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Trigonometry Summary

CSE 4303 / CSE 5365 Computer Graphics

2020 Fall Semester, Version 1.2, 2020 September 04

Literally, *trigonometry* means *the measurement of triangles* (from the Greek τρίγωνον trígōnon → triangle + μέτρον métron → measure). While we won't in general be "measuring triangles" we will be making use of two of the six fundamental trigonometric functions. You must understand and be completely comfortable with this subset.

Preliminaries

Angles may be measured in either *radians* or *degrees*. There are 2π radians or 360 degrees in a circle. Radian measurements are expressed simply as a number with or without the word "radians" (or "rad") following. Degrees are expressed as a number which *must* be followed by the degree symbol $^\circ$ or the word "degrees" (or "deg").

Example: 5 or $5\,\mathrm{rad}$ means 5 radians. 30° or $30\,\mathrm{deg}$ means 30 degrees.

An angle measurement with no mark is in *radians*.

An angle measurement θ in degrees, θ°, may be converted to radians thus,

$$\theta\,\mathrm{rad} = \frac{\theta^\circ \cdot \pi}{180^\circ}$$

An angle measurement in radians, $\theta\,\mathrm{rad}$, may be converted to degrees thus,

$$\theta^\circ = \frac{\theta\,\mathrm{rad} \cdot 180^\circ}{\pi}$$

Many kinds of calculations are significantly easier when angles are expressed in radians. On the other hand, most persons seem to be more familiar with degree measurement. Get comfortable with either usage.

Many software packages require angles to be expressed in radians. Others require degrees. Be very careful that you know the standard for the software you use. (Some packages use radians for certain operations and degrees for others. Ouch!)

Angles are measured from the positive side of the x axis with *counter-clockwise* being *positive* angle measurement. *Clockwise* is *negative* angle measurement.

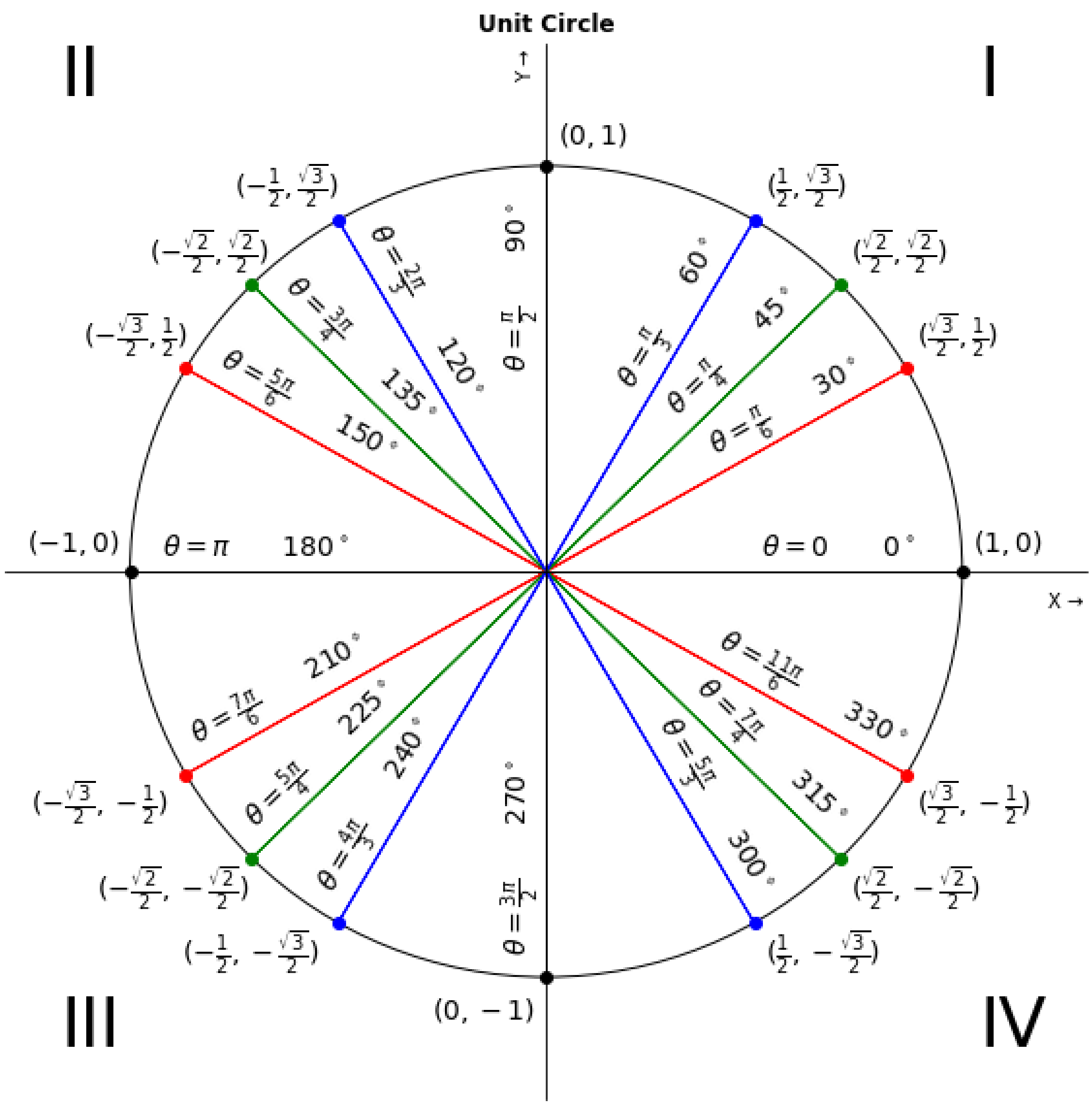
Quadrants and the Unit Circle

The *Cartesian plane* is divided into four *quadrants* by the x and y axes. These quadrants are known as the *first* through the *fourth* quadrants and are labelled with the roman numerals I, II, III, IV.

Quadrant	Coordinates
I	$x > 0, y > 0$
II	$x < 0, y > 0$
III	$x < 0, y < 0$
IV	$x > 0, y < 0$

The coordinate axes themselves are ***not*** in any quadrant; they are the boundaries *between* quadrants.

The *Unit Circle* is a circle with radius $= 1$ centered at the origin $= (0, 0)$.



The angles shown in the *Unit Circle* plot fit simple patterns (which makes them easy to remember).

While this diagram looks complicated and scary, one soon realizes that there are only *four* angles, each occuring four times in $\frac{\pi}{2}$ (90°) steps. If one learns the three angles in Quadrant I and the angle $\frac{\pi}{2}$ (90°), one knows all the rest. Just remember to use the proper $+$ and $-$ signs for the values associated with the angles in the other quadrants.

- When $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ radians $(0^\circ, 90^\circ, 180^\circ, 270^\circ)$, the angle meets the unit circle at $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. Notice that the x, y coordinate pairs all have one 0 and one of ± 1.

- For $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ $(30^\circ, 150^\circ, 210^\circ, 330^\circ)$, the angle meets the unit circle at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Notice that the x, y coordinate pairs are all $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$, or $(\pm 0.866, \pm 0.500)$ when rounded to three digits.

  (By the way, the number $\frac{\sqrt{3}}{2} = 0.866025404 \ldots$ occurs in many places in graphics. Learn to recognize it.)

- For $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ $(45^\circ, 135^\circ, 225^\circ, 315^\circ)$, the angle meets the unit circle at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Notice that the x, y coordinate pairs are all $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$, or $(\pm 0.707, \pm 0.707)$ when rounded to three digits.

  (By the way, the number $\frac{\sqrt{2}}{2} = 0.707106781 \ldots$ occurs in many places in graphics. Learn to recognize it.)

