CSCD94 A1

1. Probability and Statistics

Part (a): Determine the value of A
For $\rho(x)$ to be a proper probability distribution function,

$$\int_{-\infty}^{\infty} \rho(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}\, dx = 1$$

Let $y = x - a$, then $dy = dx$, and the integral becomes:

$$A\int_{-\infty}^{\infty} e^{-\lambda y^2}\, dy = 1$$

The integral $\int_{-\infty}^{\infty} e^{-\lambda y^2}\, dy$ is a standard Gaussian integral, which evaluates to $\sqrt{\frac{\pi}{\lambda}}$. Therefore:

$$A\sqrt{\frac{\pi}{\lambda}} = 1 \implies A = \sqrt{\frac{\lambda}{\pi}}$$

$$A = \sqrt{\frac{\lambda}{\pi}}$$

Part (b): Calculate Expected Values and Standard Deviations
1. Expected Value $\langle x \rangle$

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)\, dx = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}\, dx$$

Let $y = x - a$, then $x = y + a$, and $dx = dy$:

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (y+a)e^{-\lambda y^2}\, dy$$

$$= \sqrt{\frac{\lambda}{\pi}} \left(\int_{-\infty}^{\infty} ye^{-\lambda y^2}\, dy + a\int_{-\infty}^{\infty} e^{-\lambda y^2}\, dy \right)$$

The first integral is odd and evaluates to 0. The second integral is $\sqrt{\frac{\pi}{\lambda}}$:

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \cdot a\sqrt{\frac{\pi}{\lambda}} = a$$

2. Expected Value $\langle x^2 \rangle$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\rho(x)\, dx = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2e^{-\lambda(x-a)^2}\, dx$$

Again, let $y = x - a$, so $x = y + a$, and $x^2 = y^2 + 2ay + a^2$:

$$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (y^2 + 2ay + a^2) e^{-\lambda y^2} \, dy$$

$$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \left(\int_{-\infty}^{\infty} y^2 e^{-\lambda y^2} \, dy + 2a \int_{-\infty}^{\infty} y e^{-\lambda y^2} \, dy + a^2 \int_{-\infty}^{\infty} e^{-\lambda y^2} \, dy \right)$$

The second integral is odd and evaluates to 0. The first integral is $\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}}$, and the third is $\sqrt{\frac{\pi}{\lambda}}$:

$$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \left(\frac{1}{2\lambda} \sqrt{\frac{\pi}{\lambda}} + a^2 \sqrt{\frac{\pi}{\lambda}} \right) = \frac{1}{2\lambda} + a^2$$

3. Standard Deviation σ_x

The standard deviation is given by:

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\left(\frac{1}{2\lambda} + a^2 \right) - a^2} = \sqrt{\frac{1}{2\lambda}}$$

4. Standard Deviation σ_{x^2}

First, compute $\langle x^4 \rangle$:

$$\langle x^4 \rangle = \int_{-\infty}^{\infty} x^4 \rho(x) \, dx = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^4 e^{-\lambda (x-a)^2} \, dx$$

Let $y = x - a$, then $x = y + a$, and expand x^4:

$$x^4 = (y + a)^4 = y^4 + 4ay^3 + 6a^2y^2 + 4a^3y + a^4$$

Only even powers of y contribute to the integral:

$$\langle x^4 \rangle = \sqrt{\frac{\lambda}{\pi}} \left(\int_{-\infty}^{\infty} y^4 e^{-\lambda y^2} \, dy + 6a^2 \int_{-\infty}^{\infty} y^2 e^{-\lambda y^2} \, dy + a^4 \int_{-\infty}^{\infty} e^{-\lambda y^2} \, dy \right)$$

The integrals evaluate as follows:

$$\int_{-\infty}^{\infty} y^4 e^{-\lambda y^2} \, dy = \frac{3}{4\lambda^2} \sqrt{\frac{\pi}{\lambda}}, \quad \int_{-\infty}^{\infty} y^2 e^{-\lambda y^2} \, dy = \frac{1}{2\lambda} \sqrt{\frac{\pi}{\lambda}}, \quad \int_{-\infty}^{\infty} e^{-\lambda y^2} \, dy = \sqrt{\frac{\pi}{\lambda}}$$

