CPSC 221 Theory Assignment 2

Sample Solutions March 2017

1. Suppose x^5 is $O(x^2)$.

Then, by the definition of Big-O:

$x^5 = c x^2$ where $c \in \mathbb{R}^+$ for all $x \geq n_0$, $n_0 \in \mathbb{Z}^+$

$\Rightarrow x^3 = c$

But, there is no constant $c \in \mathbb{R}^+$ that makes this true $\forall x \in \mathbb{Z}^+$. Recall that x can grow without bound.

Thus, we have a contradiction and $x^5 \notin O(x^2)$.

2. $T(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3$

Big-O:

$$T(n) \le \underbrace{n^3 + n^3 + \cdots + n^3}_{n \text{ times}}$$

$$= n^4$$

$\therefore T(n) \in O(n^4)$ – witnesses: $c=1$ & $n_0=1$

Big-Ω:

$$T(n) = 1^3 + 2^3 + \cdots + n^3$$

$$= 1^3 + 2^3 + \cdots + \left\lfloor \frac{n}{2} \right\rfloor^3 + \left\lceil \frac{n+1}{2} \right\rceil^3 + \cdots (n-1)^3 + n^3$$

$$\ge \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n+1}{2} \right\rceil^3$$ – throw away about half of the terms, and focus on the smallest remaining term

$$\ge \frac{n}{2} \left(\frac{n}{2}\right)^3$$

$$= \frac{1}{16} n^4$$

$\therefore T(n) \in \Omega(n^4)$ – witnesses $c = \frac{1}{16}$ & $n_0 = 1$

Since $T(n) \in O(n^4)$ and $T(n) \in \Omega(n^4)$, it follows that $T(n) \in \Theta(n^4)$.

3. a) $E_1 = 0$

$E_k = E_{k-1} + k + 1 \quad \forall\, k \in \mathbb{Z}^+,\ k \geq 2$

difference

By substitution:

$$E_1 = 0$$
$$E_2 = \boxed{E_1} + 2 + 1 = 0 + 2 + 1 = 3$$
$$E_3 = \boxed{E_2} + 3 + 1 = 3 + 3 + 1 = 7$$
$$E_4 = \boxed{E_3} + 4 + 1 = 7 + 4 + 1 = 12$$
$$E_5 = \boxed{E_4} + 5 + 1 = 12 + 5 + 1 = 18$$

3

4

5

6

We see a pattern forming.

Note that: $3+4+5+6 = 18$

$$\therefore E_k = \sum_{i=3}^{k+1} i$$
$$= \sum_{i=1}^{k+1} i - 2 - 1$$
$$= \frac{(k+1)(k+2)}{2} - 3$$
$$= \frac{k^2 + 3k + 2 - 6}{2}$$
$$= \frac{k^2 + 3k - 4}{2}$$
$$= \frac{(k+4)(k-1)}{2} \quad //$$

3 b) Show that $E_k = \frac{(k+4)(k-1)}{2}$ by induction.

Base case: $k=1$

$LHS = E_1 = 0$ by definition

$RHS = \frac{(1+4)(1-1)}{2} = 0 = LHS$ ✓

Inductive hypothesis:

Assume it's true for $k=j$

i.e., $E_j = \frac{(j+4)(j-1)}{2}$

Inductive Step:

Show that it's true for $k=j+1$

i.e., $E_{j+1} = \frac{(j+5)(j)}{2}$ (*)

$LHS = E_{j+1} = E_j + (j+1) + 1$ by the definition

by Inductive Hypothesis

$$= \frac{(j+4)(j-1)}{2} + j + 1 + 1$$

$$= \frac{j^2+3j-4+2j+4}{2}$$

$$= \frac{j^2+5j}{2}$$

$$= \frac{j(j+5)}{2} = RHS^{(*)}$$

QED

4. Quicksort call = qsort(A, 0, 8);

A =

5	3	2	8	1	0	6	7	4
[0]				[4]				[8]

n=9

- swap 5 and 4 (median rule)

4	3	2	8	1	0	6	7	5
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
	3	2	1	8				
				0	8			

lo	hi	p	i
0	8	0	1
		1	2
		2	3
		3	4
		4	5
			6
			7
			8
			9

- swap [0] with [4]

$[0] = 4$, $[4] = 0$

p

result:

0	3	2	1	4	8	6	7	5

$\underbrace{0\ 3\ 2\ 1}_{\text{left partition}}$ ↑ pivot's final location $\underbrace{8\ 6\ 7\ 5}_{\text{right partition}}$

left partition qsort(A, 0, 3);

pivot's final location

right partition qsort(A, 5, 8);

4 (cont.)

