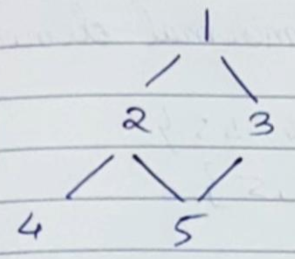
U19CS076

Tutorial - 4

Lattice and Boolean Algebra

① Let $A = \{1, 2, 3, 4, 5\}$ be ordered by following Hasse diagram. Insert correct symbol $<$, $>$ or $\parallel$ between each pair



① $1 > 2$ & $2 > 5 \Rightarrow 1 > 5$
② No relation visible $\Rightarrow 2 \parallel 3$
③. $1 > 2$ & $2 > 4 \Rightarrow 4 < 1$
④ $3 > 5$ & $2 > 5 \rightarrow \quad \Rightarrow 3 > 4$

② Consider the ordered set A in previous Hasse diagram!

1. Find all minimial & maximal elements on A

Maximal $\rightarrow \{1\}$
Minimal $\rightarrow \{4, 5\}$

2. Does A have lower and upper bound? Also discuss GLB, LUB for set A.

As $4 \parallel 5$, lower bound on A doesnt exist
Upper bound on $A = 1$
$\Rightarrow LUB = 1$, $GLB = \phi$

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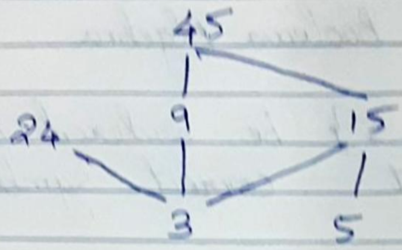
(5) For the poset [{3, 5, 9, 15, 24, 45}, divisor of],

~~1. The~~



1. The maximal and minimal elements

Maximal → {24, 45}
Minimal → {3, 5}

2. Greatest and least elements

Greatest = 45
Since 3 and 5 not related
Least element NOT EXIST

3. The upper bound and LUB of {3, 5}

{15, 45} are only related to 3 and 5
⇒ Upper bounds = {15, 45}
LUB = {15}

4. The lower bounds and GLB of {15, 45}

Lower bounds = {15, 5, 3}
GLB = {15}

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(4). If R and S are relations on $A = \{1, 2, 3\}$ represented by matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find matrices that represent

1. $M_{R \cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 1 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 0 & 0 \vee 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. $M_{R \cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \\ 0 \wedge 0 & 0 \wedge 0 & 0 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. $M_{R.S} = M_R . M_S$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. $M_{S.R} = M_S . M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

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(5) $M_{R \oplus S} = M_{R \cup S} - M_{R \cap S}$

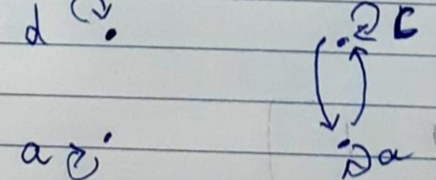
$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q5 List the ordered pairs in relations R and S whose matrix representations are given as

(1) $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (2) $M_S = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Also draw directed graphs representing R and S. Use the graph to find out if R and S are equivalence relations.

$R = \{(a,a)\ (b,b)\ (b,c)\ (c,b)\ (c,c),\ (d,d)\}$



→ All have self loop → reflexive
→ (c,b) and (b,c) → symmetric
→ Transitive

⇒ Equivalence relation

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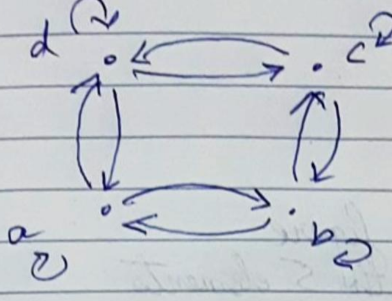
For S

$S = \{ (a,a)\ (a,b)\ (a,d)\ (b,a)\ (b,b)\ (b,c), (c,b), (c,c), (c,d), (d,a), (d,c), (d,d) \}$



