CS 381

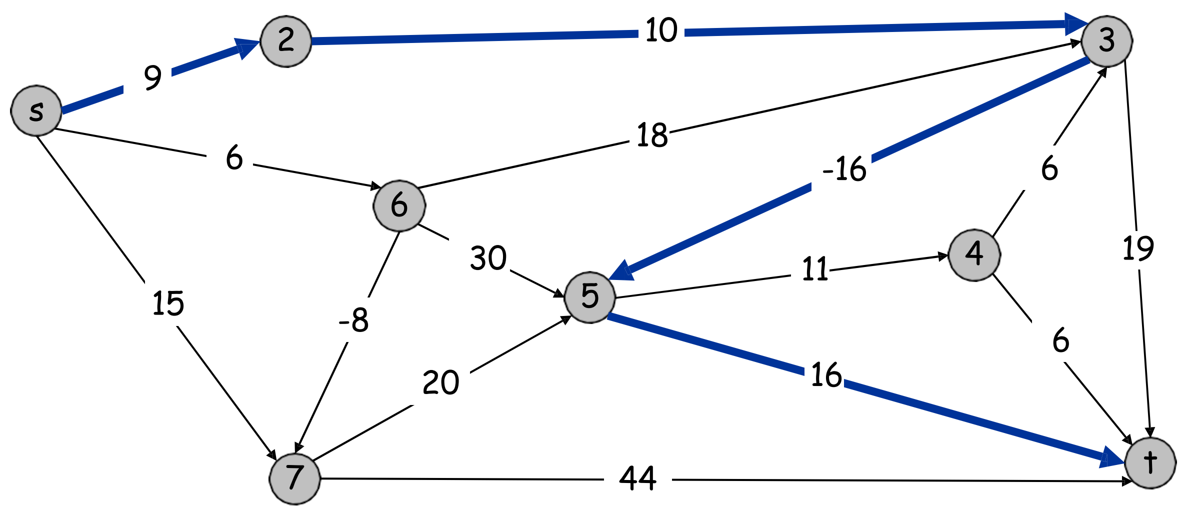
6.6 Shortest Paths
(aka Minimum Cost Paths)

Shortest Paths (Minimum Cost Paths)

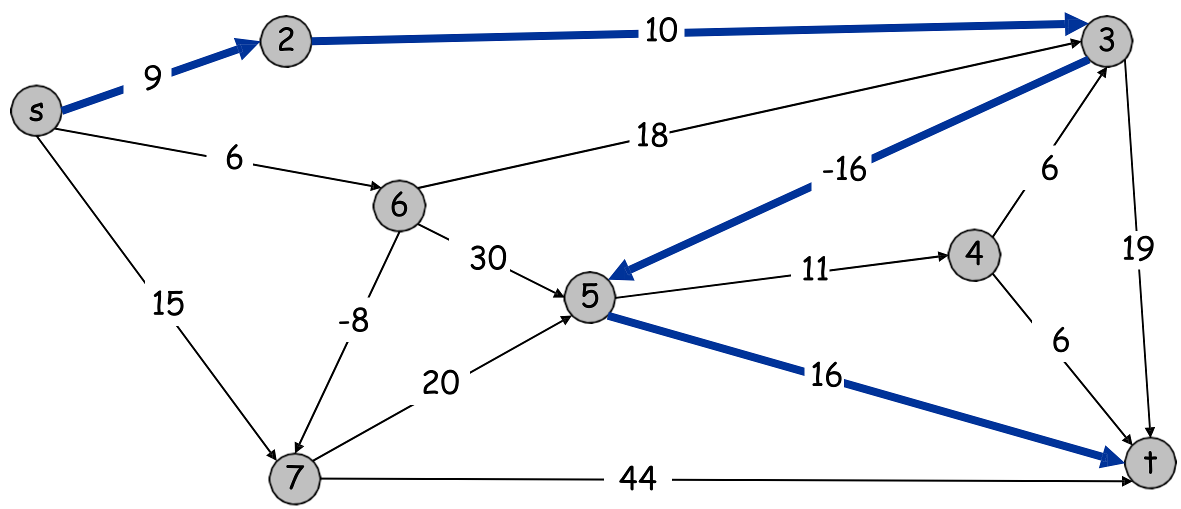
Shortest path problem. Given a directed graph $G = (V, E)$, with edge weights c_{vw}, find shortest path from node s to node t.

allow negative weights

Ex. Nodes represent agents in a financial setting and c_{vw} is cost of transaction in which we buy from agent v and sell immediately to w.

