Computational Sheet - 7

by Aleena Kriti

Q1

Given,

$$t_\ell(p,p') = V_\ell(p,p') + \overbrace{\int_0^\infty dp'' \frac{f(p'')-f(q)}{q^2-p''^2}}^{I_1} + \overbrace{f(q)\int_0^\infty dp'' \frac{1}{q^2-p''^2}}^{I_2} - \frac{i\pi f(q)}{2q} \quad \text{—(1)}$$

where,

$$f(p'') = 2\mu p'' V_\ell(p,p'') t_\ell(p'',p')$$

$$I = \int_0^\infty dp'' \frac{2\mu p'' V_\ell(p,p'') t_\ell(p'',p) - 2\mu q V_\ell(p,q) t_\ell(q,p)}{q^2-p''^2}$$

In order to discretize the equation, we choose our momentas from the momentum grid itself.

$$\therefore V_\ell(p,p') \equiv V_\ell(p_i,p_j) \equiv V_{ij}$$

$$t_\ell(p_k,p_j) \equiv t_{kj}$$

$\because$ we use gaussian quadrature

$$I_1 = \sum_{k=0}^{N-1} \left(\frac{2\mu p_k V_\ell(p,p_k) t_\ell(p_k,p') - 2\mu q V_\ell(p,q) t_\ell(q,p')}{q^2-p_k^2} \right) w_k$$

$$= \sum_{k=0}^{N-1} \frac{2\mu p_k V_{ik} t_{kj} - 2\mu q V_{iN} t_{Nj}}{q^2-p_k^2}$$

$$I_2 = f(q)\int_0^\infty dp'' \frac{1}{q^2-p''^2}$$

($\because$ I_2 is a complex integral, only one pole contributes)

$$I_2 = \frac{1}{2q} \ln\left(\frac{p_{max}+q}{p_{max}-q}\right)$$

Substituting this in equation (1)

$$t_{ij} = \sum_{k=0}^{N-1} \frac{2\mu p_k V_{ik} t_{kj} - 2\mu q V_{iN} t_{Nj}}{q^2 - p_k^2} + V_{ij} - \frac{i\pi f(q)}{2q}$$

$$= \sum_{k=0}^{N-1} \frac{2\mu p_k^2 V_{ik} t_{kj}}{q^2 - p_k^2} - \sum_{k=0}^{N-1} \frac{2\mu q^2 V_{iN} t_{Nj}}{q^2 - p_k^2} + V_{ij} - i\pi \mu q V_{iN} t_{Nj} + I_2 f(q)$$

$$\Rightarrow -V_{ij} = \sum_{k=0}^{N-1} \frac{2\mu p_k^2 V_{ik} t_{kj}}{q^2 - p_k^2} - \sum_{k=0}^{N-1} \frac{2\mu q^2 V_{iN} t_{Nj}}{q^2 - p_k^2} - i\pi \mu q V_{iN} t_{Nj} - t_{ij} + I_2 f(q) \quad \text{——(2)}$$

We want this to be of the form $A_{ik} t_{kj} = V_{ij}$.

Rewriting eqn (1)

$$V_{ij} = t_{ij} - \sum_{k=0}^{N-1} \frac{2\mu p_k^2 V_{ik} t_{kj}}{q^2 - p_k^2} + \sum_{m=0}^{N-1} \frac{2\mu q^2 V_{iN} t_{Nj}}{q^2 - p_m^2} + i\pi \mu q V_{iN} t_{Nj} - I_2 f(q)$$

$$= t_{ij} - \sum_{k=0}^{N-1} \frac{2\mu p_k^2 V_{ik} t_{kj}}{q^2 - p_k^2} + \sum_{m=0}^{N-1} \frac{2\mu q^2 V_{iN} t_{Nj}}{q^2 - p_m^2} + i\pi \mu q V_{iN} t_{Nj} - \mu q V_{iN} t_{Nj} \ln\left(\frac{p_{max} + q}{p_{max} - q}\right)$$

