Linear programming

CS 381

An Example

[Diet Problem]. An athlete wants to maximize her daily protein consumption according to the next constraints:

- At most 5 units of fat per day
- Spending at most $6 per day.
- ***Ingredients:***
  - Steak: cost $4 per pound; 2 units of protein and 1 unit of fat per pound.
  - Peanut butter: cost $1 per pound; 1 unit of protein and 2 units of fat per pound

Exercise: Write as a linear program (LP).

An Example

Let **x_1** = number of pounds of steak; **x_2** = number of pounds of peanut butter per day.

Linear Program (LP1):

$$\begin{aligned} & \max \ 2x_1 + x_2 \\ \text{subject to} \quad & 4x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

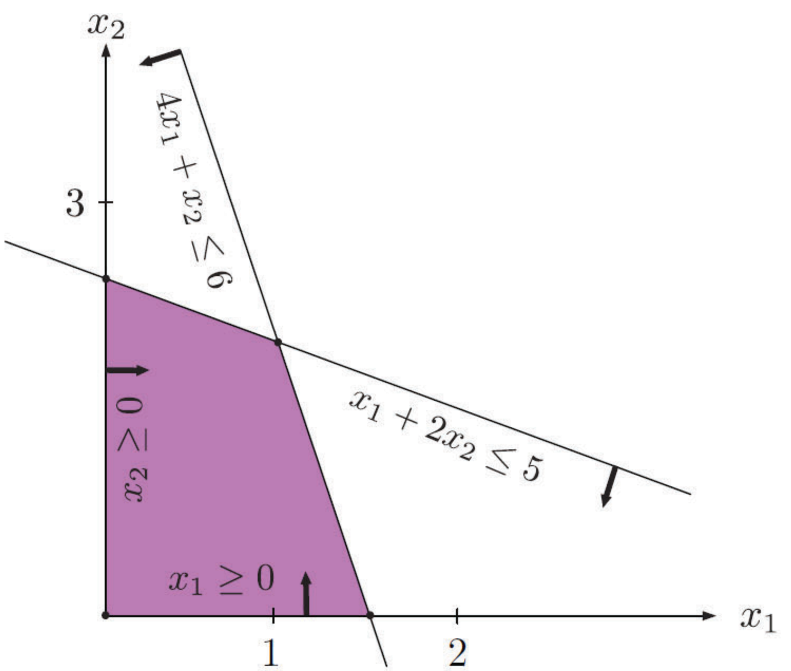
Objective function: $2x_1 + x_2$

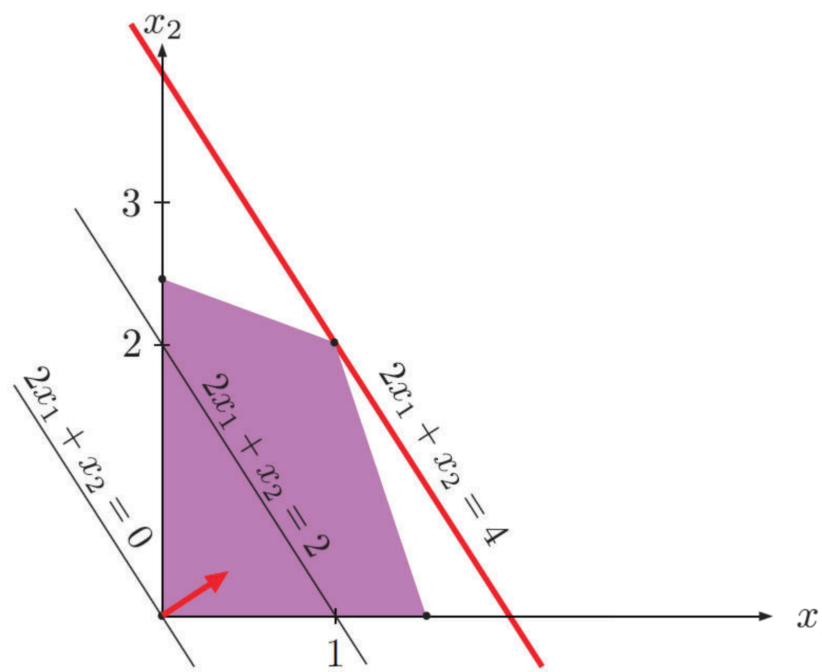
Feasible set: the set of feasible vectors that satisfy the constraints. In this example, this is the set of vectors (x_1, x_2) such that $4x_1 + x_2 \leq 6$ and $x_1 + 2x_2 \leq 5$.

An Example

Let x_1 = number of pounds of steak; x_2 = number of pounds of peanut butter per day.

In the following graphs, the purple region shows the feasible set for LP1.



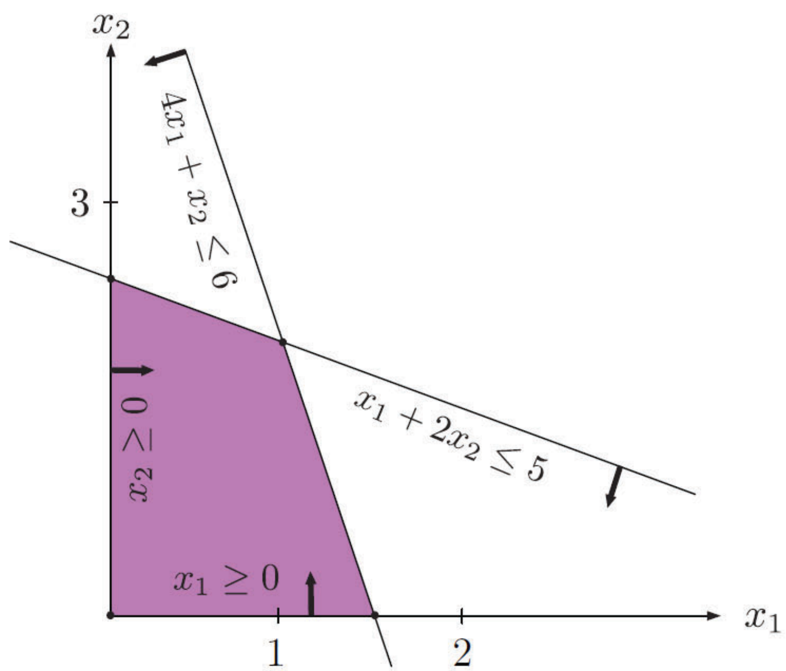


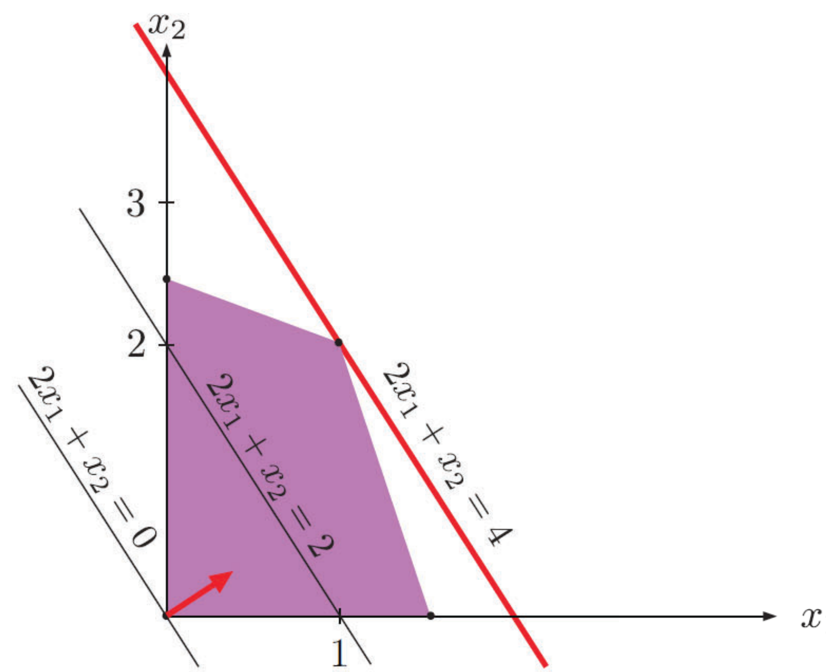
The largest c for which the line $2 \cdot x_1 + x_2 = c$ intersects the feasible set is $c = 4$. The red arrow from the origin to the right is perpendicular to the lines $2x_1 + x_2 = k$ for all k

An Example

Let x_1 = number of pounds of steak; x_2 = number of pounds of peanut butter per day.

In the following graphs, the purple region shows the feasible set for LP1.





Which point in the feasible set maximizes the objective $2x_1 + x_2$?

For LP1, this point is $(x_1, x_2) = (1,2)$; the value of the objective is $2x_1 + x_2 = 4$.

Linear Programming Duality

Linear Program (LP1):

$$\begin{aligned} & \max \;\; 2x_1 + x_2 \\ \text{subject to} \quad & 4x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- The first constraint of LP1 ensures the objective is upper bounded by 6.
- The second constraint (multiplied by 2) gives an upper bound of 10.

Multiply the first constraint by $y_1 \geq 0$ and the second constraint by $y_2 \geq 0$. We get:

$y_1(4x_1 + x_2) + y_2(x_1 + 2x_2) \leq 6y_1 + 5y_2$ **(*)**

The LHS of (*) is greater than the objective of LP1 whenever:

$$\begin{aligned} 4y_1 + y_2 &\geq 2 \\ y_1 + 2y_2 &\geq 1 \\ y_1, y_2 &\geq 0\,. \end{aligned}$$

Linear Programming Duality

Linear Program (LP1):

$$\begin{aligned} & \max\ 2x_1 + x_2 \\ \text{subject to} \quad & 4x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Constraints on y (**)

$$\begin{aligned} 4y_1 + y_2 &\geq 2 \\ y_1 + 2y_2 &\geq 1 \\ y_1, y_2 &\geq 0\,. \end{aligned}$$

Then for any (y_1, y_2) that is feasible for (**), we get: $2x_1 + x_2 \leq 6y_1 + 5y_2$ for all feasible (x_1, x_2).