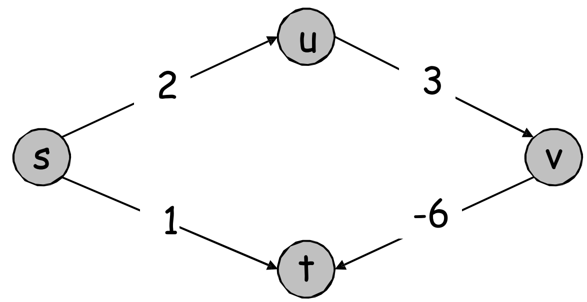


Example 2. Truck from moving company drives from city s to city t. The truck can sometimes make revenue by taking a detour by taking an edge (v,w) where there is opportunity to move some furniture from city v to w. If no move is made along an edge, then the truck loses money because it pays for gas but gets no revenue. What is a minimum cost path from s to t?

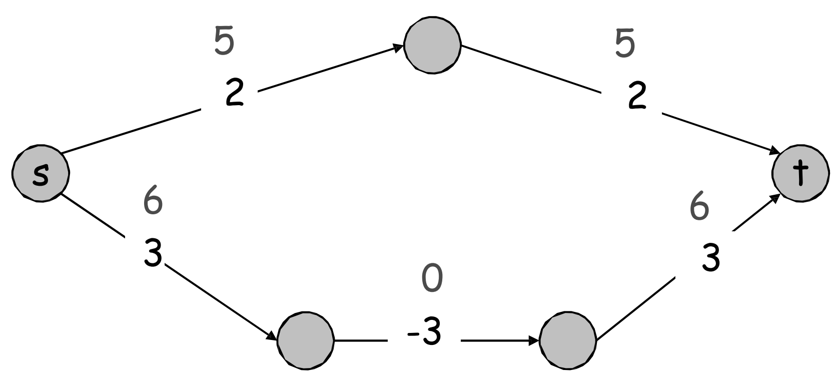


Shortest Paths: Failed Attempts

Dijkstra. Can fail if negative edge costs.

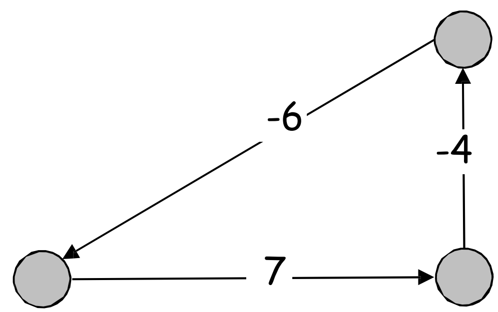


Re-weighting. Adding a constant to every edge weight can fail.

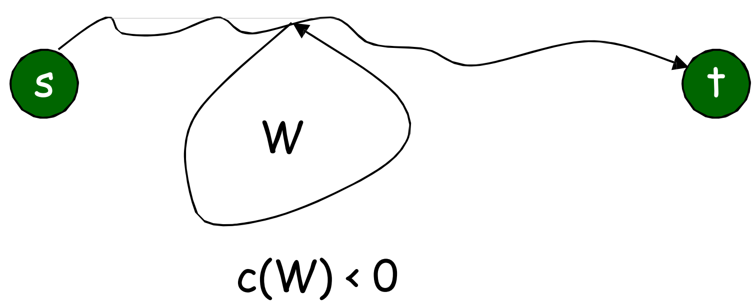


Shortest Paths: Negative Cost Cycles

Negative cost cycle.



Observation. If some path from s to t contains a negative cost cycle, there does not exist a shortest s-t path; otherwise, there exists one that is simple.



Shortest Paths: Dynamic Programming

Approach: Design algorithm using dynamic programming.

Shortest Paths: Dynamic Programming

Def. OPT(i, v) = length of shortest v-t path P using at most i edges.

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- Case 2: P uses exactly i edges.

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- Case 1: P uses at most i-1 edges.
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- Case 2: P uses exactly i edges.
  - if (v, w) is first edge, then OPT uses (v, w), and then selects best w-t path using at most i-1 edges

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$$OPT(i, v) = \begin{cases} 0 & \text{if i = 0} \\ \min\left\{ OPT(i-1, v), \quad \min\limits_{(v, w) \in E} \{ OPT(i-1, w) + c_{vw} \} \right\} & \text{otherwise} \end{cases}$$

Remark. By previous observation, if no negative cycles, then OPT(n-1, v) = length of shortest v-t path.

Shortest Paths: Implementation

```
Shortest-Path(G, t) {
   foreach node v ∈ V
      M[0, v] ← ∞
   M[0, t] ← 0

   for i = 1 to n-1
      foreach node v ∈ V
         M[i, v] ← M[i-1, v]
      foreach edge (v, w) ∈ E
         M[i, v] ← min { M[i, v], M[i-1, w] + c_vw }
}
```

Analysis. $\Theta(mn)$ time, $\Theta(n^2)$ space.

Finding the shortest paths. Maintain a "successor" for each table entry.

Shortest Paths: Practical Improvements

Practical improvements.

- Maintain only one array M[v] = shortest v-t path that we have found so far.
- No need to check edges of the form (v, w) unless M[w] changed in previous iteration.

Theorem. Throughout the algorithm, M[v] is length of some v-t path, and after i rounds of updates, the value M[v] is no larger than the length of shortest v-t path using $\leq$ i edges.

Overall impact.

- Memory: $O(m + n)$.
- Running time: $O(mn)$ worst case, but substantially faster in practice.

Bellman-Ford: Efficient Implementation

```
Push-Based-Shortest-Path(G, s, t) {
   foreach node v ∈ V {
      M[v] ← ∞
      successor[v] ← φ
   }

   M[t] = 0
   for i = 1 to n-1 {
      foreach node w ∈ V {
      if (M[w] has been updated in previous iteration){
         foreach node v such that (v, w) ∈ E {
            if (M[v] > M[w] + c_vw) {
               M[v] ← M[w] + c_vw
               successor[v] ← w
            }
         }
      }
      If no M[w] value changed in iteration i, stop.
   }
}
```

6.7 Distance Vector Protocol

Distance Vector Protocol

Communication network.

- Node ≈ router.
- Edge ≈ direct communication link.
- Cost of edge ≈ delay on link. ← naturally nonnegative, but Bellman-Ford used anyway!

Dijkstra's algorithm. Requires global information of network.

Bellman-Ford. Uses only local knowledge of neighboring nodes.

Synchronization. We don't expect routers to run in lockstep. The order in which each `foreach` loop executes in not important. Moreover, algorithm still converges even if updates are asynchronous.

Distance Vector Protocol

Distance vector protocol.

- Each router maintains a vector of shortest path lengths to every other node (distances) and the first hop on each path (directions).
- Algorithm: each router performs n separate computations, one for each potential destination node.
- "Routing by rumor.":
  - In current Bellman-Ford implementation, in each iteration i, each node v has to contact each neighbor w, and "pull" the new value $M[w]$ from it.
  - If a node w has not changed its value, then there is no need for v to get the value again; however, v has no way of knowing this, and so it must execute the pull anyway
  - if a node w had its distance change, then it "pushes" its new distance to its neighbors.

Ex. RIP, Xerox XNS RIP, Novell's IPX RIP, Cisco's IGRP, DEC's DNA Phase IV, AppleTalk's RTMP.

Caveat. Edge costs may change during algorithm (or fail completely).

Path Vector Protocols

Link state routing.

not just the distance and first hop

- Each router also stores the entire path.
- Based on Dijkstra's algorithm.
- Avoids "counting-to-infinity" problem and related difficulties.
- Requires significantly more storage.

Ex. Border Gateway Protocol (BGP), Open Shortest Path First (OSPF).

6.8 Negative Cycles in a Graph

Detecting Negative Cycles

Lemma. If OPT(n,v) = OPT(n-1,v) for all v, then no negative cycles exist.

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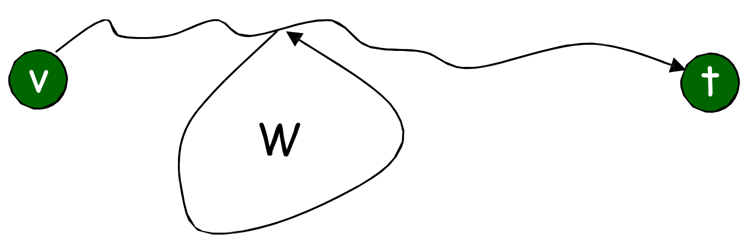
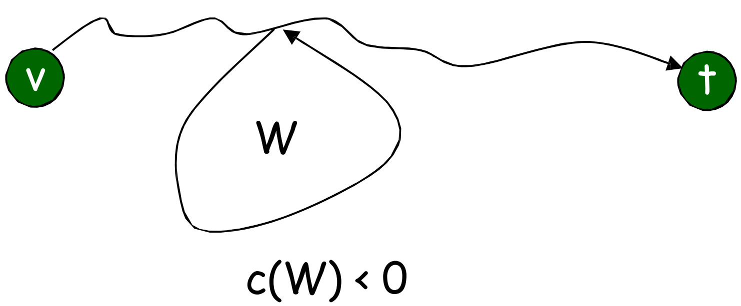
Pf. Bellman-Ford algorithm: suppose OPT(*n*, *v*) = OPT(*n* – 1, *v*) for all nodes *v*. Then OPT(n+k, v) = OPT(n-1,v) for all k. Then there cannot be a negative cycle *C* that has a path to *t*; for any node *w* on *C*, the values OPT(*i*, *w*) would have to become arbitrarily negative as *i* increased.



c(W) < 0

Detecting Negative Cycles

Lemma. If OPT(n,v) < OPT(n-1,v) for some node v, then (any) shortest path P from v to t contains a cycle W. Moreover W has negative cost.



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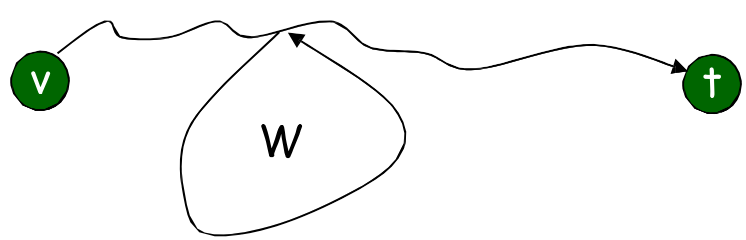
Lemma. If OPT(n,v) < OPT(n-1,v) for some node v, then (any) shortest path P from v to t contains a cycle W. Moreover W has negative cost.

Pf. (by contradiction)

- Since OPT(n,v) < OPT(n-1,v), we know P has exactly n edges.
- By pigeonhole principle, P must contain a directed cycle W.
- Deleting W yields a v-t path with < n edges $\Rightarrow$ W has negative cost.

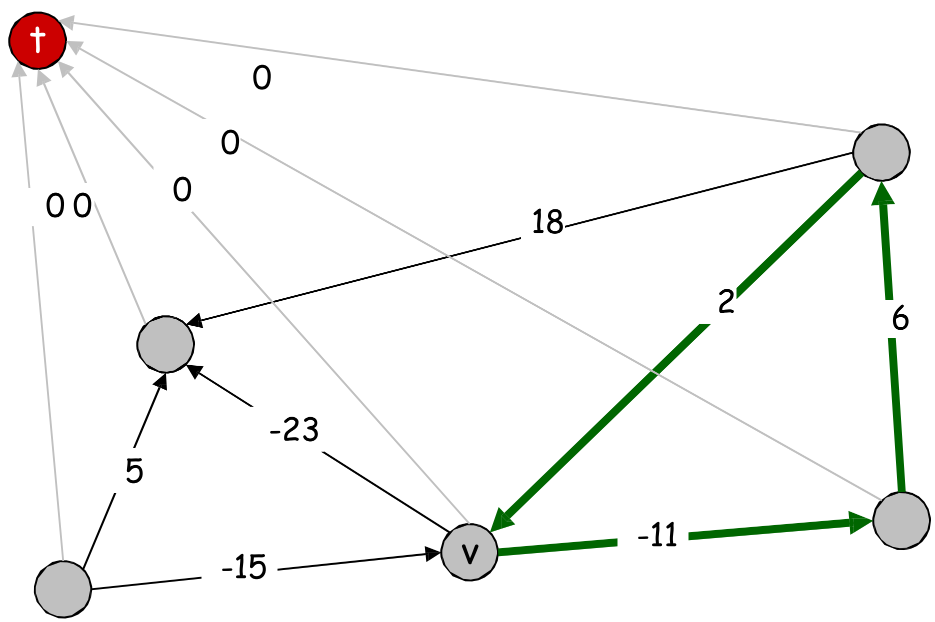


c(W) < 0

Detecting Negative Cycles

Theorem. Can detect negative cost cycle in O(mn) time.

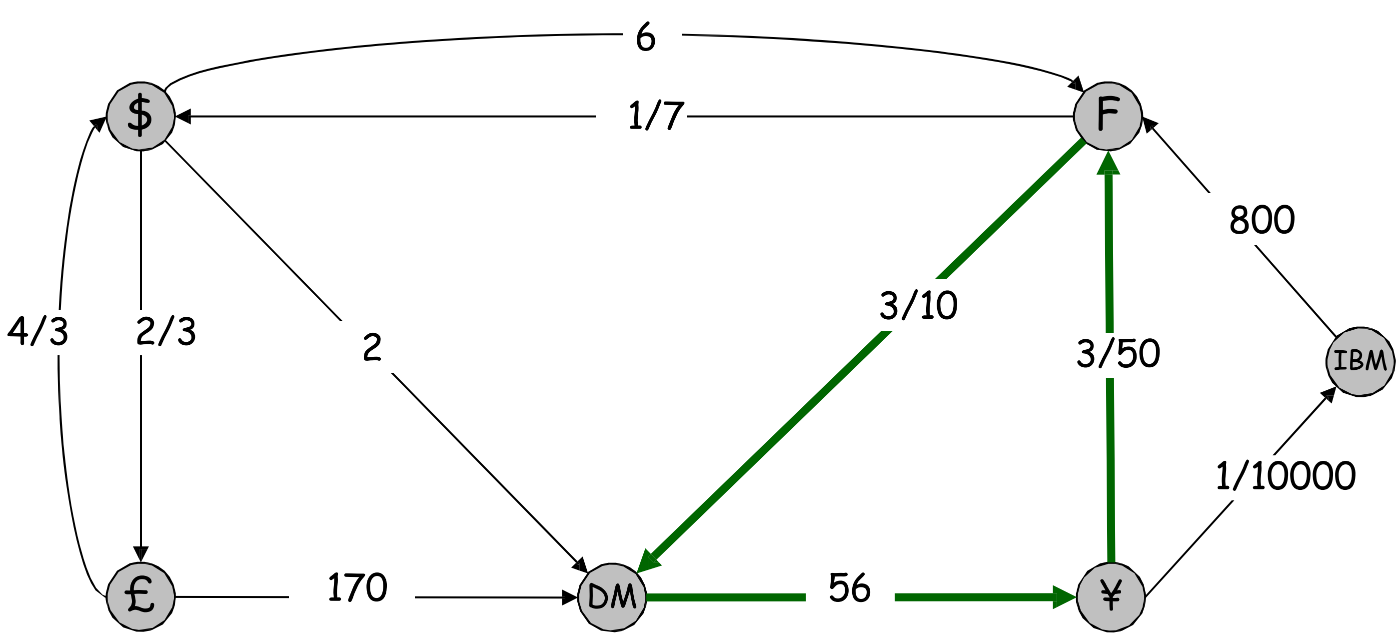
- Add new node t and connect all nodes to t with 0-cost edge.
- Check if OPT(n, v) = OPT(n-1, v) for all nodes v.
  - if yes, then no negative cycles
  - if no, then extract cycle from shortest path from v to t



Detecting Negative Cycles: Application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!



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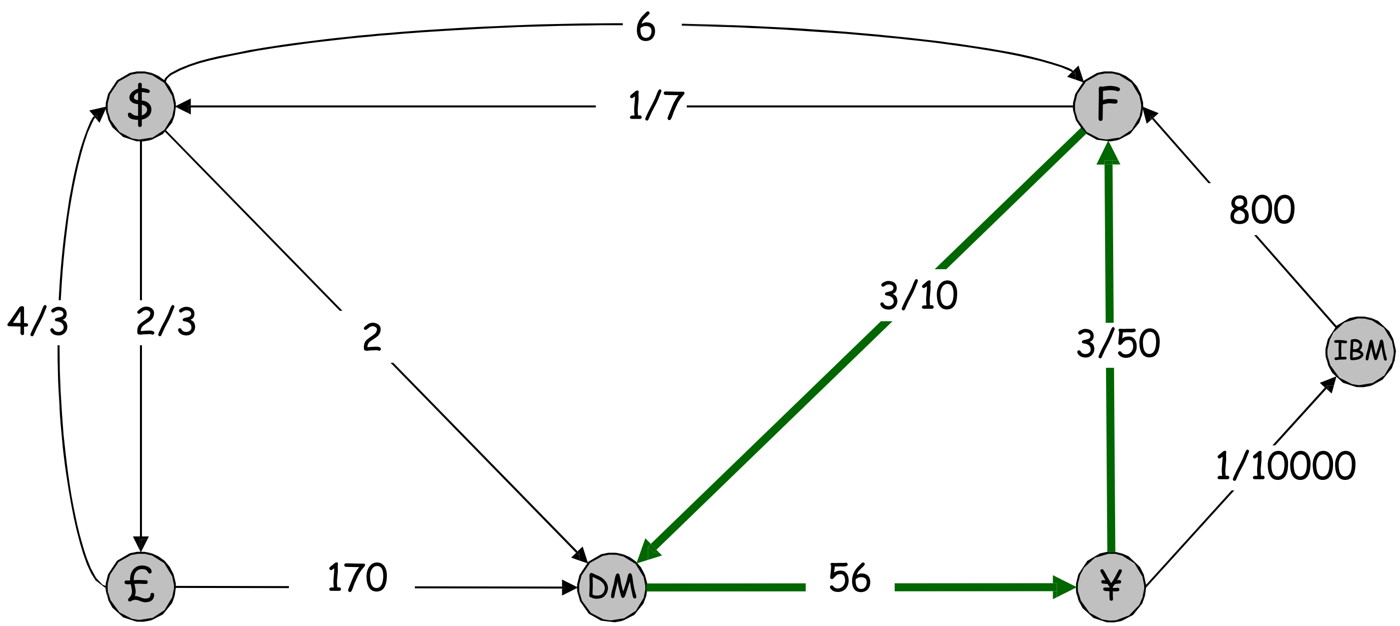

Convert each weight w_{ij} to $-\log(w_{ij})$ -> find negative cycle in the new graph.

Detecting Negative Cycles: Summary

Bellman-Ford. O(mn) time, O(m + n) space.

- Run Bellman-Ford for n iterations (instead of n-1).
- Upon termination, Bellman-Ford successor variables trace a negative cycle if one exists.
- See p. 304 for improved version and early termination rule.