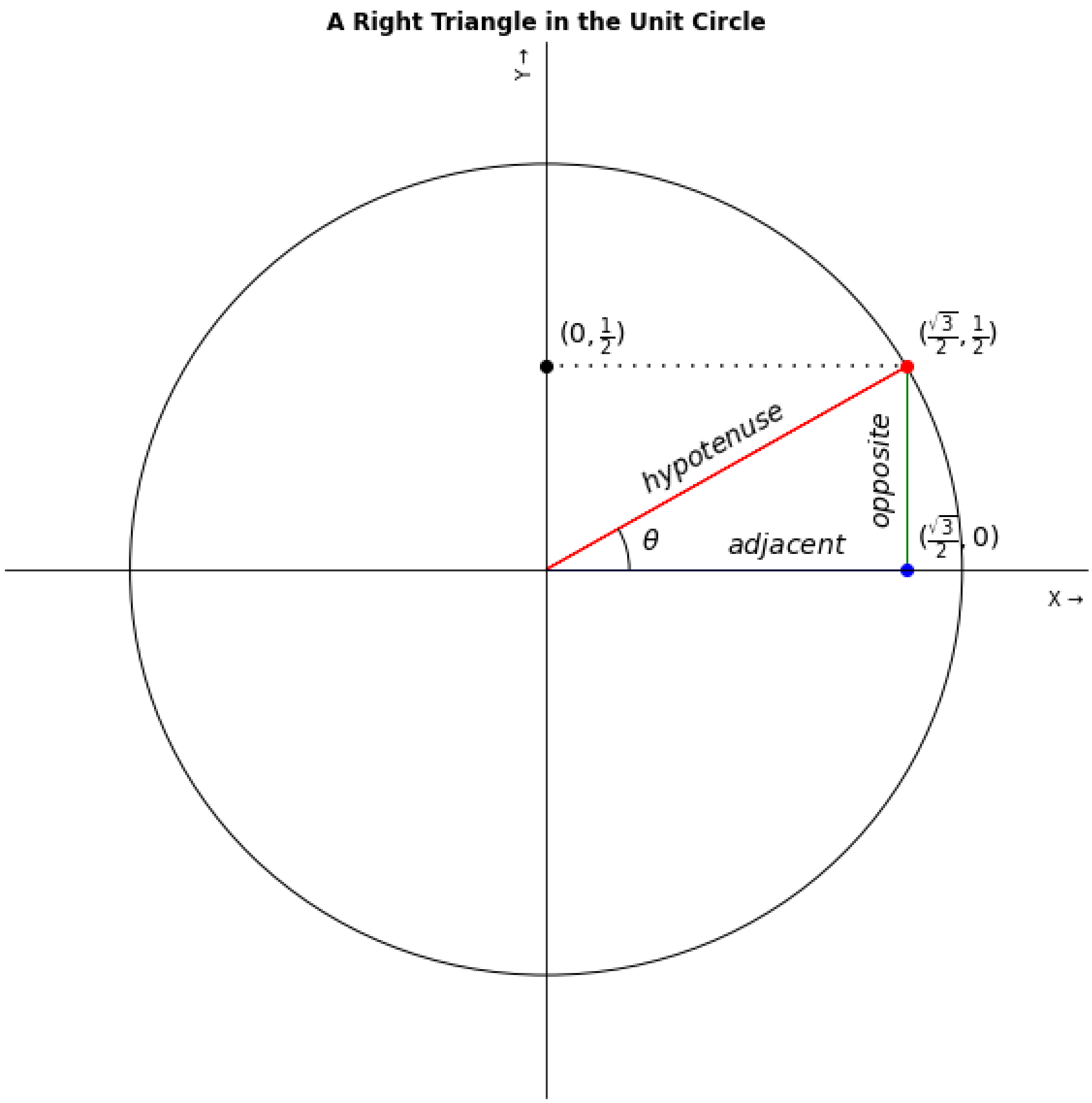
- For $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ $(60^\circ, 120^\circ, 240^\circ, 300^\circ)$, the angle meets the unit circle at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Notice that the x, y coordinate pairs are all $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, or $(\pm 0.500, \pm 0.866)$ when rounded to three digits.

Notice that the only numbers that show up as coordinates are $0, \pm\frac{1}{2} = 0.5, \pm\frac{\sqrt{2}}{2} = \pm 0.707106781 \ldots, \pm\frac{\sqrt{3}}{2} = \pm 0.866025404 \ldots$, and ± 1. It's not hard to learn these.

Angle measurement is not unique. Since a circle has 2π radians (360°), angles that are a multiple of that value apart have the same properties. For example, $\frac{\pi}{4}$ and $\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi$ $(45^\circ \text{ and } 405^\circ = 45^\circ + 360^\circ)$ both meet the unit circle at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Sine and Cosine

The trigonometric functions are customarily defined using a right triangle with its origin at the center of the Cartesian plane, $(0, 0)$.



In this example, the angle θ between the *adjacent* line and the *hypotenuse* line is $\frac{\pi}{6}$ (30°).

The *sine* of an angle is the y component of the spot where the *hypotenuse* of the right triangle touches the unit circle. In this case, the value of $\sin\frac{\pi}{6} = \frac{1}{2}$. This is also the length of the *opposite* side of the triangle.

The *cosine* of an angle is the x component of the spot where the *hypoteneuse* of the right triangle touches the unit circle. In this case, the value of $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. This is also the length of the *adjacent* side of the triangle.

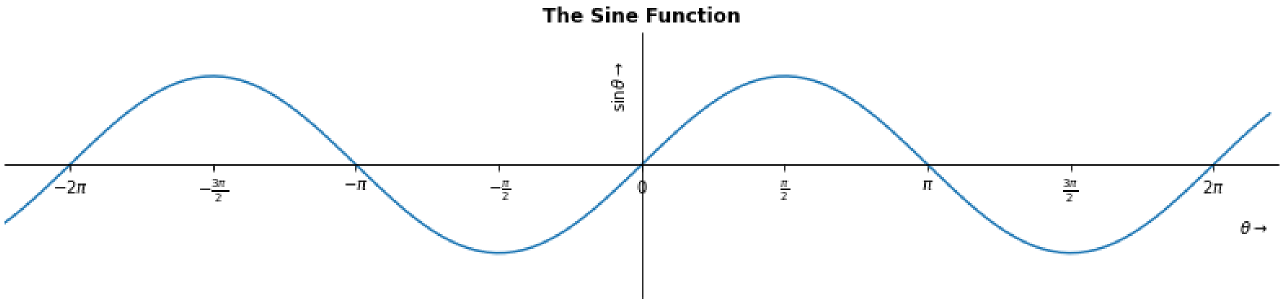
If one has a right triangle that is *not* inscribed in the unit circle (that is, the length of the hypotenuse is $\neq 1$, one can make use of these definitions:

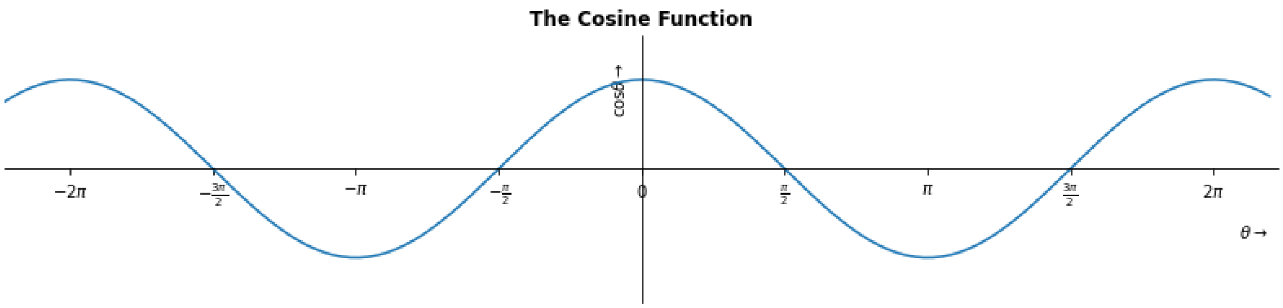
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Since $\sin$ and $\cos$ are defined on the unit circle, we have $\sin^2\theta + \cos^2\theta = 1$, by the Pythagorean theorem, for *any* angle θ. This is useful for solving various trigonometry problems and for deriving handy trigonometric identities.

There are six fundamental trigonometric functions, but the others do not come up as much in graphics as do $\sin$ and $\cos$. However, in the spirit of completeness, their definitions are given in an appendix.





As one can see in the above graphs of the *sine* and *cosine* functions, they are periodic, with a period of 2π.

Sine starts at 0 for $\theta = 0$ moves up to $+1$ at $\theta = \frac{\pi}{2}$ then back to 0 at $\theta = \pi$. From there, it goes negative, reaching -1 at $\theta = \frac{3\pi}{2}$ then back to zero at $\theta = 2\pi$.

