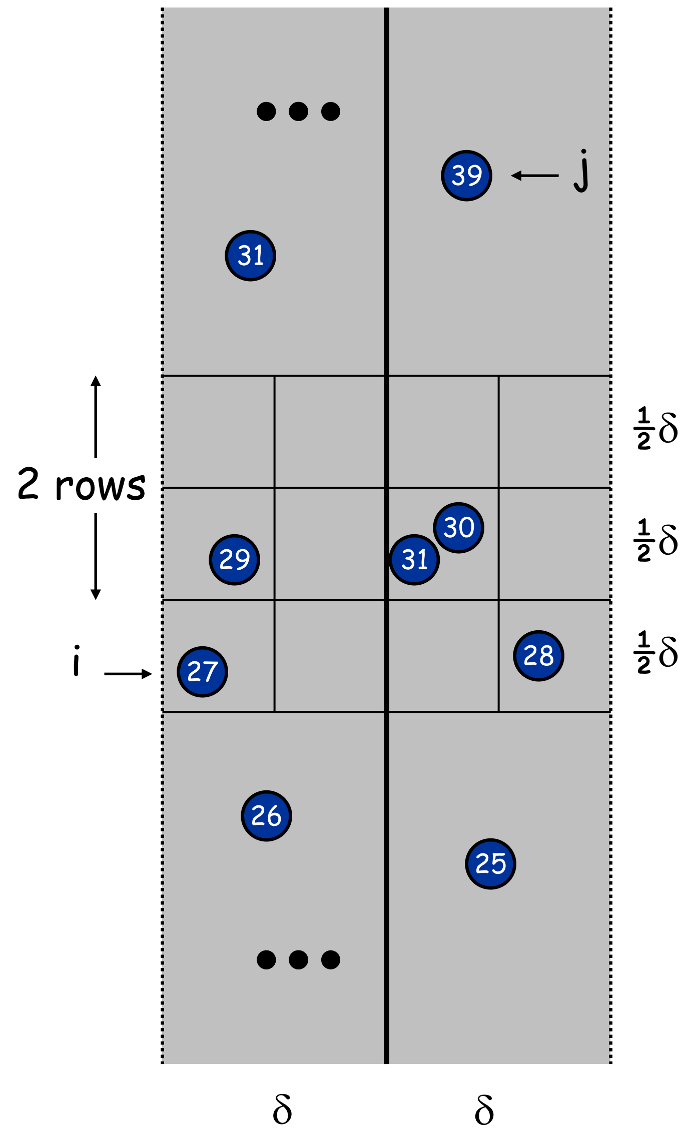
Pf.

- How many points in the same box?



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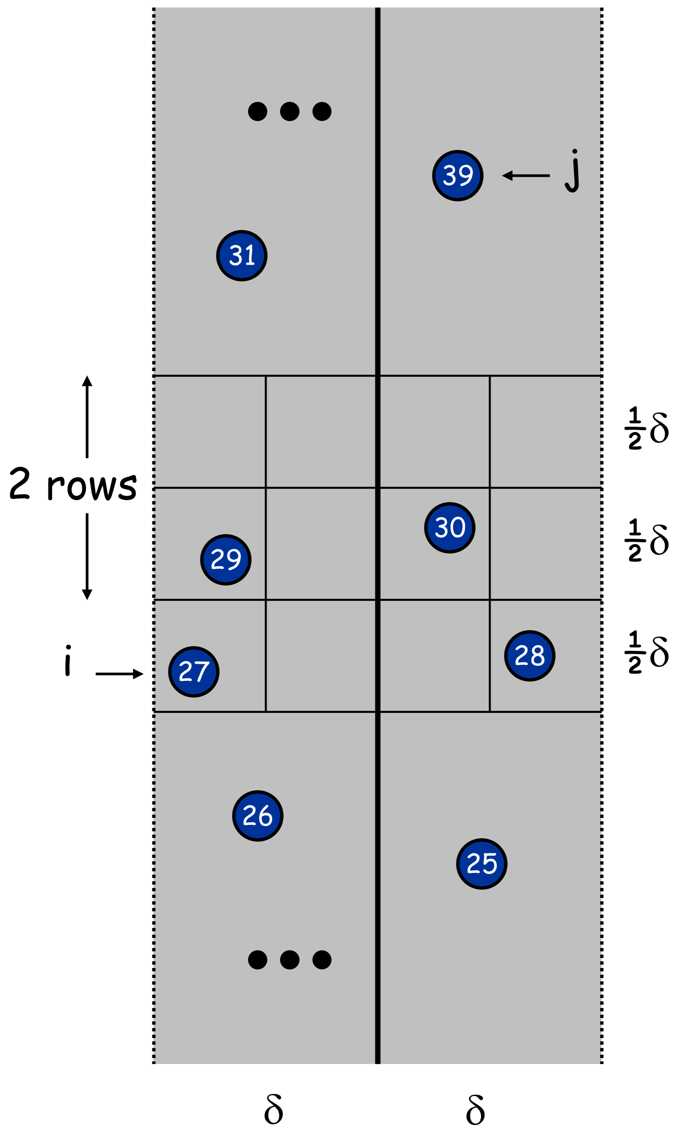
- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.