The best upper bound we can obtain this way on the optimal value of LP1 is given by the solution to the linear program:

Linear Program (LP2):

$$\begin{aligned} & \min 6y_1 + 5y_2 \\ \text{subject to} \quad & 4y_1 + y_2 \geq 2 \\ & y_1 + 2y_2 \geq 1 \\ & y_1, y_2 \geq 0\,. \end{aligned}$$

Linear Programming Duality

In general, a maximization LP in standard form is written as:

$$\left.\begin{aligned} \max\ \mathbf{c}^T\mathbf{x} \\ \text{subject to} \qquad \\ A\mathbf{x} \le \mathbf{b} \\ \mathbf{x} \ge 0 \end{aligned}\right\}, \qquad (P)$$

where $A \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$

(P) is known as the primal LP. This linear program is feasible if the feasible set is non-empty:

$$\mathcal{F}(P) := \{\mathbf{x} \mid A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge 0\}$$

Linear Programming Duality

As discussed before, if a vector $y \geq 0 \in R^m$ satisfies the inequality $y^T A \geq c^T$, then:

$$\forall \mathbf{x} \in \mathcal{F}(P), \ \mathbf{y}^T\mathbf{b} \geq \mathbf{y}^T A\mathbf{x} \geq \mathbf{c}^T\mathbf{x}.$$

This motivates introducing the following linear program, known as the dual LP:

$$\left.\begin{array}{r} \min \ \mathbf{b}^T\mathbf{y} \\ \text{subject to} \qquad \\ \mathbf{y}^T A \geq \mathbf{c}^T \\ \mathbf{y} \geq 0 \end{array}\right\} \qquad (D)$$

where $y \geq 0 \in R^m$.

Exercise: Check that the dual of the dual LP is the primal LP.

Duality Theorem

Let $A \in R^{m \times n}$; $x, c \in R^{n}$; $y, b \in R^{m}$. Recall the primal LP and its dual are:

$$\left.\begin{aligned} \max\ \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{aligned}\right\}, \qquad (P) \qquad\qquad \left.\begin{aligned} \min\ \mathbf{b}^T\mathbf{y} \\ \text{subject to} \\ \mathbf{y}^T A \geq \mathbf{c}^T \\ \mathbf{y} \geq 0 \end{aligned}\right\} \qquad (D)$$

Theorem (Duality Theorem): If $F(P)$ and $F(D)$ are non-empty, then:

- $b^T y \geq c^T x$ for all $x \in F(P)$ and $y \in F(D)$. [known as ***weak duality***]
- (P) has an optimal solution x^*, (D) has an optimal solution y^*, and $c^T x^* = b^T y^*$

Duality Theorem

Let $A \in R^{m \times n}; x, c \in R^{n};\ y, b \in R^{m}$. Recall the primal LP and its dual:

$$\left.\begin{array}{r} \max\ \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{array}\right\}, \qquad (P) \qquad\qquad \left.\begin{array}{r} \min\ \mathbf{b}^T\mathbf{y} \\ \text{subject to} \\ \mathbf{y}^T A \geq \mathbf{c}^T \\ \mathbf{y} \geq 0 \end{array}\right\} \qquad (D)$$

Corollary (Complementary slackness): Let x^* be feasible for (P) and y^* be feasible for (D). Then the following statements are equivalent:

1. x^* is optimal for (P) and y^* is optimal for (D).
2. For each i such that $\sum_{1 \leq j \leq n} a_{ij} \cdot x_j^* < b_i$, we have $y_i^* = 0$ and for each j such that $c_j < \sum_{1 \leq i \leq m} y_j^* \cdot a_{ij}$ we have $x_j^* = 0$.

Duality Theorem

Recall the primal LP and its dual:

$$\left.\begin{aligned} \max\ \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ A\mathbf{x} \le \mathbf{b} \\ \mathbf{x} \ge 0 \end{aligned}\right\}, \quad (P) \qquad \left.\begin{aligned} \min\ \mathbf{b}^T\mathbf{y} \\ \text{subject to} \\ \mathbf{y}^T A \ge \mathbf{c}^T \\ \mathbf{y} \ge 0 \end{aligned}\right\} \quad (D)$$

Corollary (Complementary slackness): Let x^* be feasible for (P) and y^* be feasible for (D). Then the following statements are equivalent:

1. x^* is optimal for (P) and y^* is optimal for (D).
2. For each i such that $\sum_{1\le j\le n} a_{ij} \cdot x_j^* < b_i$, we have $y_i^* = 0$ and for each j such that $c_j < \sum_{1\le i\le m} y_j^* \cdot a_{ij}$ we have $x_j^* = 0$.

(***)

Proof: x^* and y^* feasible, so: $\sum_j c_j x_j^* \le \sum_j x_j^* \sum_i y_i^* a_{ij} = \sum_i y_i^* \sum_j a_{ij} x_j^* \le \sum_i b_i y_i^*$

Duality Theorem

Recall the primal LP and its dual:

$$\left.\begin{aligned} \max\ \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ A\mathbf{x} \le \mathbf{b} \\ \mathbf{x} \ge 0 \end{aligned}\right\}, \qquad (P) \qquad\qquad \left.\begin{aligned} \min\ \mathbf{b}^T\mathbf{y} \\ \text{subject to} \\ \mathbf{y}^T A \ge \mathbf{c}^T \\ \mathbf{y} \ge 0 \end{aligned}\right\} \qquad (D)$$

Corollary (Complementary slackness): Let x^* be feasible for (P) and y^* be feasible for (D). Then the following statements are equivalent:

1. x^* is optimal for (P) and y^* is optimal for (D).
2. For each i such that $\sum_{1\le j\le n} a_{ij} \cdot x_j^* < b_i$, we have $y_i^* = 0$ and for each j such that $c_j < \sum_{1\le i\le m} y_j^* \cdot a_{ij}$ we have $x_j^* = 0$.

Proof: $x^*$and y^* feasible, so: $$\sum_j c_j x_j^* \le \sum_j x_j^* \sum_i y_i^* a_{ij} = \sum_i y_i^* \sum_j a_{ij} x_j^* \le \sum_i b_i y_i^* \qquad \textbf{(***)}$$

By duality theorem, optimality of $x^*$and y^* means there is equality throughout **(***)**.

Duality Theorem

Recall the primal LP and its dual:

$$\left.\begin{aligned} \max\ \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{aligned}\right\}, \qquad (P) \qquad\qquad \left.\begin{aligned} \min\ \mathbf{b}^T\mathbf{y} \\ \text{subject to} \\ \mathbf{y}^T A \geq \mathbf{c}^T \\ \mathbf{y} \geq 0 \end{aligned}\right\} \qquad (D)$$

Corollary (Complementary slackness): Let x^* be feasible for (P) and y^* be feasible for (D). Then the following statements are equivalent:

1. x^* is optimal for (P) and y^* is optimal for (D).
2. For each i such that $\sum_{1 \leq j \leq n} a_{ij} \cdot x_j^* < b_i$, we have $y_i^* = 0$ and for each j such that $c_j < \sum_{1 \leq i \leq m} y_j^* \cdot a_{ij}$ we have $x_j^* = 0$.

Proof: $x^*$and y^* feasible, so: $$\sum_j c_j x_j^* \leq \sum_j x_j^* \sum_i y_i^* a_{ij} = \sum_i y_i^* \sum_j a_{ij} x_j^* \leq \sum_i b_i y_i^* \qquad \textbf{(***)}$$

By duality theorem, optimality of $x^*$and y^* means there is equality throughout **(***)**.

By feasibility, for each j we have: $c_j x_j^* \leq x_j^* \cdot \sum_i y_i^* a_{ij}$ and for each i we have: $x_j^* \cdot y_i^* \sum_i a_{ij} \cdot x_j^* \leq b_i y_i^*$. Then equality holds in **(***)** if and only if (2) holds.

Example of Primal and Dual Pair

Advertising problem.

An advertiser wants to purchase advertising space in a set of n newspapers (ads placed over a period of time).

- ***Cost*** c_j of placing an ad in newspaper j.
- ***Target population:*** m types of readers (e.g. based on location, age, interests)
- ***Goal:*** b_i users of type i will see the ad.
- ***Suppose*** a_{ij} is the expected number of type i users that will see each ad in newspaper j.

Example of Primal and Dual Pair

Advertising problem.

An advertiser wants to purchase advertising space in a set of n newspapers (ads placed over a period of time).

- ***Cost*** c_j of placing an ad in newspaper j.
- ***Target population:*** m types of readers (e.g. based on location, age, interests)
- ***Goal:*** b_i users of type i will see the ad.
- ***Suppose*** a_{ij} is the expected number of type i users that will see each ad in newspaper j.

Exercise: How many ads of each type should the advertiser place in each newspaper to meet the demographic targets at minimum cost? [approximation/upper bound is good enough]

Example of Primal and Dual Pair

Let x_j be the number of ad slots purchased from newspaper j.

Example of Primal and Dual Pair

Let x_j be the number of ad slots purchased from newspaper j.

The advertiser solves the following LP:

$$\min \sum_{1 \le j \le n} c_j x_j$$

$$\text{subject to} \quad \sum_{1 \le j \le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$

$$x_1, x_2 \ldots, x_n \ge 0.$$

Example of Primal and Dual Pair

Let x_j be the number of ad slots purchased from newspaper j.

The advertiser solves the following LP:

$$\min \sum_{1 \le j \le n} c_j x_j$$

$$\text{subject to} \quad \sum_{1 \le j \le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$

$$x_1, x_2 \ldots, x_n \ge 0.$$

Dual LP:

$$\max \sum_{1 \le i \le m} b_i y_i$$

$$\text{subject to} \quad \sum_{1 \le i \le m} y_i a_{ij} \le c_j \quad \text{for all } 1 \le j \le n$$

$$y_1, y_2 \ldots, y_m \ge 0.$$

Example of Primal and Dual Pair

Primal LP

$$\min \sum_{1 \le j \le n} c_j x_j$$

$$\text{subject to} \quad \sum_{1 \le j \le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$

$$x_1, x_2 \ldots, x_n \ge 0.$$

Dual LP

$$\max \sum_{1 \le i \le m} b_i y_i$$

$$\text{subject to} \quad \sum_{1 \le i \le m} y_i a_{ij} \le c_j \quad \text{for all } 1 \le j \le n$$

$$y_1, y_2 \ldots, y_m \ge 0.$$

- ***Interpretation of the dual:*** Consider an online advertising exchange that wants to convince the advertiser to switch from advertising in newspapers to online.

