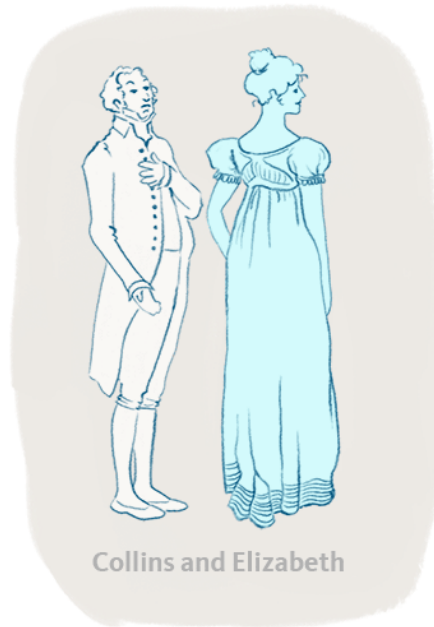


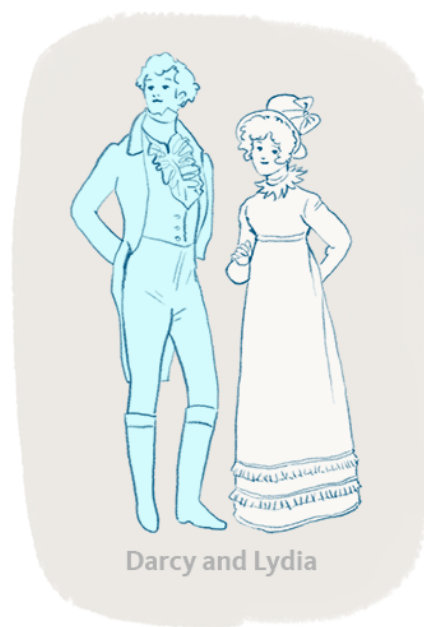
CS 381

# 1.1 A First Problem: Stable Matching

---



Collins and Elizabeth



Darcy and Lydia

**An unstable match:**

ELIZABETH AND DARCY LIKE EACH OTHER BETTER THAN THEIR PARTNERS

## REVIEW: Matching Residents to Hospitals

**Goal.** Given a set of preferences among hospitals and medical school students, design a **self-reinforcing** admissions process.

**Unstable pair:** applicant  $x$  and hospital  $y$  are **unstable** if:

- $x$  prefers  $y$  to its assigned hospital.
- $y$  prefers  $x$  to one of its admitted students.

**Stable assignment.** Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

# Stable Matching Problem

**Goal.** Given  $n$  men and  $n$  women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	favorite ↓ 1 <sup>st</sup>	2 <sup>nd</sup>	least favorite ↓ 3 <sup>rd</sup>
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓ 1 <sup>st</sup>	2 <sup>nd</sup>	least favorite ↓ 3 <sup>rd</sup>
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

# Stable Matching Problem

**Perfect matching:** everyone is matched 1-1.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

**Stability:** no incentive for some pair of participants to undermine assignment by joint action.

- In matching  $M$ , an unmatched pair  $m$ - $w$  is **unstable** if man  $m$  and woman  $w$  prefer each other to current partners.
- Unstable pair  $m$ - $w$  could each improve by eloping.

**Stable matching:** perfect matching with no unstable pairs.

**Stable matching problem.** Given the preference lists of  $n$  men and  $n$  women, find a stable matching if one exists.

# Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	favorite ↓		least favorite ↓
	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓		least favorite ↓
	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

## Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A. No. Bertha and Xavier will prefer each other to their current partners.

	favorite ↓ 1 <sup>st</sup>	2 <sup>nd</sup>	least favorite ↓ 3 <sup>rd</sup>
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓ 1 <sup>st</sup>	2 <sup>nd</sup>	least favorite ↓ 3 <sup>rd</sup>
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

## Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

	favorite ↓ 1 <sup>st</sup>		least favorite ↓ 3 <sup>rd</sup>
	2 <sup>nd</sup>		
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓ 1 <sup>st</sup>		least favorite ↓ 3 <sup>rd</sup>
	2 <sup>nd</sup>		
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile



## Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

↖ is core of market nonempty?

Stable roommate problem.