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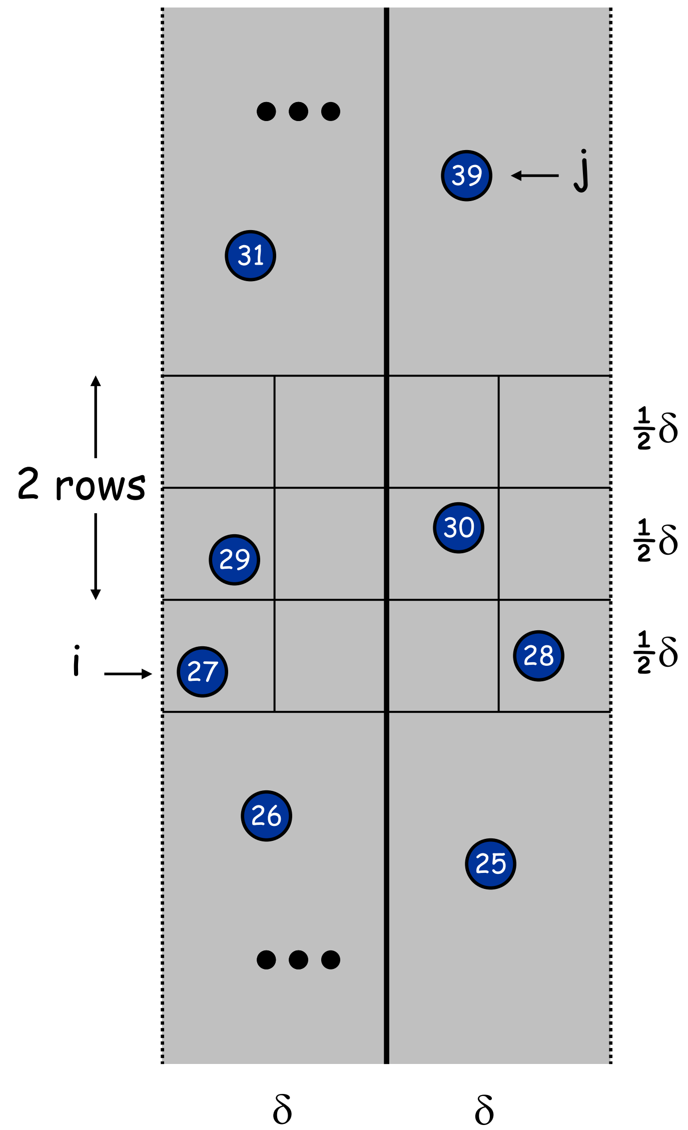
- How can we use this?



