RESEARCH SUMMARY

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I work in the field of geometric analysis and partial differential equations. My current interest includes wave equations and related fluid equations (such as magnetohydrodynamics equations and Boussinesq equations), as well as fully nonlinear elliptic equations (such as Monge-Ampère equations and k-Hessian equations).

1. Wave equations and related fluid equations

Wave equations are some of the most intensively studied topics since the beginning of the 1980s, and play a pivotal role in physics, including acoustics, electromagnetics and fluid mechanics. The goal of this program is to study the dynamics and nonlinear stability for wave equations and related fluid equations, especially magnetohydrodynamics (MHD) equations and Boussinesq equations. This part of our research is mainly based on the energy method.

1.1. Wave equations [1–3].

In [1]. We described the asymptotic behavior of small-data smooth solutions to the Cauchy problem for one-dimension quasilinear wave equations with null conditions from two standpoints: one was to provide **sharper convergence** from quasilinear waves to linear waves (free waves), the other was to prove the **rigidity from infinity for quasilinear waves**. In particular, the rigidity aspect can be summarized as:

the one-dimensional quasilinear waves must vanish everywhere
if their scattering fields vanish at infinities.

This work can be viewed as a model for the results and methods in [4,5].

In [2,3]. We studied the inverse scattering problem for $(1+1)$-dimensional semi-linear wave equations in [2]. Assuming null conditions, we first proved that small data lead to global existence of solutions. This result then allowed us to construct the scattering fields and their corresponding weighted Sobolev spaces at the infinities. Finally, we proved an **inverse scattering theorem** that the scattering operator not only describes the **scattering behavior** of the solution but also uniquely determines the solution. In [3], the above results were generalized to wave maps from Minkowski space $\mathbb{R}^{1+1}$ into general Riemannian manifolds, provided the initial data are small. We can conclude the main results in these two works that the **nonlinear scattering operators** can be linearized as the corresponding linear propagating operators, which is consistent with the following physical intuition:

the one-dimensional semilinear waves / wave maps behave exactly in the same manner as
their scattering fields detected by the faraway observers.

These two works can be viewed as models for the results and methods in [6].

1.2. **Alfvén waves governed by MHD equations [4–6].** The study of magnetohydrodynamics (MHD) concerns mutual interactions between electromagnetic fields and electrically conducting fluids, and our discussion is restricted to the ideal incompressible case. Because Alfvén waves can propagate and return, we are interested in an inverse scattering topic to recover the initial data that emanate from the plasma when given scattering fields at infinities, namely the faraway traces of solutions to the MHD system.

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In [4]. This is a collaboration work with Prof. Yu. The scattering theory (leading to rigidity) for Alfvén waves governed by the MHD system was the central issue of this work, which came as a surprise to most researchers because there were no prior analogous constructions. Precisely, we proved that the **scattering fields of Alfvén waves** are well defined by the traces of the solution at characteristic infinities, and more importantly, the **scattering rigidity for Alfvén waves** can be constructed as follows:

the Alfvén waves must vanish everywhere
if their scattering fields (in both two separate families) vanish at infinities.

This statement is rather striking due to its consistency with the following physical intuition:

there are no Alfvén waves at all emanating from the plasma
if no waves are detected by the faraway observers.

In essence, our rigidity results reflected a sense of uniqueness that one can recover the vanishing initial Alfvén waves from the vanishing scattering fields. This work also paves the way for [5,6].

In [5]. Based on [4], it was of interest to know whether a similar rigidity result still holds when the vanishing is only given for the scattering field of Alfvén wave in merely one family of the two separate families. Our purpose of [5] was to give such strengthened rigidity. In order to avoid confusion, we called the rigidity in [4] as the **bilateral rigidity** and the rigidity in [5] as the **unilateral rigidity**. We proved that

if scattering field of Alfvén wave in any family vanishes at infinity,
then Alfvén wave in this family must vanish, and moreover the fluid velocity
is equivalent in size to and collinear to the perturbation of magentic field.

In [6]. A natural extension of the rigidity phenomenon in [4] was whether we can recover the initial Alfvén waves from whatever the scattering fields are. This work also came as a surprise to most researchers because there were no prior analogous constructions: We established a set of **scattering isomorphisms** to substantially strengthen our previous rigidity results. In this way, we provided **a more satisfactory answer to the inverse scattering problem**: the scattering operator can uniquely determine solutions to the MHD system, and Alfvén waves can be accordingly reconstructed from the knowledge of their scattering fields at infinities. This answer is consistent with the physical intuition that

the Alfvén waves behave exactly in the same manner as
their scattering fields detected by the faraway observers.

1.3. **Boussinesq equations [7].** The Boussinesq system with anisotropic (full or partial) dissipation arises in atmospheric sciences and oceanic circulation, and plays an important role in the study of Raleigh-Bénard convective phenomena, mathematically and physically. It was extremely challenging to deal with the loss of partial dissipation and the boundary layer effects, especially for the bounded/unbounded domain with non-slip boundary.

