QUANTUM DOUBLE MODEL OPERATORS

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Introduction

Quantum lattice models represent interacting quantum particles that are arranged in some symmetrical way. With Kitaev's quantum double model construction, we are able to construct a quantum lattice model with some finite group, G. We study the low energy effective field theory and the excitations, or *anyons*, of this model. Depending on the abelian property of our choice of G, the operators on the anyons were different — in particular, movement operators for abelian anyons on the quantum double was more understood than the non-abelian case. The goal in this paper is to introduce generalized movement operators with Kitaev's quantum double model. First we present an overview of Kitaev's quantum double model and review the creation and splitting of anyonic theories on this model. We then introduce ***unitary movement operators*** for an arbitrary anyon, given a particular finite group — regardless of its abelian nature. Using these movement operators, we reconfirm the categorical data for non-abelian anyons of the group S_3.

1. Quantum Double Model

1.1. **Drinfeld Double.** The anyons of the quantum double model are described as irreducible representations of the Drinfeld double of the group, DG. Given a finite group, G, we define the Drinfeld Double, DG to be the algebra as the tensor product of the algebra $\mathbb{C}[G]$ with its dual, $\mathbb{C}[G]^*$, or simply $\mathbb{C}[G] \otimes \mathbb{C}[G]$. It generated by the following two sets of operators: $\{A_g \mid g \in G\}$ and $\{B_h \mid h \in G\}$. These generators obey the following relations:

$$A_{g_1} A_{g_2} = A_{g_1 g_2}$$
$$\mathrm{B}_{h_1} B_{h_2} = \delta_{h_1 = h_2} B_{h_1}$$
$$\mathrm{A}_g B_h = B_{ghg^{-1}} A_g$$

DG is $|G|^2$ dimensional as a vector space with a basis given by $\{D_{(h,g)} = B_h A_g \mid h, g \in G\}$.

While we will not be explicitly be using the co-product structure, it is important to note that DG can indeed be equipped with the structure of a quasi-triangular Hopf algebra.

1.1.1. *Representations of the Drinfeld Double.* Let $g \in G$ be an arbitrary group element and recall that we define the conjugacy class of g as $C_g = \{hgh^{-1} \mid h \in G\}$ and the centralizer of g as $Z_g = \{r \in G \mid rg = gr\}$. An irreducible representation of DG is given by a pair (C_r, π) where C_r is the conjugacy class of some group element r and π is an irreducible representation of Z_r. It is important to note that $Z_g \cong Z_h$ for $g, h \in C$, meaning that different choices of representatives of a conjugacy class lead to equivalent representations of DG.

An irrep (C_r, π) acts on the Hilbert space $V_{(C_r,\pi)} = \mathbb{C}[C_r] \otimes V_\pi$, which has an orthonormal basis given by $\{|c\rangle \otimes |j\rangle \mid c \in C_r,\ j = 1, \cdots, |V_\pi|\}$. For any $x \in C_r$, we can fix an element $k_x \in G$ such that $k_x r k_x^{-1} = x$. In the case where $x = r$ it is convention to always choose $k_r = 1$. Using this, we can define the action of DG on $V_{(C_r,\pi)}$ as follows:

$$D_{(h,g)} |c\rangle \otimes |j\rangle = \delta_{h = gcg^{-1}} \sum_{i=1}^{|V_\pi|} \pi_{ij}(k_{gcg^{-1}}^{-1} g k_c) \, |gcg^{-1}\rangle \otimes |i\rangle$$

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It is straightforward to check that $k^{-1}_{gcg^{-1}}gk_c$ is always an element of Z_r and that this action indeed defines a representation on $V_{(C_r,\pi)}$.

1.2. **Quantum Double with $G = S_3$.** For the rest of this paper, we will be focusing on the group S_3, and its Drinfeld double, DS_3.

Recall the three conjugacy classes of S_3 consist of the identity, the transpositions, and the 3-cycles. We will refer to them as C_1, C_2, and C_3, respectively. Also notice that the centralizer of a representative of C_1 is given by entire group, S_3. In a similar way, C_2 is is isomorphic to $\mathbb{Z}_2$, and that of a representative of C_3 is isomorphic to $\mathbb{Z}_3$. The irreducible representations (i.e. irreps) of the centralizer of C_1 are the trivial irrep id, sign irrep sgn, and the standard representation std. Similarly, we have the trivial and sign representation for the centralizer of C_2, and the one-dimensional representations mapping to $1, \omega, \omega^2$ corresponding to the the irreps of the centralizer of C_3. Therefore, in total, we see that we have 8 irreps of DS_3.

As a summary, we list below the irreps of DS_3, each corresponding to the types of anyons in the quantum double model with $G = S_3$.

