CS 381: Turing machines

1 Turing Machines

The word ”computer” dates back to 1613 (in written records) and used to mean “someone who computes”: a person performing mathematical calculations, before electronic computers existed. The idea was that “the human computer is supposed to be following fixed rules; he has no authority to deviate from them in any detail.” Groups of people were often employed to undertake long and tedious calculations, with the work divided so it could be finished in parallel. (see [wik]).

1.1 Definition

A Turing Machine (TM) consists of:

- An infinite tape. The cells of the tape are numbered $1, 2, \dots$ as a convenience when proving properties of a TM. These indexes are not known to the TM. [Memory of a Computer].
- A two-way read/write head [Program Counter: i.e. the processor register that indicates where a computer is in its program sequence.]
- Finite control (defined by a set of states Q and a transition function δ) [The Program].

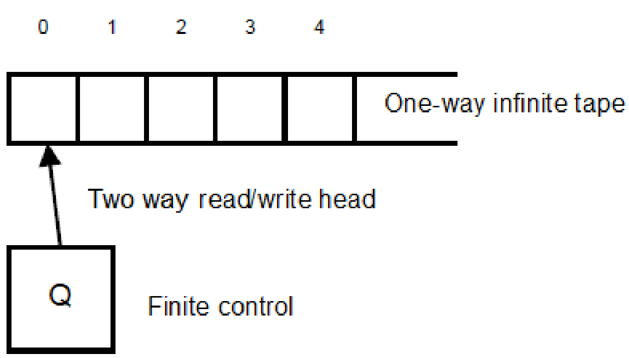

Figure 1: Turing machine

You will see some definitions where the tape is two-way (extends infinitely in both directions). These definitions do not matter (the model is robust – the classes of things computed by these different models are the same].

It is defined by:

- Q is the set of states.
- Σ is the input alphabet (not containing the blank symbol $\square$).
- Γ is the tape alphabet, with $\square \in \Gamma$, $\Sigma \in \Gamma$.
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function.
- $s \in Q$: Start state.
- "yes" $\in Q$: Accept state.
- "no" $\in Q$: Reject state.

Initially:

- The input $x_1 \dots x_n$ is placed on cells $1 \dots n$.
- All other cells contain the blank symbol $\square$. Head starts at position 1.

Transition details: $\delta(q, a) = (q', b, L)$ then:

- If the machine is in state q and the head reads symbol a, then the state changes to q', the machine writes the symbol b in the place of a, and the head moves left one cell.

How it works (high level):

- Given the input, the machine computes following the rules induced by the transition function. Computation continues until it enters the accept or reject states (then it halts). Otherwise, it goes on forever (Have you written a program with an infinite loop?).

1.2 Language of a TM; Recognizable, Decidable, and Undecidable Languages

The collection of strings that a Turing machine M accepts represents the **language of M** (or the language recognized by M), denoted by $L(M)$.

Definition 1 (Turing recognizable languages)**.** *A language A is Turing-recognizable if some Turing machine M recognizes it (i.e. $L(M) = A$). [aka recursive enumerable language]*

Definition 2 (Deciders)**.** *A machine can fail to accept a string in two ways: actually end in a rejecting configuration, or loop. A Turing machine that always halts is called a* ***decider****.*

Definition 3 (Turing decidable languages)**.** *A language A is (Turing-)decidable if some Turing machine M decides it (i.e. $L(M) = A$ and M always halts). [aka recursive language]*

Definition 4 (Undecidable language)**.** *We say a decision problem (i.e. where the question is given an input string, to output yes or no) is undecidable if there is no Turing machine that always halts with a correct yes/no answer (similar for decidable).*

Definition 5 (Time complexity)**.** *We denote by $time_M(x) = \#$ steps that M takes to halt on input x.*

2 Examples

We will skip the very low level details. Take a look at page 165 of Sipser for a complete Turing machine.

Problem 1. *Find a TM for deciding* $L = \{0^{2^n} \mid n \geq 0\}$.

Proof. Here is a Turing machine that decides the language:

1. Move the head across the tape, from left to right, crossing off every other 0.
2. If at step 1 the tape contained only one 0, accept.
3. If at step 1 the tape contained more than one 0 and the number was odd, reject.
4. Return the head to the left end of the tape.
5. Go to step 1.

$\square$

Reading: More details can be found in Chapter 3 of Sipser [Sip12].

References

[Sip12] Michael Sipser. *Introduction to the theory of computation.* Cengage Learning, 2012.

[wik] Human computer. https://en.wikipedia.org/wiki/Human_computer. Online; accessed Jan 2018.