CSCD94 A2

1. Validity and Measurements

Valid Qubit States

(a) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- **Standard Basis**:

$$P(|0\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}, \quad P(|1\rangle) = \frac{1}{2}$$

- $|+\rangle, |-\rangle$ **Basis**:

$$P(|+\rangle) = 1, \quad P(|-\rangle) = 0$$

(b) $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

- **Standard Basis**:

$$P(|0\rangle) = \left|-\frac{1}{2}\right|^2 = \frac{1}{4}, \quad P(|1\rangle) = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$$

- $|+\rangle, |-\rangle$ **Basis**:

$$P(|+\rangle) = \frac{2-\sqrt{3}}{4}, \quad P(|-\rangle) = \frac{2+\sqrt{3}}{4}$$

(d) $0.8|0\rangle + 0.6|1\rangle$

- **Standard Basis**:

$$P(|0\rangle) = 0.64, \quad P(|1\rangle) = 0.36$$

- $|+\rangle, |-\rangle$ **Basis**:

$$P(|+\rangle) = 0.98, \quad P(|-\rangle) = 0.02$$

(e) $\cos\theta|0\rangle + i\sin\theta|1\rangle$

- **Standard Basis**:

$$P(|0\rangle) = \cos^2\theta, \quad P(|1\rangle) = \sin^2\theta$$

- $|+\rangle, |-\rangle$ **Basis**:

$$P(|+\rangle) = 0.5, \quad P(|-\rangle) = 0.5$$

(g) $\left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$

- **Standard Basis**:

$$P(|0\rangle) = 0.5, \quad P(|1\rangle) = 0.5$$

- $|+\rangle, |-\rangle$ **Basis**:

$$P(|+\rangle) = 0.5, \quad P(|-\rangle) = 0.5$$

Invalid Qubit States

(c) $0.7|0\rangle + 0.3|1\rangle$ (Does not satisfy normalization: $0.49 + 0.09 = 0.58 \neq 1$)

(f) $\cos^2\theta|0\rangle - \sin^2\theta|1\rangle$ (Does not satisfy normalization in general, for example when $\theta = \pi/4$)

2. State Collapse

a) Measuring the First Qubit as '1'

After measuring the first qubit and observing '1', the state collapses to the subspace where the first qubit is '1' (i.e., $|10\rangle$ and $|11\rangle$):

$$|\psi'\rangle = -3\,|10\rangle - 4i\,|11\rangle$$

To normalize, compute the norm:

$$\| -3\,|10\rangle - 4i\,|11\rangle\| = \sqrt{(-3)^2 + (-4i)(4i)} = \sqrt{9+16} = 5$$

Thus, the normalized post-measurement state is:

$$|\psi'\rangle = \frac{-3\,|10\rangle - 4i\,|11\rangle}{5} = -\frac{3}{5}\,|10\rangle - \frac{4i}{5}\,|11\rangle$$

b) Subsequent Measurement of the Second Qubit

Given the post-measurement state:

$$|\psi'\rangle = -\frac{3}{5}\,|10\rangle - \frac{4i}{5}\,|11\rangle$$

The probability of measuring the second qubit as '1' is the square of the coefficient of $|11\rangle$:

$$P(\text{second qubit} = 1) = \left| -\frac{4i}{5} \right|^2 = \frac{16}{25}$$

3. Hadamard and Pauli Matrices

Hadamard Matrix H

The Hadamard matrix is:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

To show unitarity, we verify $H^\dagger H = I$:

$$H^\dagger = H, \quad H^\dagger H = HH = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$$

Thus, H is unitary.

Pauli Matrices

The Pauli matrices are:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- For X:
$$X^\dagger = X, \quad X^\dagger X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = I$$
- For Y:
$$Y^\dagger = Y, \quad Y^\dagger Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = I$$
- For Z:
$$Z^\dagger = Z, \quad Z^\dagger Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I$$

All Pauli matrices are unitary.

4. Matrix Representations

Given Matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

1. $I \otimes H$

$$I \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

2. $H \otimes I$

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

5. Pauli X Gate and Measurement

Initial State

$$|\psi\rangle = 0.8\,|00\rangle + 0.6\,|11\rangle$$

After Applying Pauli X to Second Qubit

$$|\psi'\rangle = 0.8\,|01\rangle + 0.6\,|10\rangle$$

Measurement Probabilities

- $P(|01\rangle) = |0.8|^2 = 0.64$ (64%)
- $P(|10\rangle) = |0.6|^2 = 0.36$ (36%)

6. Entangled States

Consider two arbitrary single-qubit states:

$$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\psi_2\rangle = \gamma|0\rangle + \delta|1\rangle,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$, $|\gamma|^2 + |\delta|^2 = 1$.

The tensor product $|\psi_1\rangle \otimes |\psi_2\rangle$ is:

$$|\psi_1\rangle \otimes |\psi_2\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle.$$

For this to equal the entangled state:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

the following must hold:

$$\alpha\gamma = \frac{1}{\sqrt{2}}, \quad \alpha\delta = 0, \quad \beta\gamma = 0, \quad \beta\delta = \frac{1}{\sqrt{2}}.$$

From $\alpha\delta = 0$ and $\beta\gamma = 0$:

- If $\alpha = 0$, then $\alpha\gamma = 0 \neq \frac{1}{\sqrt{2}}$.
- If $\delta = 0$, then $\beta\delta = 0 \neq \frac{1}{\sqrt{2}}$.
- If $\beta = 0$, then $\beta\delta = 0 \neq \frac{1}{\sqrt{2}}$.
- If $\gamma = 0$, then $\alpha\gamma = 0 \neq \frac{1}{\sqrt{2}}$.

