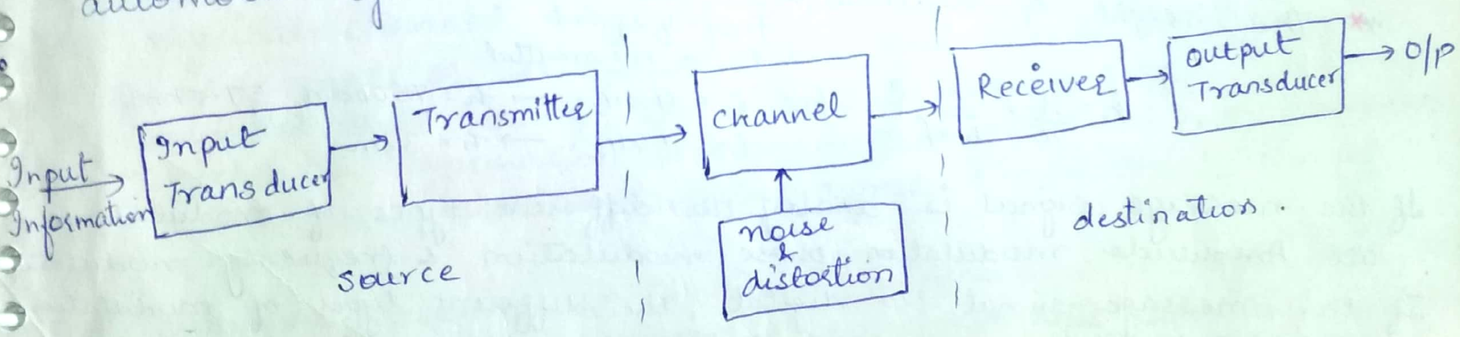
UNIT-I

Introduction to Communication System

Communication is the process of transferring information from one place to another. Information generated at source may need to travel hundred or thousands of miles via channel to reach destination. communication channel is the media by which information is sent.

- The channel could be a wired line such as copper wire or wireless such as atmosphere. Placing a physical wire from source to destination is definitely a costlier and less convenient proposition than use of feeding to channel.
- But there are issues like distortion and noise. Distortion is the process of changing shape of the communicating signal that may mislead the destination about the content of the message. This may occur due to inability of the channel to convey all frequency, phase & amplitude information thruthfully from one side to another.
- The loss in amplitude or strength of the signal as it travels through channel is called attenuation.
- Noise is random, unwanted interference on transmitted signal. Noise can be internal or external. External noise is the interference from other sources like lightning, electrical switching automobile ignition etc.



→ Transmitter is a collection of one or more electronic devices or circuits that converts the original source information to a form more suitable for transmission over a particular transmission medium.
In general a transmitter performs the operation of modulation and multiplexing.

→ Receiver is the device that receives information from channel & extracts intended electrical message from it.

→ Transducer converts the information to be transmitted to electrical equivalent message signal, as electrical sig[nal] that move with the speed of light increases the speed of communication. Hence transducers are important. Microphone is an input transducer which converts audio input like speech to an electrical signal.

→ The O/P transducer converts electrical input to a form of message required by user like speech, image, text, video etc. Loud-speaker is an example of O/P transducer where electrical I/P is converted into an audio O/P.

Modulation :- It is the process of changing one or more properties of the analog carrier in proportion with the information signal.

Modulation is necessary as

(i) It is extremely difficult to radiate low-frequency signals from an antenna in the form of EM wave.**

(ii) Information signals often occupy the same frequency band, and if two signals from more sources are transmitted at the same time, they would interfere with one another. e.g. All FM stations have audio frequency band 300 Hz to 15KHz. To avoid interference with each other, each station converts its information to a different frequency band or channel. (channel is often used to refer to a specific band of frequencies allocated to a particular service

** The height of antenna $\propto \frac{1}{\text{freq. of signal to be transmitted}}$

$$h = \frac{\lambda}{10} = \frac{c}{10 \times f}$$

for $f = 4KHz \rightarrow h = 75000m = 7.5 km$

$f = 1GHz \rightarrow h = 3cm$

If the message signal is analog the different types of modulation are Amplitude modulation, phase modulation & frequency modulation.

If the message signal is digital, the different types of modulation are Amplitude Shift Keying (ASK), Phase Shift Keying (PSK) and Frequency Shift Keying (FSK)

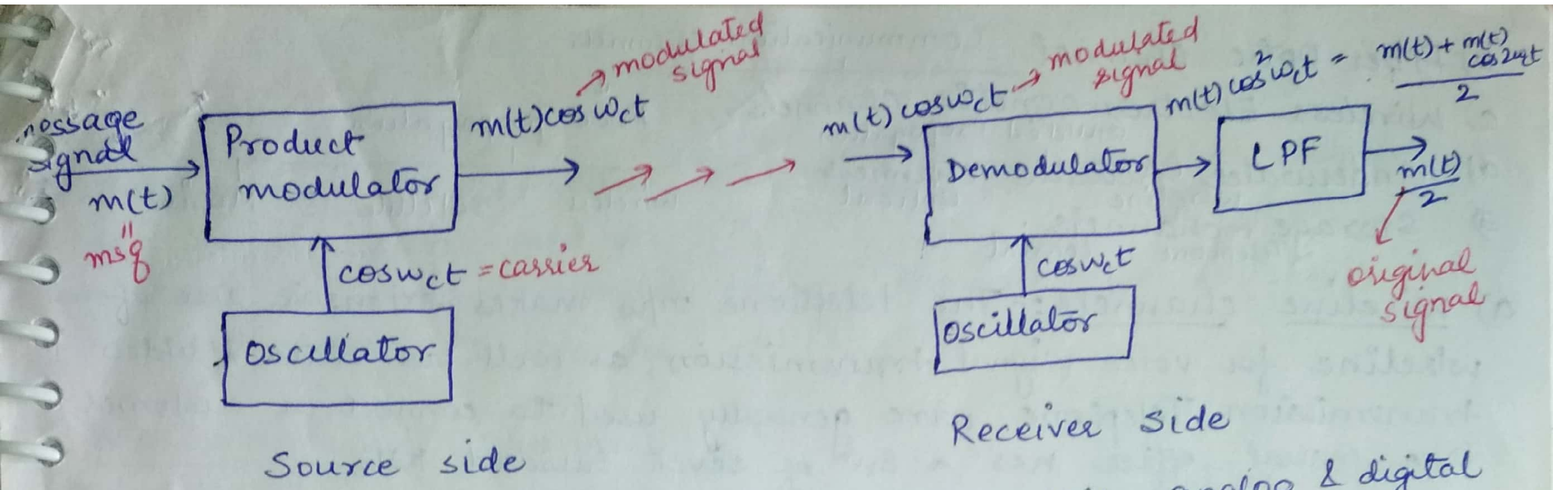
message signal

$v(t) = V \sin(2\pi f t + \theta)$

i) Analog	AM	FM	PM
ii) Digital	ASK	FSK	PSK

QAM

Quadrature Amplitude Modulation



Multiplexing :- It is the process by which multiple analog & digital signals are combined into one signal over a shared medium.
Using a carrier to shift the frequency band of a message signal for simultaneous transmission is known as frequency division multiplexing (FDM)
For digital signals, multiplexing can be done by dividing the time b/w samples of signals in various time slots and using each time slot to send one digital signal. Such a method of simultaneous transmission is known as time division multiplexing (TDM)

Communication channels

Communication channel provides the connection b/w the transmitter and the receiver.
The physical channel may be a pair of wires that carry the electrical signal, or an optical fibre that carries the information on a modulated light beam, an underwater channel ocean channel in which the information is transmitted acoustically, or free-space over which the information bearing signal is radiated by use of an antenna.
One common problem ~~the~~ in signal transmission through channel ~~noise~~ is additive noise, which is generated by components such as resistors, and solid-state devices used to implement the communication system.
The effect of noise may be minimized by increasing the power in the transmitted signal. However equipment & other practical constraints limit the power level in the transmitted signal.
Another basic limitation is the channel bandwidth. A bandwidth constraint is usually due to physical limitations of the medium and the electronic components used to implement the transmitter & the receiver.
The several communication channels used are :-
1) Wireline channels — [Telephone Wire / Coaxial Cables

Communication channels

- Guided Propogation
  - wireline
    - Telephone
    - Coaxial
  - optical
- Free propogation
  - wireless
  - satellite
  - Mobile

a) **Wireline channels**:- The telephone n/w makes extensive use of wireline for voice signal transmission, as well as data & video transmission. Telephone wire generally used to connect a customer to a central office has a BW of sevel hundred KHz.