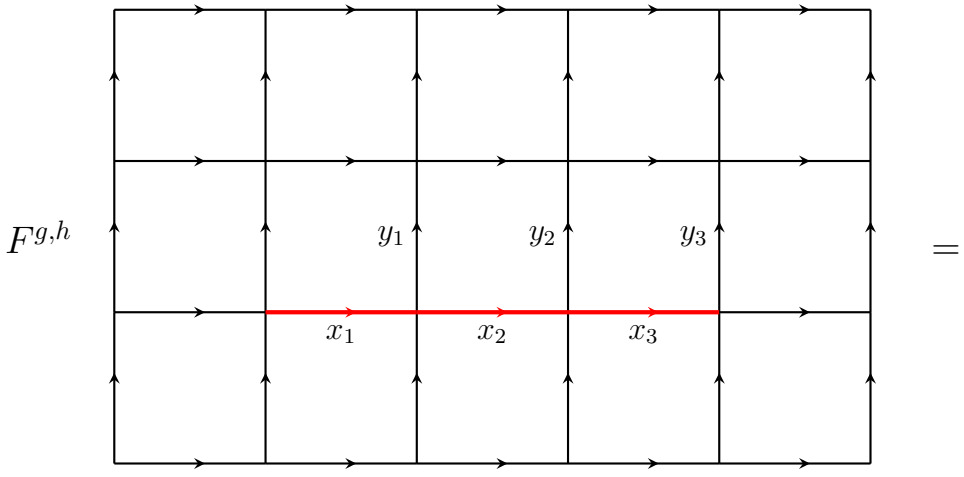
Type of Anyon	**Name of Anyon**	**(C_i, irrep of Z_i)**
vacuum	A	(C_1, id)
charge	B	(C_1, sgn)
charge	C	(C_1, std)
flux	D	(C_2, id)
flux	F	(C_3, id)
dyon	E	(C_2, sgn)
dyon	G	(C_3, ω)
dyon	H	(C_3, ω^2)

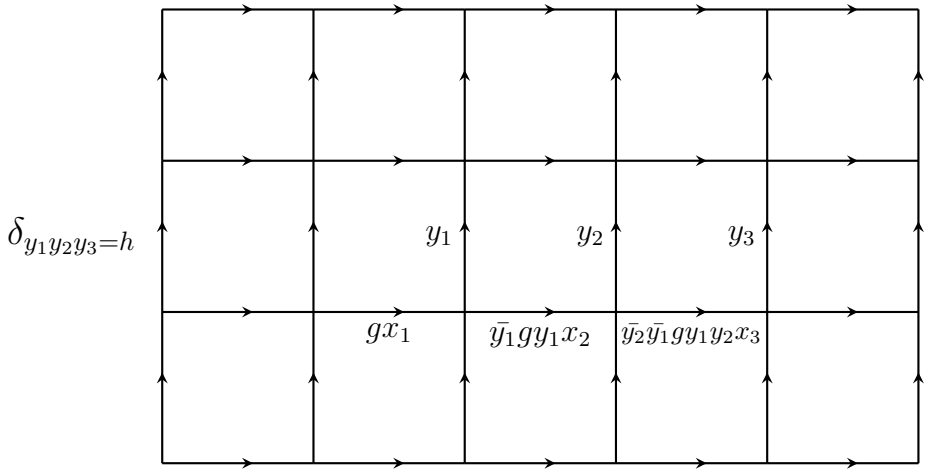
TABLE 1. Irreps of DS_3.

Notice A is essentially a trivial anyon, B and C are charges because they correspond to the trivial conjugacy class, while D and F correspond to fluxes because they have trivial irreps of the centralizer.

2. String Operators

We define string operators for the quantum double model similarly to Kitaev's ribbon operators in (reference). Note that these operators are a bit different from Kitaev's as we defined the operators in our Hamiltonian according to the dual picture.





While labelling excitations in terms of a pair of group elements allows for a nice operator algebraic theory of string operators, we really would prefer to have string operators that correspond with irreducible representations of the Drinfeld Double of our group, as this is how types of anyons are classified. Fortunately, there is a basis change that does exactly this:

$$F^{(C,\pi;i,j)} = \frac{|V_{(C,\pi)}|}{|G|} \sum_{h,g\in G} \pi_{ij}(D(h,g))F^{h,g}$$

3. Splitting Operators

We now define the **splitting operator** to split an anyon into a pair of anyons and move them. First, we will consider the case of charges. We think of charges on adjacent plaquettes as a vector in $\mathbb{C}[G]\otimes\mathbb{C}[G]$ under the regular representation of G. Then, recalling that the decomposition $\mathbb{C}[G] = \bigoplus_{\pi\in\widehat{G}} V_\pi^{\oplus dim(V_\pi)}$, we define the splitting operator of a charge π into two charges η and ρ as a unitary representation map $S_\pi(\eta,\rho) : \mathbb{C}[G]\otimes\mathbb{C}[G] \longrightarrow \mathbb{C}[G]\otimes\mathbb{C}[G]$ such that the following diagram commutes:

3.1. Splitting Operators for Charges (One-Dimensional).

To find the unitary splitting operator of charges, we will define them utilizing the natural isomorphism between **gauged** and **ungauged** theories.

For our purposes, we will define our splitting operator with our charges as elements of $\mathbb{C}[\mathbb{G}]$ defined on the domains, or plaquettes, of our lattice on the ungauged theory. We can then use the natural isomorphism to gauge, or transform, into our gauged theory of domain walls. This gauged theory acts as our quantum double model. The relative difference between the two domains on each side of a domain wall is projected on each edge of the gauged theory. This corresponds to the following map from the ungauged theory to the gauged theory:

$$P_{gh^{-1}=x} \to P_e^x$$

where e is the domain wall between elements g and h.

The isomorphism we will use between the ungauged and gauged theories corresponds to the Hilbert spaces $\mathcal{H}_{dots}^{sym} \cong \mathcal{H}_{lines}^{fluxfree}$. This isomorphism allows us to more easily work with the domains on the ungauged theory, then translate into the gauged theory (which corresponds to our quantum double model).

Now, our goal is to try to find the splitting operator of the one-dimensional anyon, B – corresponding to the sign representation charge – from the vacuum. First, we define the vector $|+\rangle$ as a vector in the trivial representation vector space (corresponding to the trivial anyon, A) in $\mathbb{C}[G]$ as the following:

$$|+\rangle := \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle$$

We know that the vector space corresponding to the B anyon is generated by the basis vector $\{|-\rangle\}$, with the group action defined as $g\,|-\rangle = sgn(g)\,|-\rangle$, and similarly, the identity representation is generated by the basis vector $\{|+\rangle\}$, where $g\,|+\rangle = |+\rangle$.

To find the splitting operator $S_A(B, B)$, we want the unitary map from the tensor product of the representations vector spaces of $A \otimes A \to B \otimes B$, and so from the ungauged theory, we have the map

$$\frac{1}{\sqrt{|G|}} \sum_{g,h \in G} |g, h\rangle \to \frac{1}{\sqrt{|G|}} \sum_{g,h \in G} sgn(gh)\,|g, h\rangle$$

.

After gauging this theory, we are left with the unitary map:

$$P_e^{gh^{-1}=x} \to sgn(x) P_e^x$$

where e represents the edge between the two anyons.

Therefore, summing over all the elements of G, we can define the unitary splitting operator, $S_A(B, B)$ as:

$$S_A(B, B) = \sum_{x \in G} sgn(x) P_e^x$$

3.2. **Splitting Operators for Charges (Higher Dimensional).** To define the splitting operator for higher dimensional charges, we consider splitting into two anyons of the type C (the anyon corresponding to standard irrep of S_3) as an example.

First, let's recall the standard 2D irrep of S_3. There are multiple choices of bases we can make for the vector space representing this representation. We will choose the following basis vectors, $|2_+\rangle$ and $|2_-\rangle$, with the following group action:

$$\tau\,|2_+\rangle = \omega\,|2_+\rangle$$
$$\tau\,|2_-\rangle = \omega^2\,|2_-\rangle$$
$$\sigma\,|2_+\rangle = |2_-\rangle$$
$$\sigma\,|2_-\rangle = |2_+\rangle$$

where $\tau = (123)$, $\sigma = (12)$, and $\omega = e^{2\pi i/n}$ is the n-th root of unity (in this case, $n = 3$).

Recall our definition for finding the unitary splitting operator. We want to find the unitary map $S_A(C, C) : V_A \otimes V_A \to V_C \otimes V_C$, where V_A is the vector space of the trivial representation, and V_C is the 2D vector space of the standard irrep of S_3. To do this, we first to write these basis vectors, $|2_+\rangle$ and $|2_-\rangle$, of the irrep's vector space V_C, in the group algebra basis, as follows:

$$|2_+\rangle = |1\rangle + \omega\,|\tau\rangle + \omega^2\,|\tau^2\rangle$$
$$|2_-\rangle = |\sigma\rangle + \omega\,|\tau\sigma\rangle + \omega^2\,|\tau^2\sigma\rangle$$

To find the unitary map, we can consider the representation map from $A \otimes A \to C \otimes C$. For this, we can write a copy of the identity in $\mathbb{C}[G] \otimes \mathbb{C}[G]$ using the 2D irrep's basis vectors, written with the group algebra basis as follows:

$$|+\rangle \otimes |+\rangle = \frac{1}{\sqrt{2}}(|2_+\rangle\,|2_-\rangle + |2_-\rangle\,|2_+\rangle)$$

(We can verify that this is indeed a copy of the identity, because if we apply a group action of any group element, we should get the original vector – as the identity should.)

Similar to what we did earlier to find the splitting operator, we can represent this map to the summation of projections, $P^{\sigma} + \omega P^{\tau\sigma} + \omega^2 P^{\tau^2\sigma}$.

Notice that this map is not unitary (recall a map U is unitary if $UU^{\dagger} = I$). To make this map unitary, we include another choice of basis so that we have projections that project every group element. If our previous basis choice was $|2_+\rangle_{(1)}$ and $|2_-\rangle_{(1)}$, we define $|2_+\rangle_{(2)}$ and $|2_-\rangle_{(2)}$ as follows:

$$|2_+\rangle_{(2)} = |\sigma\rangle + \omega^2 |\tau\sigma\rangle + \omega |\tau^2\sigma\rangle$$
$$|2_-\rangle_{(2)} = |1\rangle + \omega^2 |\tau\rangle + \omega |\tau^2\rangle$$

With this additional basis in consideration, if we rewrite our copy of the identity as:

$$|+\rangle \otimes |+\rangle = \frac{1}{\sqrt{2}}(|2_+\rangle_{(1)} |2_-\rangle_{(1)} + |2_-\rangle_{(1)} |2_+\rangle_{(1)} + |2_+\rangle_{(1)} |2_-\rangle_{(2)} + |2_-\rangle_{(1)} |2_+\rangle_{(2)})$$