- $2n$  people; each person ranks others from 1 to  $2n-1$ .
- Assign roommate pairs so that no unstable pairs.

## Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

↖ is core of market nonempty?

Stable roommate problem.

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D  $\Rightarrow$  B-C unstable  
A-C, B-D  $\Rightarrow$  A-B unstable  
A-D, B-C  $\Rightarrow$  A-C unstable

## Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

↖ is core of market nonempty?

Stable roommate problem.

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D  $\Rightarrow$  B-C unstable  
A-C, B-D  $\Rightarrow$  A-B unstable  
A-D, B-C  $\Rightarrow$  A-C unstable

**Observation.** Stable matchings do not always exist for stable roommate problem.

# Gale-Shapley Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

# Gale-Shapley Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
```

```
while (some man is free and hasn't proposed to every woman) {  
    Choose such a man m  
    w = most preferred woman on m's list to whom m has not yet proposed  
    if (w is free)  
        assign m and w to be engaged  
    else if (w prefers m to her fiancé m')  
        assign m and w to be engaged, and m' to be free  
    else  
        w rejects m  
}
```

# Analysis

What properties does this algorithm have?



## Proof of Correctness: Termination



## Proof of Correctness: Termination

**Observation 1.** Men propose to women in decreasing order of preference.

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z





## Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Can a woman become unmatched at some point?

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z



## Proof of Correctness: Termination

**Observation 1.** Men propose to women in decreasing order of preference.

**Observation 2.** Once a woman is matched, she never becomes unmatched; she only "trades up."

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

## Proof of Correctness: Termination

**Observation 1.** Men propose to women in decreasing order of preference.

**Observation 2.** Once a woman is matched, she never becomes unmatched; she only "trades up."

**Claim.** Algorithm terminates after at most  $n^2$  iterations of while loop.

**Pf.** Each time through the while loop a man proposes to a new woman.

There are only  $n^2$  possible proposals. ▀

**Question:** Is there a better asymptotic analysis? **A:** No. Input below requires  $n(n-1) + 1$  proposals

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

Tightness:  $n(n-1) + 1$  proposals required

## Proof of Correctness: Perfection

How many people remain unmatched in the worst case?

## Proof of Correctness: Perfection

**Claim.** All men and women get matched.

## Proof of Correctness: Perfection

**Claim.** All men and women get matched.

**Pf.** (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▫

## Proof of Correctness: Stability

**Claim.** No unstable pairs.

## Proof of Correctness: Stability

**Claim.** No unstable pairs.

**Pf.** (by contradiction)

- Suppose  $A-Z$  is an unstable pair: each prefers each other to partner in Gale-Shapley matching  $S^*$ .



## Proof of Correctness: Stability

**Claim.** No unstable pairs.

**Pf.** (by contradiction)

- Suppose  $A-Z$  is an unstable pair: each prefers each other to partner in Gale-Shapley matching  $S^*$ .
- Case 1:  $Z$  never proposed to  $A$ .

$S^*$

Amy-Yancey

Bertha-Zeus