→ Twisted-pair wirelines and coaxial cables are basically guided EM channel which provide relatively modest BW of several MHz (MHz). Signals transmitted through such channels are distorted in both amplitude and phase & further corrupted by additive noise.

→ Twisted pair wireline channels are also prone to cross-talks interference from physically adjacent channels.

b) **Optical Fiber Channel** :-

Optical fiber offer a channel BW that is several order of magnitude larger than coaxial cable channel. The transmitter or modulator of fiber optic communication is a LED or LASER. Information is transmitted by varying the intensity of the light source with the message signal. At the receiver, the light intensity is detected by a photodiode, whose o/p is an electrical signal.

c) **Wireless Electromagnetic Channel** :-

In radio communication system, EM wave is coupled to the propogation medium by an antenna which serves as the radiator. The mode of propogation of EM in the atmosphere and in free space may be sub divided into three categories :-

a) Ground-wave propogation → (0·3-3 MHz) → Medium Freq band → AM broadcasting

b) Sky-wave propogation → 3-30 MHz → High Freq band

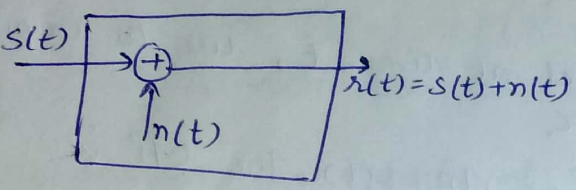
c) Line of Sight Propogation → above 30 MHz → Very HF → Terrestrial comm system.

Mathematical Models for Communication Channels

→ Mathematical model reflect the most important characteristics of the transmission medium
→ used to design channel encoder & modulator at Tx and demodulator & decoder at Rx

Additive noise Channel :-

The simplest mathematical model for a communication channel is additive noise channel.



In this the transmitted signal $s(t)$ is corrupted by an additive random noise process $n(t)$

→ The additive noise process may arise from electronic component & amplifiers at the receiver of the communication, or from interference encountered in transmission

→ If the noise is introduced primarily by electronic components and amplifiers at the receiver, it may be characterized as thermal noise. This Type of noise is characterized statistically as a Gaussian noise process.

Hence, the resulting mathematical model for the channel is usually called the additive Gaussian noise channel.

→ This is the predominant channel model used in our communication system analysis

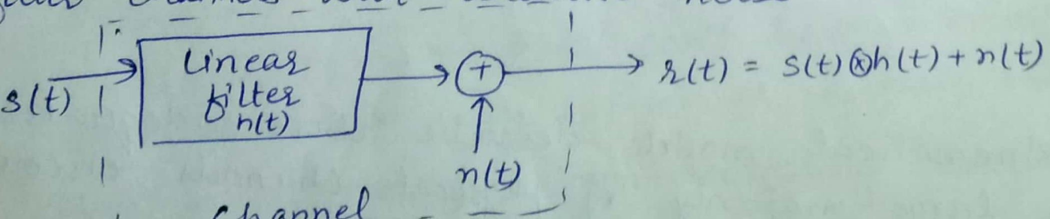
where a → attenuation factor

$$r(t) = as(t) + n(t)$$

Linear Filter Channel :-

In some physical channels such as wireline telephone channels, filters are used to ensure that the transmitted signals do not exceed specified BW limitations and thus do not interfere with one another.

Such channels are generally characterized mathematically as linear filter channels with additive noise



If the channel I/P is signal $s(t)$, then

Channel O/P = $r(t) = s(t) \otimes h(t) + n(t)$

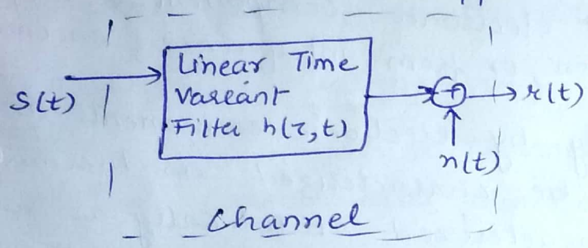
$$= \int_{-\infty}^{\infty} h(\tau)\, s(t-\tau)\, d\tau + n(t)$$

where $h(t) \rightarrow$ impulse response of filter and $\otimes$ convolution

The Linear Time-Variant Filter Channel :-

Physical channels such as under water acoustic channels and ionospheric radio channels which results in time-variant multipath propogation of the transmitted signal may be characterized mathematically as time-variant linear filters. Such linear filters are characterized by time-variant channel impulse response $h(\tau, t)$

where $h(\tau, t) \rightarrow$ response of channel at time t, due to impulse applied at $t-\tau$.



For an I/P $s(t)$, the O/P is

$$r(t) = s(t) \otimes h(\tau, t) + n(t)$$

$$= \int_{-\infty}^{\infty} h(\tau, t) \cdot s(t-\tau)\, d\tau + n(t) \quad \text{--- (1)}$$

A good model for multipath signal propogation through physical channels, such as the ionosphere & the mobile communication cellular radio channels, are a special case of (1) in which the time-variant impulse response has the form

$$h(\tau, t) = \sum_{k=1}^{L} a_k(t) \cdot \delta(\tau - \tau_k)$$

where $a_k(t)$ represents the possibly time-variant attenuation factors for the L multipath propogation paths.

So, $$r(t) = \sum_{k=1}^{L} a_k(t)\, s(t - \tau_k) + n(t)$$

Hence, the received signal consists of L multipath components where each component is attenuated by $\{a_k\}$ and delayed by τ_k

→ Three mathematical models describe above adequately characterize a large majority of physical channels encountered in practice.

Random Variables

Random Variables :- It is used to signify a rule by which a real number is assigned to each possible outcome of an experiment.
In probability, a random variable is a variable whose possible value are numerical outcomes of a random phenomenon.

Cumulative Distribution Function (CDF):-

It is defined as the probability that the outcome of an experiment will be one of the outcomes for which $X(\lambda) \le x$.

$$F_X(x) = P(\omega \in \Omega : X(\omega) \le x)$$

or $F_X(x) = P(\overset{RV}{X} \le x)$

* All Random variables will be written in boldface letters like **X**.

Ex:- Consider the experiment of tossing a coin 3 times. Let X = no. of time coin comes with head.

toss	no. of heads
TTT	0
TTH	1
THT	1
THH	2
HTT	1
HTH	2
HHT	2
HHH	3

CPF

x	p(x)	$F_X(x)$
0	1/8	$= P[X \le 0] = 1/8$
1	3/8	$F_X(1) = P[X \le 1] = 1/8 + 3/8 = 7/8$
2	3/8	
3	1/8	1

Properties

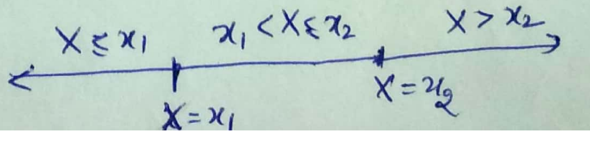
1. CDF is always bounded b/w 0 and 1
   As $F_X(x) = P[X \le x]$ and probability must have a value b/w 0 & 1, ∴ CDF is bounded b/w 0 & 1.

2. $F_X(\infty) = 1$
   → $F_X(\infty) = P[X \le \infty] \Rightarrow$ ~~all~~ probability of all possible outcomes or events. The Random variable $X \le \infty$ then becomes a certain event and therefore has a 100% probability.

3. $F_X(-\infty) = 0$
   $F_X(-\infty) = P(X \le -\infty) \Rightarrow$ RV can't have a value which is less than or equal to $-\infty$. Thus $F_X(-\infty)$ is a null event & has 0% probability

4. $F_X(x)$ is a monotone non-decreasing function
   $F_X(x_1) \le F_X(x_2)$ if $x_1 < x_2$

$\Rightarrow F_X(x_2) = P(X \le x_2)$



Since $X \le x_1$ & $x_1 < X < x_2$ are mutually exclusive

$$P(X \le x_1) \cup P(x_1 < X \le x_2) = P(X \le x_1) + P(x_1 < X \le x_2)$$

So $F_X(x_2) = P(X \le x_1) + P(x_1 < X \le x_2)$

$$F_X(x_2) = F_X(x_1) + \underbrace{P(x_1 < X \le x_2)}_{\text{non-negative since its probability}} \quad - *$$

So $\boxed{F_X(x_2) > F_X(x_1)}$

5. $P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$

from * $\quad F_X(x_2) - F_1(x) = P[x_1 < X \le x_2)$

Que:- A 3-digit message is transmitted over a noisy channel having a probability of error $P(E) = \frac{2}{5}$ per digit. Find the corresponding CDF.