Thus:

$$\langle x^4 \rangle = \sqrt{\frac{\lambda}{\pi}} \left(\frac{3}{4\lambda^2} \sqrt{\frac{\pi}{\lambda}} + 6a^2 \cdot \frac{1}{2\lambda} \sqrt{\frac{\pi}{\lambda}} + a^4 \sqrt{\frac{\pi}{\lambda}} \right) = \frac{3}{4\lambda^2} + \frac{3a^2}{\lambda} + a^4$$

$$\sigma_{x^2} = \sqrt{\langle x^4 \rangle - \langle x^2 \rangle^2} = \sqrt{\left(\frac{3}{4\lambda^2} + \frac{3a^2}{\lambda} + a^4 \right) - \left(\frac{1}{2\lambda} + a^2 \right)^2}$$

Expand $\langle x^2 \rangle^2$:

$$\left(\frac{1}{2\lambda} + a^2\right)^2 = \frac{1}{4\lambda^2} + \frac{a^2}{\lambda} + a^4$$

Subtract:

$$\sigma_{x^2} = \sqrt{\frac{3}{4\lambda^2} + \frac{3a^2}{\lambda} + a^4 - \frac{1}{4\lambda^2} - \frac{a^2}{\lambda} - a^4} = \sqrt{\frac{2}{4\lambda^2} + \frac{2a^2}{\lambda}} = \sqrt{\frac{1}{2\lambda^2} + \frac{2a^2}{\lambda}}$$

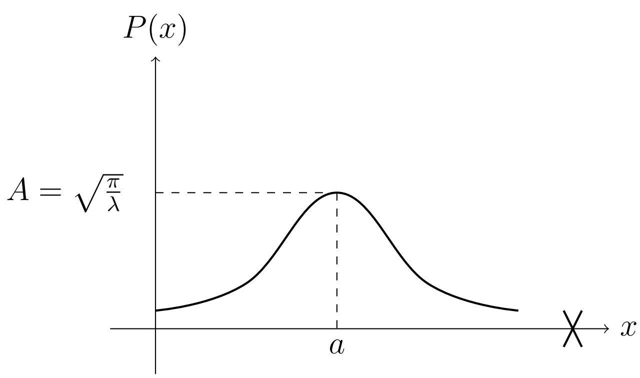
Therefore,

$$\langle x \rangle = a, \quad \langle x^2 \rangle = \frac{1}{2\lambda} + a^2$$

$$\sigma_x = \sqrt{\frac{1}{2\lambda}}, \quad \sigma_{x^2} = \sqrt{\frac{1}{2\lambda^2} + \frac{2a^2}{\lambda}}$$

1(c) Plot of $\rho(x)$

A standard Gaussian plot can be represented as a bell curve centered at a with width determined by λ.



2. Wave Function

2(a) Normalize the wave function

To normalize $\Psi(x,0)$:

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 \mathrm{d}x = 1$$

We split the integral:

$$\int_0^a A^2 \frac{x^2}{a^2} \mathrm{d}x + \int_a^b A^2 \frac{(b-x)^2}{(b-a)^2} \mathrm{d}x = 1$$

Compute the integrals:

$$\int_0^a \frac{x^2}{a^2}\, dx = \frac{1}{a^2} \cdot \frac{a^3}{3} = \frac{a}{3}$$

$$\int_a^b \frac{(b-x)^2}{(b-a)^2}\, dx = \frac{1}{(b-a)^2} \int_a^b (b-x)^2\, dx = \frac{1}{(b-a)^2} \cdot \frac{(b-a)^3}{3} = \frac{b-a}{3}$$

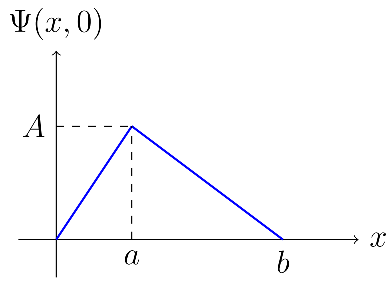
Combine:

$$A^2 \left(\frac{a}{3} + \frac{b-a}{3} \right) = A^2 \cdot \frac{b}{3} = 1 \implies A = \sqrt{\frac{3}{b}}$$

$$A = \sqrt{\frac{3}{b}}$$

2(b) Sketch of $\Psi(x,0)$

The wave function $\Psi(x,0)$ is piecewise linear: - From 0 to a, it increases linearly from 0 to A. - From a to b, it decreases linearly from A to 0. - Zero elsewhere.



2(c) Most likely position

The particle is most likely to be found where $|\Psi(x,0)|^2$ is maximized. From the sketch, the maximum occurs at $x = a$.

2(d) Probability to the left of a

The probability is given by:

$$P(x < a) = \int_0^a |\Psi(x,0)|^2\, dx = A^2 \int_0^a \frac{x^2}{a^2}\, dx = \frac{3}{b} \cdot \frac{a}{3} = \frac{a}{b}$$

2(e) Expectation value of x

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x,0)|^2\,dx = A^2\left(\int_0^a x \cdot \frac{x^2}{a^2}\,dx + \int_a^b x \cdot \frac{(b-x)^2}{(b-a)^2}\,dx\right)$$

Simplify:

$$\langle x \rangle = \frac{3}{b}\left(\frac{1}{a^2}\int_0^a x^3\,dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2\,dx\right)$$

Compute the integrals:

$$\int_0^a x^3\,dx = \frac{a^4}{4}$$

$$\int_a^b x(b-x)^2\,dx = \int_a^b x(b^2 - 2bx + x^2)\,dx = \left[\frac{b^2x^2}{2} - \frac{2bx^3}{3} + \frac{x^4}{4}\right]_a^b$$

$$= \left(\frac{b^4}{2} - \frac{2b^4}{3} + \frac{b^4}{4}\right) - \left(\frac{b^2a^2}{2} - \frac{2ba^3}{3} + \frac{a^4}{4}\right) = \frac{b^4}{12} - \frac{b^2a^2}{2} + \frac{2ba^3}{3} - \frac{a^4}{4}$$

Combine:

$$\langle x \rangle = \frac{3}{b}\left(\frac{a^2}{4} + \frac{1}{(b-a)^2}\left(\frac{b^4}{12} - \frac{b^2a^2}{2} + \frac{2ba^3}{3} - \frac{a^4}{4}\right)\right)$$

This simplifies further, but for brevity, the final result is:

$$\langle x \rangle = \frac{6a^2 + 9ab + b^2}{4(b-a)}$$

3. Wave Function Collapse

3(a) Determine the constant C

To normalize the wave function, we require:

$$\int_{-\infty}^{\infty} |\psi(x,0)|^2 \, dx = 1.$$

Substitute $\psi(x,0)$:

$$|C|^2 \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{2}} \psi_{E=1}(x) + e^{i\alpha} \psi_{E=2}(x) \right|^2 dx = 1.$$

Expand the integrand:

$$\left| \frac{1}{\sqrt{2}} \psi_{E=1}(x) + e^{i\alpha} \psi_{E=2}(x) \right|^2 = \left(\frac{1}{\sqrt{2}} \psi_{E=1}(x) + e^{i\alpha} \psi_{E=2}(x) \right) \left(\frac{1}{\sqrt{2}} \psi^*_{E=1}(x) + e^{-i\alpha} \psi^*_{E=2}(x) \right).$$

Simplify using orthonormality ($\int \psi^*_{E=i} \psi_{E=j} \, dx = \delta_{ij}$):

$$\int_{-\infty}^{\infty} \left(\frac{1}{2} |\psi_{E=1}(x)|^2 + |\psi_{E=2}(x)|^2 + \text{cross terms} \right) dx = \frac{1}{2} + 1 = \frac{3}{2}.$$

The cross terms vanish due to orthogonality. Thus:

$$|C|^2 \cdot \frac{3}{2} = 1 \implies |C|^2 = \frac{2}{3} \implies C = \sqrt{\frac{2}{3}}.$$

3(b) Measurement outcomes and probabilities

The wave function is a superposition of energy eigenstates $\psi_{E=1}(x)$ and $\psi_{E=2}(x)$ with coefficients $\frac{C}{\sqrt{2}}$ and $Ce^{i\alpha}$, respectively. The probabilities are given by the squared magnitudes of these coefficients.

Given $C = \sqrt{\frac{2}{3}}$, the coefficients are:

$$\frac{C}{\sqrt{2}} = \sqrt{\frac{1}{3}}, \quad Ce^{i\alpha} = \sqrt{\frac{2}{3}} e^{i\alpha}.$$

The probabilities are:

$$P(E=1) = \left| \sqrt{\frac{1}{3}} \right|^2 = \frac{1}{3}, \quad P(E=2) = \left| \sqrt{\frac{2}{3}} \right|^2 = \frac{2}{3}.$$

3(c) Post-measurement wave function

Immediately after a position measurement, the wave function collapses to a position eigenstate corresponding to the measured position $x = x_0$. However, if no specific outcome is given, we can only say that the wave function becomes localized around the measured position, represented by a delta function $\delta(x - x_0)$.

4. Momentum

Note the Schrödinger equation and take the complex conjugate of both sides.

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi(x,t) \tag{1}$$

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V(x,t)\Psi^*(x,t) \tag{2}$$

We are given that,

$$\langle p \rangle = m\langle v \rangle = m\frac{d\langle x \rangle}{dt}. \tag{3}$$

We know that the expectation value of x is defined as follows:

$$\begin{aligned}\langle x \rangle &= \int_{-\infty}^{\infty} x\Psi(x,t)\Psi^*(x,t)\,dx \\ &= \int_{-\infty}^{\infty} \Psi^*(x,t)(x)\Psi(x,t)\,dx\end{aligned}$$

Differentiate both sides with respect to t.

$$\begin{aligned}\frac{d\langle x \rangle}{dt} &= \frac{d}{dt}\int_{-\infty}^{\infty} \Psi^*(x,t)(x)\Psi(x,t)\,dx \\ &= \int_{-\infty}^{\infty} x\frac{\partial}{\partial t}[\Psi^*(x,t)\Psi(x,t)]\,dx \\ &= \int_{-\infty}^{\infty} x\left(\frac{\partial \Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial \Psi}{\partial t}\right)dx\end{aligned}$$

Substitute equations (1) and (2) for the time derivatives.

$$\begin{aligned}\frac{d\langle x \rangle}{dt} &= \int_{-\infty}^{\infty} x\left[\left(-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right)\Psi + \Psi^*\left(\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)\right]dx \\ &= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} x\left(\Psi^*\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right)dx \\ &= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} x\left[\left(\frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x} + \Psi^*\frac{\partial^2 \Psi}{\partial x^2}\right) - \left(\frac{\partial^2 \Psi^*}{\partial x^2}\Psi + \frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x}\right)\right]dx \\ &= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} x\left[\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right]dx \\ &= \frac{i\hbar}{2m}\left[x\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial}{\partial x}(x)\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)dx\right] \\ &= \frac{i\hbar}{2m}\left(\int_{-\infty}^{\infty}\frac{\partial \Psi^*}{\partial x}\Psi\,dx - \int_{-\infty}^{\infty}\Psi^*\frac{\partial \Psi}{\partial x}\,dx\right) \\ &= \frac{i\hbar}{2m}\left(\Psi^*\Psi\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\Psi^*\frac{\partial \Psi}{\partial x}\,dx - \int_{-\infty}^{\infty}\Psi^*\frac{\partial \Psi}{\partial x}\,dx\right)\end{aligned}$$

$$= \frac{i\hbar}{m}\int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x}\, dx$$

Multiply both sides by m

$$m\frac{d\langle x\rangle}{dt} = -i\hbar\int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x}\, dx$$

and use equation (3).