...

# Proof of Correctness: Stability

**Claim.** No unstable pairs.

**Pf.** (by contradiction)

- Suppose  $A-Z$  is an unstable pair: each prefers each other to partner in Gale-Shapley matching  $S^*$ .

- Case 1:  $Z$  never proposed to  $A$ .
  - $\Rightarrow Z$  prefers his GS partner to  $A$ .
  - $\Rightarrow A-Z$  is stable.

men propose in decreasing  
order of preference

$S^*$

Amy-Yancey

Bertha-Zeus

...

# Proof of Correctness: Stability

**Claim.** No unstable pairs.

**Pf.** (by contradiction)

- Suppose  $A-Z$  is an unstable pair: each prefers each other to partner in Gale-Shapley matching  $S^*$ .

- Case 1:  $Z$  never proposed to  $A$ .
  - $\Rightarrow Z$  prefers his GS partner to  $A$ .
  - $\Rightarrow A-Z$  is stable.

men propose in decreasing  
order of preference

$S^*$

Amy-Yancey

Bertha-Zeus

...

- Case 2:  $Z$  proposed to  $A$ .

# Proof of Correctness: Stability

**Claim.** No unstable pairs.

**Pf.** (by contradiction)

- Suppose  $A-Z$  is an unstable pair: each prefers each other to partner in Gale-Shapley matching  $S^*$ .

- Case 1:  $Z$  never proposed to  $A$ .
  - $\Rightarrow Z$  prefers his GS partner to  $A$ .
  - $\Rightarrow A-Z$  is stable.

men propose in decreasing  
order of preference

$S^*$

Amy-Yancey

Bertha-Zeus

...

- Case 2:  $Z$  proposed to  $A$ .
  - $\Rightarrow A$  rejected  $Z$  (right away or later)
  - $\Rightarrow A$  prefers her GS partner to  $Z$ .
  - $\Rightarrow A-Z$  is stable.

← women only trade up

# Proof of Correctness: Stability

**Claim.** No unstable pairs.

**Pf.** (by contradiction)

- Suppose  $A$ - $Z$  is an unstable pair: each prefers each other to partner in Gale-Shapley matching  $S^*$ .

- Case 1:  $Z$  never proposed to  $A$ .
  - $\Rightarrow Z$  prefers his GS partner to  $A$ .
  - $\Rightarrow A$ - $Z$  is stable.

men propose in decreasing  
order of preference

$S^*$

Amy-Yancey

Bertha-Zeus

...

- Case 2:  $Z$  proposed to  $A$ .
  - $\Rightarrow A$  rejected  $Z$  (right away or later)
  - $\Rightarrow A$  prefers her GS partner to  $Z$ . ← women only trade up
  - $\Rightarrow A$ - $Z$  is stable.

- In either case  $A$ - $Z$  is stable, a contradiction. ▫

# Summary

**Stable matching problem.** Given  $n$  men and  $n$  women, and their preferences, find a stable matching if one exists.

**Gale-Shapley algorithm.** Guarantees to find a stable matching for **any** problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?



# Efficient Implementation

Efficient implementation. We describe  $O(n^2)$  time implementation.

Representing men and women.

- Assume men are named  $1, \dots, n$ .
- Assume women are named  $1', \dots, n'$ .

Engagements.

- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - set entry to 0 if unmatched
  - if  $m$  matched to  $w$  then `wife[m]=w` and `husband[w]=m`

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man  $m$ .

# Efficient Implementation

## Women rejecting/accepting.

- Does woman  $w$  prefer man  $m$  to man  $m'$ ?
- For each woman, create **inverse** of preference list of men.
- Constant time access for each query after  $O(n)$  preprocessing.

Amy	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>	6 <sup>th</sup>	7 <sup>th</sup>	8 <sup>th</sup>
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4 <sup>th</sup>	8 <sup>th</sup>	2 <sup>nd</sup>	5 <sup>th</sup>	6 <sup>th</sup>	7 <sup>th</sup>	3 <sup>rd</sup>	1 <sup>st</sup>