Example of Primal and Dual Pair

Primal LP

$$\min \sum_{1 \le j \le n} c_j x_j$$

$$\text{subject to} \quad \sum_{1 \le j \le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$

$$x_1, x_2 \ldots, x_n \ge 0.$$

Dual LP

$$\max \sum_{1 \le i \le m} b_i y_i$$

$$\text{subject to} \quad \sum_{1 \le i \le m} y_i a_{ij} \le c_j \quad \text{for all } 1 \le j \le n$$

$$y_1, y_2 \ldots, y_m \ge 0.$$

- ***Interpretation of the dual:*** Consider an online advertising exchange that wants to convince the advertiser to switch from advertising in newspapers to online.
- ***The exchange agency problem:*** Determine $\mathrm{y_i}$ = how much to charge the advertiser for each ad displayed (aka impression) shown to a user of type i.

Example of Primal and Dual Pair

Primal LP

$$\min \sum_{1 \le j \le n} c_j x_j$$

subject to
$$\sum_{1 \le j \le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$
$$x_1, x_2 \ldots, x_n \ge 0.$$

Dual LP

$$\max \sum_{1 \le i \le m} b_i y_i$$

subject to
$$\sum_{1 \le i \le m} y_i a_{ij} \le c_j \quad \text{for all } 1 \le j \le n$$
$$y_1, y_2 \ldots, y_m \ge 0.$$

- ***Interpretation of the dual:*** Consider an online advertising exchange that wants to convince the advertiser to switch from advertising in newspapers to online.
- ***The exchange agency problem:*** Determine $\mathrm{y_i}$ = how much to charge the advertiser for each ad displayed (aka impression) shown to a user of type i.
- ***Note:*** $\mathrm{y_i} \cdot a_{ij}$ = expected cost of reaching the same number of type i users online that would be reached by placing a single ad in newspaper j.

Example of Primal and Dual Pair

Primal LP

$$\min \sum_{1 \le j \le n} c_j x_j$$

subject to
$$\sum_{1 \le j \le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$
$$x_1, x_2 \ldots, x_n \ge 0.$$

Dual LP

$$\max \sum_{1 \le i \le m} b_i y_i$$

subject to
$$\sum_{1 \le i \le m} y_i a_{ij} \le c_j \quad \text{for all } 1 \le j \le n$$
$$y_1, y_2 \ldots, y_m \ge 0.$$

- ***Interpretation of the dual:*** Consider an online advertising exchange that wants to convince the advertiser to switch from advertising in newspapers to online.
- ***The exchange agency problem:*** Determine $\mathrm{y_i} =$ how much to charge the advertiser for each ad displayed (aka impression) shown to a user of type i.
- ***Note:*** $\mathrm{y_i} \cdot a_{ij} =$ expected cost of reaching the same number of type i users online that would be reached by placing a single ad in newspaper j.
- If prices $\mathrm{y_i}$ are set so that $\sum_{1 \le i \le m} y_i \cdot a_{ij} \le c_j$, then the advertiser can switch from advertising in newspaper j to advertising online without increasing its cost.

Example of Primal and Dual Pair

Primal LP

$$\min \sum_{1\le j\le n} c_j x_j$$

subject to $\sum_{1\le j\le n} a_{ij}x_j \ge b_i$ for all $1 \le i \le m$

$x_1, x_2 \ldots, x_n \ge 0.$

Dual LP

$$\max \sum_{1\le i\le m} b_i y_i$$

subject to $\sum_{1\le i\le m} y_i a_{ij} \le c_j$ for all $1 \le j \le n$

$y_1, y_2 \ldots, y_m \ge 0.$

- ***The exchange agency problem:*** Determine $\mathrm{y_i}$ = how much to charge the advertiser for each ad displayed (aka impression) shown to a user of type i.

- ***Note:*** $\mathrm{y_i} \cdot a_{ij}$ = expected cost of reaching the same number of type i users online that would be reached by placing a single ad in newspaper j.

- If prices $\mathrm{y_i}$ are set so that $\sum_{1\le i\le m} y_i \cdot a_{ij} \le c_j$, then the advertiser can switch from advertising in newspaper j to advertising online without increasing its cost.
- If the advertiser completely switches from newspaper to online, then the exchange's revenue will be **… ?**

Example of Primal and Dual Pair

Primal LP

$$\min \sum_{1 \le j \le n} c_j x_j$$

subject to $$\sum_{1 \le j \le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$

$$x_1, x_2 \ldots, x_n \ge 0.$$

Dual LP

$$\max \sum_{1 \le i \le m} b_i y_i$$

subject to $$\sum_{1 \le i \le m} y_i a_{ij} \le c_j \quad \text{for all } 1 \le j \le n$$

$$y_1, y_2 \ldots, y_m \ge 0.$$

- ***The exchange agency problem:*** Determine $\mathrm{y_i}$ = how much to charge the advertiser for each ad displayed (aka impression) shown to a user of type i.
- ***Note:*** $\mathrm{y_i} \cdot a_{ij}$ = expected cost of reaching the same number of type i users online that would be reached by placing a single ad in newspaper j.
- If prices $\mathrm{y_i}$ are set so that $\sum_{1 \le i \le m} y_i \cdot a_{ij} \le c_j$, then the advertiser can switch from advertising in newspaper j to advertising online without increasing its cost.
- If the advertiser completely switches from newspaper to online, then the exchange's revenue will be $\sum_{1 \le i \le m} b_i \cdot y_i$

Example of Primal and Dual Pair

Primal LP

$$\min \sum_{1 \le j \le n} c_j x_j$$

subject to

$$\sum_{1 \le j \le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$

$$x_1, x_2 \ldots, x_n \ge 0.$$

Dual LP

$$\max \sum_{1 \le i \le m} b_i y_i$$

subject to

$$\sum_{1 \le i \le m} y_i a_{ij} \le c_j \quad \text{for all } 1 \le j \le n$$

$$y_1, y_2 \ldots, y_m \ge 0.$$

Duality theorem implies that the exchange can price the impressions so that:

- The dual LP constraints are met.
- The revenue $\sum_{1 \le i \le m} b_i \cdot y_i$ matches the total revenue of the newspapers [The exchange agency can charge a bit less than the newspapers to incentivize the advertiser to switch].

Example of Primal and Dual Pair

Primal LP

$$\min \sum_{1 \le j \le n} c_j x_j$$

subject to
$$\sum_{1 \le j \le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$
$$x_1, x_2 \ldots, x_n \ge 0.$$

Dual LP

$$\max \sum_{1 \le i \le m} b_i y_i$$

subject to
$$\sum_{1 \le i \le m} y_i a_{ij} \le c_j \quad \text{for all } 1 \le j \le n$$
$$y_1, y_2 \ldots, y_m \ge 0.$$

Complementary slackness implies that if inequality $\sum_{1 \le i \le m} a_{ij} \cdot x_j \ge b_i$ is not tight for user type i in the optimal solution of the primal, i.e. $\sum_{1 \le i \le m} a_{ij} \cdot x_j > b_i$, then:

- $y_i = 0$ in the optimal solution in the dual [Recall y_i = charge the advertiser for each ad shown to a user of type i]

Example of Primal and Dual Pair

Primal LP

$$\min \sum_{1\le j\le n} c_j x_j$$

subject to $$\sum_{1\le j\le n} a_{ij} x_j \ge b_i \quad \text{for all } 1 \le i \le m$$

$$x_1, x_2 \ldots, x_n \ge 0.$$

Dual LP

$$\max \sum_{1\le i\le m} b_i y_i$$

subject to $$\sum_{1\le i\le m} y_i a_{ij} \le c_j \quad \text{for all } 1 \le j \le n$$

$$y_1, y_2 \ldots, y_m \ge 0.$$

Complementary slackness implies that if inequality $\sum_{1\le i\le m} a_{ij} \cdot x_j \ge b_i$ is not tight for user type i in the optimal solution of the primal, i.e. $\sum_{1\le i\le m} a_{ij} \cdot x_j > b_i$, then:

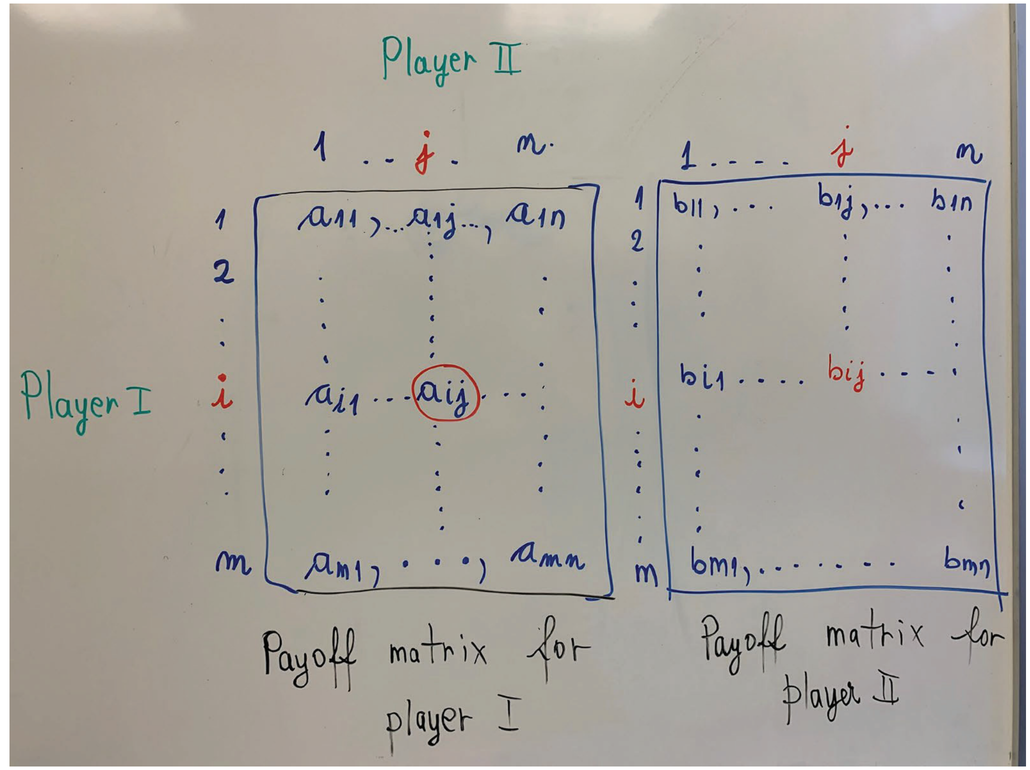
- $y_i = 0$ in the optimal solution in the dual [Recall $\mathrm{y_i}$ = charge the advertiser for each ad shown to a user of type i]

