Assignment 1

Invitation to Algorithmic Graph Theory

Theoretical Assignment

June 9, 2023

Attempt atleast 2 questions out of 4.

Problem 1

Suppose we are given a directed graph G with n vertices, and let M be the $n \times n$ adjacency matrix corresponding to G.

a. Let the product of M with itself $\left(M^2\right)$ be defined, for $1 \leq i, j \leq n$, as follows

$$M^2(i,j) = M(i,1) \odot M(1,j) \oplus \cdots \oplus M(i,n) \odot M(n,j),$$

where " $\oplus$ " is the Boolean or operator and " $\odot$ " is Boolean and. Given this definition, what does $M^2(i,j) = 1$ imply about the vertices i and j ? What if $M^2(i,j) = 0$?

b. Suppose M^4 is the product of M^2 with itself. What do the entries of M^4 signify? How about the entries of $M^5 = \left(M^4\right)(M)$? In general, what information is contained in the matrix M^p ?

c. Now suppose that G is weighted and assume the following:

(a) [1.] For $1 \leq i \leq n, M(i,i) = 0$.

(b) [2.] For $1 \leq i, j \leq n, M(i,j) =$ weight (i,j) if (i,j) is in E.

(c) [3.] For for $1 \leq i, j \leq n, M(i,j) = \infty$ if (i,j) is not in E.

Also, let M^2 be defined, for $1 \leq i, j \leq n$, as follows

$$M^2(i,j) = \min\{M(i,1) + M(1,j), \ldots, M(i,n) + M(n,j)\}.$$

If $M^2(i,j) = k$, what may we conclude about the relationship between vertices i and j ?

Problem 2

Suppose you are given a timetable, which consists of:

- A set $\mathscr{A}$ of n airports, and for each airport a in $\mathscr{A}$, a minimum connecting time $c(a)$.
- A set $\mathscr{F}$ of m flights, and the following, for each flight f in $\mathscr{F}$:
- Origin airport $a_1(f)$ in $\mathscr{A}$
- Destination airport $a_2(f)$ in $\mathscr{A}$
- Departure time $t_1(f)$
- Arrival time $t_2(f)$

Describe an efficient algorithm for the flight scheduling problem. In this problem, we are given airports a and b, and a time t, and we wish to compute a sequence of flights that allows one to arrive at the earliest possible time in b when departing from a at or after time t. Minimum connecting times at intermediate airports should be observed. What is the running time of your algorithm as a function of n and m ?

Problem 3

Consider a diagram of a telephone network, which is a graph G whose vertices represent switching centers and whose edges represent communication lines joining pairs of centers. Edges are marked by their bandwidth, and the bandwidth of a path is the bandwidth of its lowest bandwidth edge. Give an algorithm that, given a diagram and two switching centers a and b, outputs the maximum bandwidth of a path between a and b.

Problem 4

A transitive reduction of a directed graph $G = (V, E)$ is any graph G' with the same vertices but with as few edges as possible, such that the transitive closure of G' is the same as the transitive closure of G. If G is a dag, then the transitive reduction of G is unique.

Write an algorithm to compute the transitive reduction of a digraph and compute its time complexity.