On the other hand, *cosine* starts at $+1$ for $\theta = 0$ moves down to 0 at $\theta = \frac{\pi}{2}$ then continues down to -1 at $\theta = \pi$. From there, it swings back up, reaching 0 at $\theta = \frac{3\pi}{2}$ then peaks again at $+1$ for $\theta = 2\pi$.

Some Properties of $\sin$ and $\cos$

Immediately obvious from the graphs above is that $\sin$ and $\cos$ are the *same* function in that they generate the same *shape* of result, just not at the same *time*. That is, $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$ and $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$. We call this difference between the two a *phase shift*.

Being periodic, $\sin$ and $\cos$ are not one-to-one, that is there are multiple values of θ that will yield the same value of $\sin$ and $\cos$. Once again, this is clear by looking at the two graphs above. The values of $\sin$ and $\cos$ are guaranteed to repeat every 2π. That is, $\sin\theta = \sin(\theta + 2n\pi)$ and $\cos\theta = \cos(\theta + 2n\pi) \quad \forall\, n \in \mathbb{Z}$.

Though periodic, $\sin$ is also *ambiguous* within its period. Taking a look at the graph, we see that $\sin\theta$ reverse repeats the values $[0, 1)$ from the domain $\left[0, \frac{\pi}{2}\right)$ to the values $(1, 0]$ for the domain $\left(\frac{\pi}{2}, \pi\right]$. Similarly, it reverse repeats the values $[0, -1)$ from the domain $\left[\pi, \frac{3\pi}{2}\right)$ to the values $(-1, 0]$ for the domain $\left(\frac{3\pi}{2}, 2\pi\right]$.

Thus, if we want a *non-ambiguous* interpretation of $\sin$, we must stay within the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\cos$ has the same ambiguity issue as $\sin$ but because of the phase shift between the two functions, the domains are different. $\cos\theta$ reverse repeats the values $[0, 1)$ from the domain $\left[-\frac{\pi}{2}, 0\right)$ to the values $(1, 0]$ for the domain $\left(0, \frac{\pi}{2}\right]$. Similarly, it reverse repeats the values $[0, -1)$ from the domain $\left[\frac{\pi}{2}, \pi\right)$ to the values $(-1, 0]$ for the domain $\left(\pi, \frac{3\pi}{2}\right]$.

Thus, if we want a *non-ambiguous* interpretation of $\cos$, we must stay within the domain $[0, \pi]$.

Angle Negation

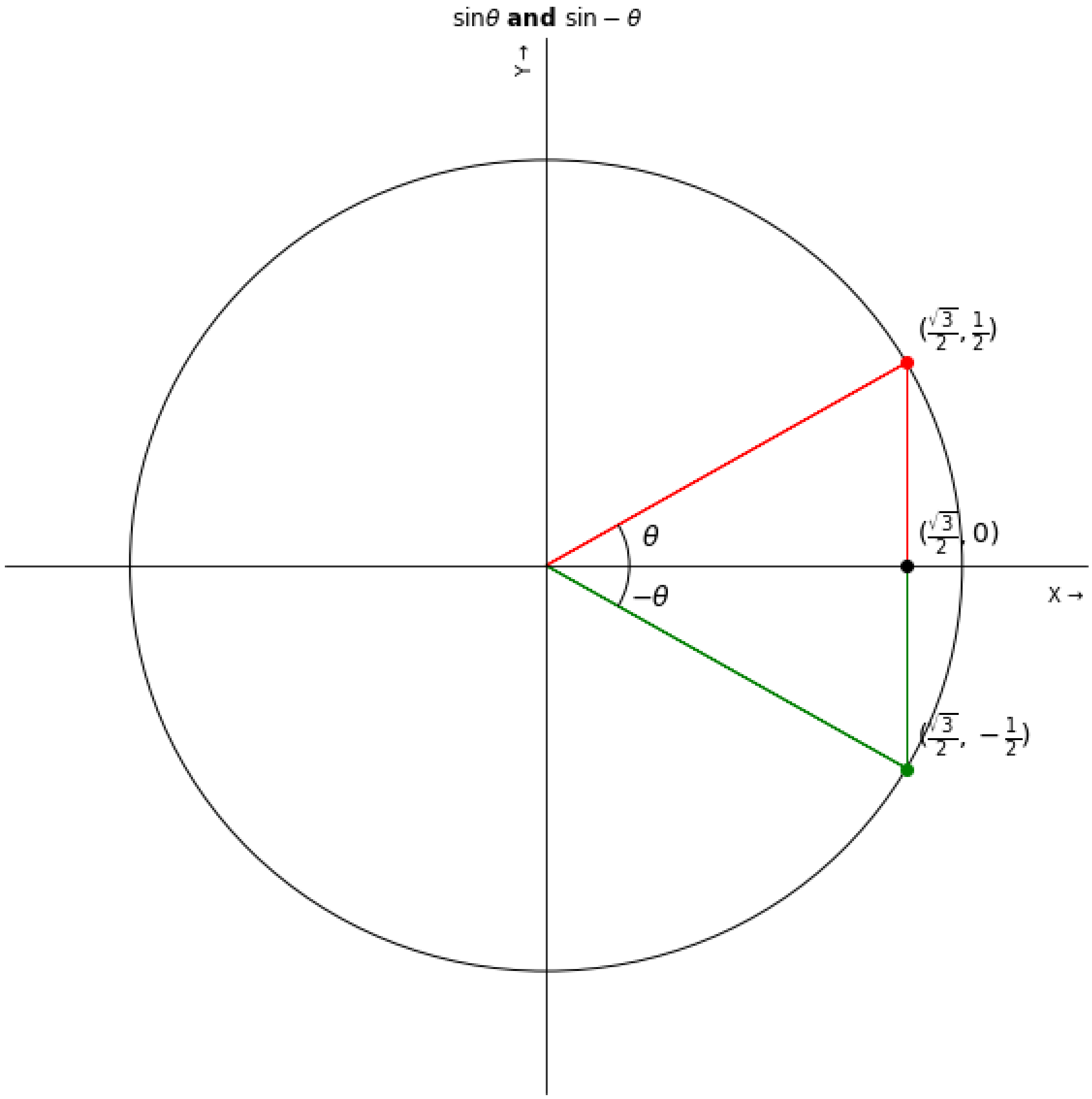
Given the periodic and ambiguous nature of the $\sin$ function, there are other interesting properties that can be discovered. One is the relationship between the value of $\sin\theta$ and the value of $\sin -\theta$.

This is easiest to visualize if we consider first angles in the first quadrant, $0 < \theta < \frac{\pi}{2}$. For these angles, $\sin\theta$ is in the range $(0, 1)$.

Since θ is *positive*, it goes *counter-clockwise*. Therefore, $-\theta$ is *negative*, so it goes *clockwise*. This angle will be in the fourth quadrant (since θ is $< \frac{\pi}{2}$, $-\theta > -\frac{\pi}{2}$, so it doesn't make it to the third quadrant). Looking at the fourth quadrant, we see that all angles here have $\sin < 0$.

Take a look at the example in the following plot. Here we have $\theta = \frac{\pi}{6}$ so $-\theta = -\frac{\pi}{6}$

The two triangles are *similar* since they are both right triangles and the sizes of their angles rooted at the origin are the same (it doesn't matter that one angle is positive measure and the other one is negative measure; the magnitude is what counts when considering similarity). In fact, the two triangles are *identical* in size, just mirrored about the x axis. Therefore the distance from the x axis to the point each triangle's hypotenuse touches the unit circle is the same, though opposite in sign.



We can therefore conclude that for angles with measurements in the first quadrant, $\sin -\theta = -\sin\theta$.

By symmetry, the same argument works for angles with measurements in the fourth quadrant. This time the original angle is negative and the negated angle is positive.

We construct a “reflected” argument for positive angles in the second quadrant. Reflected in the sense that the two triangles will be on the left of the y axis instead of the right, but the argument is otherwise the same. Additionally, the same symmetry argument works for angles with measurements in the third quadrant.

For angles that coincide with the coordinate axes (remember, the axes are *not* in any quadrant!), the argument is even simpler. $\sin \frac{\pi}{2} = 1$ and $\sin -\frac{\pi}{2} = -1$ so it's obvious that $\sin -\theta = -\sin\theta$ in this case. Finally, $\sin 0 = \sin -0 = \sin \pi = \sin -\pi = 0$, negation not mattering when the result is 0.

