1. proof: Suppose by contradiction that $T(x)=x^5$, $T(x) \in O(x^2)$.

By definition, there exists positive constant c and x_0 such that $x^5 \le cx^2$ for all $x \geq x_0$.

Then $x^5 \le cx^2$

$x^2(x^3-c) \le 0$

Since $x^2 \geq 0$ we have $x^3-c \le 0$.

Therefore, $x \le \sqrt[3]{c}$ such that $x^5 \le cx^2$.

However, the definition states that for $x \geq x_0$, the inequality $x^5 \le cx^2$ is true.

The requirement $x \le \sqrt[3]{c}$ conflicts $x \geq x_0$.

Therefore, we conclude that x^5 is not $O(x^2)$.

2. Prove that $1^3+2^3+\cdots+n^3$ is $\Theta(n^4)$

S1: prove $1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$ by induction

S2: Show $\left(\frac{n(n+1)}{2}\right)^2$ in $\Theta(n^4)$.

proof: Step 1: base case, when $n=1$, $LHS = 1^3 = 1$

$RHS = \left(\frac{1\times 2}{2}\right)^2 = 1$.

$LHS = RHS$.

Therefore $1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$ is true when $n=1$.

Inductive: suppose $1^3+2^3+\cdots+k^3 = \left(\frac{k(k+1)}{2}\right)^2$ for $n=k$, $n \geq 1$. is true.

Let $n=k+1$.

$$LHS = 1^3+2^3+\cdots+k^3+(k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= \frac{(k+1)^2[k+4(k+1)]}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$

$$= RHS.$$

Therefore $1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$ is true for $n \geq 1$.