Compound Poisson Process

Arrival rate λ:

0.01 0.5 5

0.01 0.51 1.01 1.51 2.01 2.51 3.01 3.51 4.01 4.51 5

Jump rate μ:

0.01 1 5

0.01 0.51 1.01 1.51 2.01 2.51 3.01 3.51 4.01 4.51 5

Number of simulations:

4000

- [x] Overlay theoretical pdf

Change parameters to update the plots.

Plots | Derivation

Derivation of the Compound Process

Definition

We study the compound process

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

with

$$N(t) \sim \text{Poisson}(\lambda t)$$

(exponential interarrival times) and

$$X_i \overset{\text{iid}}{\sim} \text{Exponential}(\mu).$$

Laplace transform

The Laplace transform is defined as:

$$\mathcal{L}_{S(t)}(u) = E[e^{-uS(t)}].$$

Condition on $N(t)$:

$$E[e^{-uS(t)} \mid N(t) = n] = \left(E[e^{-uX_1}]\right)^n.$$

For an exponential(μ) jump:

$$E[e^{-uX_1}] = \frac{\mu}{\mu + u}.$$

Using the PGF of Poisson, $E[s^{N(t)}] = \exp(\lambda t(s-1))$, set $s = \mu/(\mu + u)$ to get

$$\boxed{\mathcal{L}_{S(t)}(u) = \exp\Big(-\lambda t \frac{u}{\mu + u}\Big)\, .}$$

Moment Generating Function (MGF)

MGF is $M_{S(t)}(v) = E[e^{vS(t)}]$. Relate to Laplace transform by $M_{S(t)}(v) = \mathcal{L}_{S(t)}(-v)$.

Therefore:

$$\boxed{M_{S(t)}(v) = \exp\Big(\lambda t \frac{v}{\mu - v}\Big), \qquad v < \mu\, .}$$

Mean and Variance

From compound-Poisson identities:

$$E[S(t)] = E[N(t)]E[X] = \frac{\lambda t}{\mu},$$

$$\mathrm{Var}(S(t)) = \lambda t\, E[X^2] = \frac{2\lambda t}{\mu^2}.$$

At zero and continuous density

Point mass at zero:

$$P(S(t) = 0) = P(N(t) = 0) = e^{-\lambda t}.$$

For $s > 0$, the continuous pdf (Poisson–Gamma mixture) can be written as

$$f_{S(t)}(s) = e^{-\lambda t - \mu s} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \frac{\mu^n s^{n-1}}{(n-1)!}, \qquad s > 0.$$

This series has a closed form using the modified Bessel function:

$$\boxed{f_{S(t)}(s) = e^{-\lambda t - \mu s} \frac{\lambda t\, \mu}{\sqrt{\lambda t \mu s}}\, I_1\big(2\sqrt{\lambda t \mu s}\big), \quad s > 0.}$$

Normal approximation (large λt)

By the CLT, for large λt:

$$S(t) \approx \mathcal{N}\Big(\frac{\lambda t}{\mu},\ \frac{2\lambda t}{\mu^2}\Big).$$

Note

- Larger λ increases the expected number of jumps and reduces mass at zero.
- Larger μ makes jumps smaller (reduces mean and variance).
- For very large λt use the normal approximation to avoid numerical instability evaluating the Bessel form.