We have therefore covered all angles with measurements in the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We extend the argument to $\theta \in \mathbb{R}$ by remembering that any angle θ that is outside the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ may be reduced to that range by successive addition or subtraction of 2π without changing the value of $\sin\theta$.

Since $\sin -\theta = -\sin\theta$, it is known as an *odd* function. (“Odd” in the sense “not even” as opposed to “weird” :) Further, $\sin$ is symmetric about the *origin*.

Well, that takes care of $\sin$. What about the relationship between $\cos\theta$ and $\cos -\theta$?

As one might expect, we follow an amazingly similar line of reasoning. We can even use the same diagram of triangles as was used in the $\sin$ case.

What's different is that we see that for angles with measurements in the first quadrant, $\cos\theta = \cos -\theta$. That shouldn't be surprising as $\cos$ is concerned with measurement along the x axis and simply negating the angle doesn't change its position projected along the x axis, in the example being the value $\frac{\sqrt{3}}{2}$. The same symmetry and reflection argument works for angles with measurements in the fourth quadrant, second quadrant, and third quadrant.

There a bit of difference in the arguments for angles that coincide with the axes, in that now it's $\cos\frac{\pi}{2} = \cos -\frac{\pi}{2} = 0$, $\cos \pi = \cos -\pi = -1$, and $\cos 0 = \cos -0 = 1$, but we get the same support for $\cos\theta = \cos -\theta$.

We use the same reasoning as we used in the $\sin$ case to extend this $\cos$ argument to all values $\theta \in \mathbb{R}$.

Since $\cos -\theta = \cos\theta$, it is known as an *even* function. (“Even” in the sense ”not odd“ as opposed to “uniform”) Further, $\cos$ is symmetric about the y axis.

Sum and Difference Identities

Once one understands computing $\sin\theta$ and $\cos\theta$, one can move on to sums and differences of angles. Here we'll just assert the identities. (Take a look at an appendix for the derivations.)

In particular,

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

Once one knows the sum identities, the difference identities are absolutely trivial. (Why?)

> $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

> $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

There are a heck of a lot of other trigonometric identities, but the ones we've discussed so far are the ones that come up most frequently.

Inverse Trigonometric Functions

The $\sin$ and $\cos$ functions take arbitrary angles and return values in the range $[-1, +1]$. Given a value in the range $[-1, +1]$, we can also ask which angle has that value as its *sine* or *cosine*. These inverse functions are known as $\arcsin$ and $\arccos$ (sometimes abbreviated to *asin* and *acos*).

> Do *not **ever*** write $\sin^{-1}$ or $\cos^{-1}$ to mean the inverse of $\sin$ or $\cos$. The irregularity of this notation can cause confusion.
>
> The notation is “irregular” in the sense that for any number n, $\sin^n x$ should mean $(\sin x)^n$. Why make a special exception for the $n = -1$ case? This abomination of a notation was introduced by John Herschel is 1813. While Herschel was no doubt a brilliant man (take a look at his biography on Wikipedia), he was definitely smoking crack when he used $\sin^{-1}$ to mean $\arcsin$.

We decribe $\arcsin$ and $\arccos$ as “functions” — and they are. While it is true that, e.g., an infinite number of angles $\theta = \frac{\pi}{4} + 2n\pi, n \in \mathbb{Z}$ have $\sin\theta = \frac{\sqrt{2}}{2}$, there is only one angle returned by $\arcsin\frac{\sqrt{2}}{2}$, and that is $\frac{\pi}{4}$. This is known as the *principal value* of $\arcsin$. The $\arcsin$ function is *defined* to return an angle in the range $\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$.

The $\arccos$ function also returns a single value, but its range is *defined* to be $[0, \pi]$.

This difference in range between $\arcsin$ and $\arccos$ occurs because of the difference in where the ambiguities occur with $\sin$ and $\cos$. As mentioned above, $\sin$ is ambiguous on the domain $[0, \pi]$ and $\cos$ is ambiguous on the domain $\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$ so we can't pick one or the other of these and use it as the range of *both* $\arcsin$ and $\arccos$ if we want them to be unambiguous.

Appendix: The Six Fundamental Trigonometric Functions

As said previously, there are six fundamental trigonometric functions.

Hmm, why exactly six? Why not four? Or maybe eight? Or some other number?

Well, think about it for a moment.

We said “trigonometry” was all about the measuring of a triangle. The trigonometric functions assist in that measurement by representing the *ratios* of the lengths of various sides of a right triangle. Since a triangle has three sides, there are six possible ratios, three possible numerators times two possible denominators for each numerator. (No, we don't divide a length by itself. The constant 1 isn't a very useful ratio. :)

Thus, six possible fundamental trigonometric functions. Obvious, once you know the answer, huh? Anyway, here are the six possible ratios in terms of the side names we used in the figure *A Right Triangle in the Unit Circle* given previously,

