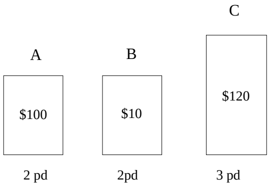
The Fractional Knapsack

The Knapsack Problem...



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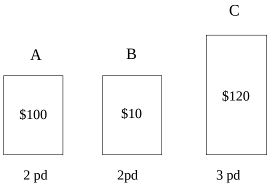

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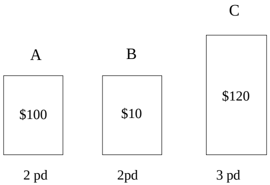

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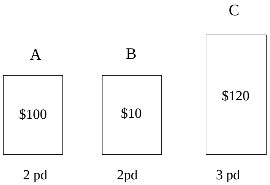

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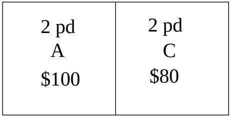

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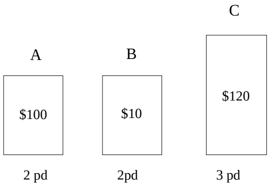

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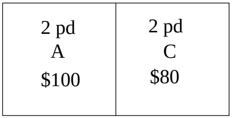

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Can only take or leave item. You can't take a fraction.

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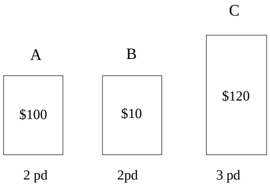

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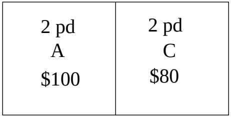

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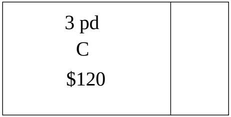

The Fractional Knapsack Problem: Formal Definition

Given K and a set of n items:

weight	w_1	w_2	...	w_n
value	v_1	v_2	...	v_n

Find: $0 \leq x_i \leq 1$, $i = 1, 2, \ldots, n$ such that

$$\sum_{i=1}^{n} x_i w_i \leq K$$

and the following is maximized:

$$\sum_{i=1}^{n} x_i v_i$$

Outline

The Knapsack problem.

Algorithm for the fractional knapsack problem?

Greedy Solution for Fractional Knapsack

Sort items by decreasing value-per-pound

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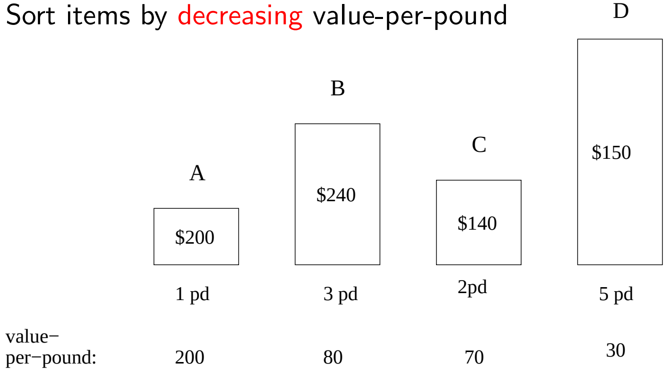

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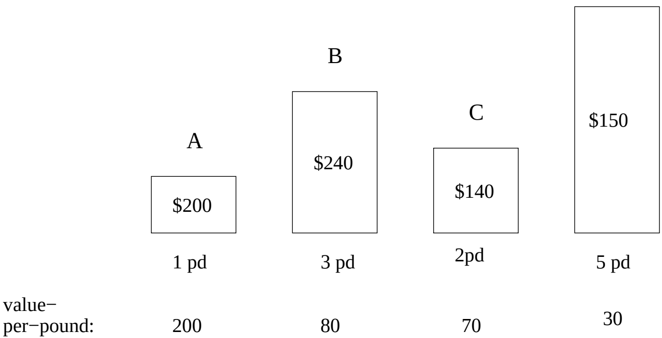

If knapsack holds $K = 5$ pd, solution is:

1	pd	A
3	pd	B
1	pd	C

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Running time: $O(n \log n)$.

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We claim that the total value for this set of items is the optimal value.

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