

2.12 (1) VECTORS

Any quantity having n -components is called a *vector of order n* . Therefore, the coefficients in a linear equation or the elements in a row or column matrix will form a vector. Thus any n numbers x_1, x_2, \dots, x_n written in a particular order, constitute a vector \mathbf{x} .

(2) Linear dependence. The vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are said to be **linearly dependent**, if there exist numbers $\lambda_1, \lambda_2, \dots, \lambda_r$ not all zero, such that

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \dots + \lambda_r \mathbf{x}_r = \mathbf{0}. \quad \dots(i)$$

If no such numbers, other than zero, exist, the vectors are said to be **linearly independent**. If $\lambda_1 \neq 0$, transposing $\lambda_1 \mathbf{x}_1$ to the other side and dividing by $-\lambda_1$, we write (i) in the form

$$\mathbf{x}_1 = \mu_2 \mathbf{x}_2 + \mu_3 \mathbf{x}_3 + \dots + \mu_r \mathbf{x}_r.$$

Then the vector \mathbf{x}_1 is said to be a linear combination of the vectors $\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_r$.

Example 2.41. Are the vectors $\mathbf{x}_1 = (1, 3, 4, 2)$, $\mathbf{x}_2 = (3, -5, 2, 2)$ and $\mathbf{x}_3 = (2, -1, 3, 2)$ linearly dependent? If so express one of these as a linear combination of the others.

Solution. The relation $\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 = \mathbf{0}$.

i.e.,
$$\lambda_1(1, 3, 4, 2) + \lambda_2(3, -5, 2, 2) + \lambda_3(2, -1, 3, 2) = \mathbf{0}$$

is equivalent to $\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0, 3\lambda_1 - 5\lambda_2 - \lambda_3 = 0,$
 $4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0, 2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$

As these are satisfied by the values $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -2$ which are not zero, the given vectors are linearly dependent. Also we have the relation,

$$\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 = 0$$

by means of which any of the given vectors can be expressed as a linear combination of the others.

Obs. Applying elementary row operations to the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, we see that the matrices

$$A = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 \end{bmatrix}$$

have the same rank. The rank of B being 2, the rank of A is also 2. Moreover $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent and \mathbf{x}_3 can be expressed as a linear combination of \mathbf{x}_1 and \mathbf{x}_2 $\left[\because \mathbf{x}_3 = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) \right]$. Similar results will hold for column operations and for any matrix. In general, we have the following results :

If a given matrix has r linearly independent vectors (rows or columns) and the remaining vectors are linear combinations of these r vectors, then rank of the matrix is r . Conversely, if a matrix is of rank r , it contains r linearly independent vectors are remaining vectors (if any) can be expressed as a linear combination of these vectors.

8. If A and B are orthogonal matrices, prove that AB is also orthogonal.

(Anna, 2000)

9. Are the following vectors linearly dependent. If so, find the relation between them :

(i) $(2, 1, 1), (2, 0, -1), (4, 2, 1)$.

(Mumbai, 2009)

(ii) $(1, 1, 1, 3), (1, 2, 3, 4), (2, 3, 4, 9)$.

(iii) $\mathbf{x}_1 = (1, 2, 4), \mathbf{x}_2 = (2, -1, 3), \mathbf{x}_3 = (0, 1, 2), \mathbf{x}_4 = (-3, 7, 2)$.

(U.P.T.U., 2003 ; Nagpur, 2001)

2.13 (1) EIGEN VALUES

If A is any square matrix of order n , we can form the matrix $A - \lambda I$, where I is the n th order unit matrix. The determinant of this matrix equated to zero,

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$
$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots + k_n = 0,$$

(2) Eigen vectors

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, then the linear transformation $Y = AX$... (i)

Let X be such a vector which transforms into λX by means of the transformation (i).

$$\lambda X = AX \quad \text{or} \quad AX - \lambda X = 0 \quad \text{or} \quad [A - \lambda I]X = 0$$
[illegible]

$X = [x_1, x_2, \dots, x_n]'$, which is known as the *eigen vector* or *latent vector*.

Example 2.42. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. (Bhopal, 2008)

i.e., $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$ or $\lambda^2 - 7\lambda + 6 = 0$

or $(\lambda - 6)(\lambda - 1) = 0 \quad \therefore \lambda = 6, 1.$

If x, y be the components of an eigen vector corresponding to the eigen value λ , then

$$[A - \lambda I] X = \begin{bmatrix} 5 - \lambda & 4 \\ 1 & 2 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Corresponding to $\lambda = 6$, we have $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\therefore \frac{x}{4} = \frac{y}{1}$ giving the eigen vector (4, 1).

Corresponding to $\lambda = 1$, we have $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $x + y = 0$.

$\therefore \frac{x}{1} = \frac{y}{-1}$ giving the eigen vector $(1, -1)$.

Example 2.43. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(Bhopal, 2009 ; Raipur, 2005)

Solution. The characteristic equation is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$, i.e., $\lambda^3 - 7\lambda^2 + 36 = 0$

Since $\lambda = -2$ satisfies it, we can write this equation as

$$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0 \quad \text{or} \quad (\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$$

Thus the eigen values of A are $\lambda = -2, 3, 6$.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ , we have

$$[A - \lambda I]X = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \dots(i)$$

Putting $\lambda = -2$, we have $3x + y + 3z = 0$, $x + 7y + z = 0$, $3x + y + 3z = 0$.

The first and third equations being the same, we have from the first two

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \quad \text{or} \quad \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

Hence the eigen vector is $(-1, 0, 1)$. Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = -2$.

Similarly, the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 6$ are the arbitrary non-zero multiples of the vectors $(1, -1, 1)$ and $(1, 2, 1)$ which are obtained from (i).

Hence the three eigen vectors may be taken as $(-1, 0, 1), (1, -1, 1), (1, 2, 1)$.

Example 2.44. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ (U.P.T.U., 2005)

Solution. The characteristic equation is

$$[A - \lambda I] = 0, \quad \text{i.e.,} \quad \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

Thus the eigen values of A are 2, 3, 5.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ , we have

$$[A - \lambda I]X = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Putting $\lambda = 2$, we have $x + y + 4z = 0$, $6z = 0$, $3z = 0$, i.e., $x + y = 0$ and $z = 0$.

$$\therefore \frac{x}{1} = \frac{y}{-1} = \frac{z}{0} = k_1 \text{ (say)}$$

Hence the eigen vector corresponding to $\lambda = 2$ is $k_1(1, -1, 0)$.

Putting $\lambda = 3$, we have $y + 4z = 0$, $-y + 6z = 0$, $2z = 0$, i.e., $y = 0$, $z = 0$.

$$\therefore \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = k_2$$

LINEAR ALGEBRA : DETERMINANTS, MATRICES

Hence the eigen vector corresponding to $\lambda = 3$ is $k_2 (1, 0, 0)$.

Similarly, the eigen vector corresponding to $\lambda = 5$ is $k_3 (3, 2, 1)$.