Sine

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

Cosine

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Tangent

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Cosecant

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

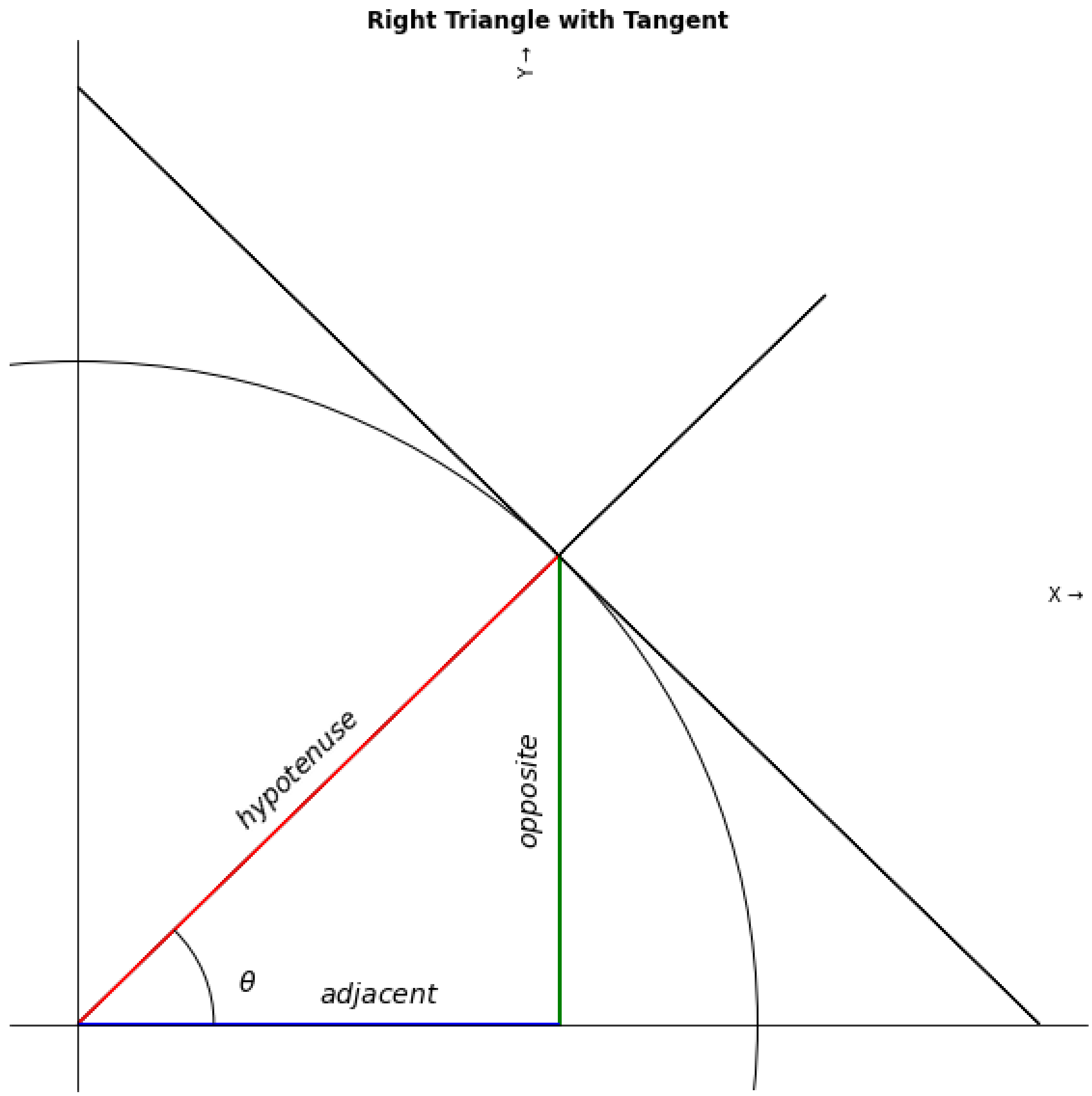
Secant

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

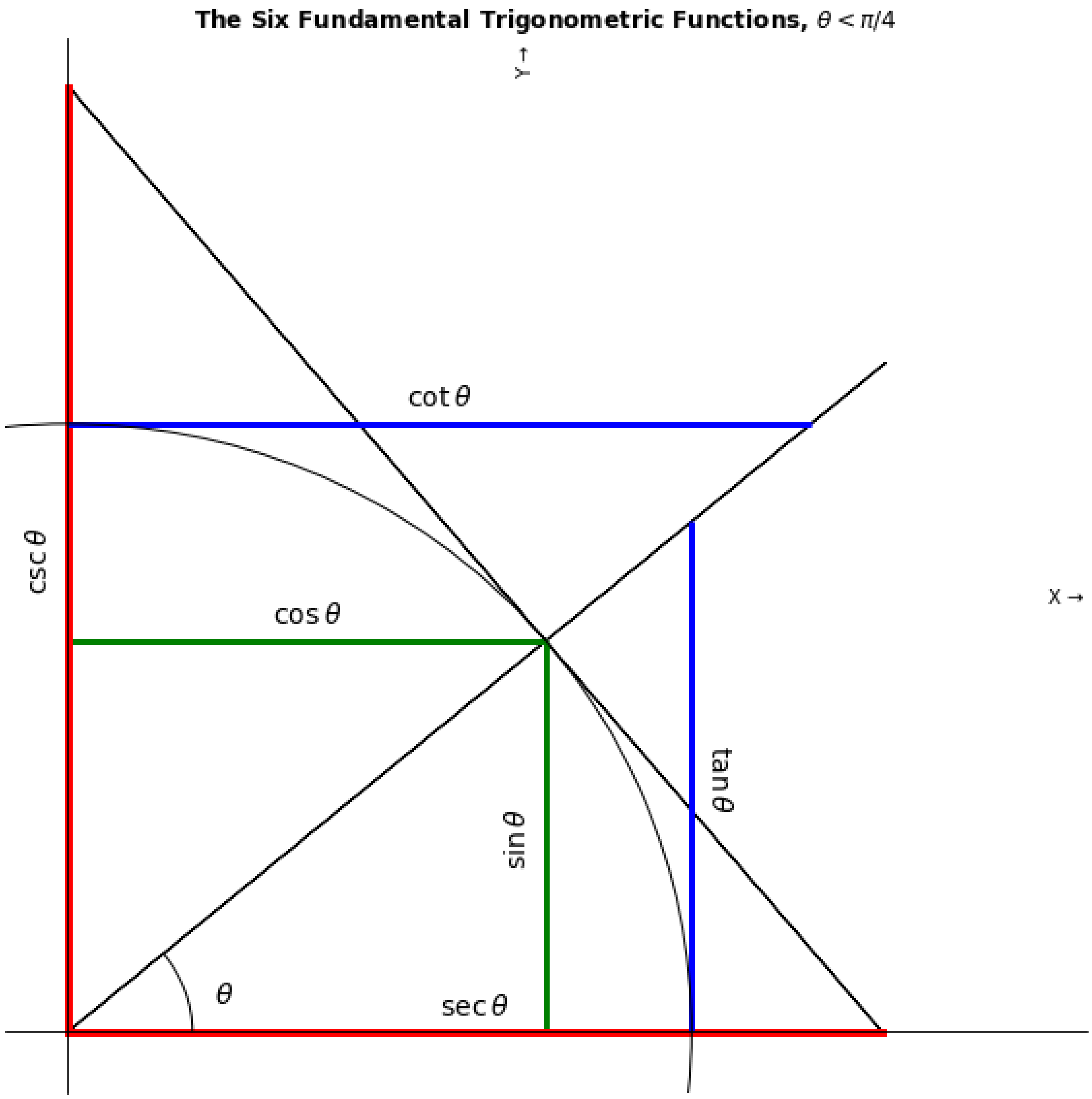
Cotangent

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Looking at these ratios, it might not be immediately obvious why they are useful in “measuring triangles”. Take a look at the following diagram, where we show the original right triangle in the unit circle but have shown a continuation of the hypotenuse line beyond the unit circle. We've also added a tangent line at the point where the hypotenuse touches the unit circle.



Once that's clear, take a look at this next diagram, where we have added line segments representing the values of each of the six fundamental trigonometric functions. This diagram has been drawn with $\theta \neq \frac{\pi}{4}, 45^\circ$ so that $\sin\theta \neq \cos\theta$ and $\tan\theta \neq \cot\theta$.



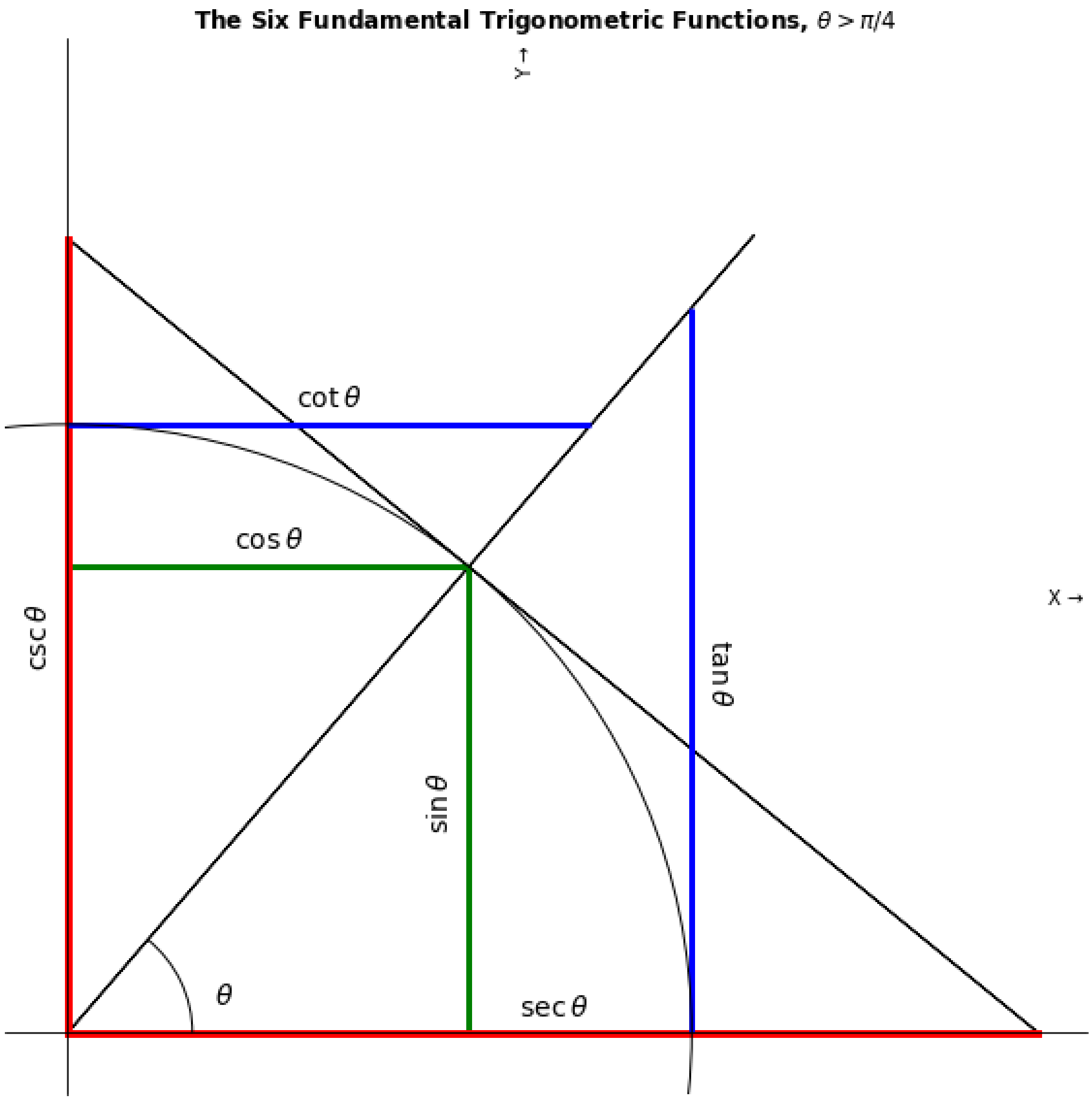
The line segments representing $\sin\theta$ and $\cos\theta$ are easily recognizable, $\sin\theta$ is the distance from the x axis to the point where the hypotenuse touches the unit circle. Similarly, $\cos\theta$ is the distance from the y axis to the point where the hypotenuse touches the unit circle.

