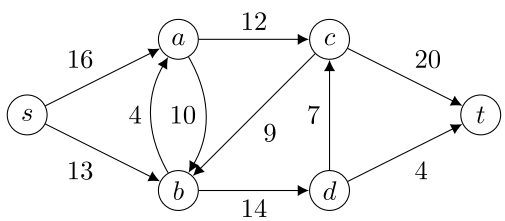
PSO #9 Solutions Sketch (Week 11)

Week of 2021-08-30

1 Network Flow

1. Consider the following flow network graph



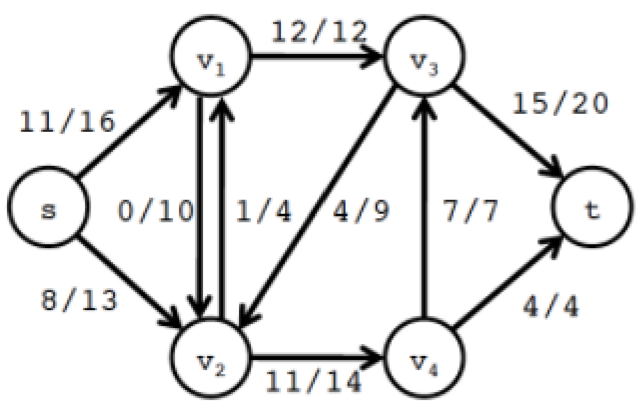
What is the maximum bottleneck capacity of an augmenting path?

Solution:

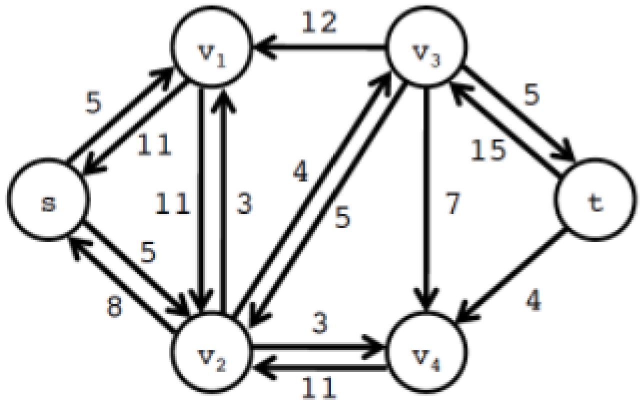
The path $s \to a \to c \to t$ has a bottleneck capacity of 12, which is maximal

2. Give example of a flow network where the augmenting path does not involve the same vertices and edges as in the original graph. (If they take too long, like over 8 minutes, just show them this example and move on. This isn't that important.)

Solution: Same graph as number 1, where if we have the following flow:



and its corresponding residual graph:



We can see that the augmenting path $s \to v_2 \to v_3 \to t$ of capacity 4 isn't the same as the path through which the capacity can be pushed through the original flow network.
Make sure to stress that the use of the residual graph to find the max flow is only designed to find the max flow/min cut, it is NOT designed to find the actual flow through the original network that produces the max flow/min cut.

3. Consider a situation where we have s students and d dorms, and we wish to determine the maximum number of student preferences that can be accommodated with regard to dorm assignments. Design a scheme to compute this number.

   Solution: Construct a flow network, where the students are the first layer from the source, the dorms are the second, and the dorms all point to the sink. The edge weights are all 1 except for the edges from the dorms to the sink, where their capacities are equal to the capacity of the dorm, and the edges from the students to the dorms correspond to the dorms the students have demonstrated a preference to live in. Then, find the max flow.

4. Given a graph G, find the number of edge disjoint paths that go from a vertex A to a vertex B in G, i.e., all the paths that do not have any edges in common.

   Solution: Take the given graph and initialize all the edge weights to 1, and then find the max flow with A as source and B as sink. Because each edge can only support one augmenting path, we can deduce the max flow is the number of paths that do not have a single edge in common, thus satisfying the definition for an edge-disjoint path, and because each augmenting path has a capacity of 1, the max flow is the number of edge-disjoint paths.

5. Given a graph G, find the number of node-disjoint paths that go from a vertex A to a vertex B in G, i.e., all the paths that do not have any vertices in common.

   Solution: This time, we don't touch the edge weights and instead take each vertex v and split it into v_1, v_2, where all the incoming edges to v go into v_1 and all the outgoing edges from v go out of v_2 (if the graph is undirected we transform each edge into two directed edges both of the same capacity as the undirected edge, one incoming and the other outgoing). We then draw a directed edge from v_1 to v_2 with weight 1, thus ensuring that any flow can only have one augmenting path through a single vertex. Thus, the number of augmenting flows is the same as the max flow, which is the number of paths that do not have any vertices in common.