PSO #10 Solutions Sketch (Week 12)

Week of 2021-11-08

1 Linear Programming

1. A farmer has 10 acres in which to plant wheat and rye. He must plant at least 7 acres worth of land, but he only has \$1200 to spend. Each acre of wheat costs \$200 to plant, and each acre of rye costs \$100. Moreover, the planting must be done in 12 hours and it takes an hour to plant an acre of wheat, 2 hours to plant an acre of rye.

   If the profit from wheat is \$500 per acre and the profit from rye is \$300 per acre, how many acres of each should be planted to maximize the farmer's profits?

   Solution:

   Let x be the acres of wheat, y be the acres of rye. We have

$$\begin{array}{ll} \text{maximize} & 500x + 300y \\ \text{subject to} & x + y \leq 10 \\ & x + y \geq 7 \\ & 200x + 100y \leq 1200 \\ & x + 2y \leq 12 \\ & x \geq 0, \quad y \geq 0 \end{array}$$

   For the max acreage constraints, we have 10 for the x and y intercepts. For the min acreage constraints, we have 7 for both intercepts. Graph these, leaving a trapezoidal strip.

   Next, consider and graph the cost constraint. It has an x-intercept of 6, y-intercept of 12.

   Next, the time constraint has an x-intercept of 12, y-intercept of 6.

   Determine the corner points to be $(5, 2), (4, 4), (2, 5)$, and evaluate the objective function at them to determine that the maximum profit of \$3200 is obtained by planting 4 acres of each crop

2. Given the following linear program, write its dual

$$\begin{array}{rc} \text{maximize over } x\text{'s} & v_1x_1 + v_2x_2 + v_3x_3 \\ \text{subject to} & a_1x_1 + x_2 + x_3 \le b_1 \\ & x_1 + a_2x_2 = b_2 \\ & a_3x_3 \ge b_3 \\ & x_1 \ge 0, x_2 \le 0, x_3 \in \mathbb{R} \end{array}$$

Solution:

Depending on how comfortable they are with this and how many examples they've seen, this may be a good problem to just walk through. Hopefully this writeup is sufficiently clear and correct

First we rewrite our objective function as a minimization, and rewrite each inequality constraint as a $\le$ or $=$, and rearrange so that the RHS is 0

$$\begin{array}{rc} \text{minimize over } x\text{'s} & -v_1x_1 - v_2x_2 - v_3x_3 \\ \text{subject to} & a_1x_1 + x_2 + x_3 - b_1 \le 0 \\ & x_1 + a_2x_2 - b_2 = 0 \\ & -a_3x_3 + b_3 \le 0 \\ & x_1 \ge 0, x_2 \le 0, x_3 \in \mathbb{R} \end{array}$$

Now we define dual variables for each constraint. Nonnegative dual variables for inequality constraints, and unrestricted equality constraints.

Associate $y_1 \ge 0$ and $y_3 \ge 0$ to the 1st and 3rd constraint respectively. To the 2nd constraint, associate y_2.

Now, for each constraint, eliminate the constraint and add (dual var) $\cdot$ (LHS) to the objective function. Maximize the result over the dual variables

$$\begin{array}{rrl} \text{maximize over } y\text{'s} & \text{minimize over } x\text{'s} & -v_1x_1 - v_2x_2 - v_3x_3 \\ & & + y_1(a_1x_1 + x_2 + x_3 - b_1) \\ & & + y_2(x_1 + a_2x_2 - b_2) \\ & & + y_3(-a_3x_3 + b_3) \\ & \text{subject to} & x_1 \ge 0, x_2 \le 0, x_3 \in \mathbb{R} \\ & & y_1, y_2, y_3 \ge 0 \end{array}$$

Rewrite to factor the primal variables out.

$$\begin{array}{rrl} \text{maximize over } y\text{'s} & \text{minimize over } x\text{'s} & -b_1y_1 - b_2y_2 - b_3y_3 \\ & & + x_1(a_1y_1 + y_2 - v_1) \\ & & + x_2(y_1 + a_2y_2 - v_2) \\ & & + x_3(y_1 - a_3y_3 - v_3) \\ & \text{subject to} & x_1 \ge 0, x_2 \le 0, x_3 \in \mathbb{R} \\ & & y_1, y_2, y_3 \ge 0 \end{array}$$

Replace terms of the form (primal var) $\cdot$ (expr. with dual vars) with an appropriate constraint of the form expr ≥ 0 if the primal is nonneg, expr ≤ 0 if the primal is nonpos, expr $= 0$ if the primal is unrestricted.

$$\begin{array}{rc} \text{maximize over } y\text{'s} & -b_1y_1 - b_2y_2 - b_3y_3 \\ \text{subject to} & a_1y_1 + y_2 - v_1 \geq 0 \\ & y_1 + a_2y_2 - v_2 \leq 0 \\ & y_1 - a_3y_3 - v_3 = 0 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Rewrite as minimization and clean it up

$$\begin{array}{rc} \text{minimize} & b_1y_1 + b_2y_2 + b_3y_3 \\ \text{subject to} & a_1y_1 + y_2 \geq v_1 \\ & y_1 + a_2y_2 \leq v_2 \\ & y_1 - a_3y_3 = v_3 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

3. Given the following optimization problem, write it as a linear program. (If they're stuck after a good deal of time without ideas, give this hint: write the absolute value in terms of inequalities)

$$\begin{array}{rc} \text{minimize} & \sum_{i=1}^{n} |x_i - c_i| \\ \text{subject to} & Ax = b \\ & A \in \mathbb{R}^{mxn},\ x, c \in \mathbb{R}^n,\ b \in \mathbb{R}^m \end{array}$$

Solution:

$$\begin{array}{rc} \text{minimize} & \sum_{i=1}^{n} y_i \\ \text{subject to} & x_i - c_i \leq y_i \\ & x_i - c_i \geq -y_i \\ & Ax = b \\ & A \in \mathbb{R}^{mxn},\ x, c \in \mathbb{R}^n,\ b \in \mathbb{R}^m \end{array}$$

4. Given the following statement about the linear program, write the forward and backwards directions of the proof, but do not prove the statement.

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^{n} \log(b_i - a_i^T x) \\ \text{subject to} & A \in \mathbb{R}^{nxn} \text{ with rows } a_i^T, \ A \text{ is invertible } , x \in \{x | b \succ Ax\}, \ b \in \mathbb{R}^m \end{array}$$

where $\succ$ means element-wise greater than. The set of possible values for x is unbounded if and only if there exists $v \neq 0$ s.t. $0 \succeq Av$, where $\succeq$ means element-wise greater than or equal to.

Solution:
One direction is that if there exists a vector $v_0 \neq 0$ s.t. $0 \succeq Av_0$, then we must prove the set of possible values for x is unbounded.
The other direction is that given the set of values of x is unbounded, we must prove there exists $v \neq 0$ s.t. $0 \succeq Av$.