In [7]. This is a collaboration work with Dr. Wang. We were interested in the **zero-viscosity limit behaviour** of 2-D Boussinesq system with partial viscosity and zero diffusivity in the half plane $\mathbb{R}^2_+$ satisfying the Navier boundary condition. To our best knowledge, this work was the first one to consider the strong zero-viscosity limit of Boussinesq equations with partial viscosity and boundaries, which is from physical consideration and involves layer effects. We proved the **nonlinear stability** of the approximate solution constructed by boundary layer expansion in conormal Sobolev space. The optimal expansion order and convergence rates for the inviscid limit were also identified in this work.

2. Fully nonlinear elliptic equations

Fully nonlinear elliptic equations arise in several geometric problems such as the Hilbert metric in convex domains, the affine spheres, the prescribed Gauss curvature equation, the L_p-Minkowski problem and the centroaffine Minkowski problem. The goal of this program is to study the existence (of solutions and of sub-solutions), uniqueness and regularity (including interior and boundary regularity) of solutions to fully nonlinear elliptic equations (especially Monge-Ampère equations and k-Hessian equations) over convex domains. This part of our research is mainly based on the sub- and super-solution method.

2.1. Boundary regularity and lower estimates [8–13].

In [8–10]. In a collaboration work [8] with Dr. Li, we were concerned with the **global regularity** of solutions to the Dirichlet problem for a class of Monge-Ampère type equations (over bounded convex domains) with zero boundary conditions. The above result was respectively generalized to the Dirichlet problem for a class of Monge-Ampère type equations (over bounded convex domains) with **nonzero boundary conditions** in [9] and to **singular** Monge-Ampère equations over convex domains being either bounded or **unbounded** in [10]. In these three works, by employing the concept of (a, η) type domain, we emphasized that the boundary regularity essentially depends on the convexity of the domain. We provided global Hölder estimates of convex solutions to our problems over convex domains based on carefully choosing auxiliary functions and constructing sub-solutions.

In [11]. This is a collaboration work with Dr. Li and Prof. Liu. We were interested in the boundary regularity for a class of k-Hessian equations which can be **degenerate and (or) singular** on the boundary of the domain. Motivated by the case of Monge-Ampère equations in [8], we first constructed sub-solutions, then applied the characteristic of the global Hölder continuity for convex functions, and finally used the maximum principle to obtain the boundary Hölder continuity for the solutions of the k-Hessian equations. However, finding such sub-solutions was rather difficult due to the complexity of the **k-Hessian operator**. In particular, we employed the symmetric mean to overcome the difficulties.

In [12, 13]. These are two collaboration works with Dr. Li. We studied the Dirichlet problem for a class of fully nonlinear elliptic equations. Under structural conditions, we established **boundary Hölder regularity (boundary estimates for the upper bound)** in [12] and **boundary estimates for the lower bound** in [13] for convex solutions to the problem. There were there major ingredients: the first was the application of an (a, η) type condition and the introduction of an (a, η, μ) type condition, which served as descriptions of convex domains and better adapted to boundary estimates; the second was the use of convex envelope of auxiliary functions over convex subdomains when constructing super-solutions; and the third was the exploration of **eigenvalue estimates** based on the observation that tangential eigenvalues and normal eigenvalues might become singular or degenerate with different speeds when the problem is singular or degenerate near the boundary. In these two works, as an application, we provided the optimal boundary regularity and lower estimates of convex solutions to hyperbolic affine sphere equations.

2.2. Existence, uniqueness and interior regularity [14].

In [14]. This is a collaboration work with Dr. Li. We focused on the Dirichlet problem for a class of Monge-Ampère type equations, which can be degenerate or singular near the boundary of convex domains. Based on the barrier functions established in [8], we further constructed viscosity subsolutions and viscosity supersolutions to this problem via comparison principle. Based on an adaption of the Perron method, we finally demonstrated the **existence, uniqueness and a series of interior regularities** (including $W^{2,p}$ with $p \in (1, +\infty)$, $C^{1,\mu}$ with $\mu \in (0, 1)$, and C^∞) of the viscosity solution to this problem.

2.3. Keller-Osserman type conditions [15, 16].

In [15, 16]. These are two collaboration works with Dr. Ji and Prof. Jiang. We provided necessary and sufficient conditions for the **solvability of entire admissible subsolutions** to the k-Hessian type equations in [15] and to the real $(n-1)$ Monge-Ampère equations in [16], which can be regarded as generalized Keller-Osserman conditions. We exploited the properties of radial symmetric functions and employed a revised Euler's break line method to derive local existence of Cauchy problems which then acted as a bridge leading to existence and nonexistence results of entire subsolutions to our problems. In particular, in [15], our results embraced the standard Hessian equation case as a typical example, and moreover the difference between the semilinear case ($k = 1$) and the fully nonlinear case ($k \geqslant 2$) was also concerned; in [16], we further investigated the conditions to ensure the existence of positive entire **radial** solutions, positive entire **large radial** solutions, and positive entire **bounded** solutions respectively, and moreover the nonexistence results were also extended to a more general type of Hessian quotient equations.

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