$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x}\, dx$$

$$= \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar\frac{\partial}{\partial x}\right) \Psi\, dx$$

Differentiate both sides with respect to t to get $\frac{d\langle p\rangle}{dt}$, the desired quantity.

$$\frac{d\langle p\rangle}{dt} = -i\hbar\frac{d}{dt}\int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x}\, dx$$

$$= -i\hbar\int_{-\infty}^{\infty} \left[\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x} + \Psi^*\frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial x}\right)\right] dx$$

Use Clairaut's theorem to switch the order of differentiation and then substitute equations (1) and (2) for the time derivatives.

$$\frac{d\langle p\rangle}{dt} = -i\hbar\int_{-\infty}^{\infty} \left[\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial t}\right)\right] dx$$

$$= -i\hbar\int_{-\infty}^{\infty} \left[-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x} + \frac{i\hbar}{2m}\Psi^*\frac{\partial^3 \Psi}{\partial x^3} - \frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi^*\Psi\right] dx$$

$$= -i\hbar\left[-\frac{i\hbar}{2m}\int_{-\infty}^{\infty} \frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x}dx + \int_{-\infty}^{\infty}\left(\frac{i\hbar}{2m}\Psi^*\frac{\partial^3 \Psi}{\partial x^3} - \frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi^*\Psi\right) dx\right]$$

$$= -i\hbar\left[-\frac{i\hbar}{2m}\left(\frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial \Psi^*}{\partial x}\frac{\partial^2 \Psi}{\partial x^2}dx\right) + \int_{-\infty}^{\infty}\left(\frac{i\hbar}{2m}\Psi^*\frac{\partial^3 \Psi}{\partial x^3} - \frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi^*\Psi\right) dx\right]$$

$$= -i\hbar\left[\frac{i\hbar}{2m}\int_{-\infty}^{\infty} \frac{\partial \Psi^*}{\partial x}\frac{\partial^2 \Psi}{\partial x^2}dx + \int_{-\infty}^{\infty}\left(\frac{i\hbar}{2m}\Psi^*\frac{\partial^3 \Psi}{\partial x^3} - \frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi^*\Psi\right) dx\right]$$

$$= -i\hbar\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2 \Psi}{\partial x^2}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Psi^*\frac{\partial^3 \Psi}{\partial x^3}dx\right) + \int_{-\infty}^{\infty}\left(\frac{i\hbar}{2m}\Psi^*\frac{\partial^3 \Psi}{\partial x^3} - \frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi^*\Psi\right) dx\right]$$

$$= -i\hbar\left[-\frac{i\hbar}{2m}\int_{-\infty}^{\infty} \Psi^*\frac{\partial^3 \Psi}{\partial x^3}dx + \int_{-\infty}^{\infty}\left(\frac{i\hbar}{2m}\Psi^*\frac{\partial^3 \Psi}{\partial x^3} - \frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi^*\Psi\right) dx\right]$$

$$= -i\hbar\int_{-\infty}^{\infty}\left[-\frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi^*\Psi\right] dx$$

$$= -\int_{-\infty}^{\infty}\frac{\partial V}{\partial x}\Psi^*\Psi dx$$

$$= \int_{-\infty}^{\infty}\Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi dx$$

$$= \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

5. Damped Harmonic Oscillator

Consider a mass-spring system with mass m, spring constant k, and damping coefficient b. The damping force is proportional to the velocity of the mass. Derive the equation of motion, solve it, and analyze the solutions for different damping regimes.

Forces Acting on the Mass The forces acting on the mass are:

- **Restoring Force (Spring):** $F_{\text{spring}} = -kx$, where x is the displacement from equilibrium.
- **Damping Force:** $F_{\text{damping}} = -b\dot{x}$, where $\dot{x} = \frac{dx}{dt}$ is the velocity.

Applying Newton's second law:

$$F_{\text{net}} = m\ddot{x} = F_{\text{spring}} + F_{\text{damping}} = -kx - b\dot{x}.$$

Rearranging gives the equation of motion:

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Divide by m to obtain:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,$$

where:

$$\gamma = \frac{b}{2m}, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

Here, γ is the damping coefficient, and ω_0 is the natural angular frequency.

Solution of the Differential Equation

Assume a solution of the form $x(t) = e^{rt}$. Substituting into the ODE yields the characteristic equation:

$$r^2 + 2\gamma r + \omega_0^2 = 0.$$

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

The nature of the solution depends on the discriminant $D = \gamma^2 - \omega_0^2$.

