Let V be the set of all $n \times n$ complex matrices with usual matrix addition and scalar multiplication. Then

(i) W consisting of all Hermitian matrices of order n forms a vector space when scalars are real numbers and does not form a vector space when scalars are complex numbers (W is not closed under scalar multiplication).

Let
$$\mathbf{A} = \begin{pmatrix} a & x+iy \\ x-iy & b \end{pmatrix} \in W.$$

Let $\alpha = i$. We get $\alpha\mathbf{A} = i\mathbf{A} = \begin{pmatrix} ai & xi-y \\ xi+y & bi \end{pmatrix} \notin W.$

Example 3.15 Let F and G be subspaces of a vector space V such that $F \cap G = \{\mathbf{0}\}$. The *sum* of F and G is written as $F + G$ and is defined by

$$F + G = \{\mathbf{f} + \mathbf{g} : \mathbf{f} \in F, \mathbf{g} \in G\}.$$

Show that $F + G$ is a subspace of V assuming the usual definition of vector addition and scalar multiplication.

Theorem 3.1 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be any r elements of a vector space V under usual vector addition and scalar multiplication. Then, the set of all linear combinations of these elements, that is the set of all elements of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_r \mathbf{v}_r \tag{3.19}$$

is a subspace of V, where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are scalars.

Proof Let W be the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Let

$$\mathbf{w}_1 = \sum_{i=1}^{r} a_i \mathbf{v}_i \quad \text{and} \quad \mathbf{w}_2 = \sum_{i=1}^{r} b_i \mathbf{v}_i$$

be any two linear combinations (any two elements of W). Then,

$$\mathbf{w}_1 + \mathbf{w}_2 = (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2 + \ldots + (a_r + b_r)\mathbf{v}_r \in W$$

$$\alpha \mathbf{w}_1 = (\alpha a_1)\mathbf{v}_1 + (\alpha a_2)\mathbf{v}_2 + \ldots + (\alpha a_r)\mathbf{v}_r \in W$$

$$\alpha \mathbf{w}_2 = (\alpha b_1)\mathbf{v}_1 + (\alpha b_2)\mathbf{v}_2 + \ldots + (\alpha b_r)\mathbf{v}_r \in W$$

and $\qquad \alpha(\mathbf{w}_1 + \mathbf{w}_2) = \alpha \mathbf{w}_1 + \alpha \mathbf{w}_2.$

Taking $\alpha = 0$, we find that $0\mathbf{w}_1 = \mathbf{0} \in W$. This implies that $\mathbf{w}_1 + \mathbf{0} = \mathbf{0} + \mathbf{w}_1 = \mathbf{w}_1$

Taking $\alpha = -1$, we find that $(-1)\mathbf{w}_1 = (-\mathbf{w}_1) \in W$. This implies that $\mathbf{w}_1 + (-\mathbf{w}_1) = \mathbf{0}$

Therefore, W is a subspace of V.

The elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are in the subspace W as

$$\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_r, \; \mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + \ldots + 0\mathbf{v}_r, \ldots$$

We say that the subspace W is *spanned* by the r elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Also, any subspace that contains the elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ must contain every linear combination of these elements.

Spanning set Let S be a subset of a vector space V and suppose that every element in V can be obtained as a linear combination of the elements taken from S. Then S is said to be the *spanning set* for V. We also say that S spans V.

Therefore, W is a ...

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$$\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_r,\ \mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + \ldots + 0\mathbf{v}_r, \ldots$$

We say that the subspace W is *spanned* by the r elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Also, any subspace that contains the elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ must contain every linear combination of these elements.

Spanning set Let S be a subset of a vector space V and suppose that every element in V can be obtained as a linear combination of the elements taken from S. Then S is said to be the *spanning set* for V. We also say that S spans V.

Example 3.16 Let V be the vector space of all 2×2 real matrices. Show that the sets

(i) $$S = \left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$$

(ii) $$S = \left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\right\}$$

span V.

Solution Let $\mathbf{X} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ be an arbitrary element of V.

(i) We write

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} = a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}.$$

Since every element of V can be written as a linear combination of the elements of S, the set S spans the vector space V.