CS 381

Deterministic Selection Algorithms

Goal: Find k-th smallest element of an array of n elements.

Deterministic Selection

Note: This was thought to be impossible for a long time (to do efficiently), until a 1973 paper, by Blum, Floyd, Pratt, Rivest and Tarjan proposed the “median-of-medians” algorithm.

Deterministic Selection

1. Divide the elements into **groups of five**, where the last group may have less than five elements in case when the input array size is not a multiple of five.

2. Compute the **median** of each group (ties can be broken arbitrarily).

3. Make a recursive call to calculate the **median of the medians**. Set x to the median.

4. Use x as the pivot and partition.

5. If the pivot is not the order statistics that is searched for, recurse on the subarray that contains it.

Deterministic Selection

Use a bound B to stop recursion: If the size of the array is less than or equal to B then use brute-force search to find the desired order statics.

Pseudocode

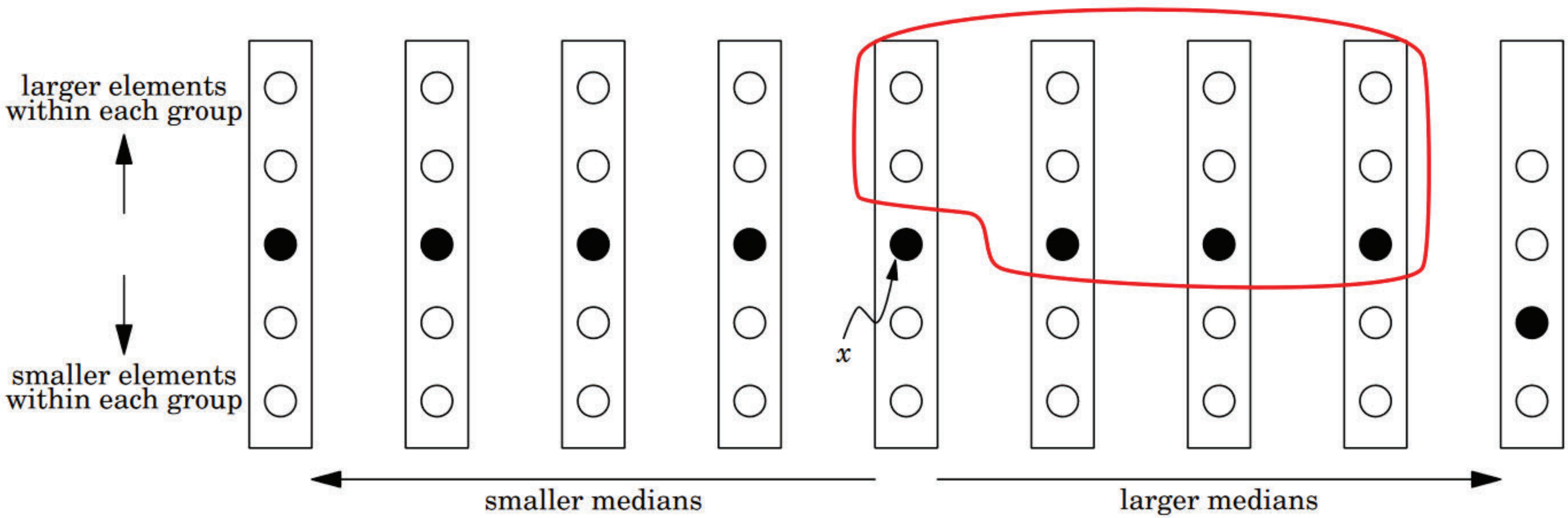
Algorithm: $\textsc{Select}(A, i)$

1. Divide the n items into groups of 5 (plus any remainder).
2. Find the median of each group of 5 (by rote). (If the remainder group has an even number of elements, then break ties arbitrarily, for example by choosing the lower median.)
3. Use $\textsc{Select}$ recursively to find the median (call it x) of these $\lceil n/5 \rceil$ medians.
4. Partition around x.[*] Let $k = \text{rank}(x)$.[†]

$$\underbrace{\boxed{\quad A_i < x \quad}}_{k-1 \text{ elements}}\underset{\substack{\uparrow \\ A_k}}{\boxed{x}}\underbrace{\boxed{\quad A_i > x \quad}}_{n-k \text{ elements}}$$

5.
   - If $i = k$, then return x.
   - Else, if $i < k$, use $\textsc{Select}$ recursively by calling $\textsc{Select}(A[1,\ldots,k-1],\ i)$.[‡]
   - Else, if $i > k$, use $\textsc{Select}$ recursively by calling $\textsc{Select}(A[k+1,\ldots,i],\ i-k)$.