Soln:- Probability of error $P(E) = \frac{2}{5}$

Probability of receiving a correct digit $P(C) = 1 - P(E) = 1 - \frac{2}{5} = \frac{3}{5}$

Sample Space {CCC, CCE, CEC, CEE, ECC, ECE, EEC, EEE}

Random variable X = no. of errors in received message

X = {No error, one error, two error, three error}

$x_i = 0, 1, 2$ and 3.

$$\text{CDF} = F_X(x) = \begin{cases} 0 & \text{for } X \le x_0 \\ \sum_{i=0}^{3} P(X = x_i) & \text{for } x_0 \le X \le x_3 \\ 1 & \text{for } X > x_3 \end{cases}$$

$P(CCC) = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$

$P(X = x_1) = P(CCE, CEC, ECC)$
$= 3\left[\frac{3}{5} \times \frac{3}{5} \times \frac{2}{5}\right]$
$= \frac{54}{125}$

$P[X = x_2] = P[CEE, ECE, EEC]$
$= 3\left[\frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}\right]$
$= \frac{36}{125}$

$P[X = x_3] = P[EEE]$
$= \frac{8}{125}$

$$\text{CDF} = F_X(x) = \begin{cases} P[X \le x_0] = P[CCC] = \frac{27}{125} & \text{for } X \le x_0 \\ \frac{81}{125} & \text{for } X \le x_1 \\ \frac{117}{125} & \text{for } X \le x_2 \\ 1 & \text{for } X \le x_3 \end{cases}$$

Que: A certain RV has a CDF as

$$F_X(x) = \begin{cases} 0 & \text{for } x \le 0 \\ Kx^2 & \text{for } 0 < x \le 10 \\ 100K & \text{for } x > 10 \end{cases}$$

a) Calculate the value of K

b) Find the value of $P(x \le 5)$ and $P(5 < x \le 7)$

c) Plot the corresponding PDF.

Solⁿ

$F_X(x) = 100K$ for $x > 10$

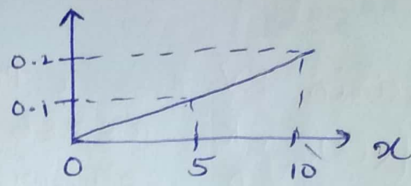
$\Rightarrow 100K = 1$ as it include all probabilities

So $= K = \frac{1}{100} = 0.01$

(ii) $P(X \le 5) = P(0 < X \le 5) = F_X(5) - F_X(0) = Kx^2\Big|_{x=0}^{x=5} = 0.01[25-0] = 0.25$

(iii) PDF $f_X(x) = \frac{d}{dx}F_X(x)$ for $0 < x \le 10$

$f_X(x) = \frac{d}{dx}Kx^2 = 2Kx = 0.02x$

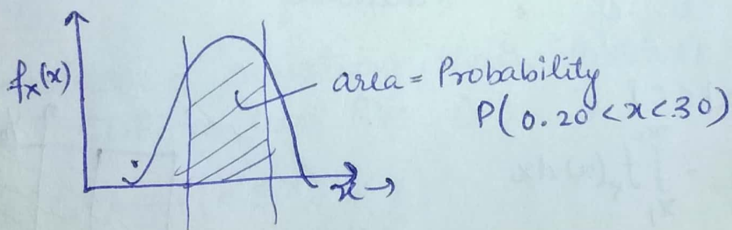


Probability Density Function [$f_X(x)$]

It is defined as the derivative of CDF

$$PDF = f_X(x) = \frac{d}{dx}F_X(x)$$

→ It is a function of a continuous random variable, whose integral across an interval gives the probability that the value of the variable lies within the same interval.



Properties

1. $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

Solⁿ:- $f_X(x) = \frac{d}{dx}F_X(x)$

Integrating both sides ← because $F_X(x) = P[X \le x]$

$$\int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} \frac{d}{dx}F_X(x)\,dx$$

$$= \int_{-\infty}^{x} f_X(x)\,dx = [F_X(x)]_{-\infty}^{x} = F_X(x) - \underbrace{F_X(-\infty)}_{0}$$

$$\int_{-\infty}^{x} f_X(x)\,dx = F_X(x)$$

ii) PDF is a non-negative function for all values of x i.e.
$f_X(x) \geq 0$ for all x

Sol^n:- CDF is a monotone Increasing function

$$PDF = \frac{d}{dx} CDF$$

$\frac{d}{dx}$ of monotone ↑ function is always +ve.

iii) The area under PDF curve is always unity

$$\therefore \int_{-\infty}^{x} f_X(x)\,dx = 1$$

Sol^n:-

$$f_X(x)dx = \frac{d}{dx}F_X(x)$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{d}{dx}F_X(x) = \left[F_X(x)\right]_{-\infty}^{\infty} = F_X(\infty) - F_X(-\infty) = 1 - 0 = 1$$

Que:- A certain RV has a PDF given by

$$f_X(x) = \begin{cases} k & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

i) Find k
b) Let $a = -1$ & $b = 2$, calculate $P(|X| \leq c)$ for $c = 1/2$

Sol^n:- Area under PDF curve is 1

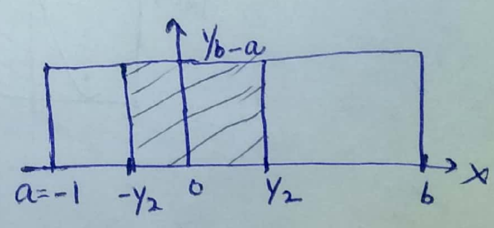
$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1, \quad \int_{a}^{b} k\,dx = 1, \quad \left[kx\right]_a^b = 1 \quad k(b-a) = 1 \quad k = \frac{1}{b-a}$$

$$\therefore f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

(ii) Value of $P(|X| \leq c)$

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$$

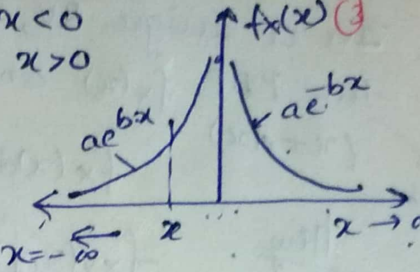
$$P\left(-\frac{1}{2} \leq X \leq \frac{1}{2}\right) = \int_{-1/2}^{1/2} \frac{1}{b-a}\,dx = \frac{1}{3}\left[x\right]_{-1/2}^{1/2} = \frac{1}{3}$$



Que:- Consider the probability density function $f_X(x) = a e^{-b|x|}$ where X is a random variable whose allowable values range from $x = -\infty$ to $x = +\infty$. Find (Sessional Feb 2015)
a) CDF
b) Relationship b/w a & b
c) Probability that the outcome lies b/w 1 & 2