The tangent and cotangent functions are related to the hypotenuse as well, but in slightly different ways than the sine and cosine functions. $\tan\theta$ is the length of the line segment dropped from the (extension of the) hypotenuse such that the segment is perpendicular to the x axis and intersects the x axis at the same point as the unit circle does, that is, at $(1,0)$.

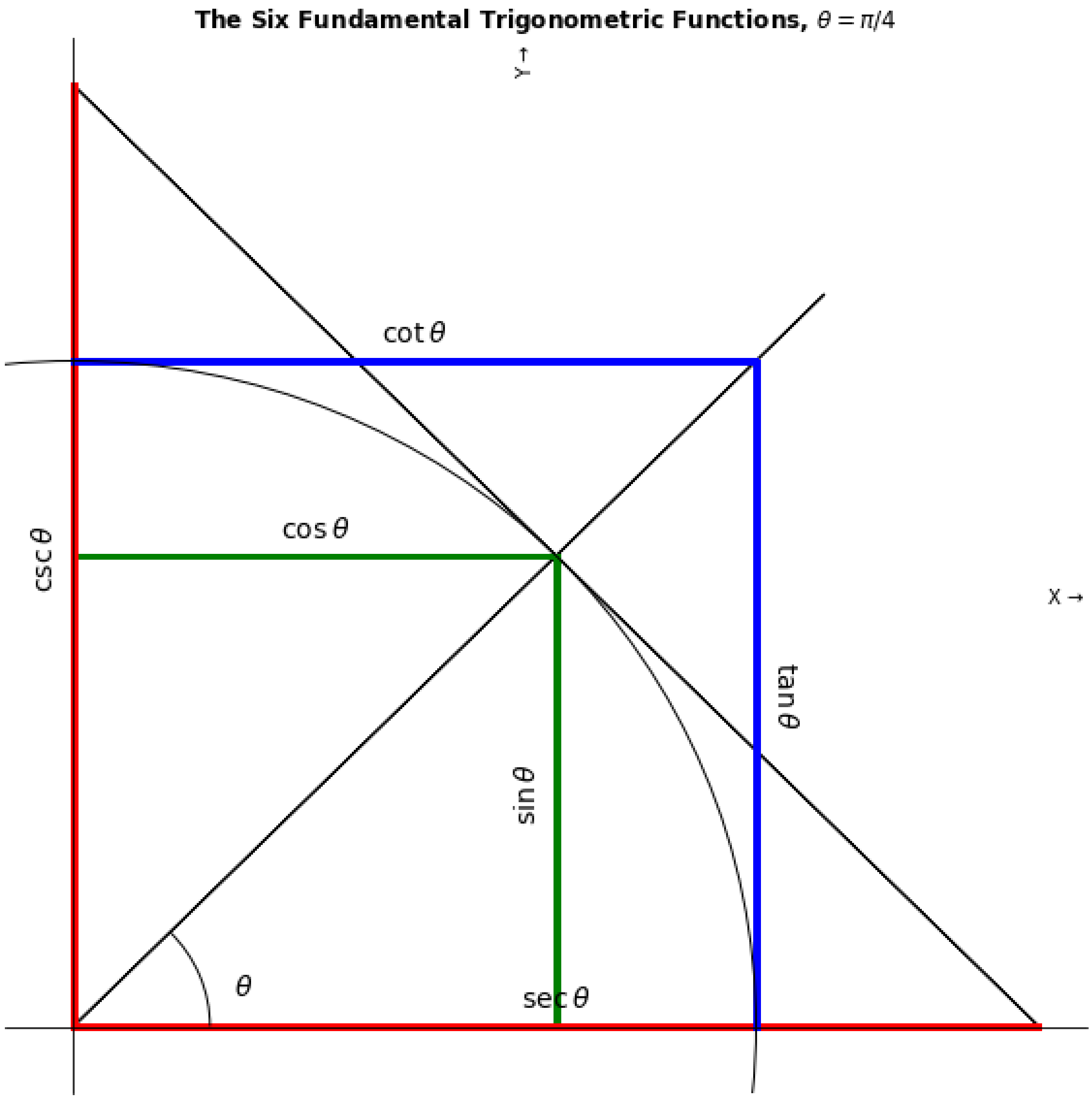
On the other hand, $\cot\theta$ is the length of the line segment dropped from the (extension of the) hypotenuse such that the segment is perpendicular to the y axis and intersects the y axis at the same point as the unit circle does, that is, at $(0, 1)$.

The secant and cosecant functions are related to the tangent line we've added. This line is tangent to the circle at the point the hypotenuse touches the unit circle and we're interested in where this tangent line intersects the x and y axes. $\sec\theta$ tells us where it intersects the x axis, $\csc\theta$ tells us where it intersects the y axis.

In the above digram, we've drawn the lines as they would look for an angle that's less than $\frac{\pi}{4}, 45^\circ$. In the following diagram, the angle is greater than $\frac{\pi}{4}, 45^\circ$. Notice the difference this makes for how $\tan\theta$ and $\cot\theta$ intersect the extention of the hypotenuse.



In the spirit of completeness, here's the diagram when $\theta = \frac{\pi}{4}$. As expected, $\sin = \cos$, $\tan = \cot$, and $\sec = \csc$.