0	3	2	1

median swap

1	3	2	0

0 3

lo	hi	p	i
0	3	0	1
		1	2
			3

$\boxed{0}$ 1 $\boxed{2 \mid 3}$

L final location R

qsort(A,0,0);

- simply returns

0

final location

qsort(A,2,3);

2	3

$\Downarrow$

3	2

$\Downarrow$

$\boxed{2}$ 3

L final location $R = \emptyset$

qsort(A,2,2);

$\Downarrow$

2

final location

qsort(A,4,3);

- returns

8	6	7	5

median swap

7	6	8	5
[5]	[6]	[7]	[8]

6

5 8

swap 7 with 5

lo	hi	p	i
5	8	5	6
		6	7
		7	8
			9

5	6	7	8

L final location R

qsort(A,5,6);

qsort(A,8,8);

4 (cont.)

qsort(A,5,6);

5	6

$\Downarrow$

median

6	5

$\Downarrow$

5 6

$\uparrow$ final location

R = ∅

qsort(A,5,5);

$\Downarrow$

5

final location

qsort(A,7,6);

-returns

qsort(A,8,8);

$\Downarrow$

8

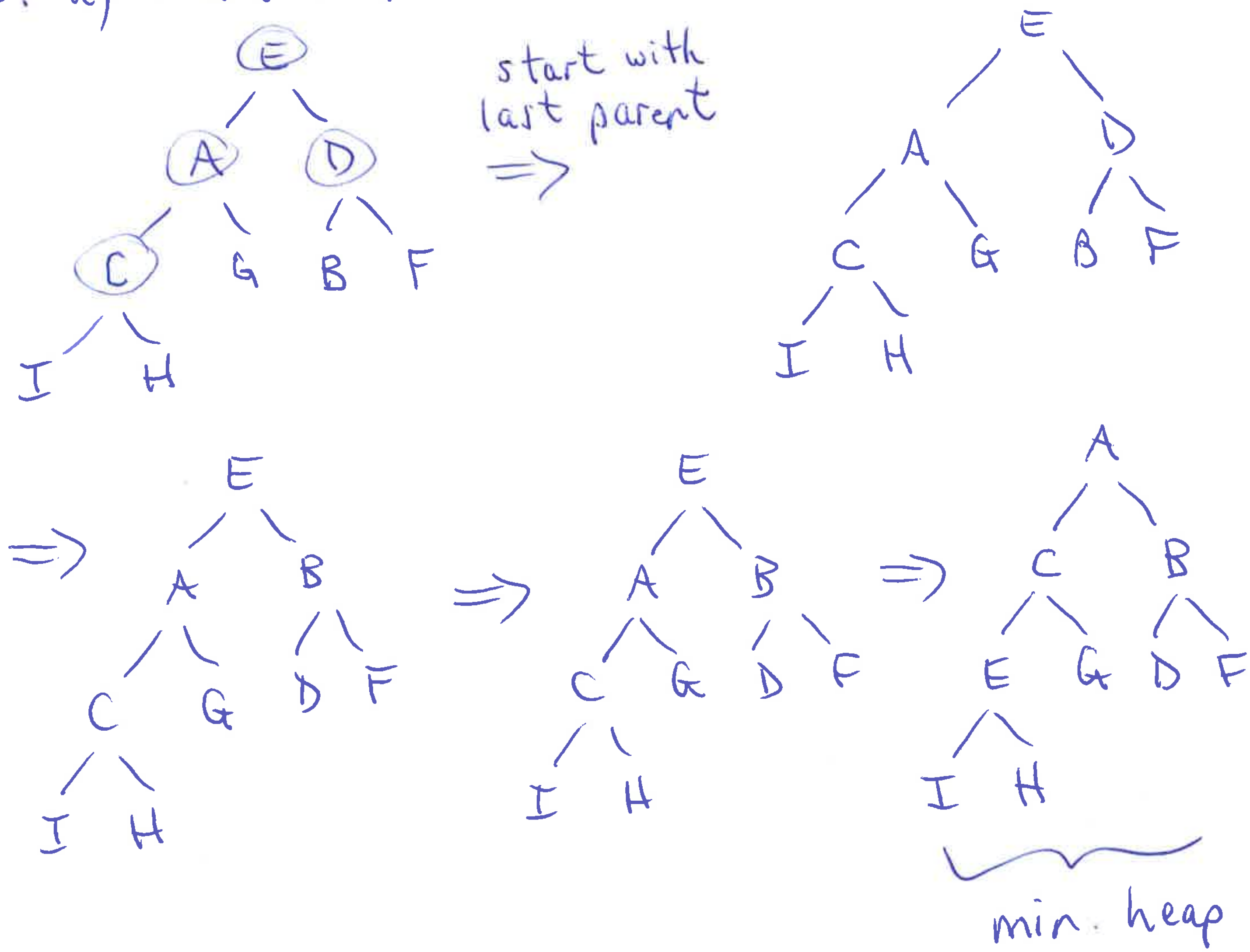
final location

Done. Read from left to right...

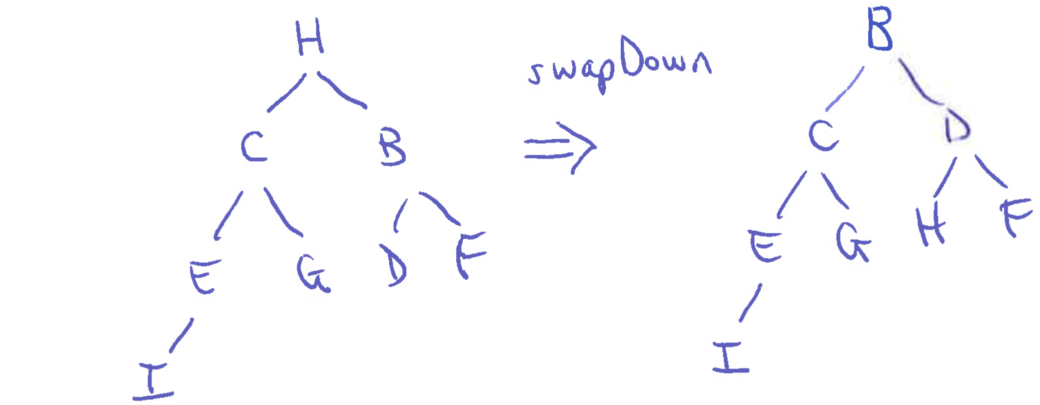
A =

0	1	2	3	4	5	6	7	8

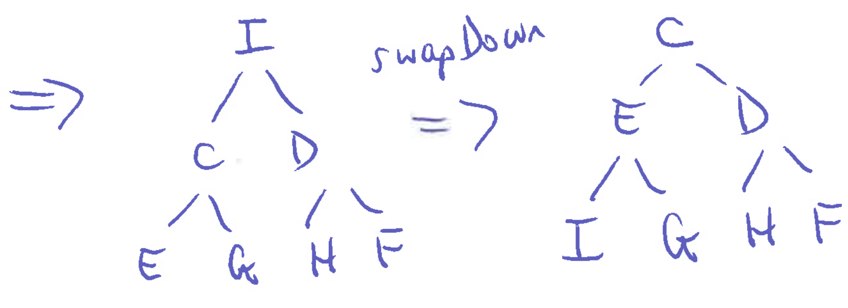
5. a) start with



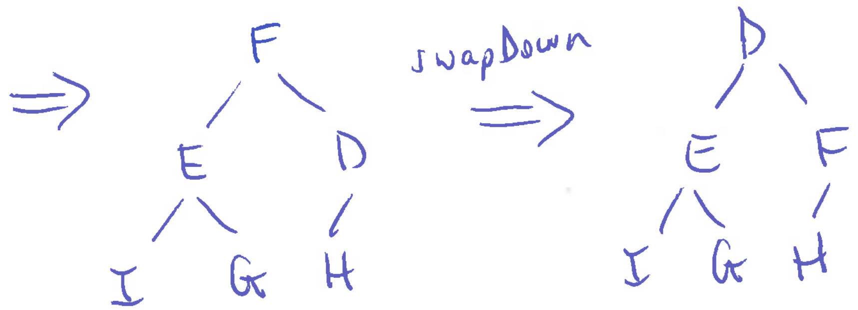
5 b) remove A (= deleteMin)