Now, the resulting sum of projections becomes $P^{\sigma} + \omega P^{\tau\sigma} + \omega^2 P^{\tau^2\sigma} + P^1 + \omega P^{\tau} + \omega^2 P^{\tau^2}$, which is now unitary, since

$$\begin{aligned}(P^{\sigma} + \omega P^{\tau\sigma} + \omega^2 P^{\tau^2\sigma} + P^1 + \omega P^{\tau} + \omega^2 P^{\tau^2})(P^{\sigma} + \overline{\omega} P^{\tau\sigma} + \overline{\omega^2} P^{\tau^2\sigma} + P^1 + \overline{\omega} P^{\tau} + \overline{\omega^2} P^{\tau^2}) \\ = P^{\sigma} + P^{\tau\sigma} + P^{\tau^2\sigma} + P^1 + P^{\tau} + P^{\tau^2} \\ = \sum_{g \in G} P^g = I\end{aligned}$$

.

Therefore, we found a unitary splitting operator for splitting two C anyons in the vacuum:

$$S_A(C, C) = P^{\sigma} + \omega P^{\tau\sigma} + \omega^2 P^{\tau^2\sigma} + P^1 + \omega P^{\tau} + \omega^2 P^{\tau^2}$$

4. Movement Operators

In this section, we will define our generalized unitary **move operators** that can move an arbitrary anyon of a finite group, without creating any other excitation on the lattice.

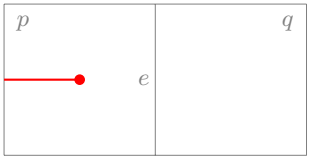
This move operator will operate by moving the anyon from one site to another. The reason that allows this operator to safely move any type of anyon, is the fact that, rather than creating unwanted excitations by extending the string in [], we handle the actual structure in the lattice. Intuitively, we can think of our move operators as "undoing" the edges around the anyon, to allow the anyon to *naturally* move through the opened edges, and then patch up the lattice back together. That way, we have preserved the structure of the lattice, but have allowed the anyon to shift through the lattice. This allows us to move anyons of any type, even non-abelian ones.

4.1. Moving Charges.

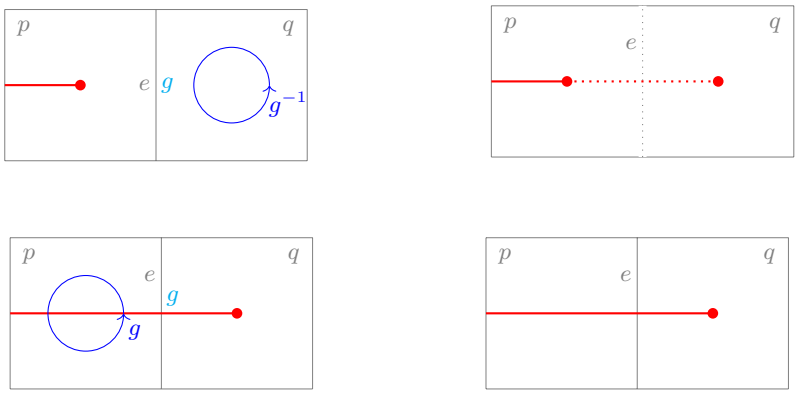
We define the action of moving charges from plaquette p to plaquette q as follows:

$$M_{pq} = \sum_{g \in G} B_p^g P_e^1 B_q^{g^{-1}}$$

We can illustrate this in the following way. Consider a charge, located on plaquette p.



We then apply the B_p operator on plaquette q with the inverse of the value of edge e, and project it to 1 – intuitively, this effectively erases edge e so that the charge can free move from plaquette p to plaquette q. Finally, we "patch up" the edge with the original value, so that our lattice is restored, but now with the violation on the other side of the edge.

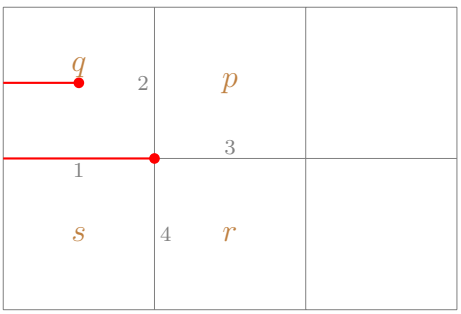


We use this idea of using the B_p^g operators around the violation to temporarily "undo" the surrounding edges of the lattice to allow the violation to slide through to another site to define our movement operators. The above example for moving charges becomes slightly more involved when we handle fluxes or dyons (note fluxes are simply dyons with trivial irreps), because we must consider all the edges corresponding to the violation on the vertex. Thus, the movement operator for fluxes and dyons will include more operators. We now will introduce the movement operators for dyons, which is very similar to the movement operators of fluxes.

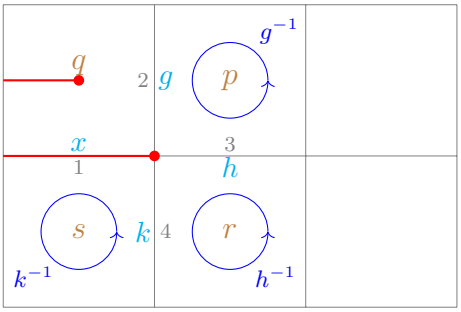
4.2. **Moving Dyons.** Our generalized move operator for a dyon is described in the following way:

$$M = \sum_{g,h,k,x \in G} B_q^g B_r^h B_s^k L_3^x P_1^x P_2^1 P_3^1 P_4^1 B_p^{g^{-1}} B_r^{h^{-1}} B_s^{k^{-1}}$$

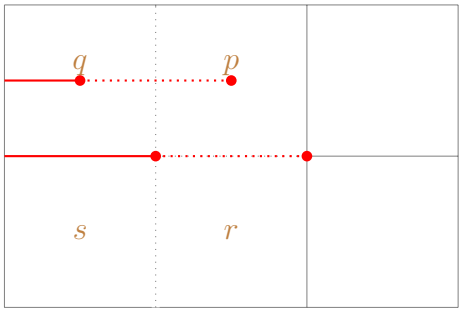
We can illustrate how this move operator operates on the following lattice. Consider some arbitrary dyon (i.e. violations or excitations on both a vertex and plaquette).



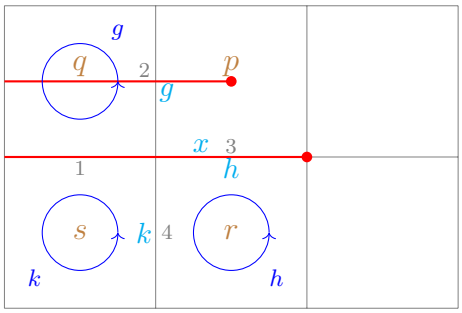
Intuitively, the first set of B_p^g operators correspond to "undoing" the lattice by multiplying each corresponding plaquette by its inverse, and projecting the edges around the charge to 1.



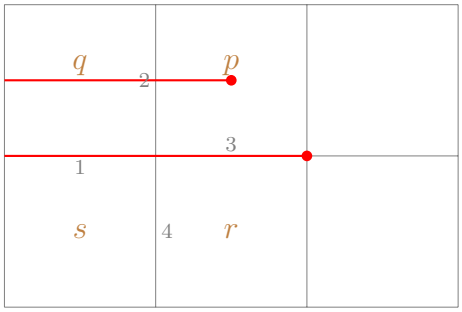
This allows the charge and flux to move freely in the vacuum, since the edges have been "undid".



Finally, we reapply the B_p^g operators on the plaquettes to build back the edges of the lattice, and multiply x onto the following edge on our lattice to extend the flux to the next vertex.



And after doing all this, we see that we have successfully moved the dyon one site to the right, without other excitations.



5. Kawagoe-Levin Procedure for Categorical Data Calculation

We begin by reviewing the procedure given in [ ] for computing R and F symbols for arbitrary non-abelian anyon theories.

Recall that a non-abelian anyon theory is defined by the following:

(1) A finite set of **anyon types**: $\mathcal{A} = \{a, b, c, \cdots\}$ (The set of irreps of DG in the case of the quantum double model)
(2) A **fusion product** of the form:

$$a \otimes b = \bigoplus_{c \in \mathcal{A}} N_c^{ab} c$$

where $N_c^{ab} \in \mathbb{N}$
(3) An **F-Symbol**, which is a collection of unitary change of basis matrices corresponding to the resulting multi-anyon states from splitting anyons in different orders.
(4) An **R-symbol**, which is a collection of unitary change of basis matrices corresponding to the phase from braiding two anyons.

5.1. **Braiding Statistics (R-Symbols).** Categorical data in the context of anyons refers to the various outcomes of braiding and fusing operations based on these different types of anyons. **Braiding** refers to the process of exchanging two anyons in a two-dimensional space. We can split two anyons to

two places in two different ways. This corresponds to the braiding of the two anyons. It is known when this braiding occurs, there is a phase difference, denoted by the **R-symbol**.

When we are calculating the braiding statistics for any arbitrary anyon, we will follow the following procedure.

This procedure follows two paths (corresponding to two states, $|1\rangle$ and $|2\rangle$) of two interacting anyons. The anyons have the same position before and after, but the paths they take differ, representing the braiding. The R-symbol calculates the phase that gets picked up through this braiding by taking the inner product of both of these states.

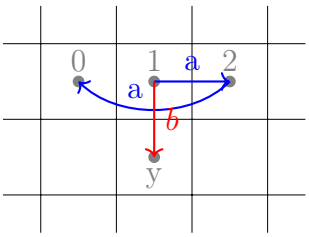


FIGURE 1. State corresponding to $|1\rangle = M_{10}^{b} M_{21}^{b} M_{1y}^{a} S(a,b)$

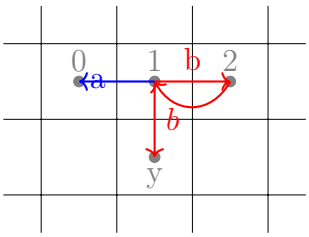


FIGURE 2. State corresponding to $|2\rangle = M_{1y}^{a} M_{21}^{a} M_{10}^{b} S(a,b)$

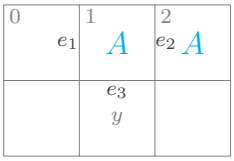
$$\langle 1|2\rangle = \langle a,b|\, S^{\dagger}(a,b) M_{1y}^{a\dagger} M_{21}^{b\dagger} M_{10}^{b\dagger} M_{1y}^{a} M_{21}^{a} M_{10}^{b} S(a,b)\, |a,b\rangle$$

Reducing this inner product will lead to the resulting phase factor, i.e. the R-symbol:

$$\langle 1|2\rangle = R(a,b)\, \langle a|b\rangle = R(a,b)$$

5.1.1. *Example: Calculating* $R_A(A,A)$. In the case of splitting the vacuum into two trivial A type anyons, one can see the splitting operator is trivial, and thus the calculation for the R-symbol is quite trivial. However, we will show $R_A^{AA} = 1$ in a more rigorous way, so that we can use this procedure for finding the R-symbols in similar ways for other anyons.