That is, if the optimal combination of ads bought from newspaper has the property that ***more ads are shown to users of type i than necessary***, then:

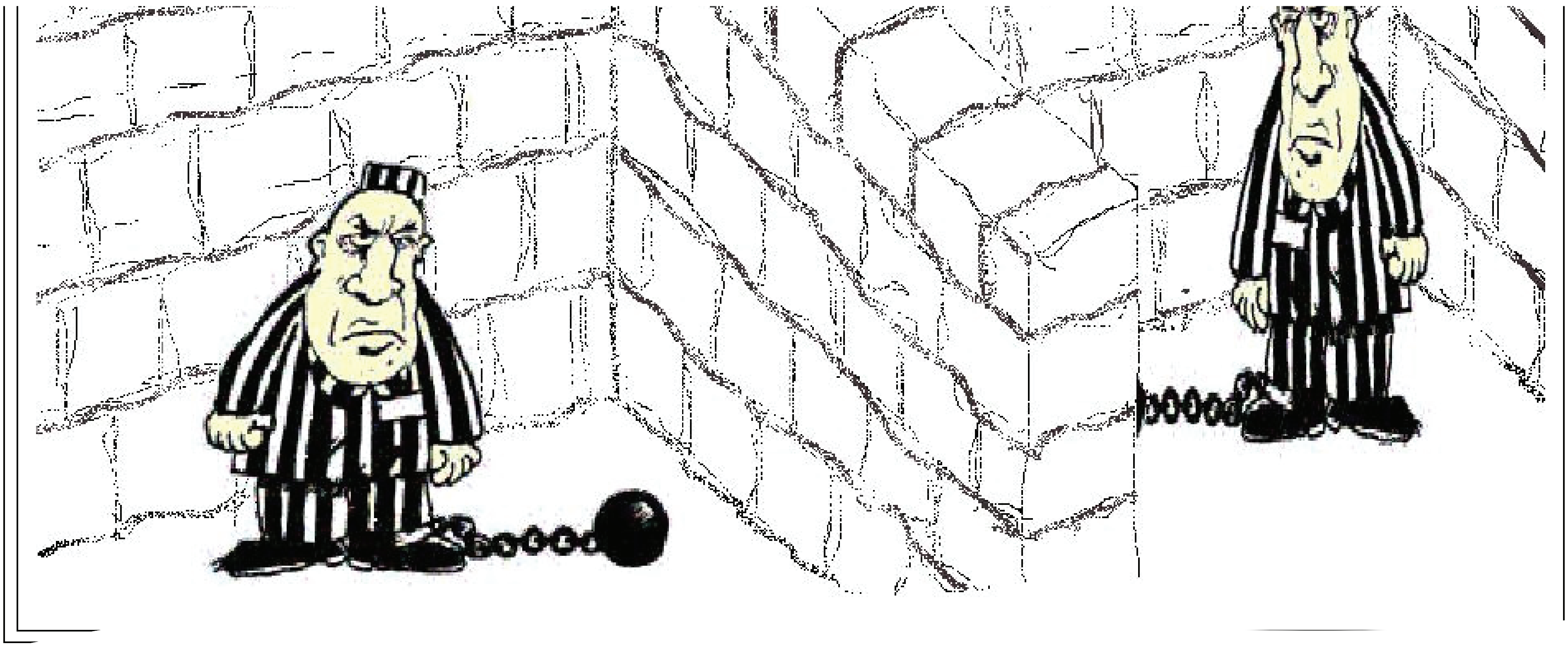
- **in the optimal pricing for the exchange, impressions shown to users of type i will be provided to the advertiser for free**.

LP and Games

A **game** is specified by two matrices A and B. If player I chooses action i and player II chooses action j, their payoffs are $\mathrm{a_{ij}}$ and $\mathrm{b_{ij}}$, respectively.



Pure Nash equilibrium: pair of strategies, one per player, such that each is a **best response** to the other.



Prisoner's Dilemma. Two suspects are imprisoned by the police who ask each of them to confess. The charge is serious, but there is not enough evidence to convict.

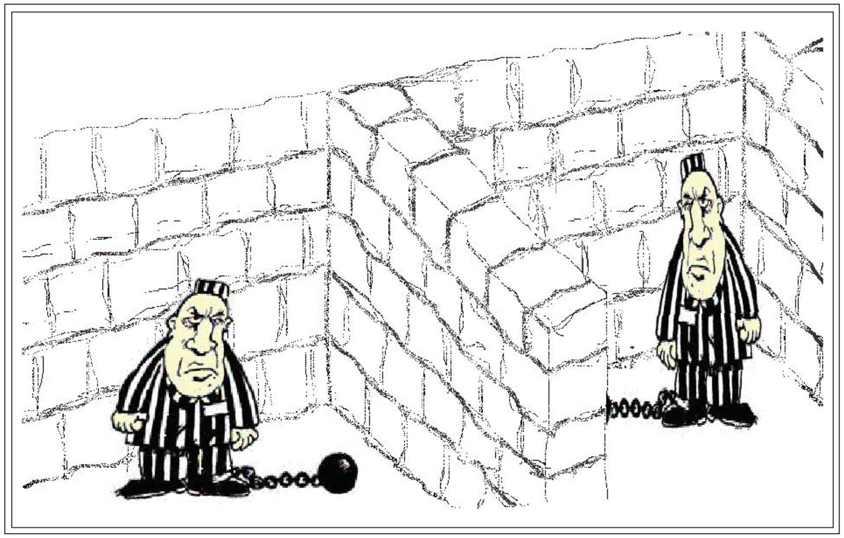
Separately, each prisoner is offered a deal by the police:

- If he confesses and the other prisoner remains silent, the confessor goes free, and his confession is used to sentence the other prisoner to ten years in jail.
- If both confess, they will both spend eight years in jail.
- If both remain silent, the sentence is one year to each for the minor crime that can be proved without additional evidence.

Prisoner's Dilemma

Payoffs written in matrix form:

		prisoner II	
		silent	confess
prisoner I	silent	$(-1, -1)$	$(-10, 0)$
	confess	$(0, -10)$	$(-8, -8)$

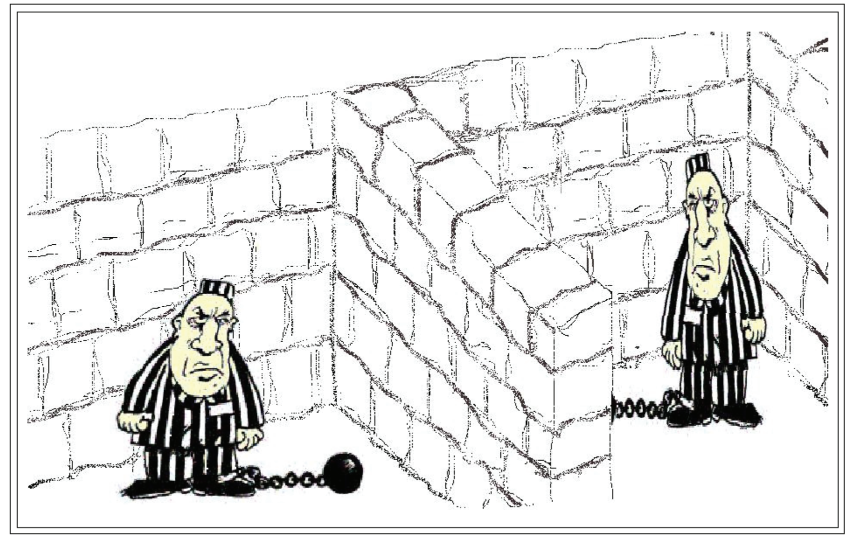


Notation: (-10,0) means that prisoner I gets payoff -10 and prisoner II gets payoff 0.

Prisoner's Dilemma

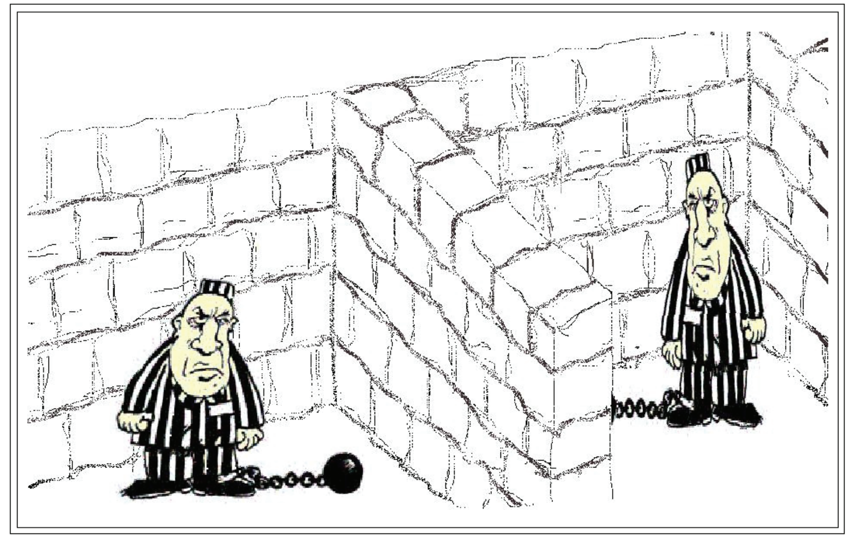
How should they play?

prisoner I	prisoner II: silent	prisoner II: confess
silent	$(-1, -1)$	$(-10, 0)$
confess	$(0, -10)$	$(-8, -8)$



Prisoner's Dilemma

How should they play?



prisoner I \ prisoner II	silent	confess
silent	$(-1,-1)$	$(-10,0)$
confess	$(0,-10)$	$(-8,-8)$