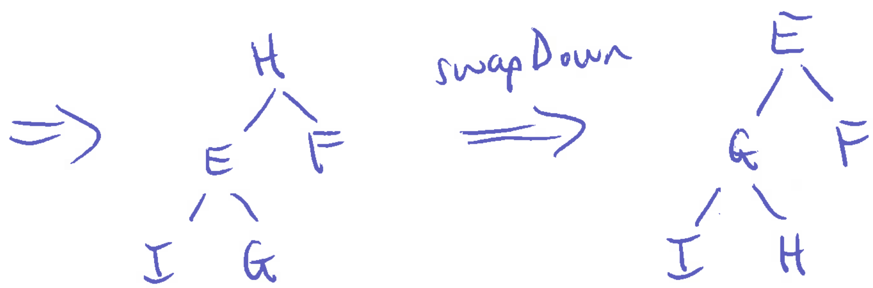
deleteMin (B)



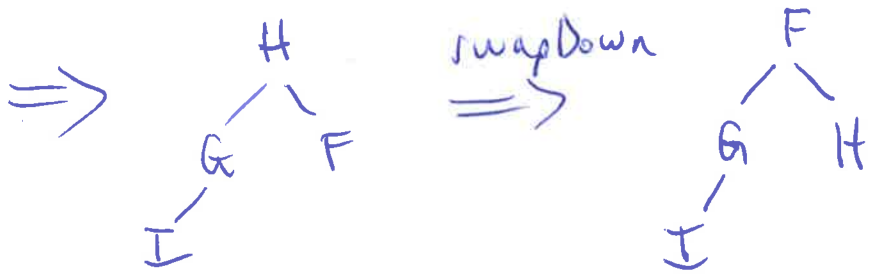
deleteMin (C)



deleteMin (D)

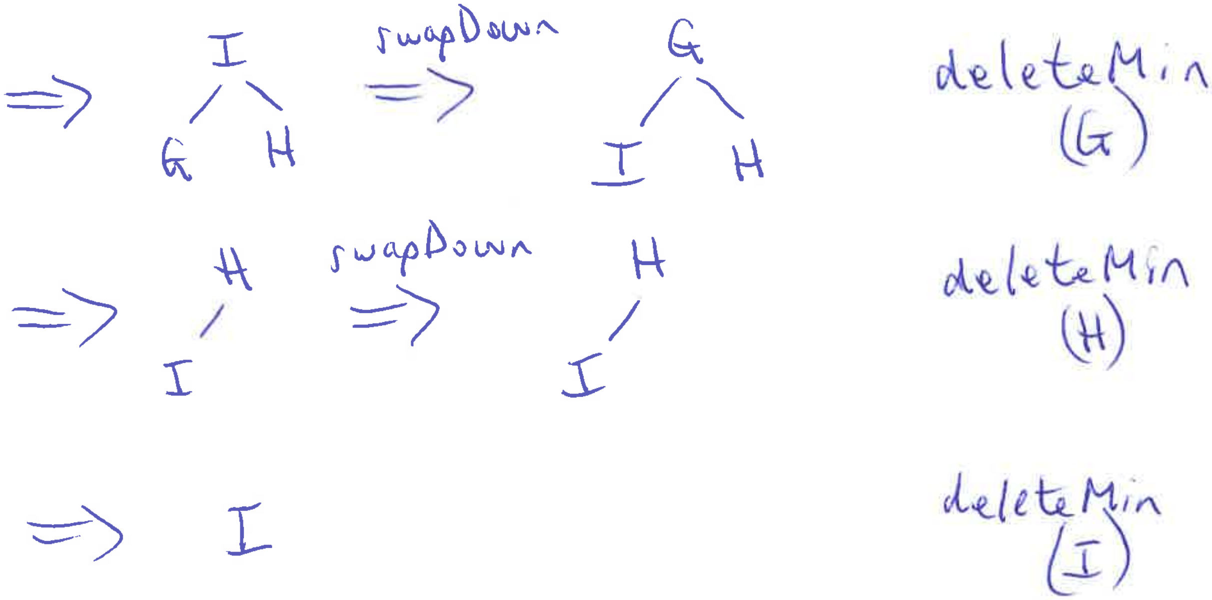


deleteMin (E)



deleteMin (F)

5 b) (cont.)