The only solution is $\alpha = \beta = \gamma = \delta = 0$, which violates normalization. Thus, $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ cannot be expressed as a tensor product of two single-qubit states and is therefore entangled.

7. Controlled-H Gate

The matrix form of the controlled-Hadamard gate would be as follows:

$$\mathrm{CH} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

8. CNOT from H and CZ

The **controlled-NOT (CNOT)** gate can be constructed using **Hadamard (H)** gates and the **controlled-Z (CZ)** gate, by applying the gates in the order H - CZ - H where H is applied to the target qubit. Since we don't change the control qubit while applying the Hadamard to the target, we can tensor the matrix with the identity matrix to make the multiplications possible. Therefore, we want to show:

$$\text{CNOT} = (I \otimes H) \cdot \text{CZ} \cdot (I \otimes H)$$

Proof by Matrix Multiplication:

1. **Hadamard Gate (H)**:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

2. **Controlled-Z (CZ) Gate**:

$$\text{CZ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

3. **First Operation:** $I \otimes H$ (applies H to the target qubit)

$$I \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

4. **Second Operation:** Multiply by CZ

$$(I \otimes H) \cdot \text{CZ} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

5. **Final Operation:** Multiply by the second $I \otimes H$

$$(I \otimes H) \cdot \text{CZ} \cdot (I \otimes H) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This matches the standard matrix form of the CNOT gate, proving the construction.

9. Bell States form an Orthonormal Basis for $\mathbb{C}^4$

To verify that the four Bell states form an orthonormal basis for $\mathbb{C}^4$, we construct a matrix where each column represents one Bell state and show that this matrix is unitary.

Matrix Representation in Computational Basis
Expressed in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

Constructing the Unitary Matrix U
We form matrix U with Bell states as columns:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & -1\\1 & -1 & 0 & 0\end{pmatrix}$$

Verifying Unitary
A matrix is unitary if $U^\dagger U = I$. First compute $U^\dagger$:

$$U^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\\1 & 0 & 0 & -1\\0 & 1 & 1 & 0\\0 & 1 & -1 & 0\end{pmatrix}$$

Now compute $U^\dagger U$:

$$U^\dagger U = \frac{1}{2}\begin{pmatrix}1\cdot 1 + 1\cdot 1 & 1\cdot 1 - 1\cdot 1 & 0 & 0\\1\cdot 1 - 1\cdot 1 & 1\cdot 1 + 1\cdot 1 & 0 & 0\\0 & 0 & 1\cdot 1 + 1\cdot 1 & 1\cdot 1 - 1\cdot 1\\0 & 0 & 1\cdot 1 - 1\cdot 1 & 1\cdot 1 + 1\cdot 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 2\end{pmatrix} = I$$

Conclusion
Since $U^\dagger U = I$, the matrix U is unitary. Therefore, its columns (the Bell states) form an orthonormal basis for $\mathbb{C}^4$.

10. No-Signaling in Bell Pair Measurements

Alice and Bob share the Bell state:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Case 1: Alice Does Not Measure

- State remains $|\Phi^+\rangle$.
- Bob's reduced density matrix:

$$\rho_B = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|.$$

- Measurement probabilities: $P(0) = P(1) = 0.5$.

Case 2: Alice Measures in $\{|0\rangle, |1\rangle\}$

- State collapses to $|00\rangle$ or $|11\rangle$ with $P = 0.5$.
- Bob's reduced density matrix remains:

$$\rho_B = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|.$$

- Measurement probabilities still: $P(0) = P(1) = 0.5$.

Conclusion

- Bob cannot distinguish whether Alice measured.
- No observable change in Bob's statistics $\Rightarrow$ **no signaling**.
- The no-signaling principle is preserved.

11. No-Cloning and No-Deletion Principle

a) No Cloning Principle

Assumption
Assume there exists a unitary operator U such that for any $|\Psi\rangle$, $U|\Psi\rangle = |\Psi\rangle|\Psi\rangle$

Contradiction for Arbitrary States
For two distinct states $|\Psi\rangle$ and $|\Phi\rangle$:

$$U\big(|\Psi\rangle \otimes |0\rangle\big) = |\Psi\rangle \otimes |\Psi\rangle.$$

$$U\big(|\Phi\rangle \otimes |0\rangle\big) = |\Phi\rangle \otimes |\Phi\rangle.$$

Which implies:

$$\begin{aligned} \langle\Psi|\Phi\rangle &= \langle\Psi|\Phi\rangle \cdot \langle 0|0\rangle \\ &= (\langle\Psi| \otimes \langle 0|)U^\dagger U(|\Phi\rangle \otimes |0\rangle) \\ &= \big(|\Psi\rangle \otimes |\Psi\rangle\big)\big(|\Phi\rangle \otimes |\Phi\rangle\big) \\ &= \big(\langle\Psi|\Phi\rangle\big)^2. \end{aligned}$$

This holds only if $\langle\Psi|\Phi\rangle = 0$ or 1. Thus, U cannot clone arbitrary states.

Conclusion
No universal quantum cloning machine exists.

b) No Deletion Principle

Assumption
Assume there exists a unitary $\tilde{U}$ such that:

$$\tilde{U}\big(|\Psi\rangle \otimes |\Psi\rangle\big) = |\Psi\rangle \otimes |0\rangle.$$

Time-Reversal Implies Cloning
Left multiplying both sides of the equation with the inverse operation $U = \tilde{U}^\dagger$ would clone $|\Psi\rangle$, which would violate the No-Cloning Principle, which we showed in part a). Therefore, no such operation exists

Conclusion
Quantum deletion is impossible, preserving reversibility.