From here, we can clearly see,

$$A_{ik} = \begin{cases} \delta_{ik} - \dfrac{2\mu V_{ik} p_k^2}{q^2 - p_k^2} & k \neq N \\[2ex] \delta_{ik} + \displaystyle\sum_{m=0}^{N-1} \frac{2\mu V_{iN} q^2}{q^2 - p_m^2} w_m - \mu q V_{iN} \ln\left(\frac{p_{max} + q}{p_{max} - q}\right) + i\pi \mu q V_{iN} & k = N \end{cases}$$

Q5 To express the differential cross section as partial waves, we have,

$$t(\vec{q}_f, \vec{q}_i) = \sum_{lm} Y_{lm}(\hat{q}_f)\, t_l(q_f, q_i)\, Y^*_{lm}(\hat{q}_i)$$

$$|t(\vec{q}_f, \vec{q}_i)|^2 = t(\vec{q}_f, \vec{q}_i)\, t^*(\vec{q}_f, q_i)$$

$$= \left[\sum_{lm} Y_{lm}(\hat{q}_f)\, t_l(q_f, q_i)\, Y^*_{lm}(\hat{q}_i)\right]\left[\sum_{l'm'} Y_{l'm'}(\hat{q}_f)\, t_{l'}(q_f, q_i)\, Y^*_{l'm'}(\hat{q}_i)\right]^*$$

$$= \sum_{\substack{l',m' \\ l,m}} \left(Y^*_{l'm'}(q_f)\, Y_{l'm'}(\hat{q}_i)\, t^*_{l'}(q_f, q_i)\right)\left(Y^*_{lm}(\hat{q}_i)\, Y_{lm}(\hat{q}_f)\, t_l(q_f, q_i)\right)$$

$\because$ t only depends on l & it's given to us

$$\sum_m Y^*_{lm}(\hat{q}_f)\, Y_{lm}(\hat{q}_i) = \frac{(2l+1)}{4\pi} P_l(\hat{q}_f \cdot \hat{q}_i)$$

$$\therefore |t(\vec{q}_f, \vec{q}_i)|^2 = \sum_{l,l'} \frac{(2l+1)}{4\pi}\, t_l(q_f, q_i)\, \frac{(2l'+1)}{4\pi}\, t^*_{l'}(q_f, q_i)\, P_{l'}(\hat{q}_f \cdot \hat{q}_i)\, P_l(\hat{q}_i \cdot \hat{q}_f)$$

$$|t(\vec{q}_f, \vec{q}_i)|^2 = \frac{1}{(4\pi)^2} \sum_{l,l'} (2l+1)(2l'+1)\, t^*_{l'}(q_f, q_i)\, t_l(q_f, q_i)\, P_l(\cos\theta)\, P_{l'}(\cos\theta)$$

$\therefore$ The differential cross section is given by.

$$\boxed{\therefore \frac{d\sigma}{dq_f} = \pi^2 \mu^2 \sum_{l,l'} (2l+1)(2l'+1)\, t^*_{l'}(q_f, q_i)\, t_l(q_f, q_i)\, P_l(\cos\theta)\, P_{l'}(\cos\theta)}$$

Computational Homework 7

by Aleena and Kriti

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
import math as m
from numpy.polynomial.legendre import leggauss
from scipy.special import legendre
```

In [2]:

```
class OBEpot:
    """Provides a method for the partial wave representation of the OBE potential.

       The matrix elements are obtained by numerical intergration.
       The mass of the exchanged boson, the strength of the
       interaction and the couter term is given on initialization.
       The interaction is regularized using a cutoff that is also
       given on init.
    """

    # this are common parameters for all instances
    hbarc=197.327

    # init interaction
    def __init__(self, cutoff=800.000, C0=2.470795*10**(-2), nx=12,mpi=138.0,A=-0.1544435):
        """Defines the one boson exchange for a given regulator, coupling strength and shor

        Parameters:
        cutoff -- regulator in MeV
        C0 -- strength of the short distance counter term (in s-wave)
        A -- strength of OBE
        nx -- number of angular grid points for numerical integration
        mpi -- mass of exchange boson in MeV"""

        self.mpi = mpi/self.hbarc
        self.cutoff = cutoff/self.hbarc
        self.C0=C0
        self.A=A
        self.nx=nx

        self.xp=np.empty((self.nx),dtype=np.double)
        self.xw=np.empty((self.nx),dtype=np.double)
        self.xp,self.xw=leggauss(self.nx)