The sorted items were removed, in order, by deleteMin:

A, B, C, D, E, F, G, H, I

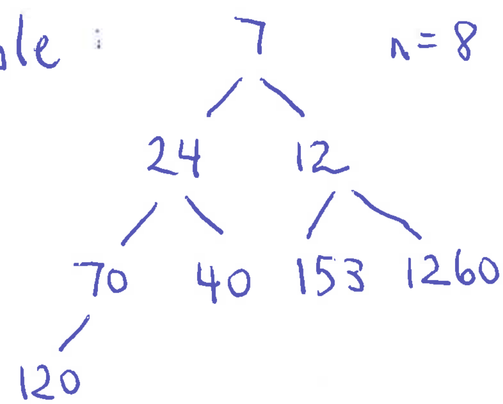
6. <u>Algorithm LessThanEq(root, q)</u> // assume min. heap

```
if (root == NULL)
    return
if (root -> key > q)
    return        // no need to check children

print root -> key
LessThanEq(root -> left, q)
LessThanEq(root -> right, q)
```

If there are k keys that are < q, then it runs in O(k) time because we only test a linear number of nodes that are unsuccessful (i.e., max 2 children per successful key/node).

Example:



if $q=70$, we print $k=4$ keys, but test 8 nodes

if $q=23$, we print $k=2$ keys, but test 5 nodes

if $q=8$, we print $k=1$ keys, but test 3

7. <u>Base case</u>: The pre-condition states that A and B are positive integers and product $=0$. So, at the start, $x=A$ and $y=B$; therefore, the loop invariant is met because:

"$xy + \text{product} = AB$"

$$LHS = AB + 0 = AB = RHS \checkmark$$

<u>Inductive Hypothesis</u>: At the end of iteration $i=k$, assume that the loop invariant holds, i.e., "$xy + \text{product} = AB$".

<u>Inductive Step</u>: Show that it's true for iteration $i=k+1$.

At the start of iteration $k+1$, assume we have values:

$$x_{OLD}\, y_{OLD} + \text{product}_{OLD} = AB$$

i.e., $x_k y_k + \text{product}_k = AB$ from iteration k.

There are two cases to check upon passing the termination check:

7 (cont.)

a) $y_k \% 2$ is 0 (because y_k is even)

If so, $\underbrace{x_{k+1}}_{\text{"}x_{NEW}\text{"}} = 2\underbrace{x_k}_{\text{"}x_{OLD}\text{"}}$ and $y_{k+1} = y_k/2$

$$\therefore\ x_{k+1}y_{k+1} + \text{product} = (2x_k)\left(\frac{y_k}{2}\right) + \text{product}_k$$
$$= x_k y_k + \text{product}_k$$
$$= AB \quad \text{by inductive hypothesis}$$

b) $y_k \% 2$ is 1 (odd case)

If so, $\text{product}_{k+1} = \text{product}_k + x_k$

and $y_{k+1} = y_k - 1$

and $x_{k+1}y_{k+1} + \text{product}_{k+1} = x_k(y_k - 1) + \text{product}_k + x_k$
$$= x_k y_k - \cancel{x_k} + \text{product}_k + \cancel{x_k}$$
$$= x_k y_k + \text{product}_k$$
$$= AB \quad \text{by the inductive hypothesis}$$

In both cases, the loop invariant is met, i.e.,

$$x_{k+1}y_{k+1} + \text{product}_{k+1} = AB.$$

7 (cont.)

Does the loop terminate?

A pre-condition of the loop is that $y = B \in \mathbb{Z}^+$.

In every iteration, y can only get smaller, and in only 2 ways:

a) if y_k is even:

$$\text{Then, } y_{k+1} = y_k/2$$

b) if y_k is odd:

$$\text{Then, } y_{k+1} = y_k - 1$$

At some finite point, y becomes 1, and then y becomes 0 in the next iteration.

The loop terminates when $y=0$, and this is at the top of the loop.

y never becomes negative.

