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Computational Assignment - 7.

Q4.

$$H_a(u) = \frac{1}{a}\sum_{i=1}^{N} (u_i - u_{i-1})^2 + a \sum_{i=1}^{N-1} \phi_i^{(a)} u_i$$

Given,
$$H_a(u^{(a)}) = H_a(\tilde{u}^{(a)}) + H_{2a}(u^{(2a)})$$

Fine to Coarse restriction : $u_i^{(2a)} = u_{2i}^{(a)}$ $\quad i = 0 \ldots N/2$

Coarse to fine prolongation : $$I_{(2a)}^{(a)} = \begin{cases} u_{i/2}^{(2a)} & i = 0, 2 \ldots N \\ \left(u_{(i-1)/2}^{(2a)} + u_{(i+1)/2}^{(2a)}\right)/2 & i = 1, 3 \ldots N-1 \end{cases}$$

$$H_{2a}(u^{(2a)}) = \frac{1}{2a}\sum_{i=1}^{N/2} \left(u_i^{(2a)} - u_{i-1}^{(2a)}\right)^2 + 2a \sum_{i=1}^{N/2-1} \phi_i^{(2a)} u_i^{(2a)}$$

$$H_a(u^{(a)}) - H_a(\tilde{u}^{(a)}) = \frac{1}{a}\sum_{i=1}^{N} \left(u_i^{(a)} - u_{i-1}^{(a)}\right)^2 - \left(\tilde{u}_i^{(a)} - \tilde{u}_{i-1}^{(a)}\right)^2$$
$$+ a \sum_{i=1}^{N-1} \phi_i^{(a)} \left(u_i^{(a)} - \tilde{u}_i^{(a)}\right)$$

comparing the terms linear in a,

$$a \sum_{i=1}^{N-1} \phi_i^{(a)} \left(u_i^{(a)} - \tilde{u}_i^{(a)}\right) = 2a \sum_{i=1}^{N/2-1} \phi_i^{(2a)} u_i^{(2a)}$$

we know $u_i^{(a)} - \tilde{u}_i^{(a)} = I_{(2a)}^{(a)} u_i^{(2a)}$

$\therefore$ LHS can be rewritten as

$$\text{LHS} = \underbrace{a \sum_{i\sim\text{even}}^{N-2} \phi_i^{(a)} u_{i/2}^{(2a)}}_{(\mathrm{I})} + \underbrace{a \sum_{i\sim\text{odd}}^{N-1} \frac{\phi_i^{(a)}}{2} \left(u_{(i-1)/2}^{(2a)} + u_{(i+1)/2}^{(2a)} \right)}_{(\mathrm{II})}$$

for I, we substitute

$$u_{i/2}^{(2a)} = u_i^{(a)}$$

for II, $\quad u_{(i-1)/2}^{(2a)} = u_{i-1}^{(a)} \quad , \quad u_{(i+1)/2}^{(2a)} = u_{i+1}^{(a)}$

$$\text{LHS} = a \sum_{i\sim\text{even}}^{N-2} \phi_i^{(a)} u_i^{(a)} + a \sum_{i\sim\text{odd}}^{N-1} \frac{\phi_i^{(a)}}{2} \left(u_{i-1}^{(a)} + u_{i+1}^{(a)} \right)$$

$$\frac{a}{2} \sum_{\text{odd } i}^{N-1} \phi_i^{(a)} \left(u_{i-1}^{(a)} \right) = \frac{a}{2} \sum_{\text{even } i}^{N-1} \phi_{i+1}^{(a)} u_i^{(a)}$$

$(i \to i+1)$

$$\frac{a}{2} \sum_{\text{odd } i}^{N-1} \phi_i^{(a)} u_{i+1}^{(a)} = \frac{a}{2} \sum_{\text{even } i}^{N-1} \phi_{i-1}^{(a)} u_i^{(a)}$$

$(i \to i-1)$

$$\Rightarrow \text{LHS} = a \sum_{\text{even } i} \left(\phi_i + \frac{1}{2} \phi_{i-1} + \frac{1}{2} \phi_{i+1} \right) u_i$$

$$\text{RHS} = 2a \sum_{i=1}^{N/2-1} \phi_i^{(2a)} u_i^{(2a)} = 2a \sum_{i=1}^{N/2-1} \phi_i^{(2a)} u_{2i}^{(a)}$$

$(2i \to k)$

$$= 2a \sum_{k\sim\text{even}}^{N-2} \phi_{k/2}^{(2a)} u_k^{(a)}$$

$\therefore$ Comparing all terms we have

$$\phi^{(2a)}_{i/2} = \frac{1}{2}\left(\frac{\phi^{(a)}_{i-1}}{2} + \phi^{(a)}_{i} + \frac{\phi^{(a)}_{i+1}}{2}\right)$$

for even i's

$$\boxed{\Rightarrow \quad \phi^{(2a)}_{k} = \frac{1}{2}\left(\frac{\phi^{(a)}_{2k-1}}{2} + \phi^{(a)}_{2k} + \frac{\phi^{(a)}_{2k+1}}{2}\right)}$$

Computational Homework Sheet 5

We have tried our best to make sense of the algorithm. Sorry if this is inconvenient for you to go through the code.
-Aleena, Kriti

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
from tqdm import tqdm
```

In [2]:

```
def Ham(u,a,N):
    H = 0
    for i in range(N):
        H += (u[i] - u[i-1])**2

    return H/a

def hastings(u,x,phi,nlevel):

    Ei= Ha(u,phi,nlevel)
    r = np.random.uniform(-1,1)
    u[x] = u[x] + delta*r
    Ef=Ha(u,phi,nlevel)
    dE=Ef-Ei
    if (dE < 0 or np.random.uniform(0,1) < np.exp(-dE)):
        u[x]=u[x]


    else:
        u[x]-=r*delta

    return u


def H2a(ucor1,phi,nlevel): #computes hamiltonian of the lattice at a particular level, a=1
    n2 = int(len(ucor1))
    he = 0
    hi = 0
    phia = np.zeros(n2)

    for i in range(n2): ##phi(2a)
        phi_n = 0.5*(phi[2*i] + 0.5*phi[2*i-1] + 0.5*phi[2*i+1])
        phia[i] = phi_n
        he += nlevel*phi_n*ucor1[i]
        hi += ((ucor1[i]-ucor1[i-1])**2)/nlevel

    Ham = he+hi
    return Ham,phia


def Ha(u,phi,nlevel):
    H = 0
    N = len(u)
    for i in range(N):
        H += ((u[i] - u[i-1])**2)/nlevel  + nlevel*phi[i]*u[i]

    return H

def autocor(marr):

    gamma = np.zeros(50)
    mavg = np.mean(marr)
    for tau in tqdm(range(50)):
        count = 0
        for k in range(len(marr)):

            for l in range(len(marr)):

                if(abs(k-l) == tau):
```

```
                    count+=1
                    gamma[tau] += (marr[k] - mavg)*(marr[l] - mavg)

        gamma[tau] *= 1/count

    Ctau = gamma/gamma[0]
    return Ctau
```

In [3]:

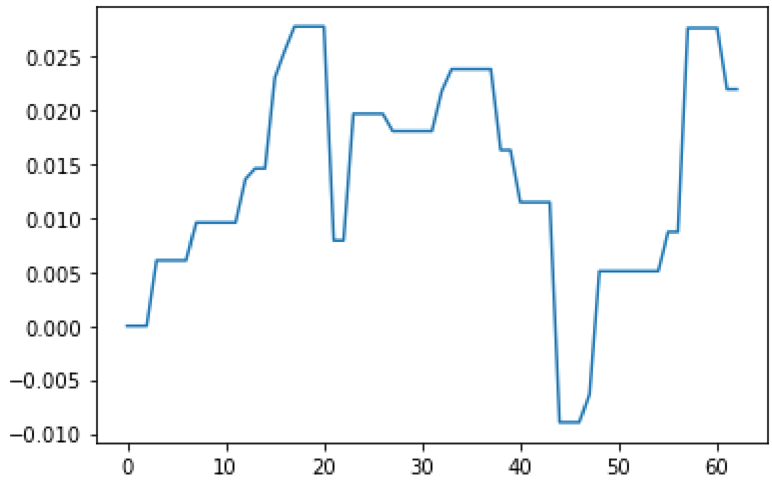
```
a =1.
N = 64
delta = 2.
Marr = []

u = np.zeros(N)
u[0],u[N-1] = 0,0
#def hastings(u,x,phi,nlevel,N):
phi = np.zeros(N)

for i in range(N-1):
    x = np.random.randint(1,N-2)
    Ei = Ha(u,phi,1)
    r = np.random.uniform(-1,1)
    u[x] += delta*r
    Ef = Ha(u,phi,1)
    dE=Ef-Ei
    if (dE < 0 or np.random.uniform(0,1) < np.exp(-dE)):
        u[x]=u[x]


    else:
        u[x]-=r*delta

    Marr.append(np.sum(u)/N)
```

In [4]:

```
plt.plot(Marr)
```

Out[4]:

