Introduction to Quantum Computing – Assignment # 2

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Problem Set #2 – Quantum-Bits and Quantum Information Basics

1. Which of the following are possible states of a qubit?

   (a) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   (b) $\frac{\sqrt{3}}{2}|1\rangle - \frac{1}{2}|0\rangle$

   (c) $0.7|0\rangle + 0.3|1\rangle$

   (d) $0.8|0\rangle + 0.6|1\rangle$

   (e) $\cos\theta|0\rangle + i\sin\theta|1\rangle$

   (f) $\cos^2\theta|0\rangle - \sin^2\theta|1\rangle$

   (g) $\left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$

   For each valid state among the above, give the probabilities of observing $|0\rangle$ and $|1\rangle$ when the system is measured in the standard computational basis. What are the probabilities of the two outcomes when the state is measured in the basis $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$?

2. A tw-qubit system is in the following state

   $$\frac{1}{\sqrt{30}}(|00\rangle + 2i|01\rangle - 3|10\rangle - 4i|11\rangle)$$

   The first qubit is measured and observed to be 1. What is the state of the system after the measurement? What is the probability that a subsequent measurement of the second qubit will observe a 1?

3. Show that the Hadamard matrix and the Pauli matrices are unitary.

4. If I is the 2-dimensional identity matrix and H is the Hadamard operator, give matrix representations of the operators $I \otimes H$ and $H \otimes I$.

5. Suppose a two-qubit system is in the state $0.8|00\rangle + 0.6|11\rangle$. A Pauli X gate (i.e. a NOT gate) is applied to the second qubit and a measurement performed (on each qubit) in the computational basis.
   What are the probabilities of the possible measurement outcomes?

6. Show that the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ cannot be expressed as a tensor product of two single qubit states.
   Hint: start with a general expression of a tensor product of two single qubit states,
   $(\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$ and multiply out.

7. Give the matrix form of the *controlled-Hadamard gate*. That is a two-qubit gate – let the first qubit be the control and the second be the target; i.e. if the first (control) qubit equals zero, then the second (target) qubitr is unchanged, while if the first (control) qubit equals one, then a Hadamard transformation is performed on the second (target) qubit.
   Hint: Look at the matrix form of the CNOT gate, in which the first quibit controls a Pauli-X (not) gate on the second qubit.

8. Show that the controlled-Not gate can be constructed from Hadamard gates and the controlled-Z gate. Demonstrate that the construction is correct by multiplying the corresponding matrices.

9. Verify that the four Bell states

$$(\frac{1}{\sqrt{2}}|00\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

   form an orthonormal basis for $\mathbb{C}^4$.

10. **The NO-Signalling Principle**
    Suppose that Alice and Bob are at different spatial locations (far enough, e.g. galaxies apart), but each of them have one-half of a Bell pair: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
    Alice can measure her qubit whenever she wants, and this collapse Bob's to the same state. So we would like to know of Bob could infer whether or not Alice has measured her qubit, if he can then this would represent a way to communicate information to Bob – aka *signalling* this information to Bob. But all that Bob can do to infer whether Alice has measured her qubit is to measure his own qubit – therefore, the question reduces to whether the measurement probabilities that Bob sees are altered by virtue of Alice having performed her measurement. Consider the possible outcomes from Alice measuring or not her qubit, and what conclusions you can draw from that regarding Bob inferring the correct measurement from Alice?

11. **The NO-Cloning Principle**
    Suppose a quantum state $|\Psi\rangle$ and a register intially set to $|0\rangle$. We would like to know whether there exist an unitary cloning operator, U, such that,

$$U|\Psi\rangle = |\Psi\rangle|\Psi\rangle$$

    Prove that such U does not exist by considering that a proper clonning-operator must clone all possible quantum states, in particular the two aforementioned. Then take the inner product of the applications of U to them, i.e. project the application of one into the other and see what could you conclude from that.

    The NO-Deletion Principle
    An immediate consequence of the *NO-Cloning Principle* is the *NO-Deletion Principle*, which arises from *time-reversal* of the no-cloning principle. In other words, there does not exist a unitary operator $\tilde{U}$ such that can delete one of two copies of a quantum state. More generally, quantum computing is reversible (except for measurement), and therefore the (im)possibility of some computation implies the (im)possibility of its reverse.