```
for i = 1 to n
    inverse[pref[i]] = i
```

Amy prefers man 3 to 6  
since  $\text{inverse}[3] < \text{inverse}[6]$   
2                      7



## Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of *Gale-Shapley* yield the same stable matching? If so, which one?

## Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z



## Understanding the Solution

**Q.** For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

**Def.** Man  $m$  is a **valid partner** of woman  $w$  if there exists some stable matching in which they are matched.



## Understanding the Solution

**Q.** For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

**Def.** Man  $m$  is a **valid partner** of woman  $w$  if there exists some stable matching in which they are matched.

**Man-optimal assignment.** Each man receives best valid partner.

**Claim.** All executions of GS yield **man-optimal** assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

## Man Optimality

**Claim.** GS matching  $S^*$  is man-optimal.

# Man Optimality

**Claim.** GS matching  $S^*$  is man-optimal.

**Pf.** (by contradiction)

- Suppose some man is paired with someone other than best partner.  
Men propose in decreasing order of preference  $\Rightarrow$  some man is rejected by valid partner.

S

Amy-Yancey

Bertha-Zeus

...

# Man Optimality

**Claim.** GS matching  $S^*$  is man-optimal.

**Pf.** (by contradiction)

- Suppose some man is paired with someone other than best partner.  
Men propose in decreasing order of preference  $\Rightarrow$  some man is rejected by valid partner.
- Let  $Y$  be **first** such man, and let  $A$  be **first valid** woman that rejects him.

$S$

Amy-Yancey

Bertha-Zeus

...

# Man Optimality

**Claim.** GS matching  $S^*$  is man-optimal.

**Pf.** (by contradiction)

- Suppose some man is paired with someone other than best partner.  
Men propose in decreasing order of preference  $\Rightarrow$  some man is rejected by valid partner.
- Let  $Y$  be **first** such man, and let  $A$  be **first valid** woman that rejects him.
- Let  $S$  be a stable matching where  $A$  and  $Y$  are matched.

$S$	
	Amy-Yancey
	Bertha-Zeus
	...



# Man Optimality

**Claim.** GS matching  $S^*$  is man-optimal.

**Pf.** (by contradiction)

- Suppose some man is paired with someone other than best partner.  
Men propose in decreasing order of preference  $\Rightarrow$  some man is rejected by valid partner.
- Let  $Y$  be **first** such man, and let  $A$  be **first valid** woman that rejects him.
- Let  $S$  be a stable matching where  $A$  and  $Y$  are matched.
- When  $Y$  is rejected,  $A$  forms (or reaffirms) engagement with a man, say  $Z$ , whom she prefers to  $Y$ .

$S$	
	Amy-Yancey
	Bertha-Zeus
	...

# Man Optimality

**Claim.** GS matching  $S^*$  is man-optimal.

**Pf.** (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference  $\Rightarrow$  some man is rejected by valid partner.
- Let  $Y$  be **first** such man, and let  $A$  be **first valid** woman that rejects him.
- Let  $S$  be a stable matching where  $A$  and  $Y$  are matched.
- When  $Y$  is rejected,  $A$  forms (or reaffirms) engagement with a man, say  $Z$ , whom she prefers to  $Y$ .
- Let  $B$  be  $Z$ 's partner in  $S$ .

$S$	
	Amy-Yancey
	Bertha-Zeus
	...

# Man Optimality

**Claim.** GS matching  $S^*$  is man-optimal.

**Pf.** (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference  $\Rightarrow$  some man is rejected by valid partner.
- Let  $Y$  be **first** such man, and let  $A$  be **first valid** woman that rejects him.
- Let  $S$  be a stable matching where  $A$  and  $Y$  are matched.
- When  $Y$  is rejected,  $A$  forms (or reaffirms) engagement with a man, say  $Z$ , whom she prefers to  $Y$ .
- Let  $B$  be  $Z$ 's partner in  $S$ .
- $Z$  not rejected by any valid partner at the point when  $Y$  is rejected by  $A$ . Thus,  $Z$  prefers  $A$  to  $B$ .

$S$	
	Amy-Yancey
	Bertha-Zeus
	...

↑  
since this is first rejection  
by a valid partner

# Man Optimality

**Claim.** GS matching  $S^*$  is man-optimal.

**Pf.** (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference  $\Rightarrow$  some man is rejected by valid partner.
- Let  $Y$  be **first** such man, and let  $A$  be **first valid** woman that rejects him.