```
[<matplotlib.lines.Line2D at 0x1e4bc7553d0>]
```



V Cycle

In [5]:

```
N = 64
vpre = int(N/2)
vpost = int(N/2)

delta = 2.
Marr = []




 #   def Ha(u,phi,nlevel,N)

u = np.zeros(N)
u[0],u[N-1] = 0,0
phia = np.zeros(N)

## Pre-coarsening Steps

for j in range(vpre):
    for i in range(N-1):
        x = np.random.randint(1,N-2)
        Ei = Ha(u,phi,1)
        r = np.random.uniform(-1,1)
        u[x] += delta*r
        Ef = Ha(u,phi,1)
        dE=Ef-Ei
        if (dE < 0 or np.random.uniform(0,1) < np.exp(-dE)):
            u[x]=u[x]


        else:
            u[x]-=r*delta
```

In [6]:

```
ucor=np.zeros(int(N/2))
ucor1=np.zeros(int(N/4))
ucor2=np.zeros(int(N/8))

for i in range(int(N/2)):
    ucor[i]=u[2*i]
H_2a,phi2a = H2a(ucor,phia,2)


for i in range(int(N/4)):
    ucor1[i]=u[4*i]
H_4a,phi4a = H2a(ucor1,phi2a,4)


for i in range(int(N/8)):
    ucor2[i]=u[8*i]
H_8a,phi8a = H2a(ucor2,phi4a,8)


Mcor =[]

#Metropolis step after coarsening 3 times

#def hastings(u,x,phi,nlevel,N):
nl = int(N/8)
for j in range(vpost):
    for i in range(nl-1):
        x = np.random.randint(1,nl-2)
        ucor2 = hastings(ucor2,x,phi8a,8)
        Mcor.append(np.sum(ucor2)/nl)
```

In [7]:

```
## coarse to fine --> 3 levels
ufine=np.zeros(N)
ufine1=np.zeros(int(N/2))
ufine2=np.zeros(int(N/4))

#Prolongation for third level
for i in range(int(N/4)-1):
    if(i%2==0):
        ufine2[i]=ucor2[int(i/2)]
    else:
        ip = int((i+1)/2)
        im = int((i-1)/2)

        ufine2[i]=(ucor2[im]+ucor2[ip])/2

#Interpolation for second level
for i in range(int(N/2)-1):
    if(i%2==0):
        ufine1[i]=ufine2[int(i/2)]
    else:
        ip = int((i+1)/2)
        im = int((i-1)/2)
        ufine1[i]=(ufine2[im]+ufine2[ip])/2


#Prolongation for first level
for i in range(N-1):
    if(i%2==0):
        ufine[i]=ufine1[int(i/2)]
    else:
        ip = int((i+1)/2)
        im = int((i-1)/2)
        ufine[i]=(ufine1[im]+ufine1[ip])/2

Mfinal=[]

for i in range(N-1):

    #Ei = Ha(ufine,x,1)
    x = np.random.randint(1,N-2)
    ufine = hastings(ufine,x,phi,1)
    Mfinal.append((np.sum(ufine))**2/N**2)
```

In [8]:

```
corr1 = autocor(Mfinal)
```

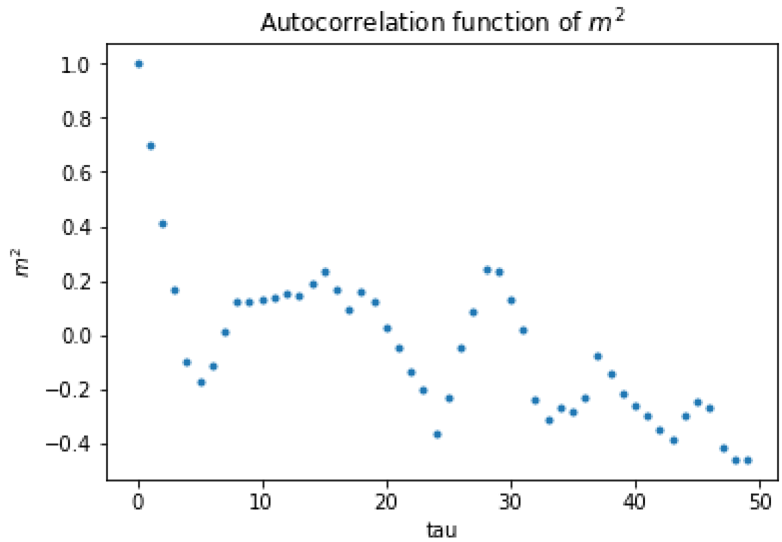
```
100%|██████████████████████████████| 50/50 [00:00<00:00, 724.50it/s]
```

In [9]:

```
plt.title('Autocorrelation function of $m^2$')
plt.xlabel('tau')
plt.ylabel('$m^2$')
plt.plot((corr1),'.')
```

Out[9]:

```
[<matplotlib.lines.Line2D at 0x1e4bc882e20>]
```



W Cycle

In [10]:

```
N = 64
vpre = int(N/2)
vpost = int(N/2)

delta = 2.
Marr = []




 #   def Ha(u,phi,nlevel,N)

u = np.zeros(N)
u[0],u[N-1] = 0,0
phia = np.zeros(N)

## Pre-coarsening Steps

for j in range(vpre):
    for i in range(N-1):
        x = np.random.randint(1,N-2)
        Ei = Ha(u,phi,1)
        r = np.random.uniform(-1,1)
        u[x] += delta*r
        Ef = Ha(u,phi,1)
        dE=Ef-Ei
        if (dE < 0 or np.random.uniform(0,1) < np.exp(-dE)):
            u[x]=u[x]


        else:
            u[x]-=r*delta
```

In [11]:

```
ucor=np.zeros(int(N/2))
ucor1=np.zeros(int(N/4))
ucor2=np.zeros(int(N/8))

for i in range(int(N/2)):
    ucor[i]=u[2*i]
H_2a,phi2a = H2a(ucor,phia,2)


for i in range(int(N/4)):
    ucor1[i]=u[4*i]
H_4a,phi4a = H2a(ucor1,phi2a,4)


for i in range(int(N/8)):
    ucor2[i]=u[8*i]
H_8a,phi8a = H2a(ucor2,phi4a,8)


#Metropolis step after coarsening 3 times

#def hastings(u,x,phi,nlevel,N):
nl = int(N/8)
for j in range(vpost):
    for i in range(nl-1):
        x = np.random.randint(1,nl-2)
        ucor2 = hastings(ucor2,x,phi8a,8)



## Uncoarsen it
ufine2=np.zeros(int(N/4))

#Prolongation for third level
for i in range(int(N/4)-1):
    if(i%2==0):
        ufine2[i]=ucor2[int(i/2)]
    else:
        ip = int((i+1)/2)
        im = int((i-1)/2)

        ufine2[i]=(ucor2[im]+ucor2[ip])/2


for j in range(vpost):
    for i in range(nl-1):
        x = np.random.randint(1,nl-2)
        ucor2 = hastings(ucor2,x,phi8a,8)


#Coarsen it again
for i in range(int(N/8)):
    ucor2[i] = ufine2[2*i]
H_8a,phi8a = H2a(ucor2,phi4a,8)


nl = int(N/8)
for j in range(vpost):
    for i in range(nl-1):
```

```
        x = np.random.randint(1,nl-2)
        ucor2 = hastings(ucor2,x,phi8a,8)
        Mcor.append(np.sum(ucor2)/nl)


## coarse to fine --> 3 levels
ufine=np.zeros(N)
ufine1=np.zeros(int(N/2))
ufine2=np.zeros(int(N/4))

#Prolongation for third level
for i in range(int(N/4)-1):
    if(i%2==0):
        ufine2[i]=ucor2[int(i/2)]
    else:
        ip = int((i+1)/2)
        im = int((i-1)/2)

        ufine2[i]=(ucor2[im]+ucor2[ip])/2

#Prolongation for second level
for i in range(int(N/2)-1):
    if(i%2==0):
        ufine1[i]=ufine2[int(i/2)]
    else:
        ip = int((i+1)/2)
        im = int((i-1)/2)
        ufine1[i]=(ufine2[im]+ufine2[ip])/2


#Prolongation for first level
for i in range(N-1):
    if(i%2==0):
        ufine[i]=ufine1[int(i/2)]
    else:
        ip = int((i+1)/2)
        im = int((i-1)/2)
        ufine[i]=(ufine1[im]+ufine1[ip])/2

MfinalW=[]

for i in range(N-1):

    #Ei = Ha(ufine,x,1)
    x = np.random.randint(1,N-2)
    ufine = hastings(ufine,x,phi,1)
    MfinalW.append((np.sum(ufine))**2/N**2)
```

In [12]:

```
corr = autocor(MfinalW)
plt.title('Autocorrelation function of $m^2$')
plt.xlabel('tau')
plt.ylabel('$m^2$')
plt.plot((corr),'.')
```

```
100%|
| 50/50 [00:00<00:00, 833.10it/s]
```

Out[12]:

```
[<matplotlib.lines.Line2D at 0x1e4bc95b8b0>]
```