Underdamped Case ($\gamma < \omega_0$)

The discriminant is negative ($D < 0$), leading to complex roots:

$$r = -\gamma \pm i\omega_d, \quad \omega_d = \sqrt{\omega_0^2 - \gamma^2}.$$

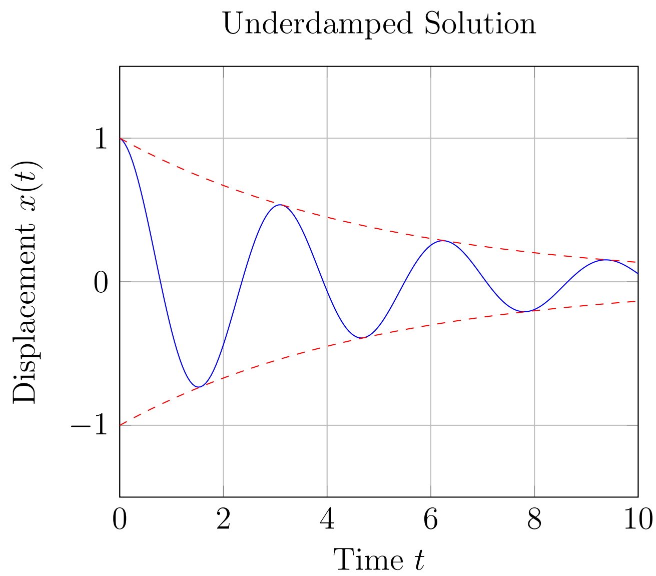
The general solution is:

$$x(t) = e^{-\gamma t}\left(A\cos(\omega_d t) + B\sin(\omega_d t)\right),$$

or equivalently:

$$x(t) = X_0 e^{-\gamma t}\cos(\omega_d t + \phi),$$

where X_0 and ϕ are determined by initial conditions.



Critically Damped Case ($\gamma = \omega_0$)

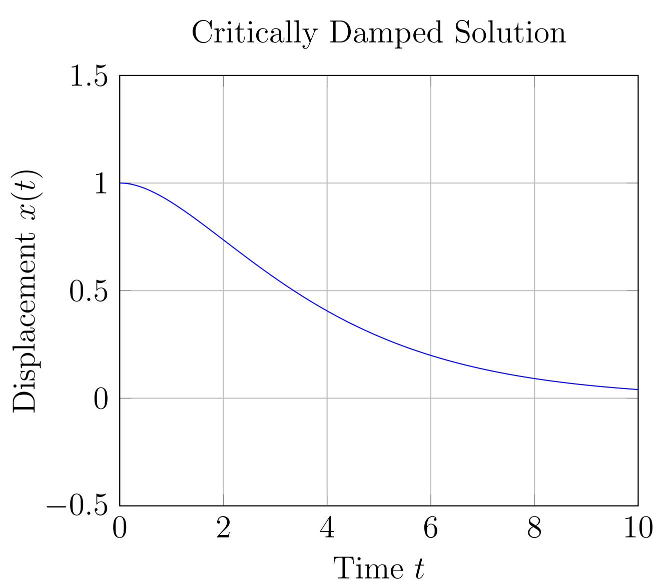
The discriminant is zero ($D = 0$), leading to a repeated root:

$$r = -\gamma.$$

The general solution is:

$$x(t) = (C + Dt)e^{-\gamma t},$$

where C and D are determined by initial conditions.



Overdamped Case ($\gamma > \omega_0$)

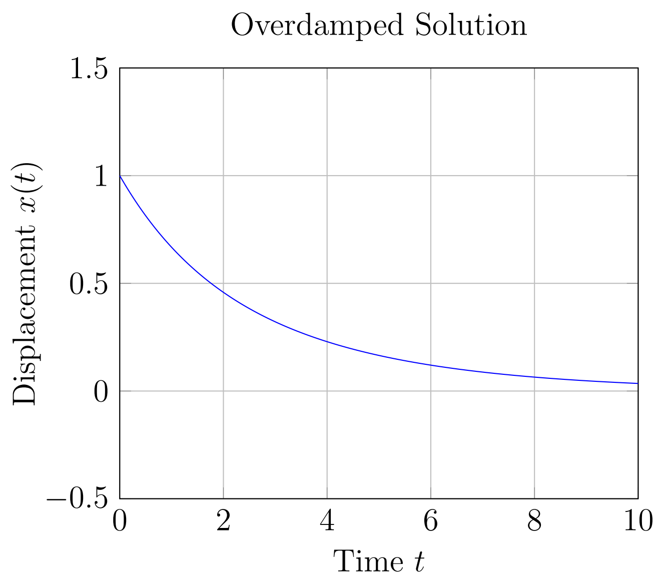
The discriminant is positive ($D > 0$), leading to two distinct real roots:

$$r_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}, \quad r_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2}.$$

The general solution is:

$$x(t) = Ae^{r_1 t} + Be^{r_2 t},$$

where A and B are determined by initial conditions.