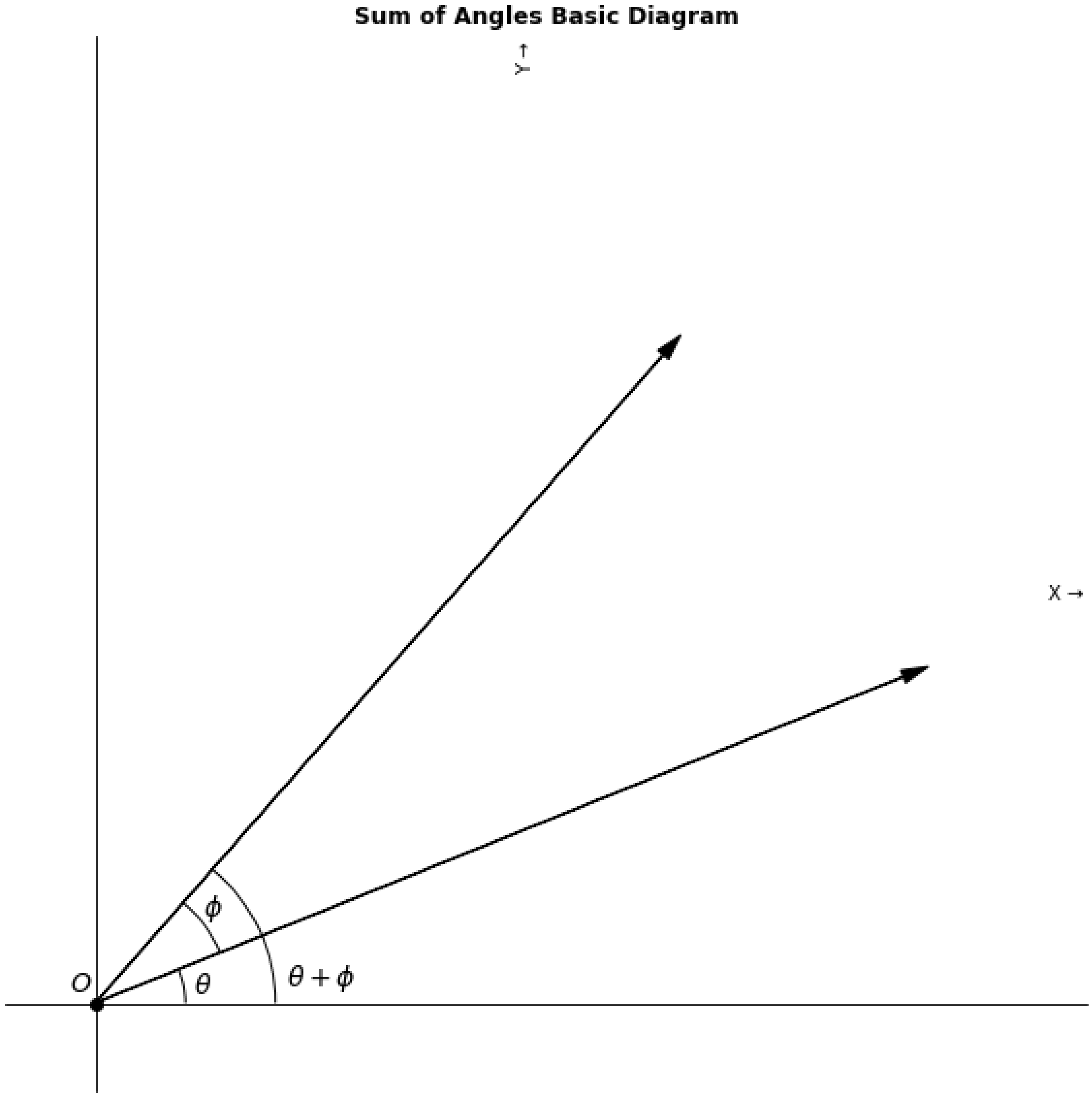


Appendix: Deriving Sum-of-Angles Rules

Above we just *claimed* that $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$, etc. These equations can be derived using simple plane geometry, if one thinks about the problem the correct way.

> The *easiest* way to derive this identity, however, is to use Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, but this equation usually doesn't come up until complex analysis, so using it might not be considered “fair”. For completeness, this derivation is included after the “simple” one.

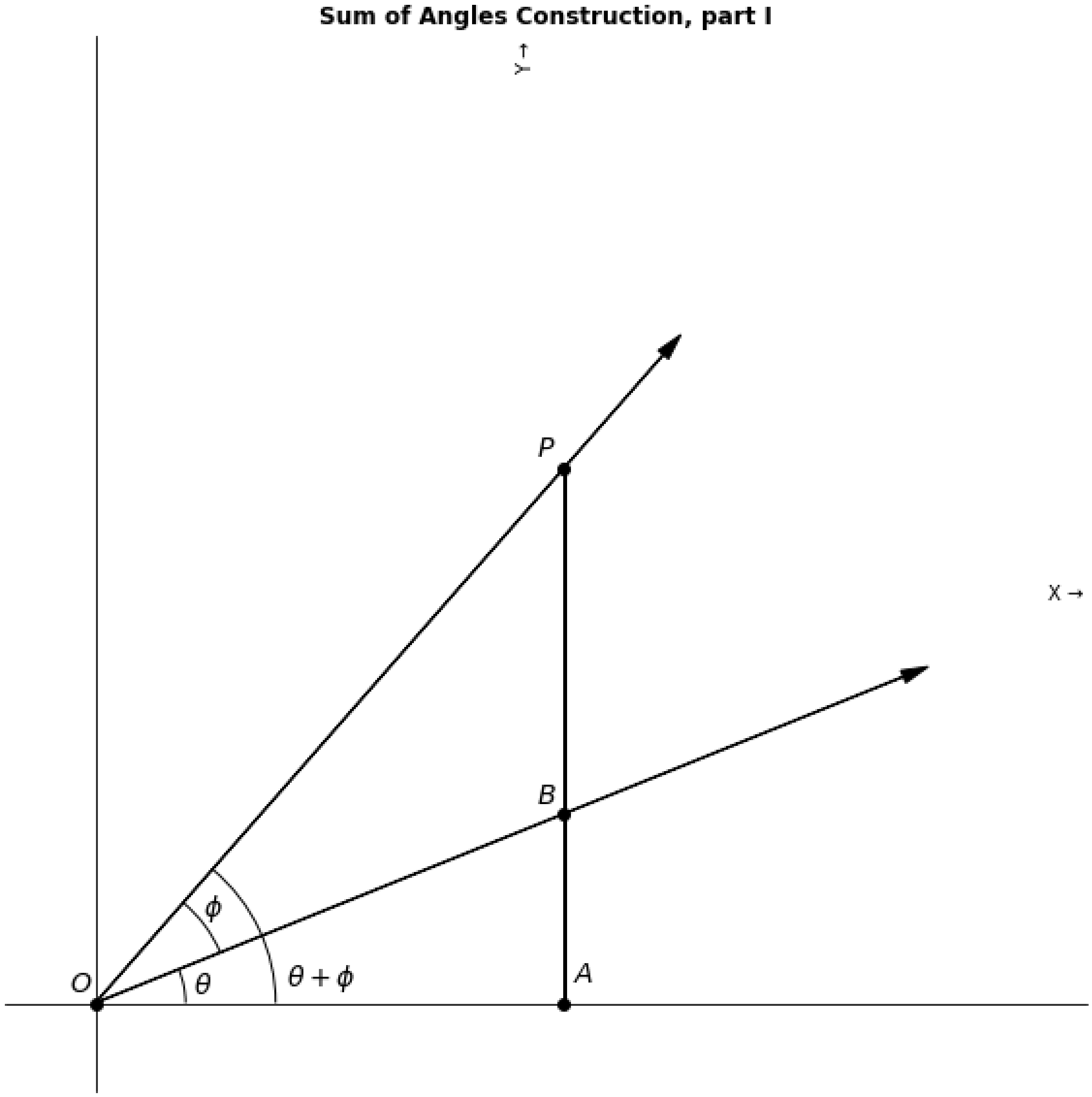
Let's derive the $\cos(\theta+\phi)$ identity. To make this derivation easier to show, we will initially assume that $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$ and that $\theta + \phi < \frac{\pi}{2}$. By doing so, we keep the problem in the first quadrant and the following diagram is completely justified. (Once this simplified case is derived, it's not hard to extend the derivation to include all possible angles.)



Given that basic diagram, we take the following construction steps,

- Pick any point P on the terminal side of the angle ϕ.
- Construct a line segment between point P and the x axis such that the line segment is perpendicular to the x axis. Call the point at which the line segment intersects the x axis A. Call the point at which the line segment intersects the terminal side of the angle θ B.