To find $R_A(A,A)$, our goal is to reduce the inner product to get $\langle 1|2\rangle = R_A(A,A)\, \langle A|A\rangle = R_A(A,A)$. Recall that the anyon A corresponds to the trivial irrep (C_1, id), so we use the trivial splitting operator, and although the movement operators are trivial as well, we will use our movement operators for charges to make our calculation more rigorous.



We proceed by reducing the inner product as shown below (note: $M_{1y}^{A\dagger} = M_{y1}^{A}$)

$$\langle 1|2\rangle = \langle A|\, S^{\dagger}(A,A) M_{1y}^{A\dagger} M_{21}^{A\dagger} M_{10}^{A\dagger} M_{1y}^{A} M_{21}^{A} M_{10}^{A} S(A,A)\, |A\rangle$$

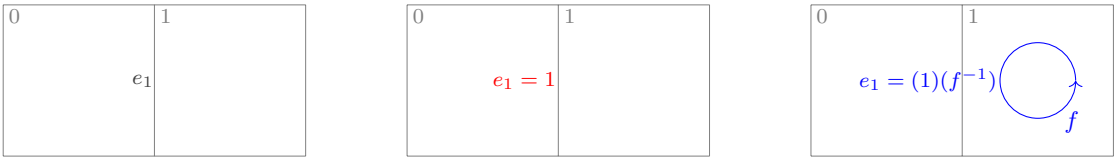
$$\langle 1|2\rangle = \langle A|\, S^{\dagger}(A,A) M_{y1}^{A} M_{12}^{A} M_{01}^{A} M_{1y}^{A} M_{21}^{A} M_{10}^{A} S(A,A)\, |A\rangle$$

Using our charge movement operator definition, we can write our first movement operator, M_{10}^{A} – which moves the anyon from plaquette 1 to 0 – as $\sum_{g\in G} B_1^g P_{e_1}^1 B_0^{g^{-1}}$. In a similar fashion, we rewrite the product movement operators in our inner product as:

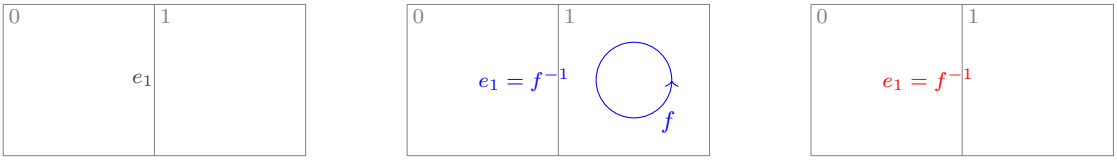
$$\sum_{a,b,c,d,e,f\in G} (B_y^a P_{e_3}^1 B_1^{a^{-1}})(B_1^b P_{e_2}^1 B_2^{b^{-1}})(B_0^c P_{e_1}^1 B_1^{c^{-1}})(B_1^d P_{e_3}^1 B_y^{d^{-1}})(B_2^e P_{e_2}^1 B_1^{e^{-1}})(B_1^f P_{e_1}^1 B_0^{f^{-1}})$$

Notice these operators are acting on the ground state, $|A\rangle$. Recall that B_p is invariant on the ground state, i.e. $B_p|A\rangle = |A\rangle$. Therefore, if we can commute all the B_p operators past all the projectors, we can reduce the product of movement operators to just a summation of projectors on the ground state.

To see how a B_p^g operator would permute with a projector, consider $B_1^f P_{e_1}^1$.



We see that applying B_1^f after the projector results in the edge e_1 to obtaining a value of f^{-1}. Therefore, if we want to permute B_1^f past the projector, we would want to have e_1 have the value f^{-1} projected to that edge.



Therefore, we have that $B_1^f P_{e_1}^1 = P_{e_1}^{f^{-1}} B_1^f$.

In this same way, we can permute all the B_p^g operators past to get the following:

$$\sum_{a,b,c,d,e,f\in G} (\prod P)(\prod B)$$

where

$$\prod_{a,b,c,d,e,f\in G} P = P_{e_3}^{a^{-1}} P_{e_2}^{ab} P_{e_1}^{cb^{-1}a^{-1}} P_{e_3}^{abcda^{-1}} P_{e_2}^{abcda^{-1}} P_{e_1}^{cf^{-1}e^{-1}d^{-1}c^{-1}b^{-1}a^{-1}}$$

$$\prod_{a,b,c,d,e,f\in G} B = B_y^a B_1^{a^{-1}} B_1^b B_2^{b^{-1}} B_0^c B_1^{c^{-1}} B_1^d B_y^{d^{-1}} B_2^e B_1^{e^{-1}} B_1^f B_0^{f^{-1}}$$