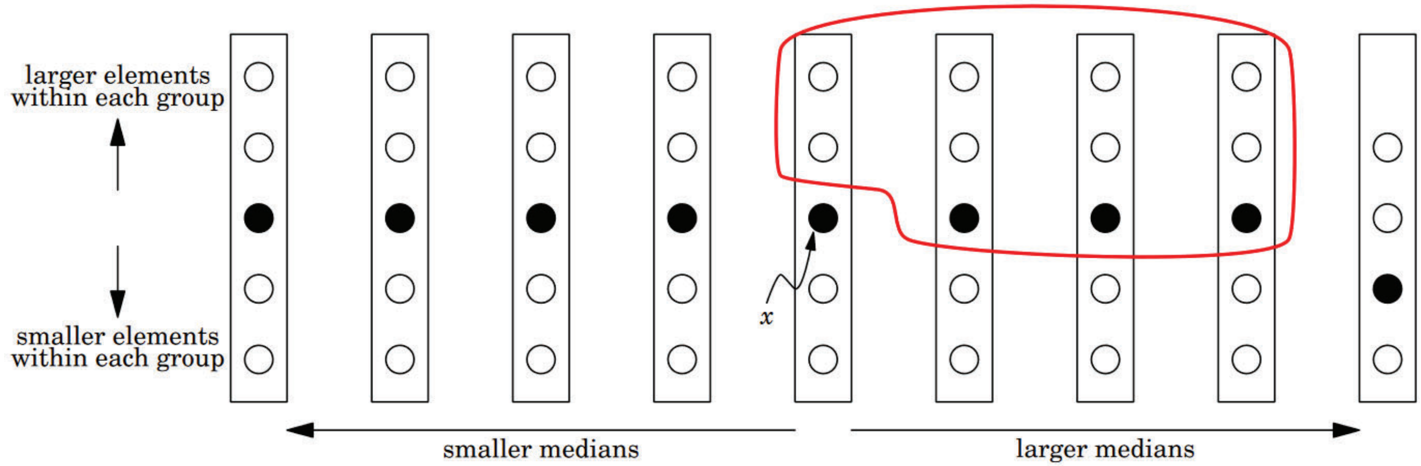
Imagine laying out the groups side by side, ordered with respect to their medians. Also imagine each group is sorted from greatest to least, top to bottom.



x : median of medians

Q: How large can the sub-array be after running Partition?

Deriving bounds on the subarrays



Each group of size 5 with median

- at most x contains at least 3 elements smaller than x.
- at least x contains at least 3 elements greater than x.