    # function defines the x integral
    def _g(self,pp,p,k):
        """Calculates g function of the partial wave decomposition of OBE.

           pp -- outgoing momentum
           p -- incoming momentum
           k -- angular momentum"""

        # define prefact
        # get the corresponding legendre polynomial
        Pk = legendre(k)
        # define momentum transfer dependent on angles
        qval=np.sqrt(p**2+pp**2-2*p*pp*self.xp)

        # build integral of regularized OBE
        return float(np.sum(Pk(self.xp)/((qval**2+self.mpi**2))*self.xw*np.exp(-(qval**2+se

    # determines complete, regularized interaction
    def v(self,pp,p,l):
        """Potential matrix element in fm**2

           pp -- outgoing momentum in fm**-1
           p -- incoming momentum in fm**-1
```

```
        l -- angular momentum"""

      # first overall prefact of 1pi exchange part  (cancel 2pi factors!)
      prefact=self.A

      mat=prefact*self._g(pp,p,l)

      if (l==0):   # add s-wave counter term
          mat+=self.C0*np.exp(-(pp**2+p**2)/self.cutoff**2)  # 4pi is take into account b

      return mat
```

In [3]:

```
pot = OBEpot()
```

In [4]:

```
def kronekar (i,k):
    if (i==k):
        return 1
    else:
        return 0

def trns(np1,np2,pa,pb,pc):
      """Auxilliary method that provides transformed Gauss-Legendre grid points and integra

         This is using a hyperbolic trafo shown in the lecture.
         Parameter:
         np1 --  grid points in ]0,pb[
         np2 --  grid points are distributed in ]pb,pc[ using a linear trafo

         pa  -- half of np1 points are in interval [0,pa]
         pb  -- interval boundary as defined above
         pc  -- upper integration boundary """

      x1grid,x1weight=leggauss(np1)
      x2grid,x2weight=leggauss(np2)

      # trafo (1.+X) / (1./P1-(1./P1-2./P2)*X) for first interval
      p1grid=(1.+x1grid) / (1./pa-(1./pa-2./pb)*x1grid)
      p1weight=(2.0/pa-2.0/pb)*x1weight / (1./pa-(1./pa-2./pb)*x1grid)**2

      # linear trafo
      p2grid=(pc+pb)/2.0 + (pc-pb)/2.0*x2grid
      p2weight=(pc-pb)/2.0*x2weight

      pgrid=np.empty((N+1),dtype=np.double)
      pweight=np.empty((N+1),dtype=np.double)

      pgrid = np.concatenate((p1grid, p2grid), axis=None)
      pweight = np.concatenate((p1weight, p2weight), axis=None)

      return pgrid,pweight
```

In [5]:

```
def matA (i,k,V_ik,q,p,w,p_max,mu=938.92):
    A=0
    if (k!=N):
        return kronekar(i,k)-2*mu*V_ik*(p[k]**2)*w[k]/(q**2-p[k]**2)
    elif(k==N):
        for m in range(N):
               A+=2*mu*V_ik*q**2*w[m]/(q**2-p[m]**2)

        return(kronekar(i,k)+A-mu*q*V_ik*np.log(p_max+q/(p_max-q))+1j*np.pi*mu*q*V_ik)
```

In [6]:

```
pot=OBEpot(nx=24,mpi=138.0)
print(pot.v)
```

```
<bound method OBEpot.v of <__main__.OBEpot object at 0x000002617EE65EE0>>
```

In [7]:

```
N=30
A=np.zeros((N+1,N+1),dtype='complex')
V=np.zeros((N+1,N+1))
p_grid,w=trns(20,10,1.0,5.0,35.0)
q=1
p_grid=np.concatenate((p_grid,[q]),axis=None)
for i in range(N+1):
          for j in range(N+1):
            V[i][j]=pot.v(p_grid[i],p_grid[j],0)


for i in range(0,len(p_grid)):
    for k in range(0,len(p_grid)):
        A[i][k]=matA(i,k,V[i][k],q,p_grid,w,20)
```

In [8]:

```
t=np.linalg.inv(A)@V
print(t[N][N])
```

```
(-0.00016663726356060 81-0.0001384414671008402j)
```

In [9]:

```
S=1-2j*np.pi*938.92*q*t[N][N]
print(np.linalg.norm(S))
```

```
1.0000000000000278
```

Since S is coming out to be one which is required

In [10]:

```
N_grid=np.linspace(10,40,7).astype(int)
Pmax_grid=np.linspace(10,30,11)
```

In [18]:

```
stab_mat=np.zeros((len(N_grid),len(Pmax_grid)))
for n in range(len(N_grid)):
    for m in range(len(Pmax_grid)):
        N=N_grid[n]
        A=np.zeros((N_grid[n]+1,N_grid[n]+1),dtype='complex')
        V=np.zeros((N_grid[n]+1,N_grid[n]+1))

        p_grid,w=trns(int(np.ceil(2*N_grid[n]/3)),int(N_grid[n]/3),1.0,5.0,Pmax_grid[m])
        q=np.sqrt(2*938.92*1)  ## we have fixed a q accordind to the conndition E=1Mev

        p_grid=np.concatenate((p_grid,[q]),axis=None)
        for i in range(N_grid[n]+1):
            for j in range(N_grid[n]+1):
                V[i][j]=pot.v(p_grid[i],p_grid[j],0)


        for i in range(0,len(p_grid)):
            for k in range(0,len(p_grid)):
                A[i][k]=matA(i,k,V[i][k],q,p_grid,w,Pmax_grid[m])
        t=np.linalg.inv(A)@V
        S=1-2j*np.pi*938.92*q*t[N_grid[n]][N_grid[n]]
        stab_mat[n][m]=np.linalg.norm(S)
```

In [ ]:

```

```

In [19]:

```
print(p_grid)
```

```
[2.39138702e-03 1.26430129e-02 3.12627706e-02 5.85613641e-02
 9.49922319e-02 1.41165742e-01 1.97863884e-01 2.66057921e-01
 3.46928718e-01 4.41888693e-01 5.52603294e-01 6.81008221e-01
 8.29315851e-01 1.00000000e+00 1.19574165e+00 1.41930883e+00
 1.67333139e+00 1.95991649e+00 2.28003864e+00 2.63263884e+00
 3.01340441e+00 3.41330785e+00 3.81719590e+00 4.20303266e+00
 4.54268635e+00 4.80510401e+00 4.96201035e+00 5.19771182e+00
 6.03002001e+00 7.48027387e+00 9.47063326e+00 1.18938406e+01
 1.46192711e+01 1.75000000e+01 2.03807289e+01 2.31061594e+01
 2.55293667e+01 2.75197261e+01 2.89699800e+01 2.98022882e+01
 2.00000000e+00]
```

In [13]:

```
### pmax --> infty
def matA1 (i,k,V_ik,q,p,w,p_max,mu=938.92):
    A=0
    if (k!=N):
        return kronekar(i,k)-2*mu*V_ik*(p[k]**2)*w[k]/(q**2-p[k]**2)
    elif(k==N):
        for m in range(N):
               A+=2*mu*V_ik*q**2*w[m]/(q**2-p[m]**2)