To simplify $\prod_{a,b,c,d,e,f\in G} P$, we utilize the fact that $\sum_{g\in G} P_e^g = 1$ and that $P_e^g P_e^h \to g = h$, to get the following:

$$P_{e_3}^{a^{-1}=abcda^{-1}} P_{e_2}^{ab=abcdb^{-1}} P_{e_1}^{cb^{-1}a^{-1}=cf^{-1}e^{-1}d^{-1}c^{-1}b^{-1}a^{-1}}$$

This corresponds to the system of equations as follows:

$$\begin{cases} a^{-1} = abcda^{-1} \\ ab = abcdb^{-1} \\ cb^{-1}a^{-1} = cf^{-1}e^{-1}d^{-1}c^{-1}b^{-1}a^{-1} \end{cases} \tag{1}$$

We are left with a simplified product of projectors, $\prod_{a,b,c,d,e,f\in G} P = P_{e_3}^1 P_{e_2}^b P_{e_1}^{b^{-1}a^{-1}}$.

We can reduce $\sum_{a,b} P_{e_2}^{b} P_{e_1}^{b^{-1}a^{-1}}$ as follows:

$$= \sum_{a,b} P_{e_2}^{1} P_{e_1}^{a^{-1}} + P_{e_2}^{(12)} P_{e_1}^{(12)a^{-1}} + P_{e_2}^{(13)} P_{e_1}^{(13)a^{-1}} ...$$
$$= P_{e_2}^{1} \sum_{a} P_{e_1}^{a^{-1}} + P_{e_2}^{(12)} \sum_{a} P_{e_1}^{a^{-1}} + P_{e_2}^{(13)} \sum_{a} P_{e_1}^{a^{-1}} ...$$

Notice the individual summations are equivalent to the identity, and so we are left with $P_{e_2}^{1} + P_{e_2}^{(12)} + P_{e_2}^{(13)}...$, which is equivalent to $\sum_{g \in G} P_{e_2}^{g}$. Therefore, we conclude that $\prod_{a,b,c,d,e,f \in G} P = 1$.

Because our movement operators are independent on the type of anyon, we can use this summation representing the product of movement operators when calculating the R-symbols of other anyons.

Therefore, because we are handling the trivial case, we have the splitting operators to be trivial, and so our inner product $\langle 1|2\rangle$ becomes $\langle A|A\rangle$.

We thus can conclude that the phase of the braiding is 1, and so $R_A(A, A) = 1$.■

5.2. **Fusion Statistics (F-Symbols).** Like braiding, we also have the fusing anyons, a characteristic that allows us to characterize anyons. The **fusion product** corresponds to the anyon that results from fusing two anyons together. For non-abelian anyons, when we fuse two anyons, a, b, the resulting fusion product becomes $a \otimes b = \bigoplus_c N_c^{ab} c$, where c is other possible anyons that $a \otimes b$ can fuse to, and N_c^{ab} is called the "*fusion multiplicities*" corresponding to the largest integer representing the number of orthonormal states such that $a \otimes b = c$. The **F-symbol** is a phase that gets picked up when we fuse three anyons in different orders. The different ways we fuse three anyons is analogous to associativity. For example, if A, B, C are three different anyons, then the state in which we fuse $A \otimes B$, and then with C (i.e. $(A \otimes B) \otimes C$) corresponds to the phase in which we fuse $B \otimes C$ first, and then with A (i.e. $A \otimes (B \otimes C)$), with some $U(1)$ phase, called the F-symbol (similar to the $U(1)$ phase we called the R-symbol resulting when we braid two anyons).

We will use a procedure similar to the one we used to calculate the R-symbols. We compare two different processes where we take some anyon, D, and compare the two ways it can split into three anyons, A, B, C, as described earlier.

In one process, we have D into C and some anyon E, an anyon that can then split into the anyons A and B. We compare the phase difference with the process where D splits into A and some other anyon F that splits into A and B. Both processes have the same starting anyon, D, and end with the same anyons, A, B, C, but the order in which the anyons fuse differ.

With non-abelian anyons, we also have to consider the additional states for the fusion products in which the fusion multiplicity is more than 1. Thus, at the splitting intersections, we include additional indices, μ, v, κ, λ, that are indexed from 1 through the corresponding fusion multiplicity. (However, because we are using anyons corresponding to the group S_3, we find that all the fusion multiplicities will be equal to 1).

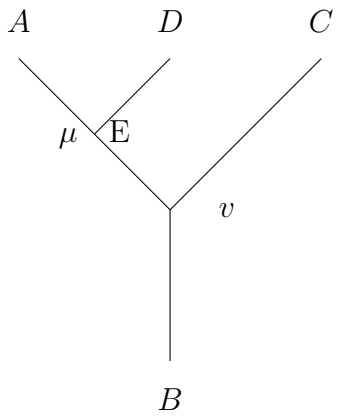


FIGURE 3. State corresponding to $|1, E, \mu, v\rangle = M_{12}^{B} M_{01}^{A} S_{E,\mu}(A, B) M_{y2}^{C} S_{D,v}(E, C) |D\rangle$

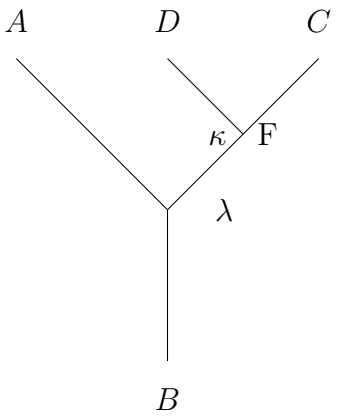


FIGURE 4. State corresponding to $|2, F, \kappa, \lambda\rangle = M_{y2}^{C} S_{F,\kappa}(B, C) M_{12}^{F} M_{01}^{A} S_{D,\lambda}(A, F) |D\rangle$

The F-matrix is defined to be the unitary matrix $F_{D}^{A,B,C}$, with elements of the matrix defined with $(F_{D,E,F}^{A,B,C})_{\mu v}^{\kappa\lambda}$, where κ, λ, μ, v correspond to the respective fusion multiplicities.

$$|1, E, \mu, v\rangle = \sum_{F,\kappa,\lambda} (F_{D,E,F}^{A,B,C})_{\mu v}^{\kappa\lambda} |2, F, \kappa, \lambda\rangle$$

Similar to the calculation of the R-symbols, we can find the F-symbol by taking the inner product of the two states, $\langle 1, E, \mu, v | 2, F, \kappa, \lambda\rangle$.

$$\langle 1, E, \mu, v | 2, F, \kappa, \lambda\rangle = \sum_{F,\kappa,\lambda} (F_{D,E,F}^{A,B,C})_{\mu v}^{\kappa\lambda}$$

5.2.1. *Example: Calculating F_{C}^{CCC}.* Recall that C is a non-abelian anyon – thus, the associated F-symbol will be a unitary matrix. We will try calculating elements, or the 6j-symbols, of the matrix corresponding to the F-matrix, $F_{C}^{CCC} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Each element of this matrix corresponds to a particular fusion state, and the fusion rule of $C \otimes C$ is $C \otimes C = A \oplus B \oplus C$. That means that the fusion of $C \otimes C$ consists of the superposition of three anyons, A, B, C.

We index each element as $F_{C:\gamma\gamma'}^{CCC}$ based on the state of the fusion superposition – since there are three possible states in the superposition of the fusion $C \otimes C$, we index the matrix's 1st, 2nd, and 3rd position for the state corresponding to the anyon A, B, C, respectively. For example, if we are considering the two states where $\gamma = A$ and $\gamma' = B$, then we consider the F-symbol that lies in the 1st row, 2nd column of our F-matrix.

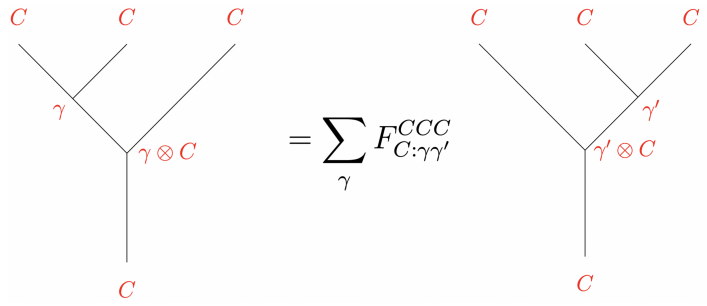


FIGURE 5. Each term, $F^{CCC}_{C:\gamma\gamma'}$ is an element of the matrix F^{CCC}_{C}.

Now, our goal is to verify this F-symbol for the fusion of the anyons $C \otimes C$. To do this, we will follow a similar procedure as calculating the R-symbols: by taking the inner product of the operators that create the two corresponding states, differing by that F-symbol phase. The only catch with finding F-symbols with non-abelian anyons is we need to find all the elements of the F-matrix, corresponding to the superposition of anyons creating the fusion rule.

Let's first show the first 6j-symbol: $F^{CCC}_{C:AA} = 1$. Note that all the fusion multiplicities for the anyons of $D(S_3)$ will be 1 – so, $\mu = \kappa = v = \lambda = 1$.

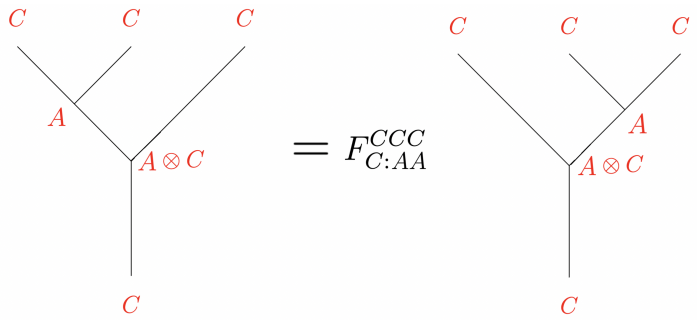


FIGURE 6. Diagrams to explain the phase difference between the two fusion states, $|1\rangle$ *and* $|2\rangle$, respectively

Thus, to find this 6j-symbol, we take the inner product of the two states as follows:

$$F^{CCC}_{C:AA} = \langle 1|2\rangle = \langle A|\, S^{\dagger}_{C}(C,A) M^{C}_{10} M^{A}_{21} S^{\dagger}_{A}(C,C) M^{C}_{2y} M^{C}_{12} M^{C}_{01} S_{A}(C,C) M^{C}_{y2} S_{C}(A,C)\, |A\rangle$$