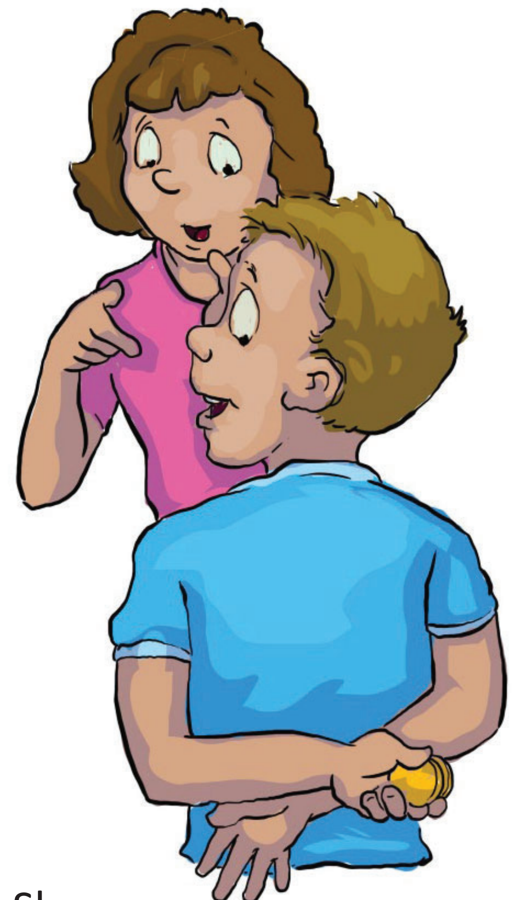
Better that they both stay silent than that they both confess.

However, coordination is difficult since actions are chosen separately (they are confined to separate cells and cannot communicate).

Note that for each possible action of one prisoner, the other is better off confessing.

Confessing is a ***pure Nash equilibrium***.

Pick a hand



There are two players, **Chooser** (player I) and **Hider** (player II).

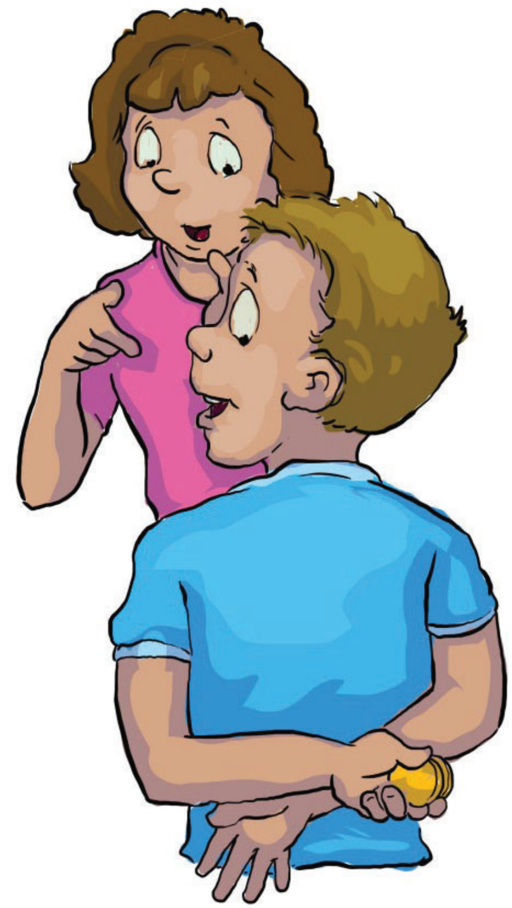
Hider has two gold coins in his back pocket. At the beginning of a turn, he puts his hands behind his back and either

- takes out one coin and holds it in his left hand (strategy L1) or
- takes out both coins and holds them in his right hand (strategy

Chooser picks a hand and wins any coins the hider has hidden there. She may get nothing (if the hand is empty), or she might win one coin, or two.

Exercise: How much should Chooser be willing to pay in order to play this game? Write this game in matrix form.

Pick a hand

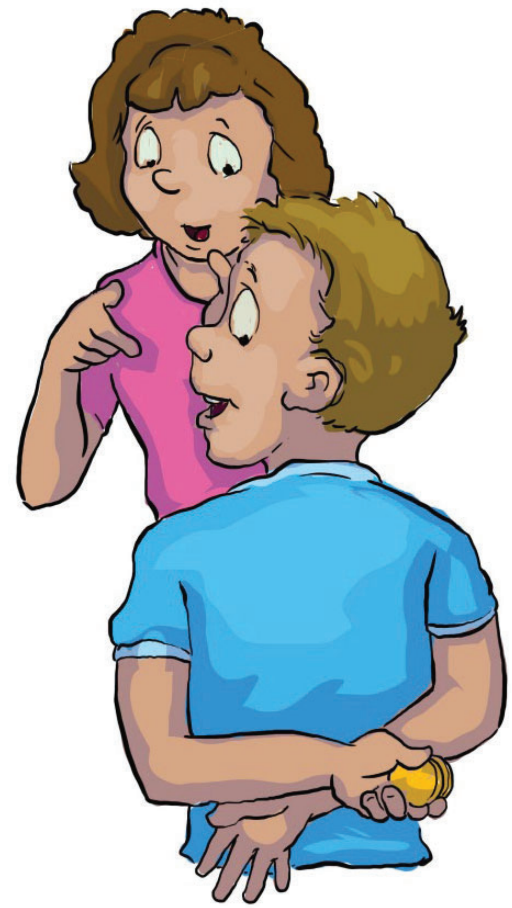


Chooser's payoffs depending on the actions taken by the players:

		Hider	
		$L1$	$R2$
Chooser	L	1	0
	R	0	2

How should Hider and Chooser play?

Pick a hand



Chooser's payoffs depending on the actions taken by the players:

Chooser \ Hider	$L1$	$R2$
L	1	0
R	0	2

How should Hider and Chooser play? Some observations:

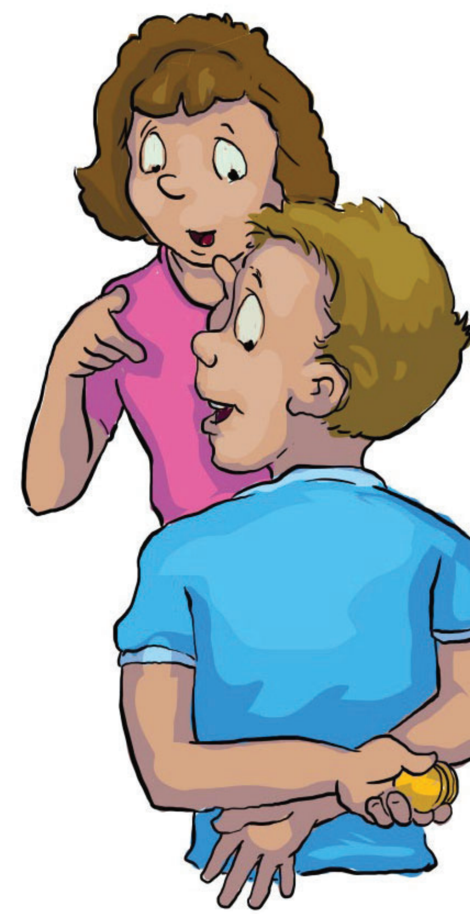
- **Hider** can guarantee himself a loss of at most 1 by selecting action L1.

  If Hider selects R2, he has the potential to lose 2.

- **Chooser** cannot guarantee herself any positive gain because:
  - if she selects L, in the worst-case, Hider selects R2,
  - if she selects R, in the worst case, Hider selects L1.

Pick a hand

Chooser \ Hider	$L1$	$R2$
L	1	0
R	0	2



The players can randomize.

Suppose Hider selects

- L1 with probability y_1 and
- R2 with probability $y_2 = 1\ - y\_1$.

Then Hider's expected loss is y_1 if Chooser plays L and $2(1 - y_1)$ if Chooser plays R.

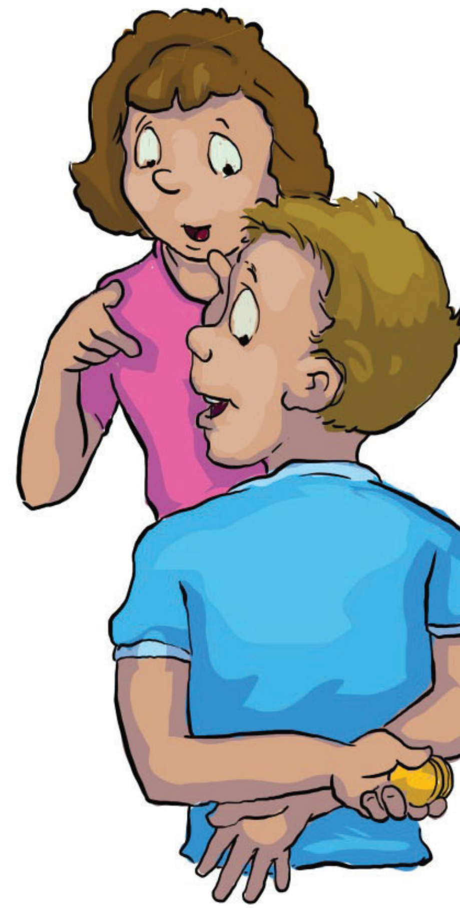
Pick a hand

Chooser \ Hider	$L1$	$R2$
L	1	0
R	0	2

The players can randomize.

Suppose Hider selects

- L1 with probability y_1 and
- R2 with probability $y_2 = 1 - y\_1$.



Then Hider's expected loss is y_1 if Chooser plays L and $2(1 - y_1)$ if Chooser plays R.

Thus Hider's worst case expected loss is $\max(y_1, 2(1 - y_1))$.

How should Hider play to minimize his worst case expected loss?

Pick a hand

		Hider	
		$L1$	$R2$
Chooser	L	1	0
	R	0	2

Suppose Hider selects

- L1 with probability y_1 and
- R2 with probability $y_2 = 1 - y\_1$.

Then Hider's expected loss is y_1 if Chooser plays L and $2(1 - y_1)$ if Chooser plays R.

Thus Hider's worst case expected loss is $\max(y_1, 2(1 - y_1))$. To minimize this, the Hider should set $y_1 = \frac{2}{3}$.

What bound on the expected loss can Hider guarantee for himself, no matter what Chooser plays?