$\therefore$ The loop terminates and the loop invariant holds:

$$\left.\underset{\substack{\uparrow \\ 0}}{x\,y} + \text{product} = AB \right\} \text{result of final iteration}$$

$\therefore$ product $= AB$

(post condition)

8. Assume that the loop invariant is the first line of the body of the while loop, so that at the start of iteration i:

"$pow = a^{i-1}$ and $i \leq n$".

Like Slide 16+ of Unit 3, we will show that the loop keeps the invariant true for the next iteration of the loop.

Finally, when the loop terminates after iteration n, $pow = a^n$ and $i \nleq n$.

<u>Base case</u>: Before the loop, $pow = 1$ and $i = 1$. The invariant holds:

$LHS = pow = 1$
$RHS = a^{i-1} = a^{1-1} = a^0 = 1 = LHS$.

<u>Inductive Hypothesis</u>:

Assume it's true for $i = k$

i.e., $pow = a^{k-1}$ and $k \leq n$ at the start of iteration k.

<u>Inductive Step</u>:

Show that it's true for the next iteration, when $i = k+1$

i.e., $pow = a^k$ and $k+1 \leq n$ at the start of iteration $k+1$.

8 (cont.)

By the I.H., at the start of iteration k, $pow_k = a^{k-1}$ and $k \leq n$.

During iteration k, $pow = pow_k * a = a^{k-1} * a$
$= a^k$ at the end of iteration k

and the loop counter increments to $k+1$.

Thus, at the start of iteration $k+1$, $pow = a^k$ and k increments to $k+1$.

Does the loop terminate?

Yes. The final iteration is when $k = n$. At the end of that iteration, $pow = a^k = a^n$, and the loop is not entered after that time (i.e., we break from the loop).

QED

because $k+1 \not\leq n$
$\uparrow$
n

9. Save for theory assignment 3

10. Save for theory assignment 3

11. Ternary tree of height h has:

$$n \le \frac{3^{h+1}-1}{2} \text{ nodes}$$

<u>Base case:</u> $n = 0 \Rightarrow h = -1$ (null case)

LHS = 0

$$\text{RHS} = \frac{3^{-1+1}-1}{2} = \frac{3^0-1}{2} = \frac{1-1}{2} = 0$$

LHS ≤ RHS ✓

<u>Base case 2:</u> $n = 1 \Rightarrow h = 0$ (optional)

LHS = 1

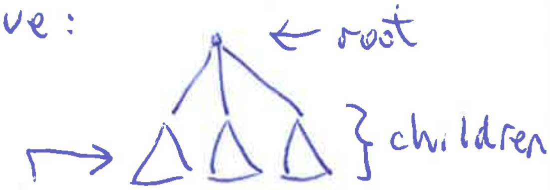
$$\text{RHS} = \frac{3^{0+1}-1}{2} = \frac{3^1-1}{2} = \frac{2}{2} = 1$$

LHS ≤ RHS.

<u>Inductive Hypothesis:</u>

Suppose the formula is true for all cases where # of nodes < n.

- Furthermore, suppose for n nodes, we have:



heights: h_1 h_2 h_3

\# nodes: n_1 n_2 n_3

i.e., $k \le \frac{3^{h+1}-1}{2}$ nodes for $k < n$

11. (cont.) Inductive Step: Show it's true for n nodes.

Let $n = 1 + n_1 + n_2 + n_3$

$h = 1 + \max\{h_1, h_2, h_3\}$

By the inductive hypothesis:

$$n_1 \leq \frac{3^{h_1+1}-1}{2} \; ; \; n_2 \leq \frac{3^{h_2+1}-1}{2} \; ; \; n_3 \leq \frac{3^{h_3+1}-1}{2}$$

$$n \leq 1 + \frac{3^{h_1+1}-1}{2} + \frac{3^{h_2+1}-1}{2} + \frac{3^{h_3+1}-1}{2}$$

$$\leq 1 + \frac{3^h-1}{2} + \frac{3^h-1}{2} + \frac{3^h-1}{2}$$

$$= 1 + \frac{3^h+3^h+3^h-1-1-1}{2}$$

$$= \frac{2+3^h+3^h+3^h-3}{2}$$

$$= \frac{3 \cdot 3^h - 1}{2}$$

$$= \frac{3^{h+1}-1}{2}$$

$$\therefore n \leq \frac{3^{h+1}-1}{2}$$

QED

12. Save for theory assignment 3.