CS 381 Introduction to the Analysis of Algorithms
Fall 2021 Simina Branzei HW 6

Due Wed Dec 1 at 11:59PM

1. (20 points) Let $G = (V, E)$ be a flow network, with source s, sink t and integer capacities. Suppose that someone shows you an integer flow f of G and claims that this is a maximum flow in G. How fast can you verify that f is indeed a maximum-value flow? How?

2. (20 points) Suppose you are given a graph $G = (V, E)$ with n vertices and positive edge weights. Let x_e be a variable for each edge where we have the constraint $x_e \geq 0$. Give a linear objective function and a set of linear constraints that correctly models the Minimum Spanning Tree Problem.

3. (20 points) In the Minimum Vertex Cover problem, we are given an undirected graph $G = (V, E)$ and our goal is to pick the smallest subset of vertices $S \subseteq V$ such that every edge in E is adjacent to some vertex in S. For example, consider the following graph on six vertices.

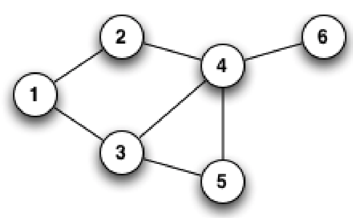

Figure 1: An Undirected Graph

In this graph, the set $S = \{1, 3, 4\}$ is a vertex cover of minimum size. Notice, that the set $S' = \{1, 2, 3, 5, 6\}$ is also a vertex cover but it is not of minimum size.

(a) Let x_v be a variable for each vertex where we have $x_v \in \{0, 1\}$. Give a linear objective function and a set of linear constraints that correctly models the Minimum Vertex Cover problem.

(b) Show that a valid vertex cover for a graph $G = (V, E)$ satisfies the constraints to the program you wrote in part (a).

(c) Now, we will relax the constraints on the x_v variables so that we have $x_v \geq 0$. Give a graph where solving this new relaxed program will give a fractional solution with a lower objective value than the best integral solution.

4. (20 Points) Answer the following questions:

[Part I; 6 points] Recall that in 2 × 2 games where all the entries are distinct, we showed that every optimal strategy pair is either pure or fully mixed. For 3 × 3 games, show one of the following:

(a) Construct an explicit 3 × 3 game where all the entries in the matrix are distinct and there is an equilibrium that is not fully mixed; or

(b) show that in every 3 × 3 game with distinct entries every equilibrium is either pure or fully mixed.

Justify your answer.

[Part II; 14 points] Player two is moving an important item in one of three cars, labeled 1, 2, and 3. Player one will capture one of the cars of her choosing. She has no chance of destroying the item if she captures the wrong car. If she chooses the right car, then her probability of destroying the item in the car depends on that car (the different cars have different drivers that are capable of avoiding capture with some probability). The probabilities for cars 1, 2, and 3 are equal to 3/4, 1/4, and 1/2. Answer the following questions and justify your answers:

(a) Write the 3×3 payoff matrix for the game, and find a pair of optimal strategies for the players. Find the value of the game.

(b) Write an LP that captures player 1's problem of finding a strategy that maximizes their gain, and also an LP for player 2's problem of minimizing their loss.

(c) Say player two plays the following mixed strategy: the item goes in each car with probability 1/3. Find a best response of player 1.

5. (20 Points) A zebra has four possible locations to cross the Zambezi River; call them a, b, c, and d, arranged from north to south. A crocodile can wait (undetected) at one of these locations. If the zebra and the crocodile choose the same location, the payoff to the crocodile (that is, the chance it will catch the zebra) is 1. The payoff to the crocodile is 1/2 if they choose adjacent locations, and 0 in the remaining cases, when the locations chosen are distinct and nonadjacent. Answer the following questions and justify your answers:

   (a) Write the payoff matrix for the game, and find a pair of optimal strategies for the players. Find the value of the game.

   (b) Write an LP that captures the zebra's problem of finding a strategy that maximizes their gain, and also an LP for the crocodile's problem of minimizing their loss.

   (c) Say the crocodile plays the following mixed strategy: wait at each location with probability 1/4. Show a best response of the zebra.

6. (20 points) Design a Turing machine for deciding the language

$$PAL = \{x_1 x_2 \dots x_n \in \{0,1\}^* \mid x_1 \dots x_n = x_n \dots x_1\}$$

   that runs in as few steps as possible asymptotically and give the asymptotic upper bound (i.e. the function $t(n)$ so that the worst case runtime is $O(t(n))$. Justify your answers.