Pick a hand

		Hider	
		$L1$	$R2$
Chooser	L	1	0
	R	0	2

Suppose Hider selects

- L1 with probability y_1 and
- R2 with probability $y_2 = 1 \ - y\_1$.

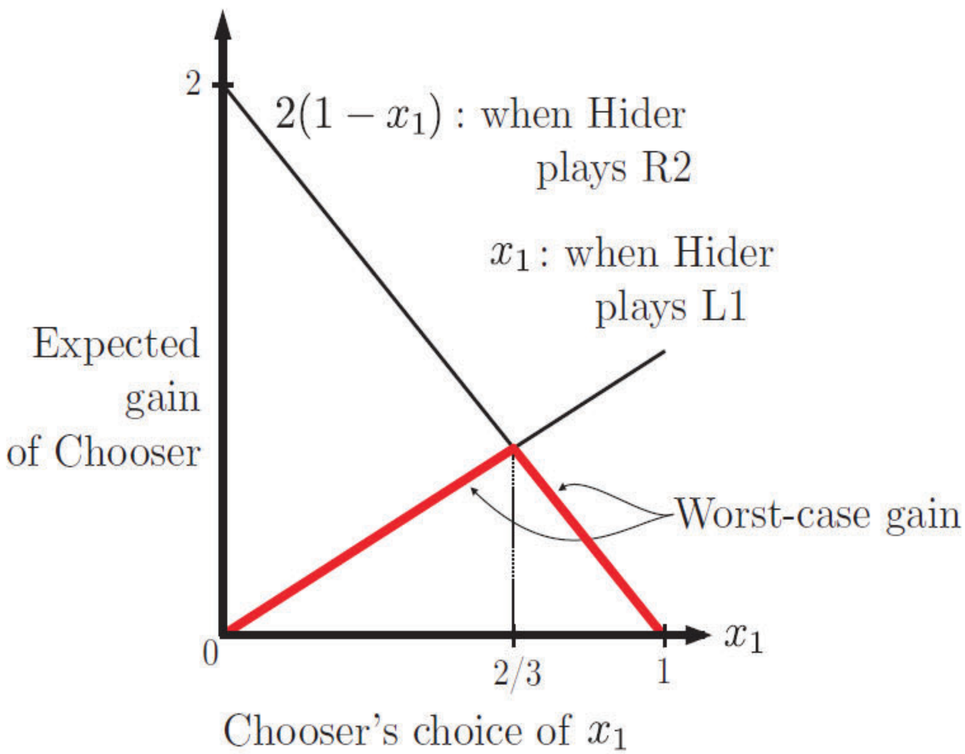
Then Hider’s expected loss is y_1 if Chooser plays L and $2(1 - y_1)$ if Chooser plays R.

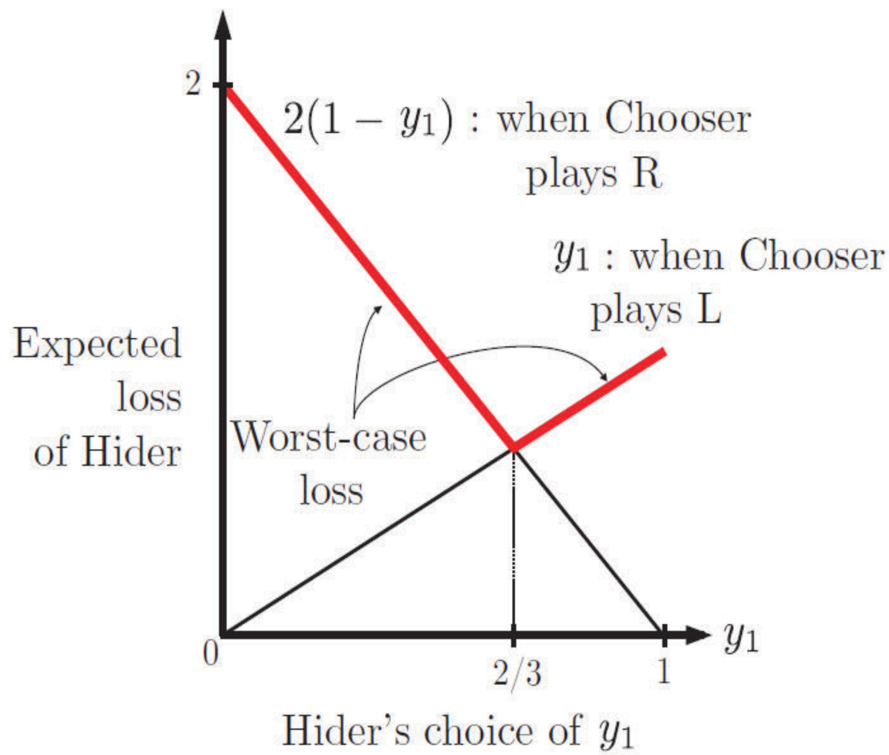
Thus Hider’s worst case expected loss is $\max\big(y_1, 2(1 - y_1)\big)$. To minimize this, the Hider should set $y_1 = \frac{2}{3}$.

From the above, no matter how Chooser plays, Hider can guarantee himself an expected loss of at most 2/3.

Pick a hand

Chooser		Hider $L1$	Hider $R2$
	L	1	0
	R	0	2





Pick a hand

		Hider	
		$L1$	$R2$
Chooser	L	1	0
	R	0	2

Similarly, we can analyze from Chooser's point of view.

Suppose that Chooser selects L with probability x_1 and R with probability $x_2 = 1 - x_1$.

What is Chooser's worst case expected gain?

Pick a hand

		Hider	
		$L1$	$R2$
Chooser	L	1	0
	R	0	2

Similarly, we can analyze from Chooser's point of view.

Suppose that Chooser selects L with probability x_1 and R with probability $x_2 = 1 - x_1$. Then Chooser's worst-case expected gain is $\min(x_1, 2(1 - x_1))$.

How can Chooser maximize her worst-case expected gain?

Pick a hand

		Hider	
		$L1$	$R2$
Chooser	L	1	0
	R	0	2

Similarly, we can analyze from Chooser's point of view.

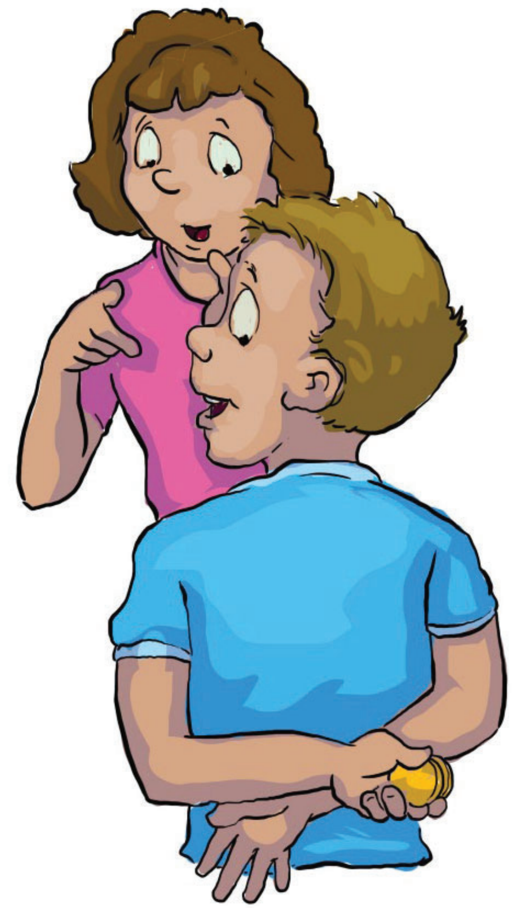
Suppose that Chooser selects L with probability x_1 and R with probability $x_2 = 1 - x_1$. Then Chooser's worst-case expected gain is $\min(x_1, 2(1 - x_1))$.

To maximize this, she will set $x_1 = \frac{2}{3}$.

Thus, no matter how Hider plays, Chooser can guarantee herself an expected gain of at least $\frac{2}{3}$.

Pick a hand

Chooser \ Hider	$L1$	$R2$
L	1	0
R	0	2



In conclusion:

Hider can only lose by playing the game.

To be interested in playing the game, Hider will need to be paid at least $\frac{2}{3}$. Conversely, Chooser should be willing to pay any sum below $\frac{2}{3}$ to play the game. Thus, we say that the value of this game is $\frac{2}{3}$.

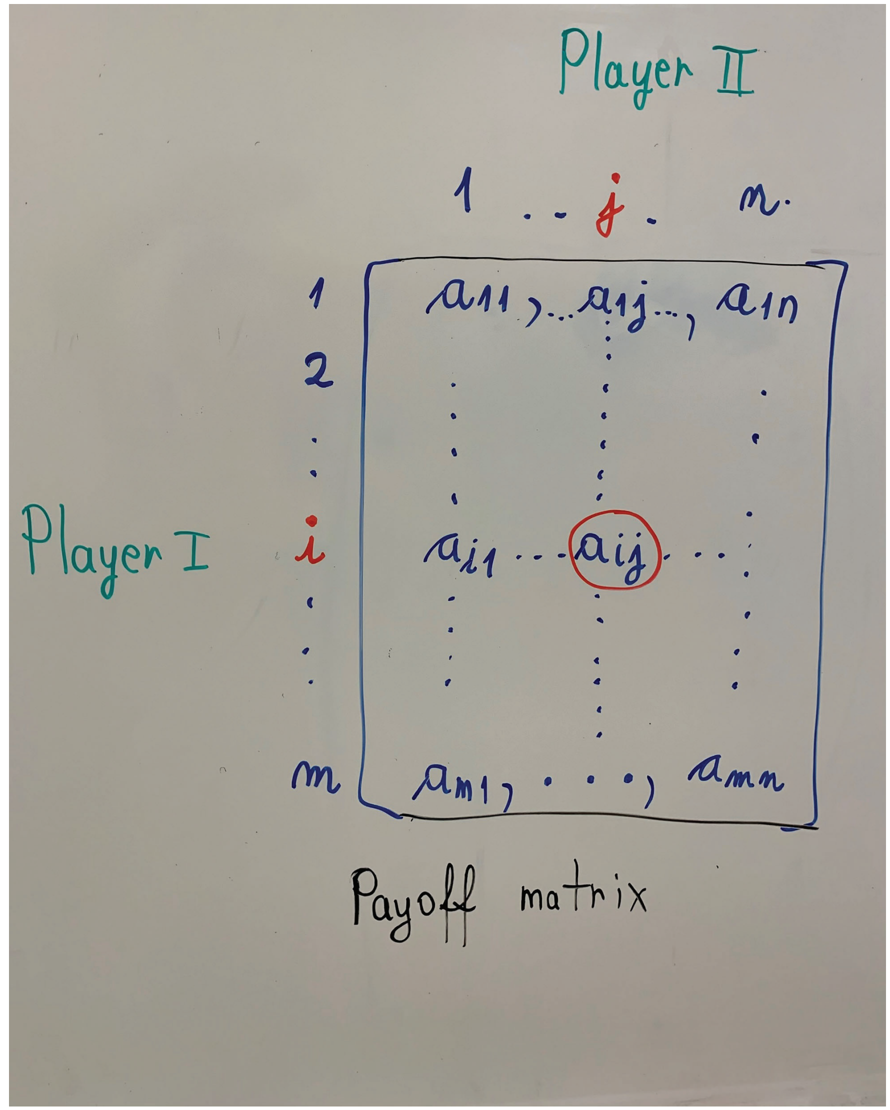
Zero-sum games

A two-person zero-sum game can be represented by an $m \times n$ payoff matrix $A = (a_{ij})$, where:

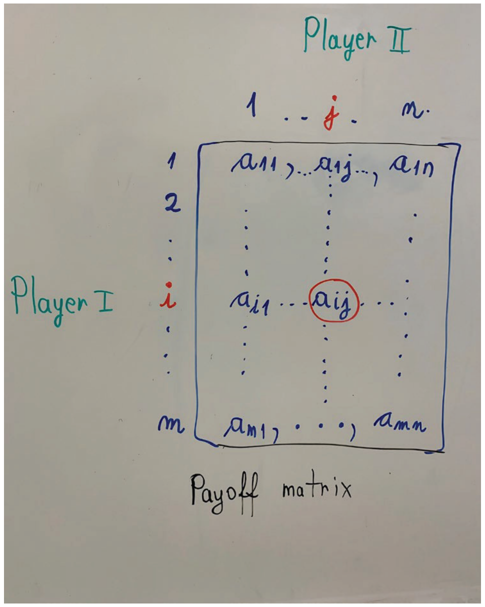
- the rows are indexed by the m possible actions of player I
- the columns are indexed by the n possible actions of player II.

Player I selects an action i and player II selects an action j, each unaware of the other's selection.

Their selections are then revealed and player II pays player I the amount a_{ij}.



Pure Strategies



If player I selects action i, in the worst case her gain will be $\min_j a_{ij}$,

and thus the largest gain she can guarantee is $\max_i \min_j a_{ij}$.

If player II selects action j, in the worst case his loss will be $\max_i a_{ij}$, and thus the smallest loss

he can guarantee is $\min_j \max_i a_{ij}$.

Player I cannot gain in the best case more than player II's smallest loss, so:

$$\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}.$$

Mixed Strategies

A strategy in which each action is selected with some probability is a mixed strategy.

A mixed strategy for player I is determined by a vector $x = (x_1, \dots, x_m)$, where x_i represents the probability of playing action i. The set of mixed strategies for player I is denoted by

$$\Delta_m = \left\{ \mathbf{x} \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \right\}$$

Similarly, the set of mixed strategies for player II is: $\Delta_n = \left\{ \mathbf{y} \in \mathbb{R}^n : y_j \geq 0, \sum_{j=1}^{n} y_j = 1 \right\}$

A mixed strategy in which a particular action is played with probability 1 is called a **pure strategy.**

Mixed Strategies

If player I uses strategy x and player II uses strategy y, then the expected gain of player I is

$$\mathbf{x}^T A\mathbf{y} = \sum_i \sum_j x_i a_{ij} y_j$$

If player I uses strategy x, she can guarantee herself an expected gain of

$$\min_{\mathbf{y}\in\Delta_n} \mathbf{x}^T A\mathbf{y} = \min_j (\mathbf{x}^T A)_j$$

Since for any vector $z \in R^n$, we have: $\min_{\mathbf{y}\in\Delta_n} \mathbf{z}^T \mathbf{y} = \min_j z_j$

Safety Strategies

A mixed strategy $x^* \in \Delta_m$ is a **safety strategy** for player I if the maximum over $x \in \Delta_m$ of the function

$$\mathbf{x} \mapsto \min_{\mathbf{y} \in \Delta_n} \mathbf{x}^T A \mathbf{y}$$

is attained at x^*. The value of this function at x^* is the **safety value** of player I.

Similar definition for player II: a mixed strategy $y^* \in \Delta_m$ is a safety strategy for player II if the maximum over $y \in \Delta_m$ of the function

$$\mathbf{y} \mapsto \max_{\mathbf{x} \in \Delta_m} \mathbf{x}^T A \mathbf{y}$$

is attained at y^*. The value of this function at y^* is the **safety value** of player II.

Minimax Theorem

Theorem (Von Neumann's Minimax Theorem). For any two-person zero-sum game with $\mathrm{m} \times n$ payoff matrix A, there is a number V, called the value of the game, satisfying

$$\max_{\mathbf{x}\in\Delta_m} \min_{\mathbf{y}\in\Delta_n} \mathbf{x}^T A\mathbf{y} = V = \min_{\mathbf{y}\in\Delta_n} \max_{\mathbf{x}\in\Delta_m} \mathbf{x}^T A\mathbf{y}.$$

John von Neumann

John von Neumann in the 1940s

Minimax Theorem

Theorem (Von Neumann's Minimax Theorem). For any two-person zero-sum game with $\mathrm{m} \times n$ payoff matrix A, there is a number V, called the value of the game, satisfying

$$\max_{\mathbf{x}\in\Delta_m} \min_{\mathbf{y}\in\Delta_n} \mathbf{x}^T A\mathbf{y} = V = \min_{\mathbf{y}\in\Delta_n} \max_{\mathbf{x}\in\Delta_m} \mathbf{x}^T A\mathbf{y}.$$

John von Neumann

John von Neumann in the 1940s

Proof: One direction is easy:

$$\max_{\mathbf{x}\in\Delta_m} \min_{\mathbf{y}\in\Delta_n} \mathbf{x}^T A\mathbf{y} \le \min_{\mathbf{y}\in\Delta_n} \max_{\mathbf{x}\in\Delta_m} \mathbf{x}^T A\mathbf{y}.$$

The other direction uses the hyperplane separator theorem.

See Game Theory Alive book, Section 2.6 (by Karlin-Peres)

Minimax Theorem

Suppose player I is trying to find a mixed strategy x that guarantees a payoff of at least V. Then it suffices to find x which guarantees her a payoff of at least V for every pure strategy that player II might play. More formally,

$$x_1 a_{1j} + x_2 a_{2j} + \ldots + x_m a_{mj} \geq V \text{ for } 1 \leq j \leq n.$$

In vector notation, where $\mathbf{e}$ is an all 1 vector, the inequalities are: $\mathbf{x}^T A \geq V \mathbf{e}^T$

Exercise: Write an LP to capture player I's expected payoff maximization problem.

Minimax Theorem

Suppose player I is trying to find a mixed strategy x that guarantees a payoff of at least V. Then it suffices to find x which guarantees her a payoff of at least V for every pure strategy that player II might play. More formally,

$$x_1 a_{1j} + x_2 a_{2j} + \ldots + x_m a_{mj} \geq V \text{ for } 1 \leq j \leq n.$$

In vector notation, where **e** is an all 1 vector, the inequalities are: $\mathbf{x}^T A \geq V\mathbf{e}^T$

To maximize her expected payoff, player I should find x ∈ R^m and V ∈ R that

$$\begin{aligned} & \text{maximize } V \\ \text{subject to} \quad & x^T A \geq V\mathbf{e}^T \\ & \sum_{1 \leq i \leq m} x_i = 1 \\ & x_i \geq 0 \text{ for all } 1 \leq i \leq m. \end{aligned}$$

Minimax Theorem

Exercise: Write an LP to capture player II's expected payoff maximization problem.

Minimax Theorem

To maximize her expected payoff, player II should find $y \in R^n$ and $V \in R$ that

$$\begin{aligned} & \text{minimize } V \\ \text{subject to} \quad & A\mathbf{y} \leq V\mathbf{e} \\ & \sum_{1 \leq j \leq n} y_j = 1 \\ & y_j \geq 0 \text{ for all } 1 \leq j \leq n. \end{aligned}$$

Minimax Theorem

To maximize her expected payoff, player I should find $x \in R^m$ and $V \in R$ that

$$
\begin{aligned}
& \text{maximize } V \\
\text{subject to} \quad & x^T A \geq V \mathbf{e}^T \\
& \sum_{1 \leq i \leq m} x_i = 1 \\
& x_i \geq 0 \text{ for all } 1 \leq i \leq m.
\end{aligned}
$$

To maximize her expected payoff, player II should find $y \in R^n$ and $V \in R$ that

$$
\begin{aligned}
& \text{minimize } V \\
\text{subject to} \quad & A\mathbf{y} \leq V \mathbf{e} \\
& \sum_{1 \leq j \leq n} y_j = 1 \\
& y_j \geq 0 \text{ for all } 1 \leq j \leq n.
\end{aligned}
$$

Minimax Theorem

To maximize her expected payoff, player I should find $x \in R^m$ and $V \in R$ that

value of the game

$$\begin{aligned} &\text{maximize } V \\ \text{subject to} \quad & x^T A \geq V \mathbf{e}^T \\ & \sum_{1 \leq i \leq m} x_i = 1 \\ & x_i \geq 0 \text{ for all } 1 \leq i \leq m. \end{aligned}$$

primal LP

To maximize her expected payoff, player II should find $y \in R^n$ and $V \in R$ that

$$\begin{aligned} &\text{minimize } V \\ \text{subject to} \quad & A\mathbf{y} \leq V \mathbf{e} \\ & \sum_{1 \leq j \leq n} y_j = 1 \\ & y_j \geq 0 \text{ for all } 1 \leq j \leq n. \end{aligned}$$

dual LP

Strong duality guarantees these LPs have same OPT value

Minimax Theorem

Theorem (Von Neumann's Minimax Theorem). For any two-person zero-sum game with $m \times n$ payoff matrix A, there is a number V, called the value of the game, satisfying

$$\max_{\mathbf{x}\in\Delta_m} \min_{\mathbf{y}\in\Delta_n} \mathbf{x}^T A\mathbf{y} = V = \min_{\mathbf{y}\in\Delta_n} \max_{\mathbf{x}\in\Delta_m} \mathbf{x}^T A\mathbf{y}.$$

Minimax theorem played a pivotal role in developing linear programming.

