NAME: _____

CS 381 (LE1), Fall 2021
Instructor: Simina Brânzei
September 28, 2021

CS 381, MIDTERM EXAM
120 minutes

This exam contains 14 pages. The last pages are intentionally left blank.

INSTRUCTIONS

1. Remember to write your name on the first page and initials on every other page.
2. You are allowed to use a one page, two-sided cheat-sheet written in reasonable font size.
3. The exam is closed book/notes. You cannot use any electronic devices, books, notes, or other written materials.
4. If you get stuck, state what you are trying to do for partial credit.
5. If you need extra space for a problem, use the blank pages of the booklet.

Problem	Max	Score
1	10	
2	15	
3	10	
4	10	
5	15	
6	10	
7	20	
8	10	
	100	

INITIALS: _____

Problem 1 (10 points) *Answer the following questions by circling True or False (1 points). No explanation is necessary.*

1. $n^2 = O(n^2 - \sqrt{n})$ ***True*** *or* ***False***
2. $n^3 = \Omega(n^3 + n)$ ***True*** *or* ***False***
3. $n \log_2 n = O(n^2)$ ***True*** *or* ***False***
4. $n \log_2 n = \Omega(n^2 - \log_2 n)$ ***True*** *or* ***False***
5. $2^n = \Omega(4^n + n^4)$ ***True*** *or* ***False***
6. $(\log_2 n)^{10} = O(n^{0.01})$ ***True*** *or* ***False***
7. $2^{n+1} = O(2^n)$ ***True*** *or* ***False***
8. $2^{2n} = O(2^n)$ ***True*** *or* ***False***
9. $6^n = O(n5^n)$ ***True*** *or* ***False***
10. $n \log_2 n = O\left((\log_2 n)^{\log_2 n}\right)$ ***True*** *or* ***False***

INITIALS: _____

Problem 2 (15 points) *1. (5 points) Describe the Merge Sort algorithm.*

2. *(5 points) Write and justify the recurrence relation used for estimating its worst case number of comparisons.*

3. *(5 points) Solve the recurrence and deduce an upper bound (sharp up to constant factors) for the worst case number of comparisons made by Merge Sort.*

INITIALS: _____

Problem 3 (10 points) *Write the recurrence relation that allows analyzing and upper bounding the expected number of comparisons of the randomized Quicksort algorithm done in class. Justify your answer. No need to solve the recurrence.*

INITIALS: _____

Problem 4 *Answer the following questions.*

1. *(5 points) Let A_1 be an algorithm that receives as input a positive integer x written in binary and is described by the pseudocode below. Note $|x|$ is the number of bits in the binary representation of x and x_i is the i-th bit of x.*

   Does A_1 run in logarithmic, polynomial, or exponential time? Briefly justify your answer.

Algorithm 1

```
1: for i = 1 to |x| do
2:     print x_i
3: end for
```

2. *(5 points) Let A_2 be the following algorithm that receives as input a positive integer x written in binary and is described in the pseudocode below. Does A_2 run in logarithmic, polynomial, or exponential time? Briefly justify your answer.*

Algorithm 2

```
1: for i = 1 to x do
2:     print i
3: end for
```

INITIALS: _____

Problem 5 (15 points) *Consider the function* $T : \mathbb{N} \to \mathbb{R}$ *given by the recurrence*

$$T(n) = 5n + T(\lfloor n/5 \rfloor) + T(\lfloor 3n/8 \rfloor) + 2,$$

where $T(0) = 0$. *Prove by induction that* $T(n) = O(n)$.

INITIALS: _____

Problem 6 (10 points) *For each pseudo-code below, find the function $t(n)$ so that the number of times the function $F()$ is called is $\Theta(t(n))$. Justify your answer.*

1. *(5 points) Code segment:*

Algorithm 1

```
1: for i = 1 to n do
2:     j = i
3:     while j < n do
4:         F(j)
5:         j = j + 5
6:     end while
7: end for
```

2. *(5 points) Code segment:*

Algorithm 2

```
1: for i = 1 to n do
2:     j = 2
3:     while j < i do
4:         F(j)
5:         j = 2 · j
6:     end while
7: end for
```

INITIALS: _____

INITIALS: _____

Problem 7 (20 points) *Consider the undirected* line graph *on n vertices $\{1, 2, \ldots, n\}$, which has edges $(1, 2)$, $(2, 3)$, ..., $(n-1, n)$. Each vertex is colored red or blue, but you don't know these colors, except you are given that vertex 1 is red and n is blue. The coloring can be determined by probing: to find the color of vertex i, we query vertex i and get back the color. An example is given below.*

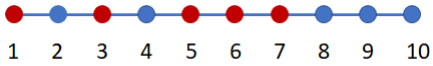

Figure 1: Example for $n = 10$.

1. *(12 points) An edge is bichromatic if one endpoint is red and the other is blue. Devise an algorithm for finding a bichromatic edge, if one exists, by probing as few vertices as possible. Justify the correctness and runtime of the algorithm.*

2. *(8 points) Can there be 10 bichromatic edges? Justify your answer.*

INITIALS: _____

INITIALS: _____

Problem 8 (10 points) *You are managing the construction of billboards on a highway that extends for M miles. The possible sites for the billboards are given by numbers $x_1, x_2, \ldots, x_n$, each in the interval $[0, M]$, specifying their position on the highway. A billboard at location x_i gives revenue $r_i > 0$. No two of the billboards can be within less than or equal to 5 miles of each other due to county regulations.*

You would like to place billboards at a subset of the sites so as to maximize your total revenue, subject to the distance restriction above. Design as efficient of an algorithm as possible for this problem. Justify its correctness and runtime.

***Example**. Suppose $M = 20$, $n = 4$, with $x_1 = 6$, $x_2 = 7$, $x_3 = 12$, $x_4 = 14$, and $r_1 = 5$, $r_2 = 6$, $r_3 = 5$, $r_4 = 1$. The optimal solution in this case is to place billboards at locations x_1 and x_3, for a total revenue of 10.*

INITIALS: _____

INITIALS: _____

INITIALS: _____