

**Example 3.14** Let  $V$  be the set of all ordered pairs  $(x, y)$ , where  $x, y$  are real numbers. Let  $\mathbf{a} = (x_1, y_1)$  and  $\mathbf{b} = (x_2, y_2)$  be two elements in  $V$ . Define the addition as

$$\mathbf{a} + \mathbf{b} = (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

and the scalar multiplication as

$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1).$$

Show that  $V$  is not a vector space. Which of the properties are not satisfied?

**Solution** Note that  $(1, 1)$  is an element of  $V$ . From the given definition of vector addition, we find that

$$(x_1, y_1) + (1, 1) = (x_1, y_1).$$

and this is true only for the element  $(1, 1)$ . Therefore, the element  $(1, 1)$  plays the role of  $\mathbf{0}$  element as defined in property 4. Now, there is no element in  $V$  for which  $(\mathbf{a}) + (-\mathbf{a}) = \mathbf{0} = (1, 1)$ , since

$$(x_1, y_1) + (-x_1, -y_1) = (-x_1^2, -y_1^2) \neq (1, 1).$$

Therefore, property 5 is not satisfied.

Now, let  $\alpha = 1, \beta = 2$  be any two scalars. We have

$$(\alpha + \beta)(x_1, y_1) = 3(x_1, y_1) = (3x_1, 3y_1)$$



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and 
$$\alpha(x_1, y_1) + \beta(x_1, y_1) = 1(x_1, y_1) + 2(x_1, y_1) = (x_1, y_1) + (2x_1, 2y_1) = (2x_1^2, 2y_1^2)$$

Therefore,  $(\alpha + \beta)(x_1, y_1) \neq \alpha(x_1, y_1) + \beta(x_1, y_1)$  and property 7 is not satisfied.

Similarly, it can be shown that property 9 is not satisfied. Hence,  $V$  is not a vector space.

#### 3.3.1 Subspaces

Let  $V$  be an arbitrary vector space defined under a given vector addition and scalar multiplication. A non-empty subset  $W$  of  $V$ , such that  $W$  is also a vector space under the same two operations of vector addition and scalar multiplication, is called a *subspace* of  $V$ . Thus,  $W$  is also closed under the two given algebraic operations on  $V$ . As a convention, the vector space  $V$  is also taken as a subspace of  $V$ .

#### Remark 8

To show that  $W$  is a subspace of a vector space  $V$ , it is not necessary to verify all the 10 properties as given in section 3.3. If it is shown that  $W$  is closed under the given definition of vector addition and scalar multiplication, then the properties 2, 3, 7, 8, 9 and 10 are automatically satisfied because these properties are valid for all elements in  $V$  and hence are also valid for all elements in  $W$ . Thus, we need to verify the remaining properties, that is, the existence of the zero element and the additive inverse in  $W$ .



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and  $\alpha(x_1, y_1) + \beta(x_1, y_1) = 1(x_1, y_1) + 2(x_1, y_1) = (x_1, y_1) + (2x_1, 2y_1) = (2x_1^2, 2y_1^2)$

Therefore,  $(\alpha + \beta)(x_1, y_1) \neq \alpha(x_1, y_1) + \beta(x_1, y_1)$  and property 7 is not satisfied.

Similarly, it can be shown that property 9 is not satisfied. Hence,  $V$  is not a vector space.

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#### Remark 8

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Consider the following examples:

1. Let  $V$  be the set of  $n$ -tuples  $(x_1 \ x_2 \ \dots \ x_n)$  in  $\mathbb{R}^n$  with usual addition and scalar multiplication. Then
  - (i)  $W$  consisting of  $n$ -tuples  $(x_1 \ x_2 \ \dots \ x_n)$  with  $x_1 = 0$  is a subspace of  $V$ .
  - (ii)  $W$  consisting of  $n$ -tuples  $(x_1 \ x_2 \ \dots \ x_n)$  with  $x_1 \geq 0$  is not a subspace of  $V$ , since  $W$  is not closed under scalar multiplication ( $\alpha x$ , when  $\alpha$  is a negative real number, is not in  $W$ ).