John von Neumann

John von Neumann in the 1940s

Minimax Theorem

Theorem (Von Neumann's Minimax Theorem). For any two-person zero-sum game with $m \times n$ payoff matrix A, there is a number V, called the value of the game, satisfying

$$\max_{\mathbf{x} \in \Delta_m} \min_{\mathbf{y} \in \Delta_n} \mathbf{x}^T A \mathbf{y} = V = \min_{\mathbf{y} \in \Delta_n} \max_{\mathbf{x} \in \Delta_m} \mathbf{x}^T A \mathbf{y}.$$

John von Neumann

John von Neumann in the 1940s

Minimax theorem played a pivotal role in developing linear programming. **In fact, zero-sum games and linear programming are equivalent!**

- Given any zero-sum game, we can write the problem of computing optimal strategies as a linear program (and its dual),
- Given any LP, we can write a zero-sum game

(the equilibrium of the game will correspond to optimal solutions of the LP).

Known for:

Abelian von Neumann algebra
Affiliated operator
Amenable group
Arithmetic logic unit
Artificial viscosity
Axiom of regularity
Axiom of limitation of size
Backward induction
Birkhoff–von Neumann theorem
Blast wave (fluid dynamics)
Taylor–von Neumann–Sedov blast wave
Bounded set (topological vector space)
Carry-save adder
Cellular automata
Class (set theory)
Computer virus
Commutation theorem
Continuous geometry
Coupling constants
Decoherence theory (quantum mechanics)
Density matrix
Dirac–von Neumann axioms
Direct integral
Doubly stochastic matrix
Duality Theorem
Durbin–Watson statistic
EDVAC
Enveloping von Neumann algebra
Ergodic theory
Finite von Neumann algebra
Explosive lenses
Game theory
Hilbert's fifth problem
Hyperfinite type II factor
Inner model
Inner model theory
Interior point method
John von Neumann (sculpture)
Koopman–von Neumann classical mechanics
Lattice theory
Lifting theory
Merge sort
Middle-square method
Minimax theorem
Monte Carlo method
Mutual assured destruction
Normal-form game
Operation Greenhouse
Operator theory
Pointless topology
Polarization identity
Pseudorandomness
Pseudorandom number generator
Quantum logic
Quantum mutual information
Quantum statistical mechanics
Radiation implosion
Rank ring
Self-replication
Software whitening
Sorted array
Spectral theory
Standard probability space
Stochastic computing
Stone–von Neumann theorem
Subfactor
Ultrastrong topology
Von Neumann algebra
Von Neumann architecture
Von Neumann bicommutant theorem
Von Neumann cardinal assignment
Von Neumann cellular automaton
Von Neumann–Wigner interpretation
Von Neumann measurement scheme
Von Neumann ordinals
Von Neumann universal constructor
Von Neumann entropy
Von Neumann equation
Von Neumann neighborhood
Von Neumann paradox
Von Neumann probes
Von Neumann programming languages
Von Neumann regular ring
Von Neumann–Bernays–Gödel set theory
Von Neumann universe
Von Neumann spectral theorem
Von Neumann conjecture

John von Neumann

John von Neumann in the 1940s

Von Neumann's inequality
Von Neumann's theorem
Von Neumann's trace inequality
Von Neumann stability analysis
Von Neumann extractor
Von Neumann ergodic theorem
Von Neumann–Morgenstern utility theorem
Weyl–von Neumann theorem
Wold–von Neumann decomposition
ZND detonation model

The Minimax Theorem played a key role in the development of linear programming. George Dantzig, one of the pioneers of linear programming, relays the following story about his first meeting with John von Neumann [Dan82].

Von Neumann explaining duality to Dantzig.

On October 3, 1947, I visited him (von Neumann) for the first time at the Institute for Advanced Study at Princeton. I remember trying to describe to von Neumann, as I would to an ordinary mortal, the Air Force problem. I began with the formulation of the linear programming model in terms of activities and items, etc. Von Neumann did something which I believe was uncharacteristic of him. "Get to the point," he said impatiently. Having at times a somewhat low kindling-point, I said to myself "O.K., if he wants a quicky, then thats what he will get." In under one minute I slapped the geometric and algebraic version of the problem on the blackboard. Von Neumann stood up and said "Oh that!" Then for the next hour and a half, he proceeded to give me a lecture on the mathematical theory of linear programs.

At one point seeing me sitting there with my eyes popping and my mouth open (after I had searched the literature and found nothing), von Neumann said: "I don't want you to think I am pulling all this out of my sleeve at the spur of the moment like a magician. I have just recently completed a book with Oskar Morgenstern on the theory of games. What I am doing is conjecturing that the two problems are equivalent. The theory that I am outlining for your problem is an analogue to the one we have developed for games." Thus I learned about Farkas Lemma, and about duality for the first time.

Solving zero-sum games

		player II	
		action 1	action 2
player I	action 1	2	3
	action 2	1	0

Solving zero-sum games

		player II	
		action 1	action 2
player I	action 1	2	3
	action 2	1	0

Strategy pair $(1,1)$ is an optimal pure strategy pair.

Definition: A **saddle point** of a payoff matrix A is a pair (i^*, j^*) such that

$$\max_i a_{ij^*} = a_{i^*j^*} = \min_j a_{i^*j}$$

What can we infer if the payoff matrix A has a saddle point?

Solving zero-sum games

		player II	
		action 1	action 2
player I	action 1	2	3
	action 2	1	0

Strategy pair $(1,1)$ is an optimal pure strategy pair.

Definition: A **saddle point** of a payoff matrix A is a pair (i^*, j^*) such that

$$\max_i a_{ij^*} = a_{i^*j^*} = \min_j a_{i^*j}$$

Observation: If (i^*, j^*) is a saddle point, then $a_{i^*j^*}$ is the value of the game. A saddle point is also called a pure Nash equilibrium.

Solving zero-sum games

What about the following game?

		player II	
		y_1	$1 - y_1$
player I	x_1	3	0
	$1 - x_1$	1	4

Solving zero-sum games

Some games have a pair of strategies (x^*, y^*) of optimal strategies that are fully mixed, i.e. where each player plays each action with strictly positive probability.

		player II	
		y_1	$1-y_1$
player I	x_1	3	0
	$1-x_1$	1	4

Solving zero-sum games

Some games have a pair of strategies (x^*, y^*) of optimal strategies that are fully mixed, i.e. where each player plays each action with strictly positive probability.

If so, player I must get, when playing against y^*, the same payoff from each strategy it is playing.

		player II	
		y_1	$1 - y_1$
player I	x_1	3	0
	$1 - x_1$	1	4

Solving zero-sum games

Some games have a pair of strategies (x^*, y^*) of optimal strategies that are fully mixed, i.e. where each player plays each action with strictly positive probability.

If so, player I must get, when playing against y^*, the same payoff from each strategy it is playing. *[Otherwise, if say, $(Ay^*)_1 > (Ay^*)_2$, then player I could increase her payoff by moving probability from action 2 to action 1 => contradicts optimality of x^*.]*

		player II	
		y_1	$1 - y_1$
player I	x_1	3	0
	$1 - x_1$	1	4

Solving zero-sum games

Some games have a pair of strategies (x^*, y^*) of optimal strategies that are fully mixed.

If so, then player I must get, when playing against y^*, the same payoff from each strategy it is playing.

Equalize the gains for player I's actions. We get: $3y_1 = y_1 + 4(1 - y_1)$. Then $y_1 = \frac{2}{3}$.

Thus if player II plays $\left(\frac{2}{3}, \frac{1}{3}\right)$, his loss will not depend on player I's actions (it will be 2 no matter what I does).

Solving zero-sum games

Some games have a pair of strategies (x^*, y^*) of optimal strategies that are fully mixed.

If so, then player I must get, when playing against y^*, the same payoff from each strategy it is playing.

Equalize the gains for player I's actions. We get: $3y_1 = y_1 + 4(1 - y_1)$. Then $y_1 = \frac{2}{3}$.

Thus if player II plays $\left(\frac{2}{3}, \frac{1}{3}\right)$, his loss will not depend on player I's actions (it will be 2 no matter what I does).

Similarly, equalizing the losses for player II's actions, we get

$$3x_1 + (1 - x_1) = 4(1 - x_1)$$

		player II	
		y_1	$1 - y_1$
player I	x_1	3	0
	$1 - x_1$	1	4

Thus $x_1 = \frac{1}{2}$. Then if player I plays $\left(\frac{1}{2}, \frac{1}{2}\right)$, her gain will not depend on player II's actions => it will be 2 no matter what II does. Thus 2 is the value of the game.

Nash Equilibrium

A mixed strategy pair (x^*, y^*), where $\mathrm{x}^* \in \Delta_m$ and $\mathrm{y}^* \in \Delta_n$ is a **Nash equilibrium** if no player can improve their expected payoff by unilaterally deviating. That is, x^* is a best response to y^* and viceversa:

$$\min_{\mathbf{y}\in\Delta_n} (\mathbf{x}^*)^T A\mathbf{y} = (\mathbf{x}^*)^T A\mathbf{y}^* = \max_{\mathbf{x}\in\Delta_m} \mathbf{x}^T A\mathbf{y}^*$$

Observation: The safe strategies in zero-sum games are in Nash equilibrium.

Minimax Theorem for 2X2 Games

Exercise: Prove the Minimax Theorem for 2X2 Games.

More precisely, the following stronger statement holds: any 2 X 2 game (i.e., a game in which each player has exactly two strategies) has a pair of optimal strategies that are both pure or both fully mixed.

		Player II	
		L	R
Player I	T	a	b
	B	c	d