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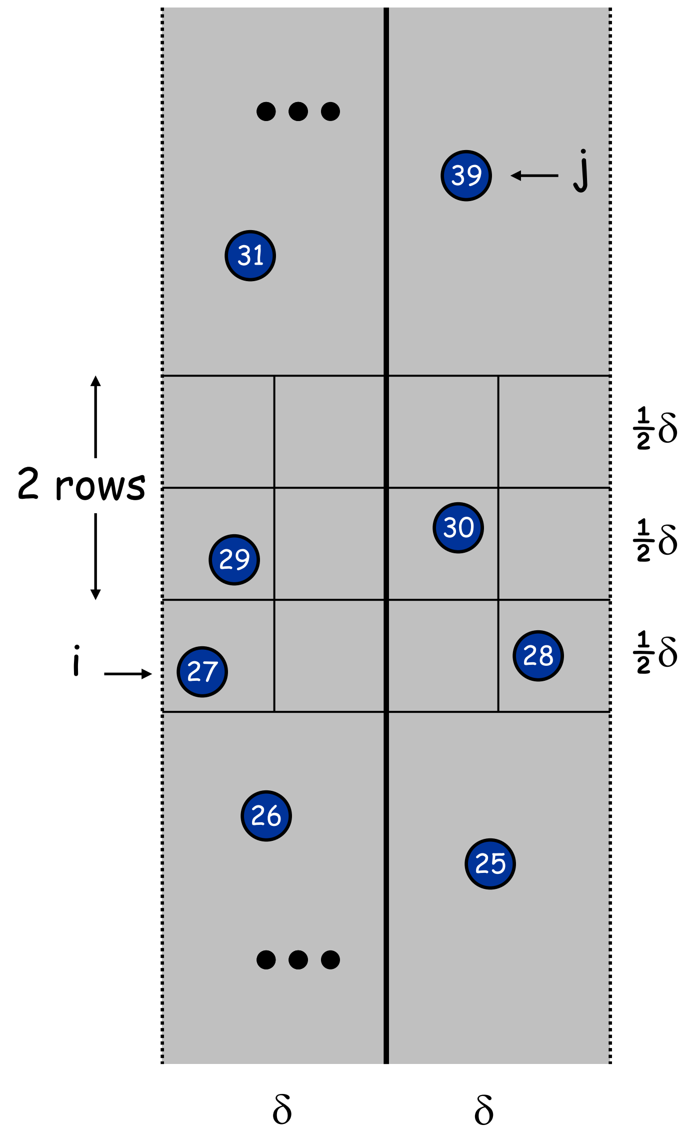
- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪



Fact. Still true if we replace 12 with 7.

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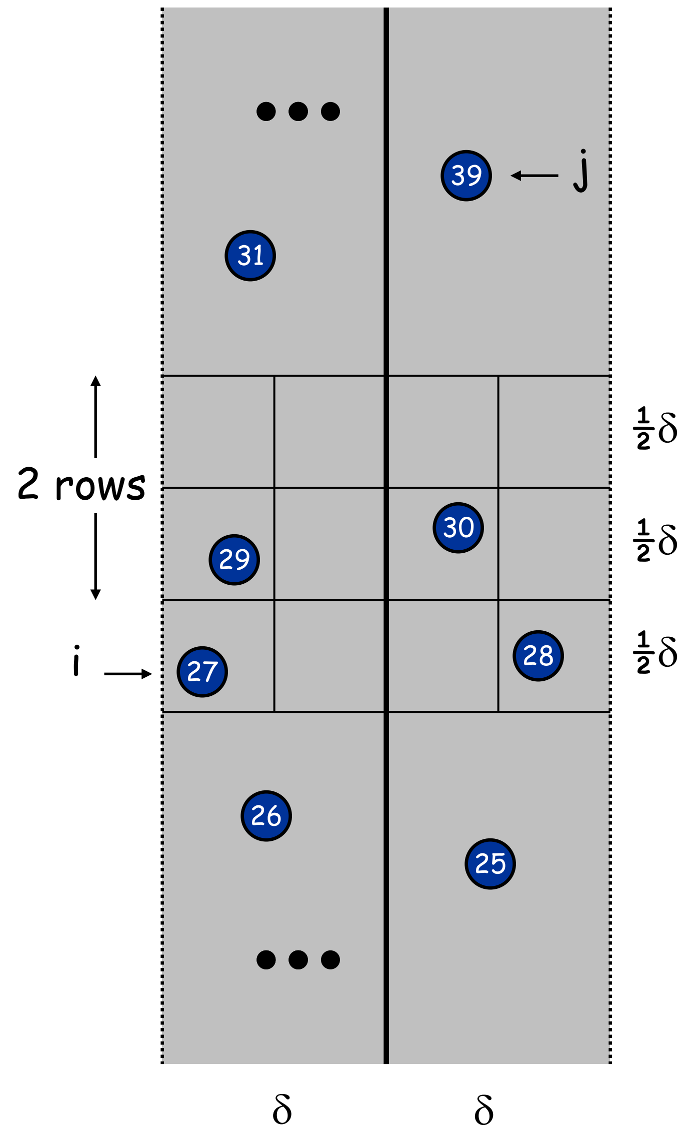

Fact. Still true if we replace 12 with 7.

Q: How can we use this?

