Examining Posterior Probability of Hypotheses Calculation

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9/9/2019

First, Review Posterior Probabilities of Hypotheses

Posterior Probabilities of Hypotheses

- Posterior Probability of model $H_\ell, \ell \in 1, ..., M$:

$$P(H_\ell|y, X) = \frac{\pi_\ell f(y|H_\ell, X)}{\sum_{m=1}^{M} \pi_m f(y|H_m, X)}$$

  where π_m is the prior on H_m (typically $\pi_m = \frac{1}{M}$), and $f(y|H_m, X)$ is the model evidence.
- $P(H_\ell|y, X)$ tells which hypothesis is more likely to give the correct model.
- $E[P(H_\ell|y, X)|H_r, X]$ may be estimated using MC approximation from simulated responses y under a chosen hypothesis H_r.
- $E[P(H_\ell|y, \mathbf{D})|H_r, \mathbf{D}]$ can be used to evaluate a design $\mathbf{D}$'s ability to distinguish hypotheses

Estimate Expected Posterior Probability of a Hypothesis

Estimate the expected posterior probability of hypothesis H_ℓ for J simulations of Y under H_r, given design $\mathbf{D} = \{x_1, ..., x_N\}$:

1. For $j = 1, \dots, J$:
   1. Draw $\beta \sim N(\mu_r, \nu_r^2)$
   2. Draw $y_i^{(j)}|\mathbf{D} \sim N(\mathbf{x}_i\beta, \sigma_r^2)$, $\forall \mathbf{x}_i \in \mathbf{D}$
   3. $\forall m \in \{1, ..., M\}$, calculate model evidences $f(y|H_m, \mathbf{D})$
      - $f(y|H_m, \mathbf{D})$ is the marginal likelihood $N(\mathbf{D}\mu_m, \sigma^2 I + \mathbf{D}V\mathbf{D}^T)$ evaluated at y and $\mathbf{D}$.
   4. Calculate the posterior probability of H_ℓ, $P(H_\ell|y^{(j)}, \mathbf{D})$, from simulation j

$$P(H_\ell|y^{(j)}, \mathbf{D}) = \frac{\pi_\ell f(y^{(j)}|H_\ell, \mathbf{D})}{\sum_{m=1}^{M} \pi_m f(y^{(j)}|H_m, \mathbf{D})}$$

2. Average the estimated posterior probabilities of H_ℓ over $\forall j$ to obtain MC estimate of $E[P(H_\ell|y, \mathbf{D})|H_r, \mathbf{D}]$

What We Expect

- We expect that, by how MED is constructed, $E[P(H_\ell|y, \mathbf{D}_{MED})|H_\ell, \mathbf{D}_{MED}] > E[P(H_\ell|y, \mathbf{D}_{space})|H_\ell, \mathbf{D}_{space}]$. In other words, we want to show that the expected posterior probability of the true hypothesis (i.e. the hypothesis from which the simulated data was generated) is higher when the design is the MED design, $\mathbf{D}_{MED}$, than when the design is another design, such as the space-filling design, $\mathbf{D}_{space}$.
- We expect that this is the case because we expect that the average density $E[f(y|H_\ell, \mathbf{D}_{MED})]$ over simulated data $y|H_\ell, \mathbf{D}_{MED}$ is higher than the average density $E[f(y|H_\ell, \mathbf{D}_{space})]$ over simulated data $y|H_\ell, \mathbf{D}_{space}$.