→ All have self loop → reflexive

→ Symetric

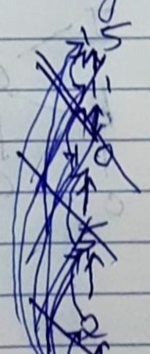
→ $(b,a) \in R$, ~~$(b,b) \in R$~~ $(a,d) \in R$

$\Rightarrow (b,d) \notin R$

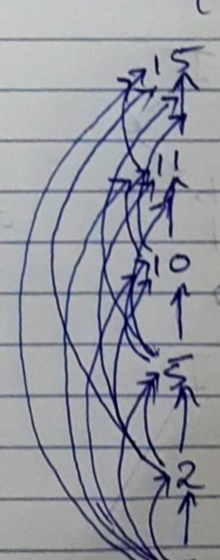
Not transitive

⇒ NOT equivalence relation.

6. Draw Hasse diagram for '≤' relation on {0, 2, 5, 10, 11, 15} starting from digraph.



Digraph



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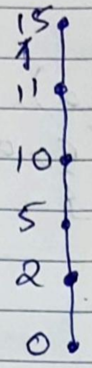
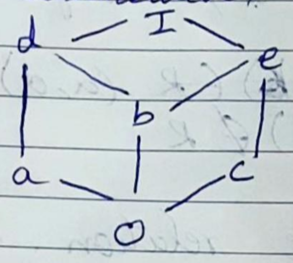
Hasse Diagram



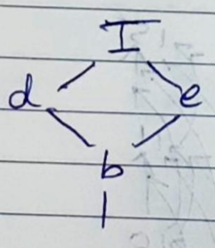
Q7

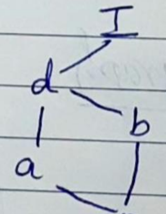
(7) Consider the lattice L in figure

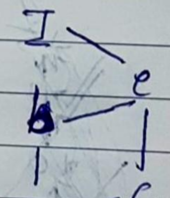
1. Find all sublattice with 5 elements
2. Find complements of a and b if they exist
3. Is L distributive? Complements?

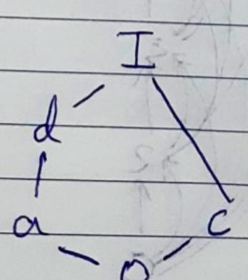


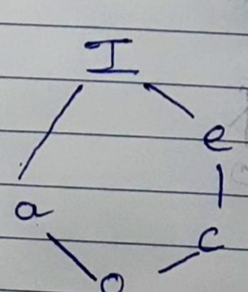
(1). Since I and 0 are meet, joint points











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→ (2) As UB = I, LB = ϕ.

$a \vee c = I$, $a \wedge c = 0$

$a \vee e = I$, $a \wedge e = 0$

→ $a^c = c, e$ (b cannot be as LUB = d for (a, b))

$I^c = 0$, $0^c = I$

as $I \vee 0 = I$, $I \wedge 0 = 0$

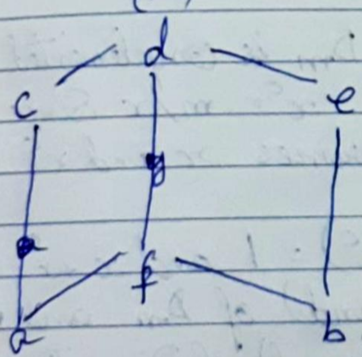
For complement of b,

(3) Since a has multiple complement, L is not distributive,

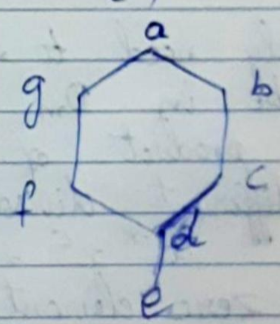
L is not complemented lattice.

Q8 Decide which of following Hasse diagram define a lattice on {a, b, c, d, e, f, g}

(1)



(2)



For (1) For every pair of elements join exists so it is a join semilattice. The meet of a, b → $a \wedge b$ = ~~d~~ $a \wedge b$ → does not exist. It is not a meet semi lattice. It is not a lattice.