$f_x(x) = a e^{-b|x|}$ or $f_x(x) = a e^{+bx}$ for $x < 0$

$= a e^{-bx}$ for $x > 0$



$$F_x(x) = \int_{-\infty}^{x} a e^{-b|x|} dx$$

for $x > 0$

$$= \int_{-\infty}^{0} a e^{+bx} + \int_{0}^{x} a e^{-bx} dx$$

$$= \frac{a}{b}\left[e^{bx}\right]_{-\infty}^{0} + \frac{a}{-b}\left[e^{-bx}\right]_{0}^{x}$$

$$= \frac{a}{b}[1 - 0) - \frac{a}{b}(e^{-bx} - 1)$$

$$= \frac{a}{2a} - \frac{a}{2a}(e^{-bx} - 1)$$

$$= \frac{1}{2} - \frac{1}{2}(e^{-bx} - 1) = 1 - \frac{1}{2}e^{-bx}$$

for $x < 0$

$$F_x(x) = \int_{-\infty}^{0} a e^{+bx}$$

$$= \frac{a}{b}\left[e^{bx}\right]_{\infty}^{x} = \frac{a}{b}e^{bx}$$

$$\therefore F_x(x) = \begin{cases} \frac{1}{2}e^{bx} & \text{for } x < 0 \\ 1 - \frac{1}{2}e^{-bx} & \text{for } x > 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f_x(x) dx = 1$$

$$= \int_{-\infty}^{0} a e^{\pm b|x|} + \int_{0}^{\infty} a e^{-bx} dx = 1$$

$$= \frac{a}{b}\left[e^{bx}\right]_{-\infty}^{0} + \frac{a}{b}\left[e^{-bx}\right]_{0}^{\infty} = 1$$

$$\frac{a}{b}(1-1) + \frac{a}{b}(0-1) = 1 \Rightarrow \frac{a}{b} + \frac{a}{b} = 1 \Rightarrow \boxed{2a = b}$$

$$P(+1 \le X \le 2) = \int_{1}^{2} f_x(x) dx$$

$$= \int_{1}^{2} a e^{-bx} dx = \frac{a}{-b}\left[e^{-bx}\right]_{1}^{2} = \frac{b/2}{-b}\left[e^{-2b} - e^{-b}\right] = \frac{1}{2}\left[e^{-b} - e^{-2b}\right]$$

Joint CDF :-

Let us consider a situation that involves two RVs X and Y then joint CDF for two RVs is :-

$$F_{XY}(x, y) = P(X \le x, Y \le y)$$

perties

1) $F_{XY}(x, y) \ge 0$

ii) CDF is always a continuous function in the XY plane

(iii) The joint CDF is a non-decreasing function of both X & Y.

Joint PDF :-

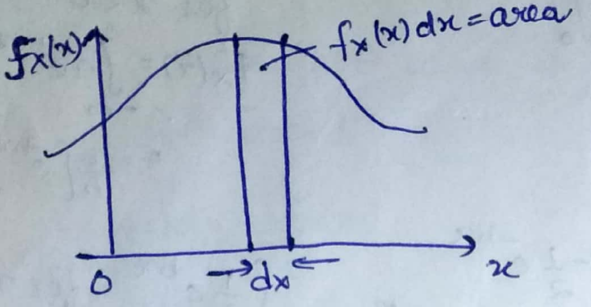
$$F_{XY}(x, y) = \frac{\partial^2 [F_{XY}(x, y)]}{\partial x \partial y}$$

Joint PDF is the partial derivative of the joint CDF with respect to two dummy variables x & y. Partial differentiation is done because two variables are simultaneously involved.

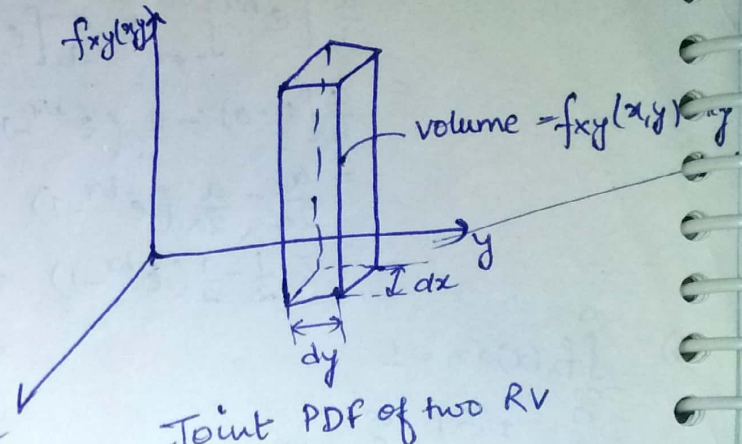
In a single RV, X the following eqⁿ tell us the relation b/w the PDF $f_X(x)$ and probability that X lies in range from x to $(x+dx)$

$$f_X(x)dx = P(x \le X \le x+dx)$$

Illy $\quad f_{XY}(x,y)\,dx\,dy = P(x < X \le x+dx,\ y < Y \le y+dy)$



PDF of a single RV.



Joint PDF of two RV

Properties

1) $f_{XY}(x,y) \ge 0$

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

(iii) $P(x_1 < X \le x_2,\ y_1 < Y \le y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy$

When we are dealing with two RVs X and Y, the individual probabilities densities $f_X(x)$ and $f_Y(y)$ can be obtained from the joint PDF $f_{XY}(x,y)$. Then these individual densities are called marginal densities or marginal PDFs.

CDF of a RV $\quad F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

joint CDF is $F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy$

If we are intrested in only $F_X(x)$ i.e. $P(X \le x)$ it can be expressed as $F_X(x) = P(X \le x, -\infty < Y \le \infty)$

$$F_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{XY}(x,y)\,dx\,dy$$

$$\text{or } \int_{-\infty}^{x}\underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}_{f_X(x)}\,dx$$

marginal density → So, $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

When X and Y are independent

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

$$P(x_1 \le X \le x_2, y_1 \le Y \le y_2) = \left[\int_{x_1}^{x_2} f_X(x)\,dx\right]\left[\int_{y_1}^{y_2} f_Y(y)\,dy\right]$$

So $f_X(x) = \underline{f(x)}$, $f_Y(y) = \underline{f(y)}$, $f_{XY}(x,y) = \underline{f(x,y)}$

Thus ( ) represents PDF in general. and not some particular PDF.

However, if the RV X and Y are not independent, then the dependence of X & Y is expressed by the <u>conditional</u> <u>PDF</u>

$$f_X(x|Y=y) = f_X(x|y) = \frac{f_{XY}(x,y) \rightarrow \text{joint PDF}}{f_Y(y) \rightarrow \text{marginal PDF}}$$

$f_X(x|y)$ → conditional PDF

|||ly

$$f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

<u>Properties</u>

(i) $f_Y(y|x) \ge 0$

(ii) $\int_{-\infty}^{\infty} f_Y(y|x)\,dy = 1$ or $\int_{-\infty}^{\infty} f_X(x|y)\,dx = 1$

(iii) $f_Y(y|x) = f_Y(y)$ & $f_X(x|y) = f_X(x)$ if X & Y are independent

i.e. Conditional PDF reduces to marginal PDF.

Que:- The joint PDF of RV X & Y is given by

$$f_{XY}(x,y) = \frac{1}{4} e^{-|x|-|y|} \qquad -\infty < x < \infty \text{ \& } -\infty < y < \infty$$

Determine (a) whether the RV X & Y are statistically Independent

(b) The probability that $X \le 1$ and $Y \le 0$.

Sol^n :- $f_{XY}(x,y) = \frac{1}{4} \cdot e^{-|x|-|y|} = \frac{1}{2} e^{-|x|} \cdot \frac{1}{2} e^{-|y|} = f_X(x) \cdot f_Y(y)$

Since joint PDF is equal to the product of PDFs of individual RV X and Y, the RVs are statistically independent

(b) Probability that $X \le 1$ and $Y \le 0$

$$P(X \le 1, Y \le 0) = \int_{-\infty}^{1}\int_{-\infty}^{0} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{1} \frac{1}{4} \cdot e^{-|x|} e^{-|y|}\,dx\,dy$$

$$= \int_{-\infty}^{1} \frac{1}{2} e^{-|x|}\,dx \cdot \int_{-\infty}^{0} \frac{1}{2} e^{-|y|}\,dy$$

$$= \int_{-\infty}^{0} \frac{e^{-|y|}}{2} dy \cdot \frac{1}{2}\left[\int_{-\infty}^{0} e^{-x} dx + \int_{0}^{1} e^{-x} dx\right]$$

$$= \frac{1}{2}\left[e^{-|y|}\right]_{-\infty}^{0} \times \frac{1}{2}\left[\left[\frac{e^{-x}}{-1}\right]_{-\infty}^{0} + \left[\frac{e^{-|x|}}{-1}\right]_{0}^{1}\right]$$

$$= \frac{1}{2}\left[e^{-0} - e^{-\infty}\right] \times -\frac{1}{2}\left[e^{-0} - e^{-\infty} - e^{-1} + e^{-0}\right]$$

$$= \frac{1}{4}[1] \times [2 - e^{-1}] = \frac{1}{4}[2 - e^{-1}]$$

Que:- The joint PDF of the random variables X and Y is given by

$$f_{XY}(x,y) = C \cdot e^{-(ax+by)} u(x) \cdot u(y)$$

where a and b are constant. Find the value of C.