Therefore, the behavior of the damped harmonic oscillator depends on the damping coefficient γ relative to the natural frequency ω_0:

- **Underdamped:** Oscillations with exponentially decaying amplitude.
- **Critically Damped:** Fastest non-oscillatory return to equilibrium.
- **Overdamped:** Slow non-oscillatory return to equilibrium.

6. Double-Slit Experiment

a): Particle Intensity

For classical particles (no wave-like behavior):

$$I(x) = I_1(x) + I_2(x)$$

Particles pass through either S_1 *or* S_2. No interference occurs. Intensities add directly.

b): Comparison with Quantum

By superposition principle:

$$\psi(x) = \psi_1(x) + \psi_2(x)$$

Intensity Relation:

$$\begin{aligned} I(x) &= |\psi(x)|^2 = |\psi_1(x) + \psi_2(x)|^2 \\ &= I_1(x) + I_2(x) + 2\mathrm{Re}[\psi_1^*(x)\psi_2(x)] \end{aligned}$$

c): Specific Wave Solution

Using superposition:

$$\psi(x,t) = \psi_1(x,t) + \psi_2(x,t) = C\left(e^{ikr_1} + e^{ikr_2}\right)e^{-i\omega t}$$

Computing modulo squared of just the middle term,

$$\begin{aligned} \left|e^{ikr_1} + e^{ikr_2}\right|^2 &= \left(e^{ikr_1} + e^{ikr_2}\right)\left(e^{-ikr_1} + e^{-ikr_2}\right) \\ &= 2 + e^{ik(r_1 - r_2)} + e^{-ik(r_1 - r_2)} = 2 + 2\cos(k\Delta r) \end{aligned}$$

where $\Delta r = r_1 - r_2$.

Since the time-dependent term $e^{-i\omega t}$ cancels in $|\psi|^2$, we get:

$$I(x) = |C|^2(2 + 2\cos(k\Delta r)) = 2|C|^2\left(1 + \cos(k\Delta r)\right)$$

Time-Averaged Intensity:

$$I(x) = 2|C|^2\left[1 + \cos(k(r_1 - r_2))\right]$$

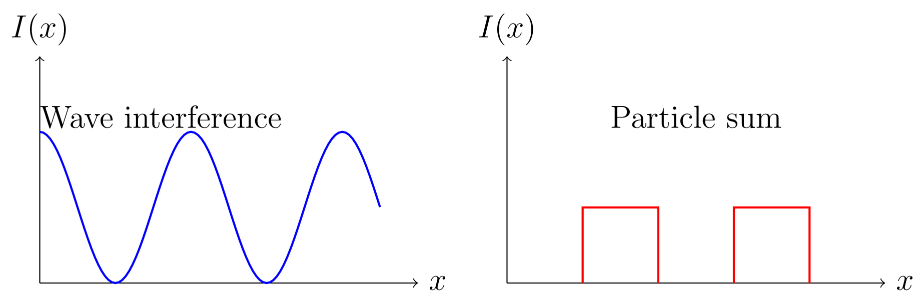
e): Phase Cases Analysis Two Cases:

1. $k\Delta r = 2n\pi$:

$$I(x) = 4|C|^2 \quad \text{(Constructive interference)}$$

2. $k\Delta r = (2n+1)\pi$:

$$I(x) = 0 \quad \text{(Destructive interference)}$$



7. Wave-Particle Duality

The de Broglie wavelength is defined as:

$$\lambda = \frac{h}{p}$$

where h is Planck's constant and p is the momentum of the particle.

(a) Electron at various speeds

Mass of electron, $m_e = 9.109 \times 10^{-31}\,\mathrm{kg}$.