For (2): For every pair of elements meet semilattice and join semilattice exists.

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(9). Write the duals of each boolean eq

1. $(a * 1) * (0 + a')$
2. $a + a'b = a + b$

1 → $(a * 1) * (0 + a')$

Dual = $(a + 0) + (1 * a')$

2 → $a + a'b = a + b$

Dual = $a * (a' + b) = a * b$

(10) Given the set D_m of divisors of m is a bounded distributive lattice with $a + b = a \vee b = \text{lcm}(a, b)$ and $a * b = a \wedge b = \gcd(a, b)$

1. Show D_m is boolean algebra if m is a square free i.e, m is a product of distinct prime.
2. Find atoms of D_m.

1. We only need to show that D_m is complemented. Let x be in D_m and let $x^c = m/x$. Since m is product of distinct primes x and x^c have different divisor.

Hence $x * x^c = \gcd(x, x^c) = 1$ and $x + x^c = \text{lcm}(x, x^c) = m$ is zero element (lower bound) of D_m and m is the identity element (upper bound) of D_m. Thus, x^c is a complement of x, and so D_m is boolean algebra.

2. The atoms of D_m are prime divisor of m.

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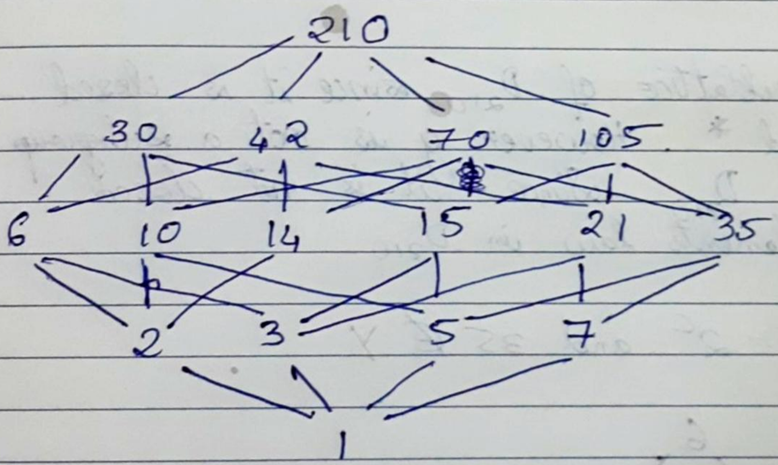
Q11) Consider the Boolean Algebra D_{210}

1. List all elements and draw diagram.
2. Find set A of all atoms.
3. Find two subalgebras with 8 elements
4. Is $x = \{1, 2, 6, 20\}$ a sublattice of D_{210}? A subalgebra?
5. Is $y = \{1, 2, 3, 6\}$ a sublattice of D_{210}? A subalgebra?

1. Divisors of 210 → 210, 105, 70, 42, 35, 30, 21, 15, 14, 10, 7, 6, 5, 3, 2, 1



2. Set of all atoms of 210 → prime divisors
$A = \{2, 3, 5, 7\}$.

3. 2 subgroups with 8 elements

$B_1 = \{1, 2, 3, 35, 6, 70, 105, 210\}$

$B_2 = \{1, 5, 6, 7, 30, 35, 42, 210\}$

B_1, B_2 are two subalgebra of D_{210}

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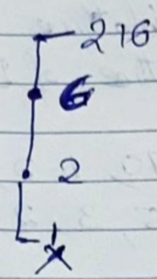
4. X is sublattice since it is linearly order.
But its not a subalgebra because complement of 2 and 6 doesn't exist in X.
35 is the complement of 2 in D_{210} but it is not present in X.



5. Y is a sublattice of D_{210} since it is closed under + and *. However y is not a ~~subgroup~~ subalgebra of D_{210}. Since it is not closed under complements law in D_{210}

Eg. $35 = 2^c$ and $35 \notin Y$