Sol^n:

$$F_{XY}(\infty,\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$$

$$1 = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(ax+by)} u(x) u(y)\, dx\, dy$$

$$1 = C\int_{0}^{\infty}\int_{0}^{\infty} e^{-ax} dx \cdot e^{-by} dy$$

$$1 = \frac{C}{ab}\left[e^{-ax}\right]_{0}^{\infty} \cdot \left[e^{-by}\right]_{0}^{\infty}$$

$$1 = \frac{C}{ab}\left[e^{-\infty} - e^{0}\right]\left[e^{-\infty} - e^{-0}\right]$$

$$1 = \frac{C}{ab}\left[(-1)(-1)\right], \quad \boxed{C = ab}$$

Optimal Receiver

- A channel is used to transmit digital information i.e. a sequence in time of bits - logical 0's and 1's. Thus in successive intervals we want to transmit one of the two possible messages, message m_0 that bit 0 is intended, or the message m_1 that bit 1 is intended

→ The two possible messages might be represented at the transmitting end by two distinct waveforms each limited in time duration to the interval allocated ~~by~~ to a bit.

→ At the receiving end we may devise a system whereby the message m_0, when received generates some voltage, say r_0, which might be as simple as dc voltage, while m_1 when received generates r_1.

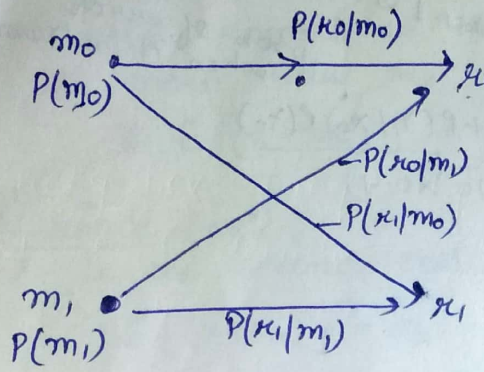
In, the absence of noise, m_0 generates r_0 and m_1 generates r_1, each with complete certainty. However because of noise, error may occur.

$P(r_0|m_0)$ = Prob. that r_0 is received given than m_0 is sent

$P(r_1|m_0)$ = " " r_1 " " " m_0 " "

$P(r_0|m_1)$ = " " r_0 " " " m_1 "

$P(r_1|m_1)$ = " " r_1 " " " m_1 "



messages m_1 and m_0 do not occur with equal frequency.
So $P(m_1)$ & $P(m_0)$ are the probabilities that m_1 & m_0 are the messages respectively intended in an message interval.
So, $P(m_1)$ & $P(m_0)$ are a priori probabilities

Consider that r_0 is received, we have to compare the conditional probabilities, called a posteriori probabilities

$P(m_0/r_0)$ = prob. that m_0 is the message given that r_0 is received

$P(m_1/r_0)$ = prob. that m_1 is the message given that r_0 is received

So, if r_0 is received

choose m_0 if $P(m_0/r_0) > P(m_1/r_0)$

or m_1 if $P(m_1/r_0) > P(m_0/r_0)$

If r_1 is received choose

m_0 if $P(m_0/r_1) > P(m_1/r_1)$

m_1 if $P(m_1/r_1) > P(m_0/r_1)$

A receiver which operates in accordance to this algorithm is said to "maximize the a posteriori probabilities" of a correct decision and is called an optimum receiver.

$$P(m_0/r_0) \cdot P(r_0) > P(m_1/r_0) \cdot P(r_0)$$

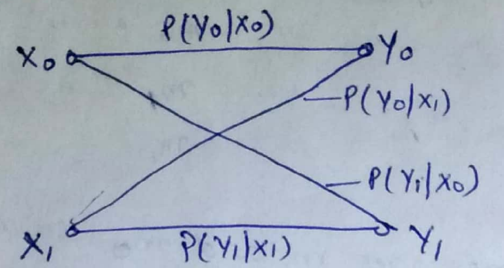
or

$$P(r_0/m_0) P(m_0) > P(r_0/m_1) P(m_1)$$

Correspondingly if r_1 is received we choose m_1 only if

$$P(r_1/m_1) P(m_1) > P(r_1/m_0) P(m_0)$$

Que :- Over a certain binary communication channel, symbol '0' is transmitted with a probability of 0.4 and symbol '1' is transmitted with a probability of 0.6. If P(E) when '1' is transmitted is 10^{-4} and when a 0 is transmitted is 10^{-6}. Find P(E) in channel.



$P(X_0) = 0.4$
$P(X_1) = 0.6$
$P(Y_1|X_0) = 10^{-6}$
$P(Y_0|X_1) = 10^{-4}$

P(E) when 1 is transmitted

prob. of error when 0 is transmitted

$$P(E) = P(Y_0|X_1)P(X_1) + P(Y_1|X_0)P(X_0)$$
$$= (10^{-4} \times 0.6) + (10^{-6} \times 0.4)$$
$$P(E) = 0.604 \times 10^{-4}$$

Mean / Expected Value :-

Let us consider a discrete RV X that has possible values $x_1, x_2 \ldots$ with the probability of occurrence to be $P(x_1), P(x_2) \ldots$

Now if there are N independent observations of the RV X, then we would expect that the outcome $X = x_1$ occurs $NP(x_1)$ times, $X = x_2$ occurs $NP(x_2)$ times etc.

∴ the arithmetic sum of all the independent observations would be:

$$x_1P(x_1)N + x_2P(x_2)N + x_3P(x_3)N + \ldots = N\sum_{i=1}^{k} x_i P(x_i)$$

$$\therefore \text{Mean value of } X = \frac{N}{N}\sum_{i=1}^{k} x_i P(x_i)$$

m_x or $E[x]$ or $\bar{x}$

$$\text{So, mean value of } X = m_x = E[x] = \bar{x} = \sum_{i=1}^{k} x_i P(x_i)$$

For a continuous RV

$$P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_x(x)\,dx$$

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} x f_x(x)\,dx$$

n^{th} moment of RV :-

$$E[x^n] = \bar{x}^n = \int_{-\infty}^{\infty} x^n f_x(x)\,dx$$

moment of X (Mean value)

$$E[X] = \bar{X} = \int_{-\infty}^{\infty} x f_X(x) dx = m_x$$ → First moment of X is always its mean or average value.

2nd moment of X (Mean Square value)

$$E[X^2] = \bar{X}^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

Central Moment

Moment means expected value.
'Central Moment' is the expected value of the difference b/w RV X and its mean value m_X

$$E[(X-m_X)^n] = \int_{-\infty}^{\infty} (x-m_x)^2 f_X(x) dx$$

Variance (σ_X^2)

It is the second central moment of RV X.

$$E[(X-m_X)^2] = \sigma_X^2 = \int_{-\infty}^{\infty} (x-m_x)^2 f_X(x) dx$$

$$\sigma_X^2 = E[X^2 - 2mX + m_X^2]$$
$$= E[X^2] - 2m_X E[X] + m_X^2$$
$$= E[X^2] - 2m_X^2 + m_X^2$$
$$\sigma_X^2 = E[X^2] - m_X^2$$
$$\boxed{\sigma_X^2 = E[X^2] - E[X]^2}$$

Standard Deviation (σ_X)

It is defined as the square root of variance σ_X^2

$$\sigma_X = \sqrt{E[X^2] - m_X^2}$$

→ It is a measure of width of the pdf. The larger the value of σ_X, the wider the pdf.

→ It indicates the deviation in the value of RV from its mean value. Small values of σ_X show that all possible values of RV are clustered around its mean, whereas, larger values of σ_X mean RV will deviate considerably from its mean value.

Cauchy-Schwarz Inequality

$$\{E[XY]\}^2 \le E[X^2] \cdot E\{Y^2\}$$

Soln:- Mean ~~value~~ square value of a RV can never be negative.

$$E[(X-aY)^2] \ge 0$$

where a is constant

$$E[X^2 - 2aXY + a^2Y^2] \ge 0$$

$$E[X^2] - 2aE[XY] + a^2 E[Y^2] \ge 0 \quad -(1)$$

Let the value of a be
$$a = \frac{E[XY]}{E[Y^2]}$$

Put in ①

$$E[x^2] - 2\frac{E[XY]\cdot E[XY]}{E[Y^2]} + \frac{\{E[XY]\}^2}{[E[Y^2]]^{\cancel{2}}}\cdot \cancel{E[Y^2]} \geq 0$$

$$E[x^2]\cdot E[Y^2] - 2\{E[XY]\}^2 + E[XY]^2 \geq 0$$

or $E[x^2]E[Y^2] - E[XY]^2 \geq 0$

$\Rightarrow E[XY]^2 \leq E[x^2]\cdot E[Y^2]$

Chebyshev Inequality

Standard deviation of a RV(X) is a measure of the width of its PDF. With increase in σ_x, The PDF becomes more and more wide. Chebyshev puts this inequality into statement form.