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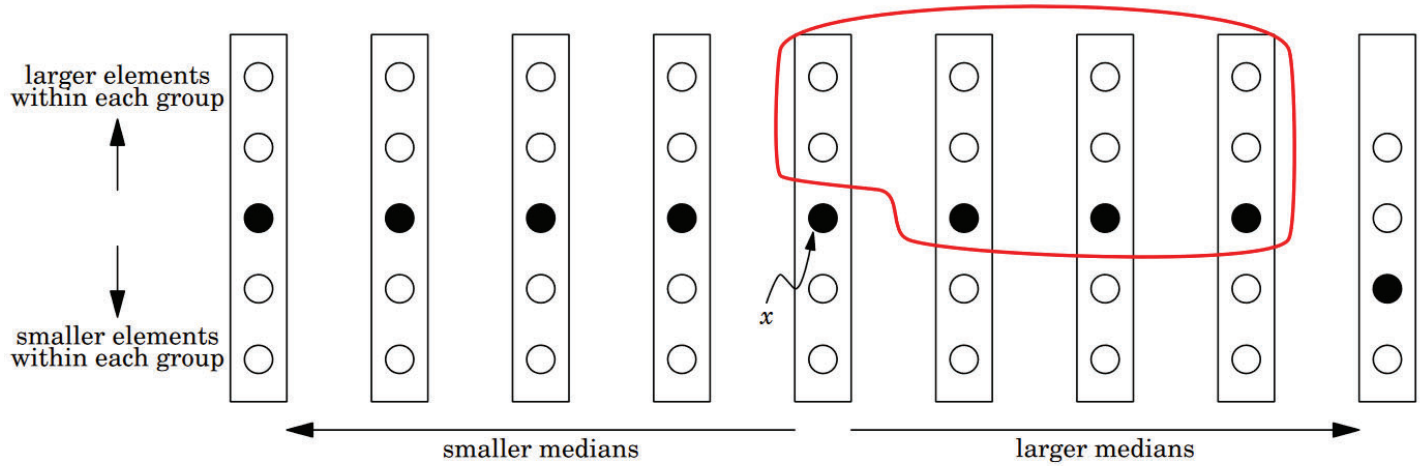

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There are $\geq \lceil n/5 \rceil - 1$ groups of size 5:

- at least $\lfloor \left(\frac{1}{2} \cdot \lceil n/5 \rceil - 2\right) \rfloor$ of them have median smaller than x
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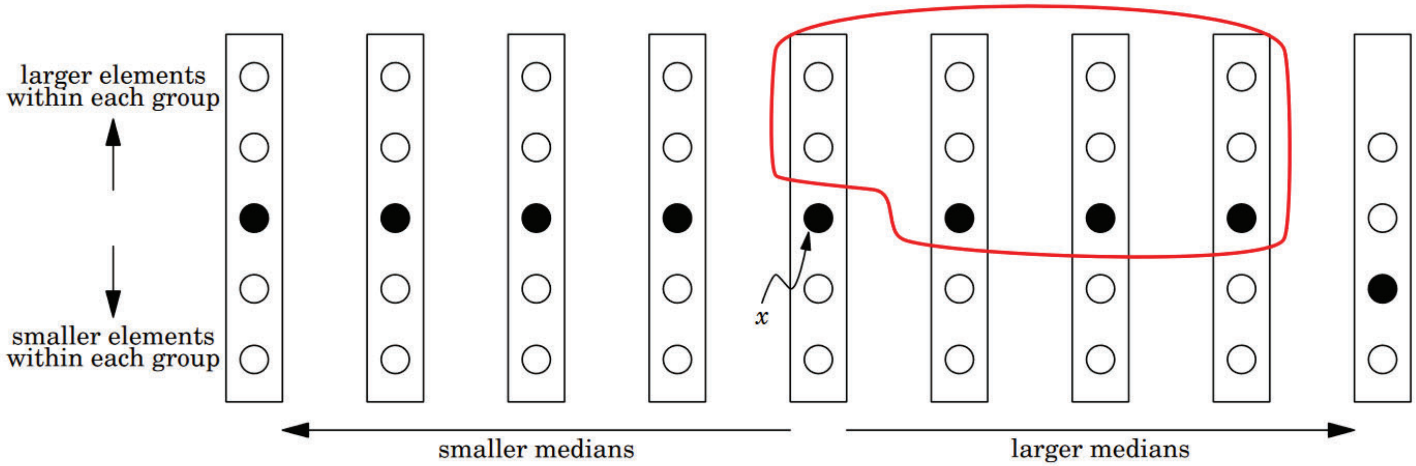

