CS 381

Review: Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).

Quicksort an *n*-element array

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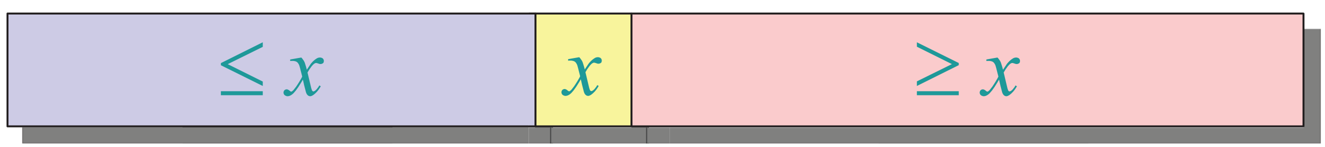

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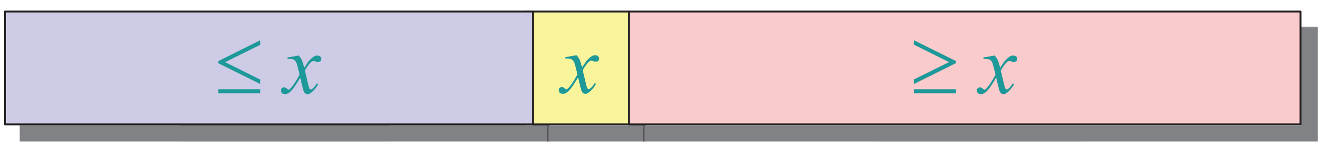


3. Recursively apply the above steps to the sub-array of elements with smaller values and separately to the sub-array of elements with greater values.

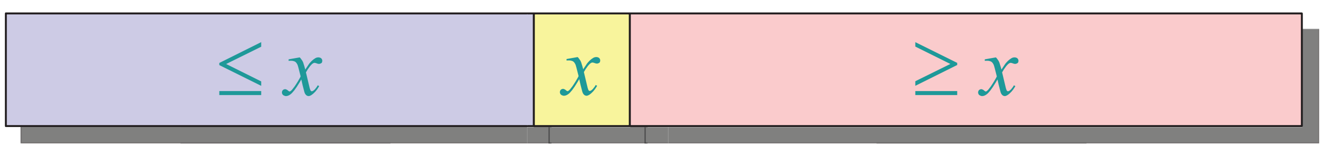
Quicksort an n-element array

Popular choice of pivot: x is a random element of the array

Exercise: write down recurrence for this choice of pivot and prove the expected run time is O(n log n).

- Assume all input elements are distinct.

(following analysis by Daniel Gildea).



The Algorithm

Quicksort(A, n)

Quicksort$'(A, 1, n)$

Quicksort$'(A, p, r)$

if $p \geq r$ **then return**

$q = \text{Partition}(A, p, r)$

Quicksort$'(A, p, q - 1)$

Quicksort$'(A, q + 1, r)$

Partition(A, p, r)

$x = A[r]$
$i \leftarrow p - 1$
for $j \leftarrow p$ **to** $r - 1$ **do**
    if $A[j] \leq x$ **then** {
        $i \leftarrow i + 1$
        Exchange $A[i]$ and $A[j]$ }
Exchange $A[i + 1]$ and $A[r]$
return $i + 1$

Worst Case Runtime

Worst-case analysis

Let T be the worst-case running time of Quicksort. Then

$$T(n) = T(1) + T(n-1) + \Omega(n).$$

By unrolling the recursion we have

$$T(n) = nT(1) + \Omega(\sum_{i=2}^{n} n).$$

Since $T(1) = O(1)$, we have

$$T(n) = \Omega(n^2).$$

Thus, we have:

Theorem A The worst-case running time of Quicksort is $\Omega(n^2)$.

Since each element belongs to a region in which **Partition** is carried out at most n times, we have:

Theorem B The worst-case running time of Quicksort is $O(n^2)$.

Best Case Runtime

The Best Cases

The best cases are when the array is split half and half. Then each element belongs to a region in which **Partition** is carried out at most $\lceil \log n \rceil$ times, so it's $O(n \log n)$.

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Recurrence:

T(n) = 2 T(n/2) + n
T(0) = T(1) = 0 (best case)
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Randomized-Quicksort

The idea is to **turn pessimistic cases into good cases by picking up the pivot randomly.**

```
Quicksort(A,n)
    Quicksort'(A,1,n)

Quicksort'(A,p,r)
    Pick t uniformly at random from {p, p + 1, ..., r}
    Exchange A[r] and A[t]
    if p ≥ r then return
    q = Partition(A, p, r)
    Quicksort'(A,p,q-1)
    Quicksort'(A,q+1,r)
```

Expected Running Time of Randomized-Quicksort

Let n be the size of the input array. Suppose that the elements are pairwise distinct.*

Let $T(n)$ be the expected running time of Randomized-Quicksort on inputs of size n. By convention, let $T(0) = 0$.

Let x be the pivot. Note that the size of the left subarray after partitioning is the rank of x minus 1.

*A more involved analysis is required if this condition is removed.

Making a hypothesis

We claim that the expected running time is at most $cn \log n$ for all $n \geq 1$. We prove this by induction on n. Let a be a constant such that partitioning of a size n subarray requires at most an steps.

For the base case, we can choose a value of c so that the claim hold.

For the induction step, let $n \geq 3$ and suppose that the claim holds for all values of n less than the current one.

Making a hypothesis

We claim that the expected running time is at most $cn \log n$ for all $n \geq 1$. We prove this by induction on n. Let a be a constant such that partitioning of a size n subarray requires at most an steps. **Note:** Convenient to work with log in base e (equivalent by changing the constant c).

For the base case, we can choose a value of c so that the claim hold.

For the induction step, let $n \geq 3$ and suppose that the claim holds for all values of n less than the current one.

The expected running time satisfies the following:

$$
\begin{aligned}
& T(n) \\
& \le \quad an + \frac{\sum_{k=0}^{n-1}(T(k) + T(n-1-k))}{n} \\
& = \quad an + \frac{2}{n}\sum_{k=1}^{n-1} T(k).
\end{aligned}
$$

By our induction hypothesis, this is at most

$$an + \frac{2c}{n}\sum_{k=1}^{n-1} k \log k.$$

Note that

$$\sum_{k=1}^{n-1} k \lg k \leq \int_1^n x \lg x dx.$$

The integration is equal to

$$\frac{1}{2}n^2 \lg n - \frac{n^2}{4} + \frac{1}{4}.$$

This is at most

$$\frac{1}{2}n^2 \lg n - \frac{n^2}{8}.$$

Note that $\int x \log(x) = \frac{1}{2}x^2 \log(x) - \frac{x^2}{4} + D$ for any constant D

By plugging this bound, we have

$$T(n) \leq cn \lg n + (a - \frac{c}{4})n.$$

Choose c so that $c > 4a$. Then,

$$T(n) \leq cn \lg n.$$

Thus, we have proven:

Theorem C Randomized Quicksort has the expected running time of $\Theta(n \lg n)$.

Alternative proof without using integrals?

Exercise:

Bound $\sum_{k=1,n-1} k \cdot \log(k)$ by $\frac{1}{2}\mathrm{n}^2\log(\mathrm{n}) - \mathrm{c}_1 \cdot n^2$