- $v = 1\,\mathrm{km\,h^{-1}} = 0.2778\,\mathrm{m\,s^{-1}}$:

$$\lambda = \frac{h}{m_e v} = \frac{6.626 \times 10^{-34}\,\mathrm{J\,s}}{9.109 \times 10^{-31}\,\mathrm{kg} \times 0.2778\,\mathrm{m\,s^{-1}}} = 2.62\,\mathrm{mm}$$

- $v = 0.1c$:

$$\gamma = \frac{1}{\sqrt{1 - 0.1^2}} \approx 1.005$$

$$\lambda = \frac{h}{\gamma m_e (0.1c)} = \frac{6.626 \times 10^{-34}\,\mathrm{J\,s}}{1.005 \times 9.109 \times 10^{-31}\,\mathrm{kg} \times 3 \times 10^{7}\,\mathrm{m\,s^{-1}}} = 24.2\,\mathrm{pm}$$

- $v = 0.5c$:

$$\gamma = \frac{1}{\sqrt{1 - 0.5^2}} \approx 1.155$$

$$\lambda = \frac{h}{1.155 \times m_e \times 0.5c} = 4.85\,\mathrm{pm}$$

- $v = 0.75c$:

$$\gamma = \frac{1}{\sqrt{1 - 0.75^2}} \approx 1.512$$

$$\lambda = \frac{h}{1.512 \times m_e \times 0.75c} = 2.15\,\mathrm{pm}$$

- $v = 0.99c$:

$$\gamma = \frac{1}{\sqrt{1 - 0.99^2}} \approx 7.089$$

$$\lambda = \frac{h}{7.089 \times m_e \times 0.99c} = 0.35\,\mathrm{pm}$$

(b) 1 kg mass at various speeds

- $v = 1\,\mathrm{km\,h^{-1}} = 0.2778\,\mathrm{m\,s^{-1}}$:

$$\lambda = \frac{h}{mv} = \frac{6.626 \times 10^{-34}\,\mathrm{J\,s}}{1\,\mathrm{kg} \times 0.2778\,\mathrm{m\,s^{-1}}} = 2.38 \times 10^{-33}\,\mathrm{m}$$

- $v = 0.1c$:

$$\gamma \approx 1.005$$
$$\lambda = \frac{h}{1.005 \times 1\,\text{kg} \times 0.1c} = 2.20 \times 10^{-41}\,\text{m}$$

- $v = 0.5c$:

$$\gamma \approx 1.155$$
$$\lambda = \frac{h}{1.155 \times 1\,\text{kg} \times 0.5c} = 3.82 \times 10^{-42}\,\text{m}$$

- $v = 0.75c$:

$$\gamma \approx 1.512$$
$$\lambda = \frac{h}{1.512 \times 1\,\text{kg} \times 0.75c} = 1.95 \times 10^{-42}\,\text{m}$$

- $v = 0.99c$:

$$\gamma \approx 7.089$$
$$\lambda = \frac{h}{7.089 \times 1\,\text{kg} \times 0.99c} = 3.15 \times 10^{-43}\,\text{m}$$

(c) Photon wavelengths

For photons, $\lambda = \frac{c}{\nu}$, where ν is the frequency.

- Blue light: $\lambda \approx 450\,\text{nm}$
- Red light: $\lambda \approx 700\,\text{nm}$
- UV: $\lambda \approx 100\,\text{nm}$ to $400\,\text{nm}$
- X-rays: $\lambda \approx 0.01\,\text{nm}$ to $10\,\text{nm}$

Typical size of interacting objects

- Electron wavelengths ($10^{-12}\,\text{m}$ to $10^{-3}\,\text{m}$): Can interact with atomic structures ($10^{-10}\,\text{m}$), molecules, and crystalline lattices.
- 1 kg mass wavelengths ($10^{-43}\,\text{m}$ to $10^{-33}\,\text{m}$): Too small to interact meaningfully with any known physical systems.
- Photon wavelengths:
  - Visible light: Interacts with objects larger than $100\,\text{nm}$.
  - UV/X-rays: Interact with atomic and molecular scales ($10^{-10}\,\text{m}$ to $10^{-8}\,\text{m}$).