- Let  $S$  be a stable matching where  $A$  and  $Y$  are matched.
- When  $Y$  is rejected,  $A$  forms (or reaffirms) engagement with a man, say  $Z$ , whom she prefers to  $Y$ .
- Let  $B$  be  $Z$ 's partner in  $S$ .
- $Z$  not rejected by any valid partner at the point when  $Y$  is rejected by  $A$ . Thus,  $Z$  prefers  $A$  to  $B$ .
- Also  $A$  prefers  $Z$  to  $Y$ .

$S$	
	Amy-Yancey
	Bertha-Zeus
	...

↑  
since this is first rejection  
by a valid partner

# Man Optimality

**Claim.** GS matching  $S^*$  is man-optimal.

**Pf.** (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference  $\Rightarrow$  some man is rejected by valid partner.
- Let  $Y$  be **first** such man, and let  $A$  be **first valid** woman that rejects him.
- Let  $S$  be a stable matching where  $A$  and  $Y$  are matched.
- When  $Y$  is rejected,  $A$  forms (or reaffirms) engagement with a man, say  $Z$ , whom she prefers to  $Y$ .
- Let  $B$  be  $Z$ 's partner in  $S$ .
- $Z$  not rejected by any valid partner at the point when  $Y$  is rejected by  $A$ . Thus,  $Z$  prefers  $A$  to  $B$ .
- Also  $A$  prefers  $Z$  to  $Y$ .
- Thus  $A$ - $Z$  is unstable in  $S$ . ▀

$S$	
	Amy-Yancey
	Bertha-Zeus
	...

↑  
since this is first rejection  
by a valid partner

# Stable Matching Summary

**Stable matching problem.** Given preference profiles of  $n$  men and  $n$  women, find a **stable** matching.

↖  
no man and woman prefer to be with each other than assigned partner

**Gale-Shapley algorithm.** Finds a stable matching in  $O(n^2)$  time.

**Man-optimality.** In version of *GS* where men propose, each man receives best valid partner.

↖  
 $w$  is a valid partner of  $m$  if there exist some stable matching where  $m$  and  $w$  are paired

**Q.** Does man-optimality come at the expense of the women?

## Woman Pessimality

**Woman-pessimal assignment.** Each woman receives worst valid partner.

**Claim.** GS finds **woman-pessimal** stable matching  $S^*$ .

# Woman Pessimality

**Woman-pessimal assignment.** Each woman receives worst valid partner.

**Claim.** GS finds **woman-pessimal** stable matching  $S^*$ .

**Pf.**

- Suppose  $A$ - $Z$  matched in  $S^*$ , but  $Z$  is not worst valid partner for  $A$ .

$S$	
	Amy-Yancey
	Bertha-Zeus
	...



# Woman Pessimality

**Woman-pessimal assignment.** Each woman receives worst valid partner.

**Claim.** GS finds **woman-pessimal** stable matching  $S^*$ .

**Pf.**

- Suppose  $A$ - $Z$  matched in  $S^*$ , but  $Z$  is not worst valid partner for  $A$ .
- There exists stable matching  $S$  in which  $A$  is paired with a man, say  $Y$ , whom she likes less than  $Z$ .

$S$	
Amy	Yancey
Bertha	Zeus
...	

# Woman Pessimality

**Woman-pessimal assignment.** Each woman receives worst valid partner.

**Claim.** GS finds **woman-pessimal** stable matching  $S^*$ .

**Pf.**

- Suppose  $A$ - $Z$  matched in  $S^*$ , but  $Z$  is not worst valid partner for  $A$ .
- There exists stable matching  $S$  in which  $A$  is paired with a man, say  $Y$ , whom she likes less than  $Z$ .
- Let  $B$  be  $Z$ 's partner in  $S$ .

$S$	
Amy	Yancey
Bertha	Zeus
...	

# Woman Pessimality

**Woman-pessimal assignment.** Each woman receives worst valid partner.

**Claim.** GS finds **woman-pessimal** stable matching  $S^*$ .

**Pf.**

- Suppose  $A$ - $Z$  matched in  $S^*$ , but  $Z$  is not worst valid partner for  $A$ .
- There exists stable matching  $S$  in which  $A$  is paired with a man, say  $Y$ , whom she likes less than  $Z$ .
- Let  $B$  be  $Z$ 's partner in  $S$ .
- $Z$  prefers  $A$  to  $B$ . ← man-optimality

$S$	
Amy	Yancey
Bertha	Zeus
...	

# Woman Pessimality

**Woman-pessimal assignment.** Each woman receives worst valid partner.

**Claim.** GS finds **woman-pessimal** stable matching  $S^*$ .

**Pf.**

- Suppose  $A-Z$  matched in  $S^*$ , but  $Z$  is not worst valid partner for  $A$ .
- There exists stable matching  $S$  in which  $A$  is paired with a man, say  $Y$ , whom she likes less than  $Z$ .
- Let  $B$  be  $Z$ 's partner in  $S$ .
- $Z$  prefers  $A$  to  $B$ . ← man-optimality
- Thus,  $A-Z$  is an unstable in  $S$ . ▫

S	
Amy	Yancey
Bertha	Zeus
...	

## Matchings as a random process