Chebyshev Inequality states that for a RV with zero mean

$$P(|x| \leq k\sigma_x) \geq 1 - \frac{1}{k^2}$$ where k is constant

i.e. probability of observing the RV X within a few standard deviation is very high.

e.g. if $\sigma_x = 1$ and $k=3$

then $P(|x| \leq 3) \geq 0.88$

Proof :- $\sigma_x^2 = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx$

$f_x(x)$ is a +ve function

& x^2 will always be +ve

$$\sigma_x^2 \geq \int_{|x| \geq k\sigma_x} x^2 f_x(x)\,dx$$

The smallest value of X is $k\sigma_x$. Even if we replace X by its smallest value, the inequality of above expression will hold true

$$\therefore \quad \sigma_x^2 \geq k^2\sigma_x^2 \geq \int_{|x| \geq k\sigma_x} f_x(x)\,dx$$

But $\int f_x(x)\,dx = CDP \Rightarrow P(|x| \geq k\sigma_x)$.

So, $\sigma_x^2 \geq k^2\sigma_x^2 \geq P(|X| \geq k\sigma_x)$

we have $P(|X| \geq k\sigma_x) \leq \frac{1}{k^2}$

$$P(\bar{A}) = 1 - P(A)$$

So, $P(|X| \geq k\sigma_x) = 1 - P(|X| \leq k\sigma_x)$

Thus $1 - P(|X| \leq k\sigma_x) = \frac{1}{k^2}$

So, $\boxed{P(|X| \leq k\sigma_x) \geq 1 - \frac{1}{k^2}}$

Correlation b/w Random Variables

Let us consider a random experiment with two outcomes described by the random variables X and Y.

Covariance is a measure of dependence b/w X and Y.

The covariance is mathematically given by

$$\sigma_{xy} = E\{(X - m_x)(Y - m_y)\}$$

$$\sigma_{xy} = E\{XY - Xm_y - m_xY + m_xm_y\}$$

$$= E[XY] - m_yE[X] - m_xE[Y] + E[m_xm_y]\}$$

$$= E[XY] - m_ym_x - m_xm_y + m_xm_y$$

$$\sigma_{xy} = E[XY] - m_xm_y$$

So, if $E[XY] = m_xm_y$, then $\sigma_{xy} = 0$

and then RV X and Y are said to be uncorrelated.

Correlation coefficient : ρ_{xy}

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x\sigma_y}$$

Thus if X and Y are uncorrelated then $\rho_{xy} = 0$

** Independent Random variables

$$E\{XY\} = m_xm_y \text{ and } \sigma_{xy} = 0$$

So, independent random variables are uncorrelated

But the converse may not always be true.

Probability Models

The various type of Random variables, we come across in communication, will have their own PDFs which are different from one another. Hence, PDFs of all RVs are approximated to some standard PDFs.

Such commonly used PDFs are :-

a) Binomial Distribution] - used for Discrete RV
b) Poisson Distribution
c) Gaussian Distribution
d) Rayleigh Distribution - For continuous RV
e) Uniform Distribution

Binomial Distribution :-

→ used to describe an integer valued RV associated with repeated trials of an experiment.

→ applied to the digital communication systems. e.g. to find the numbers of errors in the message signal of 'n' digits.

$$P(X=K) = P(K) = {}^nC_K\, p^K (1-p)^{n-K}$$

where ${}^nC_K$ → Total no. of words containing K correct bits in total n bits

p^K → probability of having K correct bit

$(1-p)^{n-K}$ → " " " $(n-K)$ incorrect bit

$${}^nC_K = \frac{n!}{K!(n-K)!} \Rightarrow \text{binomial coefficient}$$

The probability density function of X

$$f_X(x) = \sum_{K=0}^{n} P(X=K)\,\delta(x-K)$$

as X is a discrete RV and PDF is in the form of summation of all possible terms

$\delta(x-K)$ → depicts that PDF is not a continuous function.

Infact it is the sum of weighted probabilities at the sample points

$$f_X(x) = \sum_{K=0}^{n} {}^nC_K\, p^K (1-p)^{n-K}\,\delta(x-K)$$

$$\text{CDF} = \sum_{K=0}^{x} {}^nC_K\, p^K (1-p)^{n-K}$$

Mean :- $E[X] = \sum_{x=0}^{n} x \cdot {}^nC_x \, p^x (1-p)^{n-x}$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!} p^x (1-p)^{n-x}$$

Let $y = x-1 \rightarrow x = y+1$, $x=1 \rightarrow y=0$

$x=0$ vanishes and $m = n-1 \rightarrow n = m+1$, $x = n-1 \rightarrow y = m$

$$E[X] = \sum_{y=0}^{m} \frac{(m+1)!}{y!\,(m-y)!} p^{y+1} \cdot (1-p)^{m-y}$$

$$= (m+1)p \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y \cdot (1-p)^{m-y}$$

$$= np \sum_{y=0}^{m} {}^mC_y \, p^y (1-p)^{m-y}$$

Binomial theorem

$$(a+b)^m = \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} \cdot a^y b^{m-y}$$

$a=p$, $b = 1-p$

So, $(a+b)^m = 1$

So, $\boxed{E[X] = np}$

Variance $= \sigma_x^2 = E[X^2] - E[X]^2$

$$x \cdot {}^nC_x = \frac{n!}{x!\,(n-x)!} x = \frac{n!}{(x-1)!\,(n-x)!} = \frac{(n-1)!\,n}{(x-1)!\,((n-1)-(x-1))!} = n \cdot {}^{n-1}C_{x-1}$$

Hely $x^2 \, {}^nC_x = x \cdot \frac{n!}{x!\,(n-x)!} = n \cdot x \cdot {}^{n-1}C_{x-1}$ $[nx - n + n]$

$$= n(x-1)\,{}^{n-1}C_{x-1} + n\,{}^{n-1}C_{x-1}$$

$$= n(n-1)\,{}^{n-2}C_{x-2} + n\,{}^{n-1}C_{x-1}$$

$$E[X^2] = \sum_{K=0}^{n} K^2 P[X=K] = \sum_{K=2}^{n} n(n-1)\,{}^{n-2}C_{x-2} \, p^K (1-p)^{n-K} + \sum_{K=1}^{n} n \, {}^{n-1}C_{x-1} \, p^K (1-p)^{n-K}$$

$$= n(n-1)p^2 + np$$

$$\sigma_x^2 = E[X^2] - E[X]^2$$

$$= n(n-1)p^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

Standard deviation

$$\sigma_x = \sqrt{np(1-p)}$$

Que:- Binary data is transmitted over a noisy communication channel. It is being transmitted in the form of blocks with each block consisting of 16 binary digits. The probability that a received binary digit is in error due to noise is 0.005. Assume that the errors occurring in different digit position in a data block are independent

(a) Find the average no. of errors per block

(b) Find the variance of no. of errors per block

(c) Find the probability that the no of errors per block is greater than or equal to 03.