Now the diagram looks like the following,



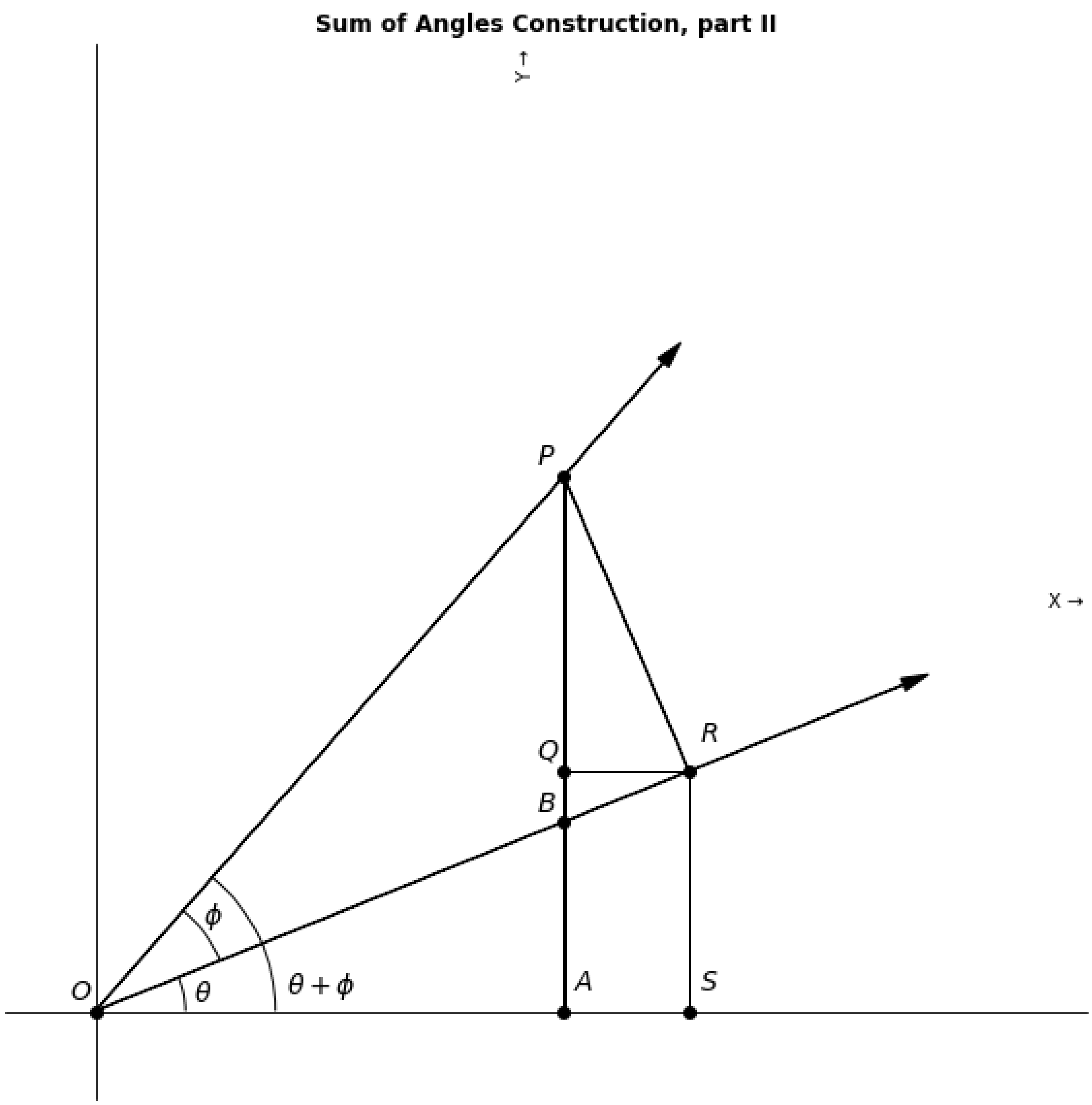
By construction, $\angle OAB$ and $\angle OAP$ are right angles. Therefore, $\triangle OAB$ and $\triangle OAP$ are right triangles.

The next steps are,

- Construct a line segment from the point P to the ray $\overrightarrow{OB}$ such that the line segment is perpendicular to the ray $\overrightarrow{OB}$. Call the point R at which the line segment intersects the ray $\overrightarrow{OB}$.

- Construct a line segment from the point R to the line segment $\overline{PA}$ such that the line segment is perpendicular to the line segment $\overline{PA}$. Call the point Q at which the line segment intersects the line segment $\overline{PA}$.
- Construct a line segment between point R and the x axis such that the line segment is perpendicular to the x axis. Call the point S at which the line segment intersects the x axis.

Now the diagram looks like this,



Some observations about this enhanced diagram,

- By construction, $\angle RQA$, $\angle QAS$, and $\angle ASR$ are all right angles. Since $\square RQAS$ is clearly a quadrilateral and three of its internal angles are right angles, its fourth internal angle $\angle SRQ$ must also be a right angle, making $\square RQAS$ a rectangle. Therefore, the length of $\overline{QR}$ equals the length of $\overline{AS}$.
- $\overline{PA}$ and $\overline{OR}$ intersect at B. Therefore opposite angles $\angle OBA$ and $\angle PBR$ have the same size. Since that's true and $\angle OAB = \angle BRP = \frac{\pi}{2}$, that means that $\angle BOA = \angle BPR = \theta$.
- $\triangle OSR$ is a right triangle with $\overline{OS} = \overline{OR} \cdot \cos\theta$.

- $\triangle ORP$ is a right triangle with $\overline{OP} = \overline{OR} \ / \ \cos\phi = \overline{RP} \ / \ \sin\phi$.
- $\triangle PQR$ is a right triangle with $\overline{QR} = \overline{PR} \cdot \sin\theta$.

> ***Do not go one step further in this derivation until you are absolutely comfortable with all of the preceding observations.***

OK, enough setup. Let's now do the derivation.

We are to determine $\cos(\theta + \phi)$. Well, here it is,

$$
\begin{aligned}
\cos(\theta+\phi) &= \frac{\text{adjacent}}{\text{hypotenuse}} && \text{(definition of cosine)} \\
&= \frac{OA}{OP} && \triangle OAP \\
&= \frac{OS - AS}{OP} && \overline{OA} \text{ is part of } \overline{OS} \\
&= \frac{OS - QR}{OP} && \overline{AS} \text{ and } \overline{QR} \text{ are} \\
& && \quad \text{parallel sides of } \square RQAS \\
&= \frac{OS}{OP} - \frac{QR}{OP} && \text{(algebra)} \\
&= \frac{OR \cdot \cos\theta}{OP} - \frac{QR}{OP} && \text{(above observation)} \\
&= \frac{OR \cdot \cos\theta}{OR \ / \ \cos\phi} - \frac{QR}{OP} && \text{(above observation)} \\
&= \cos\theta\cos\phi - \frac{QR}{OP} && \text{(algebra)} \\
&= \cos\theta\cos\phi - \frac{PR \cdot \sin\theta}{OP} && \text{(above observation)} \\
&= \cos\theta\cos\phi - \frac{PR \cdot \sin\theta}{RP \ / \ \sin\phi} && \text{(above observation)} \\
&= \cos\theta\cos\phi - \sin\theta\sin\phi && \text{(algebra)}
\end{aligned}
$$

Ta-da!

That was pretty easy, wasn't it?

This basic derivation can be extended to deal with arbitrary angles (that is, not constrained to acute angles summing into an angle in the first quadrant) as well as the *difference* of two angles as opposed to their sum, that is $\cos(\theta - \phi)$.

Further, the same general idea is used to derive the similar forms for the $\sin$ function, that is, $\sin(\theta + \phi)$ and $\sin(\theta - \phi)$ for arbitrary angles.

Can you derive all of these using only plane geometry?

(No, of course not. Everyone always jumps directly to Euler's Formula. *Sigh*)

Using Euler's Formula to Derive Sum-of-Angles Rules

OK, this is so fast that if you blink, you might miss it. Pay close attention ... :)

The definition of Euler's Formula is $e^{ix} = \cos x + i \sin x$.

OK, let's figure out what $e^{i(\theta+\phi)}$ is,

$$\begin{aligned}
e^{i(\theta+\phi)} &= e^{i\theta}e^{i\phi} && \text{(definition of exponents)} \\
&= (\cos\theta + i\sin\theta)\, e^{i\phi} && \text{(Euler's Formula)} \\
&= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) && \text{(Euler's Formula)} \\
&= \cos\theta\cos\phi + i\cos\theta\sin\phi \\
&\qquad + i\sin\theta\cos\phi + i^2\sin\theta\sin\phi && \text{(algebra)} \\
&= \left(\cos\theta\cos\phi + i^2\sin\theta\sin\phi\right) \\
&\qquad + (i\cos\theta\sin\phi + i\sin\theta\cos\phi) && \text{(algebra)} \\
&= (\cos\theta\cos\phi - \sin\theta\sin\phi) \\
&\qquad + (i\cos\theta\sin\phi + i\sin\theta\cos\phi) && \text{(algebra)} \\
&= (\cos\theta\cos\phi - \sin\theta\sin\phi) \\
&\qquad + i\,(\cos\theta\sin\phi + \sin\theta\cos\phi) && \text{(algebra)}
\end{aligned}$$

OK, we now know by our derivation that,

$$e^{i(\theta+\phi)} = (\cos\theta\cos\phi - \sin\theta\sin\phi) + i\,(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

But we also know by Euler's Formula directly that,

$$e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

By equating the real and imaginary parts of these two representations (which we are allowed to do by the definition of complex numbers), we find that,

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

and

$$\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$$

Ta-da!

That was so easy, it almost feels like cheating, ha!

If you want some exercise, prove Euler's Formula. It's not hard. Here's a massive hint: Think about the series expansions for $\cos\theta$, $\sin\theta$, and e^{ix}. That's how Euler proved it, and he was no dummy.