Closest Pair Algorithm

```
Closest-Pair(p₁, …, pₙ) {
   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L

   Sort remaining points by y-coordinate.

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these
   distances is less than δ, update δ.

   return δ.
}
```

Runtime?

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$O(n \log n)$

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   Delete all points further than δ from separation line L    O(n)

   Sort remaining points by y-coordinate.                     O(n log n)

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Closest Pair of Points: Analysis

Running time.

$$T(n) \leq 2T(n/2) + O(n \log n)$$ **Bound?**

Closest Pair of Points: Analysis

Running time.

$$T(n) \le 2T(n/2) + O(n \log n) \quad \Rightarrow \quad T(n) = O(n \log^2 n)$$

Closest Pair of Points: Analysis

Q. Can we achieve $O(n \log n)$?

Closest Pair of Points: Analysis

Q. Can we achieve O(n log n)?

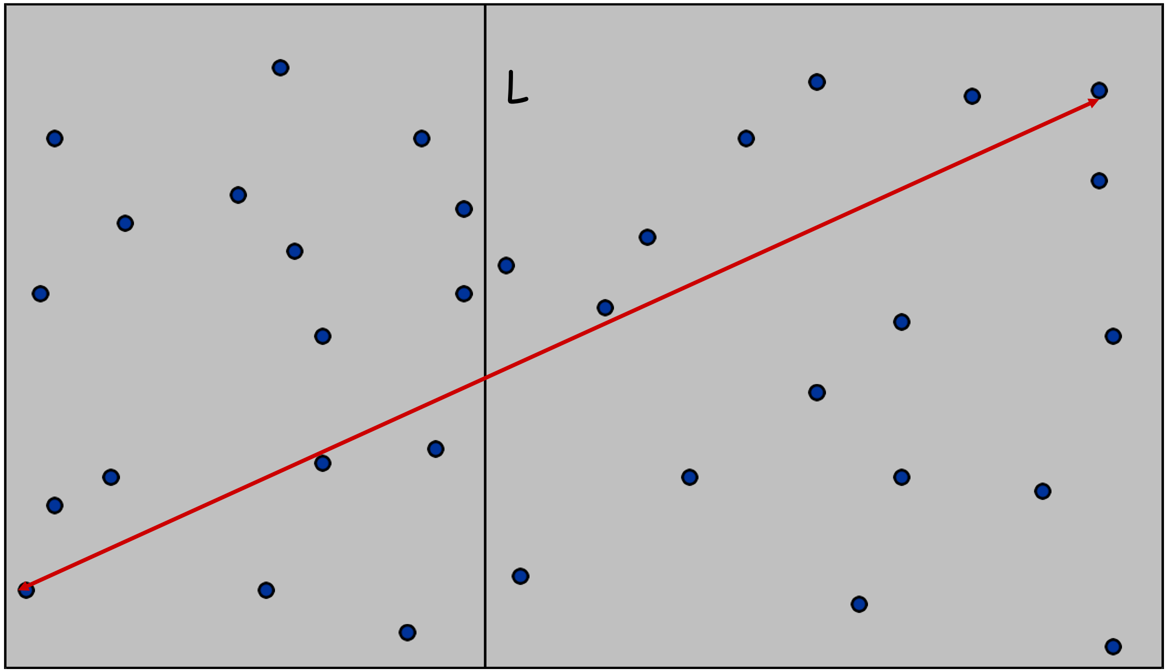
A. Yes. Don't sort points in strip from scratch each time.

- Each recursive call returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate.
- Sort by merging two pre-sorted lists.

$$T(n) \leq 2T(n/2) + O(n) \quad \Rightarrow \quad T(n) = O(n \log n)$$

Farthest Pair of Points

Given data set, find the pair of points that are farthest.



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Exercise: Suppose the data is 1D.

Farthest Pair of Points

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Exercise: For 2D data, what is the runtime?

Farthest Pair of Points

Given data set, find the pair of points that are farthest.

Exercise: For 2D data, can we ***find a faster 2-approximation algorithm***?

I.e. Find pair of points (p_i, p_j) so that

$$d(p_i, p_j) \geq \frac{1}{2} * d(p_k, p_l),$$

where (p_k, p_l) is the farthest pair of points.

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Consider this algorithm: Project each point on the X and Y axes and pick the farthest pair among these.

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Exercise (in class): Show that this algorithm gives a $\sqrt{2}$ -approximation to the optimal solution.