Sol^n: Let X be a RV which represents the no. of errors per block

so, $n = 16$

$p = 0.005$

(i) Avg. no. of errors per block $= E[X] = np = 16 \times 0.005 = 0.08$

(ii) $\sigma_x^2 = np(1-p) = 16 \times 0.005 \times (1 - 0.005) = 0.0796$

(iii) Probability that no. of errors per block is greater than or equal to 3 is given by

$$P(X \geq 3) = 1 - P(X \leq 2)$$

$$P(X \leq 2) = \sum_{K=0}^{2} {}^{16}C_K (0.005)^K (0.995)^{16-K}$$

$$P(X \leq 2) = {}^{16}C_0 (0.005)^0 (0.995) + {}^{16}C_1 (0.005)(0.995)^{15} + {}^{16}C_2 (0.005)^2 (0.995)^{14}$$

$$= 0.9229 + 0.07420 + 2.79 \times 10^{-3}$$

$$= 0.999933$$

$$P(X \geq 3) = 1 - 0.999933$$

$$= 6.667 \times 10^{-5}$$

Poisson Distribution

As the no. n increases, the binomial distribution becomes difficult to handle. If 'n' is very large, 'p' is very small and the mean value 'np' is finite

Then Binomial distribution can be approximated by Poisson distribution.

when $n \to \infty$, $p \to 0$ and such that $np \neq 0$ & $np \to m$ = mean

then $\frac{n!}{k!(n-k)!} p^k q^{n-k} \xrightarrow[n\to\infty]{} e^{-m} \cdot \frac{m^k}{k!} = \frac{e^{-m} \cdot m^k}{k!}$

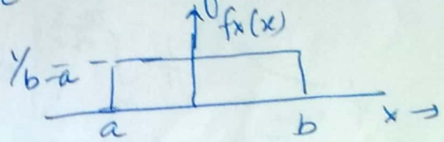
Mean $\boxed{m = np}$

variance $\boxed{\sigma_x^2 = np}$

Uniform Distribution

When a RV X is equally likely to be observed in a finite range and is likely to have zero value outside this finite range, then the RV is said to have a uniform distribution.

$$f_x(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$



Mean :- $E[x] = m = \int_{-\infty}^{\infty} x \cdot f_x(x)\,dx$

$$= \int_a^b \frac{1}{b-a} \cdot x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] = \frac{a+b}{2}$$

$$\boxed{E[x] = m = \frac{a+b}{2}}$$

Variance :- $\sigma_x^2 = E[x^2] - E^2[x]$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x) = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right] = \frac{1}{3}(a^2+b^2+ab)$$

$$\sigma_x^2 = \frac{1}{3}(a^2+b^2+ab) - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{3}(a^2+b^2+ab) - \left(\frac{a^2+b^2+2ab}{4}\right)$$

$$\boxed{\sigma_x^2 = \frac{1}{12}(b-a)^2}$$

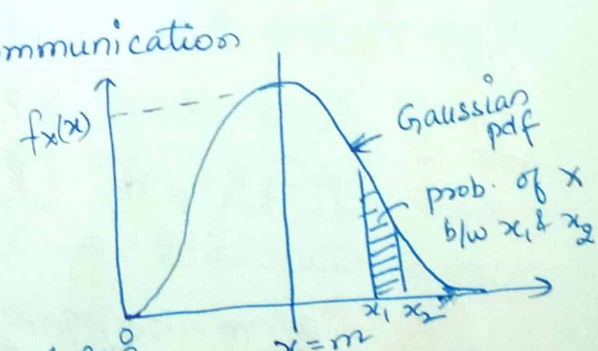
Gaussian Distribution (Normal Distribution)

→ used for continuous RV

→ most imp. PDF in the area of communication

PDF :-

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\frac{-(+x-m)^2}{2}$$



→ Gaussian PDF is also known as Normal PDF

→ Shape of Gaussian PDF is bell.

→ peak at $x = m$

→ even symmetry about peak

i.e. $P(x \le m) = P(x > m) = 1/2$

Gaussian CDF

$$CDF = \int_{-\infty}^{x} f_X(x)\,dx$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

Let $\frac{m-x}{\sqrt{2}\sigma} = y$ $\qquad x \to -\infty \quad y \to \infty$, $\quad x \to x \quad y \to \frac{m-x}{2\sqrt{\sigma}}$

$$-\frac{dx}{\sigma\sqrt{2}} = dy \Rightarrow dx = -\sigma\sqrt{2}\,dy$$

$$F_X(x) = \int_{\infty}^{\frac{m-x}{\sqrt{2}\sigma}} \frac{1}{\cancel{\sigma}\sqrt{2}\pi} e^{-y^2} (-\cancel{\sigma}\cancel{\sqrt{2}})\,dy$$

$$= -\frac{1}{\pi} \int_{\infty}^{\frac{m-x}{\sqrt{2}\sigma}} e^{-y^2} dy$$

Interchanging the limits

$$= \frac{1}{\pi} \int_{\frac{m-x}{\sqrt{2}\sigma}}^{\infty} e^{-y^2} dy = \frac{1}{2} \cdot \underbrace{\left[\frac{2}{\pi} \int_{\frac{m-x}{\sqrt{2}\sigma}}^{\infty} e^{-y^2} dy \right]}_{\text{complementary error function}}$$

$$erfc = \frac{2}{\pi} \int_{x}^{\infty} e^{-u^2} du$$

So, $\boxed{F_X(x) = \frac{1}{2} erfc\left[\frac{m-x}{\sqrt{2}\sigma}\right]}$

erfc is a monotonically decreasing function. Hence with ↑ in value of $\frac{m-x}{2\sqrt{\sigma}}$, the value of complementary error function goes on ↓.

Que:- A RV X is Gaussian distributed with mean $m_x = 5$ and variance $\sigma_x^2 = 64$. What is the probability of obtaining X between -3 and 13.

Soln:
$$P(-3 < X \le 13) = P(m_x - k\sigma < X \le m + k\sigma)$$
$$P(-3 < X \le 13) = P(5 - 8k < X \le 5 + 8k)$$
$$5 - 8k = -3 \quad \text{or} \quad 8k = 8 \quad \text{and} \quad k = 1$$

$$P(m - k\sigma) < X \le P(m + k\sigma) = erf\left(\frac{k}{\sqrt{2}}\right)$$

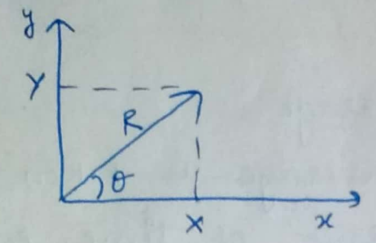
$$P(-3 < x \le 13) = erf\left(\frac{1}{\sqrt{2}}\right)$$
$$= 0.6778$$

taking value of erfc from standard table

Rayleigh's Distribution

The Rayleigh's distribution is used for continuous RVs
It describes a continuous RV produced from two Gaussian RVs
Let X & Y be two RVs having

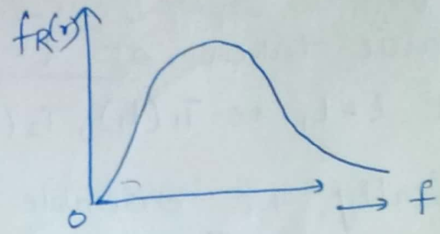
$m_x = m_y = m$ and $\sigma_x = \sigma_y = \sigma$



$R = \sqrt{x^2 + y^2}$

$\phi = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{\begin{aligned} f_R(r) &= \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \quad \text{for } r \geq 0 \\ &= 0 \quad \text{otherwise} \end{aligned}}$$



Rayleigh distribution is specifically used to analyze the narrowband Gaussian noise.

$$\boxed{\text{Mean} = \sigma\sqrt{\frac{\pi}{2}}}$$

$$\boxed{\text{Variance} = \left[2 - \frac{\pi}{2}\right]\sigma^2}$$

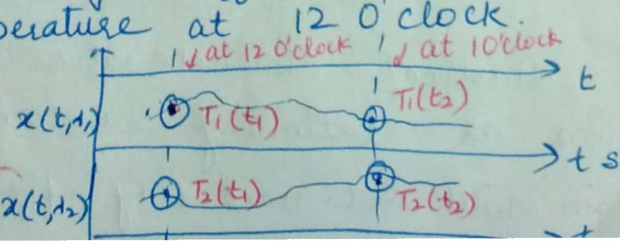
Random Processes

Random process can be defined as an ensemble of Random variables which are functions of time and hence denote the random process as X(t).

Ex:- Let us consider a random variable X which represents the temp of a city at 12 O'clock.
So, $s_1 \rightarrow$ corresponds to sample temperature at 12 O'clock on day 1
$s_2 \rightarrow$ day 2 and so on

But the temperature is not just the function of day, it is also a function of time. i.e. the temperature at 1 P.m. will have a completely different distribution then the distribution of temperature at 12 O'clock.

$\lambda \rightarrow$ represents day



Ensemble :-

Ensemble means family or collection.
Hence collection of all possible sample functions is called an ensemble.