        return(kronekar(i,k)+A+1j*np.pi*mu*q*V_ik)
```

In [14]:

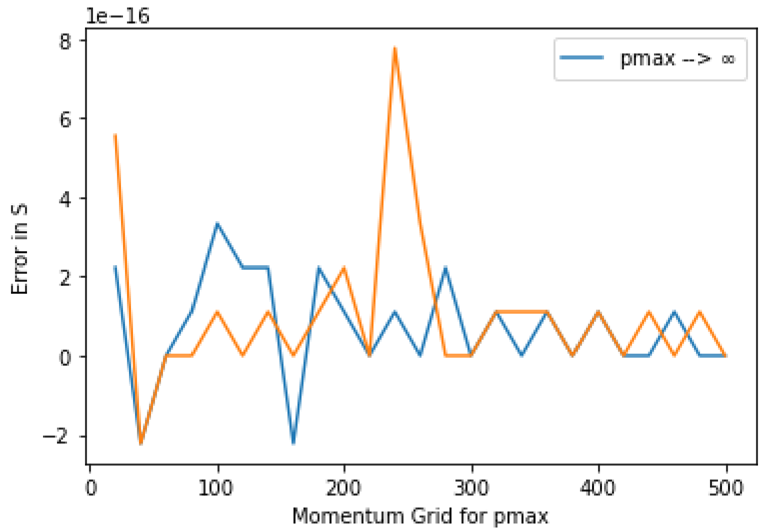
```
N=30
A=np.zeros((N+1,N+1),dtype='complex')
A1=np.zeros((N+1,N+1),dtype='complex')
V=np.zeros((N+1,N+1))
pmax_grid=np.linspace(20,500,25)
pinf=np.zeros(len(pmax_grid))
pfinite=np.zeros(len(pmax_grid))
for m in range(len(pmax_grid)):
    p_grid,w=trns(20,10,1.0,5.0,pmax_grid[m])
    q=10
    p_grid=np.concatenate((p_grid,[q]),axis=None)
    for i in range(N+1):
        for j in range(N+1):
            V[i][j]=pot.v(p_grid[i],p_grid[j],0)   ## l=0


    for i in range(0,len(p_grid)):
        for k in range(0,len(p_grid)):
            A[i][k]=matA(i,k,V[i][k],q,p_grid,w,pmax_grid[m])
            A1[i][k]=matA1(i,k,V[i][k],q,p_grid,w,pmax_grid[m])

    t=np.linalg.inv(A)@V
    t1=np.linalg.inv(A1)@V
    S1=1-2j*np.pi*938.92*q*t1[N][N]
    S=1-2j*np.pi*938.92*q*t[N][N]
    pinf[m]=1.-np.linalg.norm(S1)
    pfinite[m]=1.-np.linalg.norm(S)
```

In [21]:

```
plt.xlabel('Momentum Grid for pmax')
plt.ylabel('Error in S')
plt.plot(pmax_grid,pinf,label = 'pmax --> $\infty$')
plt.plot(pmax_grid,pfinite)
plt.legend()
```

Out[21]:

```
<matplotlib.legend.Legend at 0x2610043b250>
```



we expect the difference between the two plots to reduce as we increase pmax which we observe in our plot. Also the initial value S for the A matrix where pmax tend to ∞ is less than the the value of S where Pmax doesn't tend to ∞ as expected. as for the trend that we observe we couldn't make much sense of it

In [28]:

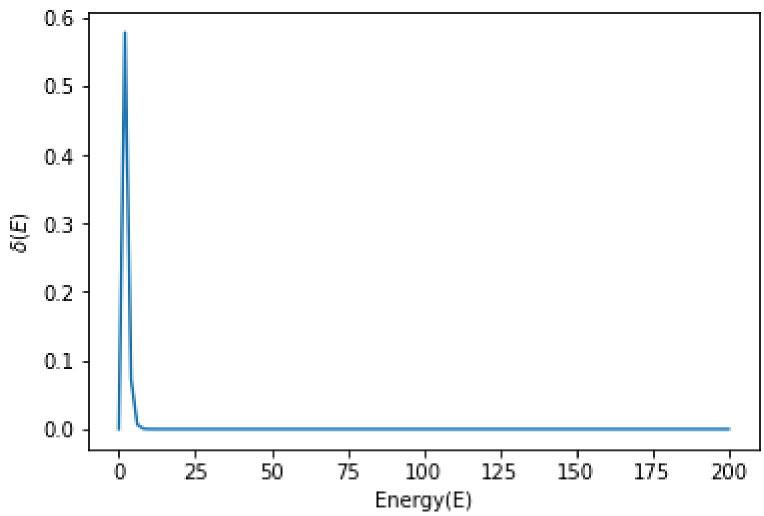
```
## Plotting the phase shift
Earr = np.linspace(0,200,100) #MeV
qarr = np.sqrt(2*938.92*Earr)
dlist = np.zeros(len(Earr))#dtype='complex')