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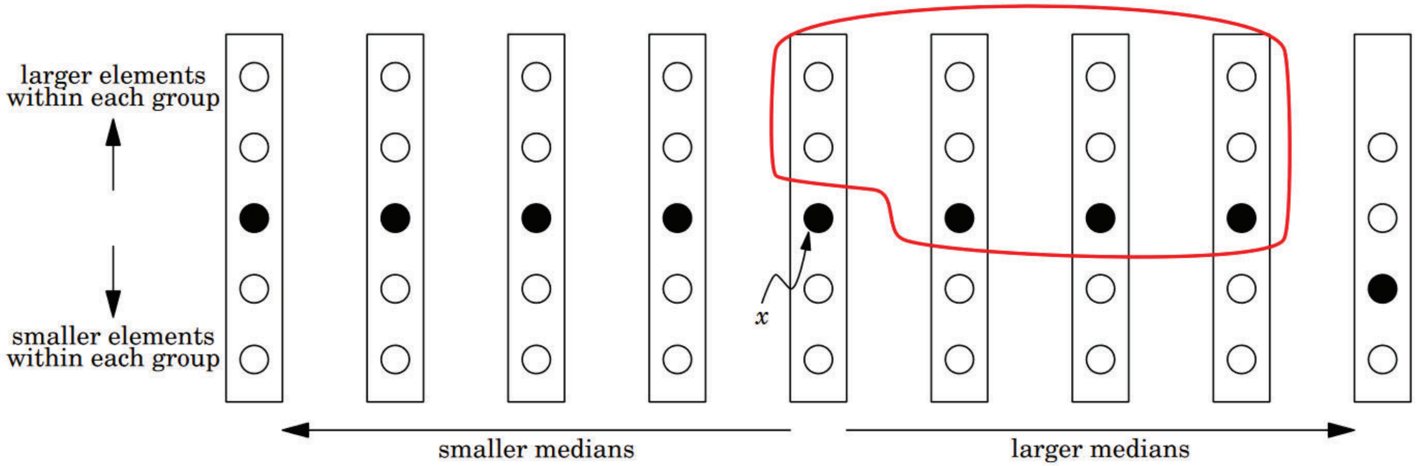
The group with x as median has 2 elements greater than x and 2 smaller than x

Deriving bounds on the subarrays



Combining these observations, we get:

$$(\#\text{ elts of } A \text{ less than } x) \geq 3\left(\left\lfloor \tfrac{1}{2}\lceil n/5\rceil - 2\right\rfloor\right) + 2 \geq 3\left(\tfrac{1}{2}\lceil n/5\rceil - 2 - \tfrac{1}{2}\right) + 2 > \tfrac{3}{10}n - 6.$$

$$(\#\text{ elts of } A \text{ greater than } x) \geq \tfrac{3}{10}n - 6.$$

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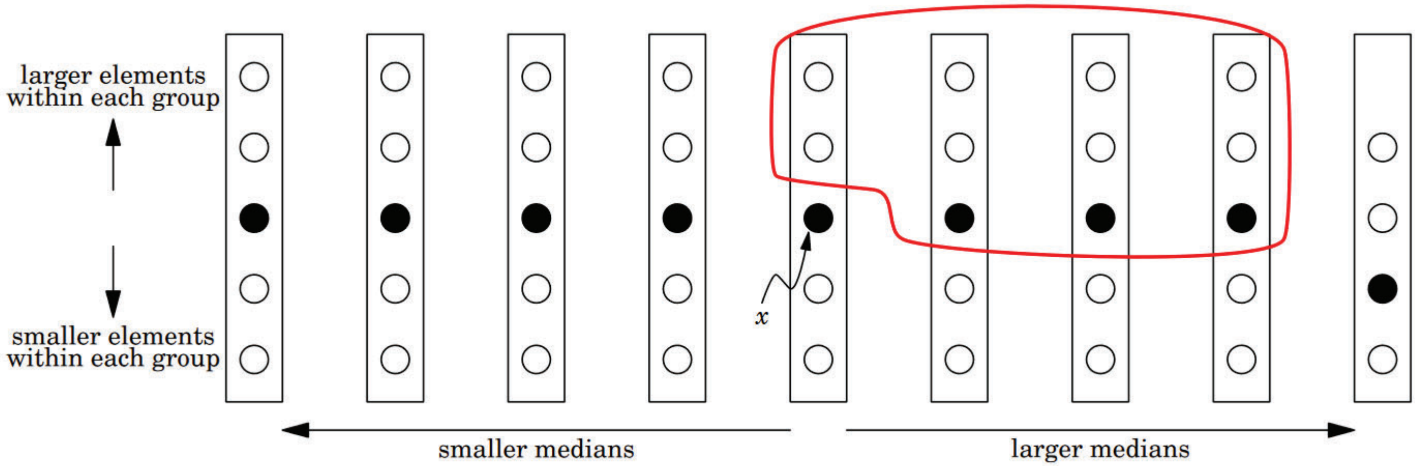

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$$(\#\text{ elts of } A \text{ greater than } x) \geq \frac{3}{10}n - 6.$$

Lemma: The larger subarray has at most $\frac{7}{10}n + 6$ elements.

Runtime Analysis

Assume the input numbers are pairwise distinct.

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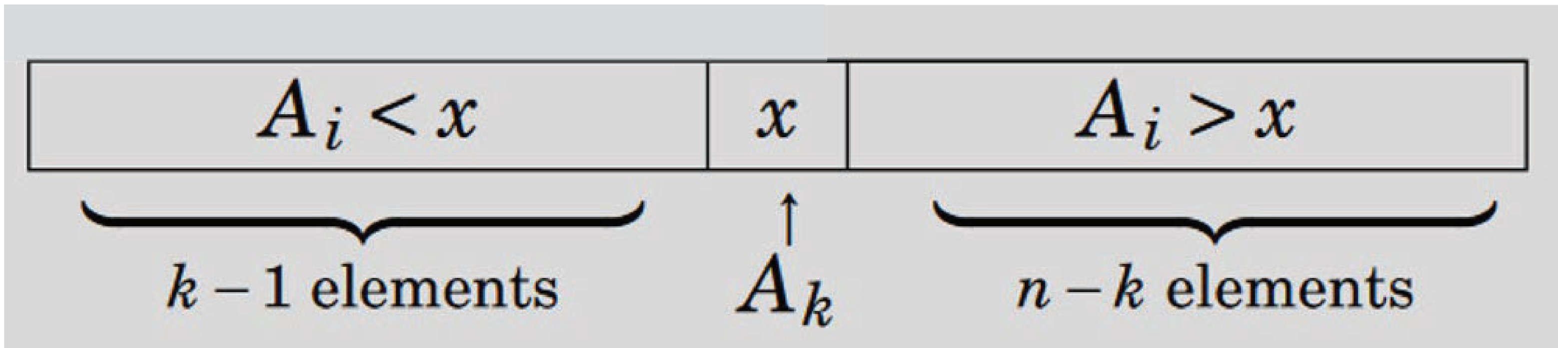
Recall B is the bound to stop the recursion: if the array has size $\leq$ B, compute the median by brute force.

As long as B is set to a constant, we can adjust a value of α so that the claim holds for all $n \leq B$.

To show the claim when $n \geq B$:



By previous **Lemma**, the larger subarray has at most $\frac{7}{10}n + 6$ elements.

Let β be a constant so that the partition operation and division into groups of five takes at most βn comparisons.

Q: How can we upper bound $T(n)$?

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By applying the induction hypothesis we get:

$$T(n) \leq \beta n + \alpha\left(\frac{n}{5} + 1 + \frac{7n}{10} + 6\right)$$

This is

$$T(n) \le \beta n + \frac{9\alpha}{10} n + 7\alpha$$

$$T(n) \le \alpha n + \beta n - \frac{\alpha}{10} n + 7\alpha$$

which is $\le \alpha n$ if

$$\beta n - \frac{\alpha}{10} n + 7\alpha \le 0$$

$$\beta n - \frac{\alpha}{10} n + 7\alpha \leq 0$$

$$-10\beta n + (n - 70)\alpha \geq 0$$

$$\alpha \geq 10\beta \frac{n}{n - 70}$$

Let $B = 140$, choose $\alpha \geq 20\beta$ to show $T(n) \leq \alpha n$.