Now, for the problem…

What We See

- The problem is that we don't see $E[P(H_\ell|y, \mathbf{D}_{MED})|H_\ell, \mathbf{D}_{MED}] > E[P(H_\ell|y, \mathbf{D}_{space})|H_\ell, \mathbf{D}_{space}]$ (in fact we see the opposite, $E[P(H_\ell|y, \mathbf{D}_{MED})|H_\ell, \mathbf{D}_{MED}] < E[P(H_\ell|y, \mathbf{D}_{space})|H_\ell, \mathbf{D}_{space}]$, which we don't want!).
  - Notice that the first row is $E[P(H_0|y, \mathbf{D})|H_0, \mathbf{D}]$ where MED does worse than space-filling
  - And the second-to-last row is $E[P(H_1|y, \mathbf{D})|H_1, \mathbf{D}]$ and again MED does worse than space-filling
  - This isn't at all what we expected or wanted!

  ```
  ##                      Fast   Space
  ## E[P(H0|Y,D)|H0,D] 0.84439 0.92671
  ## E[P(H1|Y,D)|H1,D] 0.84203 0.94727
  ```
- (In case you're wondering,) the full table of posterior probabilities is:

```
##                        Fast      Space
## E[P(H0|Y,D)|H0,D]   0.84439    0.92671
## E[P(H1|Y,D)|H0,D]   0.15561    0.07329
## E[BF01 | H0,D]     99.60020 1369.71754
## E[P(H0|Y,D)|H1,D]   0.15797    0.05273
## E[P(H1|Y,D)|H1,D]   0.84203    0.94727
## E[BF01|H1,D]        0.47709    0.10636
```

- This matters, because the evidence for the other hypothesis shouldn't be higher for MED than for space-filling design. At least, that is not an intended result.

Why This Might Be Happening

- I think this is explained by the strange occurrence that, while $E[f(y|H_\ell, \mathbf{D}_{MED})] > E[f(y|H_\ell, \mathbf{D}_{space})]$, we also see $E[f(y^{(j)}|H_{-\ell}, \mathbf{D}_{MED})] > E[f(y^{(j)}|H_{-\ell}, \mathbf{D}_{space})]$ as well! For example, let $\ell = 0$:
  1. We simulate some $Y|H_0, \mathbf{D}$ for $\mathbf{D} = \mathbf{D}_{MED}$
  2. We calculate the model evidence for each of the data, given each of the models from the hypotheses H_0, H_1 for this simulated data
  3. We simulate some $Y|H_0, \mathbf{D}$ for $\mathbf{D} = \mathbf{D}_{space}$
  4. We calculate the model evidence for each of the data, given each of the models from the hypotheses H_0, H_1 for this simulated data, as well.

```r
# fast MED
fast_simY_H0 = simulateY(fast_S5, N, mean_beta0, var_mean0, var_e, numSims, type = type[1])
fast_simEvidenceH0_YH0 = rep(NA, numSims) # added for bf01 calc
fast_simEvidenceH1_YH0 = rep(NA, numSims)
for(j in 1:numSims){
  Y = fast_simY_H0[, j]
  fast_simEvidenceH0_YH0[j] = model_evidence(Y, fast_S5, N, mean_beta0,
                                             var_mean0, var_e, type = type[1])
  fast_simEvidenceH1_YH0[j] = model_evidence(Y, fast_S5, N, mean_beta1,
                                             var_mean1, var_e, type = type[2])
}
# space-filling
space_simY_H0 = simulateY(space_filling, N, mean_beta0, var_mean0, var_e, numSims, type = type[1])
space_simEvidenceH0_YH0 = rep(NA, numSims) # added for bf01 calc
space_simEvidenceH1_YH0 = rep(NA, numSims)
for(j in 1:numSims){
  Y = space_simY_H0[, j]
  space_simEvidenceH0_YH0[j] = model_evidence(Y, space_filling, N, mean_beta0,
                                              var_mean0, var_e, type = type[1])
```

```r
  space_simEvidenceH1_YH0[j] = model_evidence(Y, space_filling, N, mean_beta1,
                                              var_mean1, var_e, type = type[2])
}
```

- So $E[f(y|H_\ell, \mathbf{D}_{MED})] > E[f(y|H_\ell, \mathbf{D}_{space})]$:

```r
mean(fast_simEvidenceH0_YH0) > mean(space_simEvidenceH0_YH0)
```

```
## [1] TRUE
```

```r
mean(fast_simEvidenceH0_YH0) * 10e-55 # E[f(y | H0, D_MED)] for y generated from H0
```

```
## [1] 287.6496
```

```r
mean(space_simEvidenceH0_YH0) * 10e-55 # E[f(y | H0, D_space)] for y generated from H0
```

```
## [1] 97.9379
```

- But also $E[f(y|H_{-\ell}, \mathbf{D}_{MED})] > E[f(y|H_{-\ell}, \mathbf{D}_{space})]$, which shouldn't be happening.

```r
mean(fast_simEvidenceH1_YH0) > mean(space_simEvidenceH1_YH0)
```

```
## [1] TRUE
```

```r
mean(fast_simEvidenceH1_YH0) * 10e-55 # E[f(y | H1, D_MED)] for y generated from H0
```

```
## [1] 17.30612
```

```r
mean(space_simEvidenceH1_YH0) * 10e-55 # E[f(y | H1, D_space)] for y generated from H0
```

```
## [1] 1.084449
```

- And hence the expected posterior probability of H_0 from data y generated from H_0, i.e. $E[P(H_0|y, \mathbf{D})|H_0, \mathbf{D}]$, is

```r
# MED
fast_logsimpostH0_YH0 = log(fast_simEvidenceH0_YH0) - log(fast_simEvidenceH0_YH0
                                                          + fast_simEvidenceH1_YH0)
fast_expected_postH0_YH0 = (1 / numSims) * exp(logSumExp(fast_logsimpostH0_YH0))
# space-filling
space_logsimpostH0 = log(space_simEvidenceH0_YH0) - log(space_simEvidenceH0_YH0
                                                        + space_simEvidenceH1_YH0)
space_expected_postH0_YH0 = (1 / numSims) * exp(logSumExp(space_logsimpostH0))
# E[P(H0|Y,D)|H0,D]
mean(fast_expected_postH0_YH0) # posterior prob of H0
```

```
## [1] 0.8443935
```

```r
mean(space_expected_postH0_YH0) # posterior prob of H0
```

```
## [1] 0.9267057
```

- which aren't what we want them to be.

Figuring Out Why

Two Other Designs

- To understand why this is happening, let's look at the much simpler designs $\mathbf{D}_0 = \{(0,0), \ldots, (0,0)\}$ and $\mathbf{D}_1 = \{(0.5, 0.5), \ldots, (0.5, 0.5)\}$.

- At $\mathbf{D}_0$, the simulated Y should have a hard time distinguishing the models, whereas $\mathbf{D}_1$ should have the easiest time, since it is located where the models are most different from each other.

```
##                        Fast   Space  D0      D1
## E[P(H0|Y,D)|H0,D] 0.84439 0.92671 0.5 0.63333
## E[P(H1|Y,D)|H1,D] 0.84203 0.94727 0.5 0.63107
```

- These results make me certain that the issue is in the calculation.
- Looking at the evidence calculation,

```
# D0, (0, 0)s
d0_simY_H0 = simulateY(d0, N, mean_beta0, var_mean0, var_e, numSims, type = type[1])
d0_simEvidenceH0_YH0 = rep(NA, numSims) # added for bf01 calc
d0_simEvidenceH1_YH0 = rep(NA, numSims)
for(j in 1:numSims){
  Y = d0_simY_H0[, j]
  d0_simEvidenceH0_YH0[j] = model_evidence(Y, d0, N, mean_beta0,
                                           var_mean0, var_e, type = type[1])
  d0_simEvidenceH1_YH0[j] = model_evidence(Y, d0, N, mean_beta1,
                                           var_mean1, var_e, type = type[2])
}
# D1, (0.5, 0.5)s
d1_simY_H0 = simulateY(d1, N, mean_beta0, var_mean0, var_e, numSims, type = type[1])
d1_simEvidenceH0_YH0 = rep(NA, numSims) # added for bf01 calc
d1_simEvidenceH1_YH0 = rep(NA, numSims)
for(j in 1:numSims){
  Y = d1_simY_H0[, j]
  d1_simEvidenceH0_YH0[j] = model_evidence(Y, d1, N, mean_beta0,
                                           var_mean0, var_e, type = type[1])
  d1_simEvidenceH1_YH0[j] = model_evidence(Y, d1, N, mean_beta1,
                                           var_mean1, var_e, type = type[2])
}
```

- And looking at some comparisons,

Comparing Evidences from Each Hypothesis Model, Given Data

- Surprisingly, they are similar for $E[f(y|H_0, \mathbf{D})]$. Is this because the null hypothesis is true, so their evidences

```
mean(d1_simEvidenceH0_YH0) > mean(d0_simEvidenceH0_YH0)
```

```
## [1] TRUE
```

```
mean(d1_simEvidenceH0_YH0) * 10e-57 # E[f(y | H0, D1)] for y generated from H0
```

```
## [1] 19.88898
```

```
mean(d0_simEvidenceH0_YH0) * 10e-57 # E[f(y | H0, D0)] for y generated from H0
```

```
## [1] 19.45612
```

- And unlike with the comparison between MED and space-filling design, $E[f(y|H_1, \mathbf{D}_1)] < E[f(y|H_1, \mathbf{D}_0)]$. This seems right, and it also makes sense that $E[f(y|H_1, \mathbf{D}_0)] \approx E[f(y|H_0, \mathbf{D}_0)]$ above, since it cannot distinguish between H_0 and H_1 very well at the point $\mathbf{x} = (0, 0)$, where $f_0(\mathbf{x}) = f_1(\mathbf{x})$.

```
mean(d1_simEvidenceH1_YH0) > mean(d0_simEvidenceH1_YH0)
```

```
## [1] FALSE
```

```r
mean(d1_simEvidenceH1_YH0) * 10e-57 # E[f(y | H1, D1)] for y generated from H0
```

```
## [1] 5.923874
```

```r
mean(d0_simEvidenceH1_YH0) * 10e-57 # E[f(y | H1, D0)] for y generated from H0
```

```
## [1] 19.45612
```

- But why $E[P(H_0|y, \mathbf{D}_{space})|H_0, \mathbf{D}_{space}] > E[P(H_0|y, \mathbf{D}_1)|H_0, \mathbf{D}_1]$?
- Comparing $\mathbf{D}_1$ to $\mathbf{D}_{space}$,
- $E[f(y|H_0, \mathbf{D})]$: it looks that the evidence for $\mathbf{D}_1$ is higher.

```r
mean(d1_simEvidenceH0_YH0) > mean(space_simEvidenceH0_YH0)
```

```
## [1] TRUE
```

```r
mean(d1_simEvidenceH0_YH0) * 10e-57 # E[f(y | H0, D1)] for y generated from H0
```

```
## [1] 19.88898
```

```r
mean(space_simEvidenceH0_YH0) * 10e-57 # E[f(y | H0, D_space)] for y generated from H0
```

```
## [1] 0.979379
```

- $E[f(y|H_1, \mathbf{D})]$: it looks that the evidence for $\mathbf{D}_1$ is higher for H_1 as well.

```r
mean(d1_simEvidenceH1_YH0) > mean(space_simEvidenceH1_YH0)
```

```
## [1] TRUE
```

```r
mean(d1_simEvidenceH1_YH0) * 10e-57 # E[f(y | H1, D1)] for y generated from H0
```

```
## [1] 5.923874
```

```r
mean(space_simEvidenceH1_YH0) * 10e-57 # E[f(y | H1, Dspace)] for y generated from H0
```

```
## [1] 0.01084449
```

- Perhaps it doesn't do any good to compare these likelihood values, though, since they aren't normalized.

Comparing Average Posterior Probabilities of Hypotheses

- The expected posterior probability of H_0 from data y generated from H_0, i.e. $E[P(H_0|y, \mathbf{D})|H_0, \mathbf{D}]$ for $\mathbf{D}_0$ and $\mathbf{D}_1$ are calculated as follows:

```r
# MED
d0_logsimpostH0_YH0 = log(d0_simEvidenceH0_YH0) - log(d0_simEvidenceH0_YH0
                                                       + d0_simEvidenceH1_YH0)
d0_expected_postH0_YH0 = (1 / numSims) * exp(logSumExp(d0_logsimpostH0_YH0))
# space-filling
d1_logsimpostH0 = log(d1_simEvidenceH0_YH0) - log(d1_simEvidenceH0_YH0
                                                   + d1_simEvidenceH1_YH0)
d1_expected_postH0_YH0 = (1 / numSims) * exp(logSumExp(d1_logsimpostH0))
# E[P(H0|Y,D)|H0,D]
mean(d0_expected_postH0_YH0) # posterior prob of H0
```

```
## [1] 0.5
```

```r
mean(d1_expected_postH0_YH0) # posterior prob of H0
```

```
## [1] 0.6333313
```