N=30
for m in range(len(Earr)):
    A=np.zeros((N+1,N+1),dtype='complex')
    V=np.zeros((N+1,N+1))
    p_grid,w=trns(20,10,1.0,5.0,35.0)


    p_grid=np.concatenate((p_grid,[qarr[m]]),axis=None)
    for i in range(N+1):
            for j in range(N+1):
              V[i][j]=pot.v(p_grid[i],p_grid[j],0)


    for i in range(0,len(p_grid)):
        for k in range(0,len(p_grid)):
            A[i][k]=matA(i,k,V[i][k],qarr[m],p_grid,w,20)

    t=np.linalg.inv(A)@V
    S=1-2j*np.pi*938.92*qarr[m]*t[N][N]
    delta = 0.5*np.arctan2(np.imag(S),np.real(S))
    dlist[m] = delta
```

In [30]:

```
plt.plot(Earr,dlist)
plt.xlabel('Energy(E)')
plt.ylabel('$\delta (E)$')
```

Out[30]:

```
Text(0, 0.5, '$\\delta (E)$')
```



In [36]:

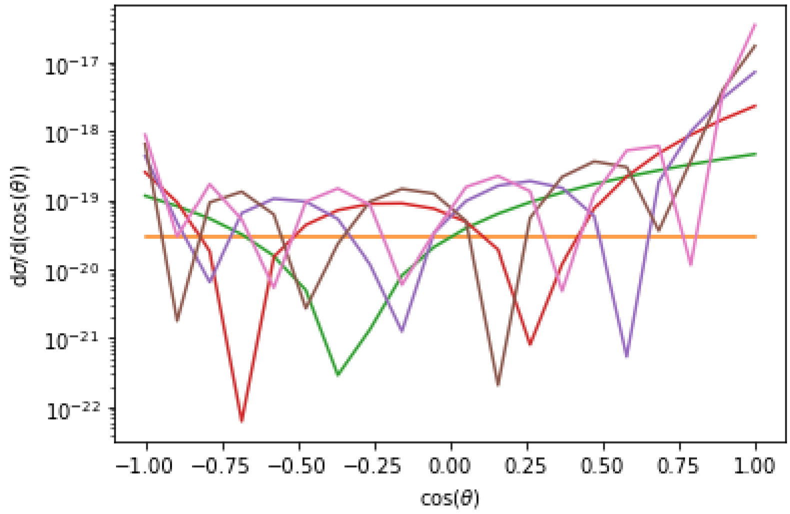
```
from scipy.special import eval_legendre

q1 = np.sqrt(2*938.92*10)
tarr = np.zeros((7),dtype = 'complex')

for m in range(7):
    A=np.zeros((N+1,N+1),dtype='complex')
    V=np.zeros((N+1,N+1))
    p_grid,w=trns(20,10,1.0,5.0,35.0)


    p_grid=np.concatenate((p_grid,[q1]),axis=None)
    for i in range(N+1):
        for j in range(N+1):
            V[i][j]=pot.v(p_grid[i],p_grid[j],m)


    for i in range(0,len(p_grid)):
        for k in range(0,len(p_grid)):
            A[i][k]=matA(i,k,V[i][k],q1,p_grid,w,20)

    t=np.linalg.inv(A)@V
    tarr[m] = t[N][N]

sumlist = []
carr = np.linspace(-1,1,20)
for n in range(7):
    sum = np.zeros(len(carr))
    for l in range(n):
        for l_ in range(n):

            sum += np.real((2*l+1)*(2*l_+1)*tarr[l]*np.conjugate(tarr[l_])*eval_legendre(l,
    sumlist.append(sum)
```

In [44]:

```
for i in range(7):

    plt.semilogy(carr,sumlist[i])

plt.xlabel(r'cos($\theta$)')
plt.ylabel(r'd$\sigma$/d(cos($\theta$))')
```

Out[44]:

```
Text(0, 0.5, 'd$\\sigma$/d(cos($\\theta$))')
```



In [ ]:

In [ ]: