Solvability of linear algebraic systems of equations

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Consider a matrix A of size $m \times n$, and a system of linear algebraic equations given by $Ax = b$. We are considering a matrix whose columns are non-zero vectors. This document guides us through the thought process in understanding when such a system of linear algebraic equations is solvable and when it is not.

1 When is a system solvable?

Note that the set of vectors that can be spanned (or represented) by the matrix-vector product Ax is given by $\sum\limits_k x_k a_k$, where a_k is the k^{th} column vector of A. This is essentially the column space of matrix A, the space spanned by the columns of matrix A. Therefore, in order to examine whether a system $Ax = b$ can be solved, it boils down to finding if the vector b belongs to the column space (spanned by the columns of the matrix) of A. Equivalently, this requires to compare the dimension of the column space of matrix A and the augmented matrix $A|b$.

The dimension of the space spanned by columns of A is defined as its *rank* (denoted as r_A here). Equivalently, the rank of a matrix is the number of linearly independent columns (or rows) of the matrix. A set of vectors $\{a_k\}_{k=1}^n$ is said to be linearly independent if and only if $\sum\limits_{k=1}^{n} c_k a_k = 0$, where c_k are scalars, implies $c_k = 0$ for all k. One method to determine the rank of a matrix is via Gaussian Elimination (very likely you will work through an assignment problem that delineates this idea).

Consider now the augmented matrix $A|b$. If a solution to $Ax = b$ needs to exist, then b has to be represented as a linear combination of the columns of A (essentially the coefficients of the linear combination are given by the solution vector x). This requires that the the dimension of the space spanned by the columns of the augmented matrix be same as that of A.

Equivalently put, if the rank of the augmented matrix (denoted as $r_{A|b}$) is same as r_A, then we are guaranteed the existence of a solution (unique or infinitely many solutions). Such a system is called a *consistent system*. It follows that *no solution* exists if $r_{A|b}$ differs from r_A. A system with no solution is also called an *inconsistent system*. Note that $r_{A|b}$ can at most differ from r_A by 1, i.e, there are only two possibilities: $r_{A|b} = r_A$ or $r_{A|b} = r_A + 1$. (Why? Think about it).

2 Given solution exists, when can we say that unique solution or infinitely many solutions exist?

Consider the scenario when a solution is guaranteed to exist for $Ax = b$, i.e, $r_{A|b} = r_A$.

1. For unique solution, the columns of A must be linearly independent.

2. For infinitely many solution, the columns of A must be linearly dependent.

Note that if u and v are distinct solutions to $Ax = b$, then $\alpha u + (1 - \alpha)\, v$ is also a solution to $Ax = b$, where α can be any scalar.

3 Rank of a matrix and the solvability criteria

Let r denote the maximum possible value for the rank of a matrix. Note that $r = \min(m, n)$. This is because when

(i) $m = n$, the dimension of the column space can utmost be n (or m),

(ii) $m > n$, with n column vectors you cannot span a space larger than n, and

(iii) $m < n$, given that there are only m degrees of freedom for each vector, once again you cannot space whose dimension is larger than m.

When the rank of a matrix is equal to r, it is said to be of *full rank*. The matrix is said to *rank-deficient* if $r_A < r$.

Now, let us pick specific cases and come up with the criteria for the existence of unique, infinitely many, or no solutions to $Ax = b$.

4 Criteria for unique, infinitely-many, and no solutions

System Size	Rank relation	Nature of solution
$m = n$	$r_A = n (\Rightarrow r_{A\|b} = n)$	Unique
$m = n$	$r_A < n; r_{A\|b} = r_A$	Infinitely-many
$m = n$	$r_A < n; r_{A\|b} \neq r_A$	No solutions
$m > n$	$r_A = n; r_{A\|b} = n$	Unique
$m > n$	$r_A = n; r_{A\|b} \neq n$	No solution
$m > n$	$r_A < n; r_{A\|b} = r_A$	Infinitely-many
$m > n$	$r_A < n; r_{A\|b} \neq r_A$	No solutions
$m < n$	$r_A \leq m; r_{A\|b} = r_A$	Infinitely many
$m < n$	$r_A < m; r_{A\|b} \neq r_A$	No solutions

Table 1: Solvability criteria

These criteria can be condensed further as:

- $r_A = r_{A|b} = n$, then the linear system has a unique solution
- $r_A = r_{A|b} < n$, then the linear system has infinite solutions
- $r_A \neq r_{A|b}$, then the linear system is inconsistent and has no solution