Sample space is the collection of all possible sample points and ensemble is the collection of all possible sample functions

Ensemble Mean or Ensemble Average :-

Ensemble mean or ensemble average is taken over the ensemble of waveforms at a fixed instant of time. e.g. the ensemble mean value taken at $t=t_1$ will consist of all the values taken at $t=t_1$ i.e. $T_1(t_1), T_2(t_1), T_3(t_1) \ldots$

Mathematically, the ensemble mean $\overline{X(t)}$ or m_x is defined as

$$m_x = \int_{-\infty}^{\infty} x\, f_X(x;t)\, dx$$

Here time t is treated as a constant because we are obtaining the ensemble mean at a fixed time instant. Hence the only variable in this equation is x.

The values of ensemble mean at different time instants will be different from each other.

Time average :-

$$m_x(T) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$$

The ensemble average are obtained at constant values of t, whereas the time averages are obtained by changing time t.

Classification of Random Processes

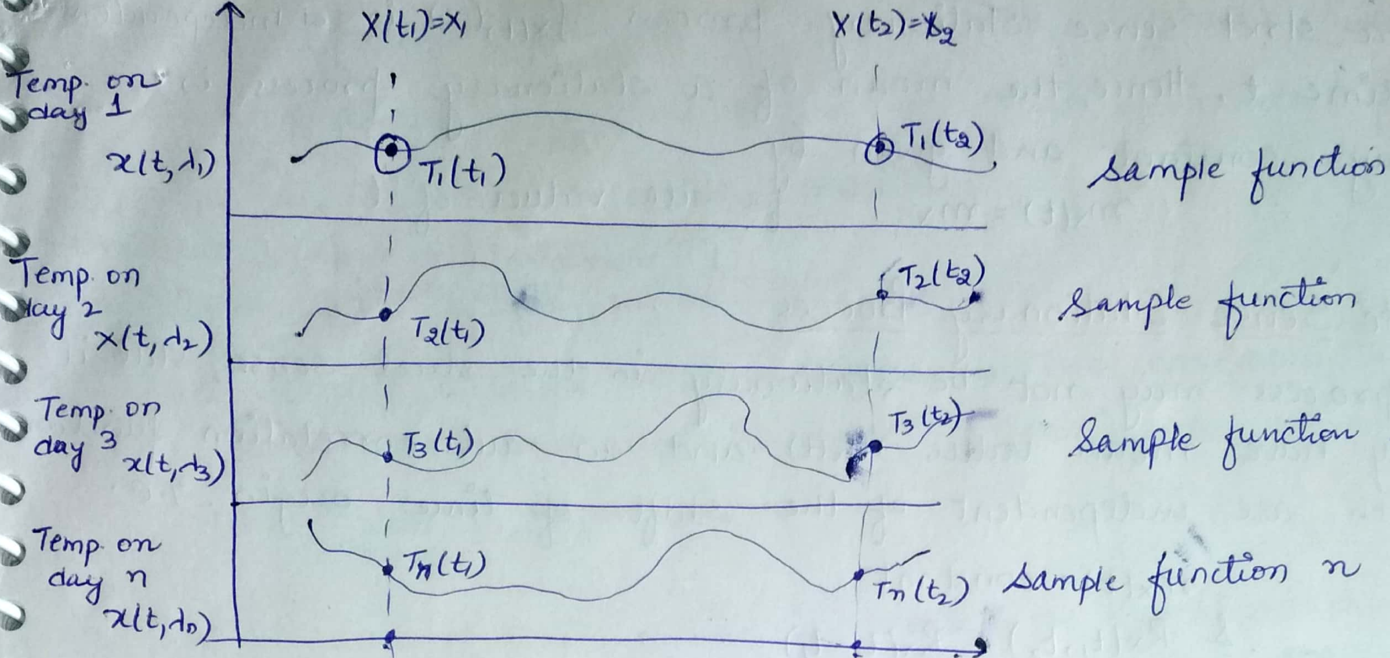
The random processes are broadly categorized as :-

(a) Wide-Sense Stationary

(b) Strict Sense Stationary → joint CDF does not change when shifted in time. Mean & variance of present does not change with time

(c) Non - stationary

A random process whose stastical characteristics do not change with time is known as stationary random process. Hence, the shift of time origin will not have any effect on the stationary random process



$X(t_1), X(t_2) \ldots X(t_n) \rightarrow$ Random variables obtained by observing the random process $X(t)$ at instants $t = t_1, t_2 \ldots t_n$

For the sake of simplicity, we can denote these RVs as $X_1, X_2 \ldots X_n$.

Hence the CDF (joint CDF) of this set of RV is denoted by

$F_{X(t_1), X(t_2), X(t_3)}(X_1, X_2, X_3)$ or

$F_{X_1, X_2, X_3}(X_1, X_2, X_3)$ in simplified notation

Let us assume that each instant of observation i.e. $t_1, t_2, t_3 \ldots$ is shifted by a fixed amount τ. Hence a new set of random variables containing $X(t_1+\tau), X(t_2+\tau) \ldots X(t_n+\tau)$. The joint CDF of this new set of RV is given by

$F_{X(t_1+\tau), X_2(t_2+\tau), X_3(t_3+\tau)}(x_1, x_2, x_3)$

Strict Sense Stationary

The random process $X(t)$ is said to be stationary in the strict sense if the joint CDF of the original set of random variables is equal to that of the new set of RV obtained after a time shift of τ.

i.e. $F_{X(t_1), X(t_2) \ldots X(t_n)}(x_1, x_2 \ldots x_n) = F_{X(t_1+\tau), X(t_2+\tau) \ldots X(t_n+\tau)}(x_1, x_2 \ldots x_n)$

In other words, a random process $X(t)$ is strictly stationary if the joint distribution of any set of variables obtained by observing the random process $X(t)$ is invariant with respect to the location of the origin $t=0$

For a strict sense stationary process $f_{X(t)}(x)$ is ~~a~~ independent of time t. Hence, the mean of a stationary process is always constant and given by

$$m_X(t) = m_X \quad \text{for all values of } t$$

Wide Sense Stationary Process

1st momonent (i.e. mean) & autocorrelation do not vary w.r.t time.

A process may not be stationary in the strict sense, still it may have mean value $m_X(t)$ and an auto correlation function which are independent of the shift of time origin. I.e.

$$m_X(t) = \text{constant}$$
$$\& \; R_X(t_1, t_2) = R_X(t_2 - t_1)$$

Such a process is known as wide-sense stationary or weakly stationary process

→ All the stationary processes are wide-sense stationary but every wide sense stationary process may not be strictly stationary. However, a truly stationary process (strict-sense stationary) process can't occur in real life.

Ergodic process has the same behaviour averaged over time as averaged over the space of all the system state

Ergodicity | Ergodic Process

Let us consider the sample function $x(t)$ of a wide sense stationary process $X(t)$. Let the observation interval be $-T \le t \le T$. Then, the time average or dc value of sample function $x(t)$ is

$$m_X(T) = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

The value of time average $m_X(T)$ depends on the observation interval & which sample function $x(t)$ of random process $X(t)$ is being chosen.

Hence, time average $m_X(T)$ is itself a RV

The mean value of the time average $m_X(T)$ is given by

$$E[m_X(T)] = E\left[\frac{1}{2T}\int_{-T}^{T} x(t)\,dt\right]$$

Interchanging the operation of expectation and integration,

$$E[m_X(T)] = \frac{1}{2T}\int_{-T}^{T} E[x(t)]\,dt$$

The term $E[x(t)]$ represents the mean value of $x(t)$ & when we $\int_{-T}^{T}$ it and divide by $\frac{1}{2T}$, we obtain the mean of random process $X(t)$ i.e. m_x or the ensemble mean.

Condition for Ergodicity in Mean

(a) The time average $m_x(T)$ approaches the ensemble average m_x with the observation interval T tending to infinity,

i.e. $$\lim_{T\to\infty} m_x(T) = m_x$$

(b) The variance of $m_x(T)$ which is treated as RV approaches zero with the observation interval T tending to infinity

$$\lim_{T\to\infty} var[m_x(T)] = 0$$

Conditions for Ergodicity in Autocorrelation functions :-

(a) $\lim_{T\to\infty} R_x(\tau,T) = R_x(\tau)$

The time averaged autocorrelation function approaches the autocorrelation function of the system with observation interval T tending to ∞.

(b) $\lim_{T\to\infty} Var[R_x(\tau,T)] = 0$

This means that the variance of $R_x(\tau,T)$ which is a random variable, approaches zero with the observation interval tending to ∞.

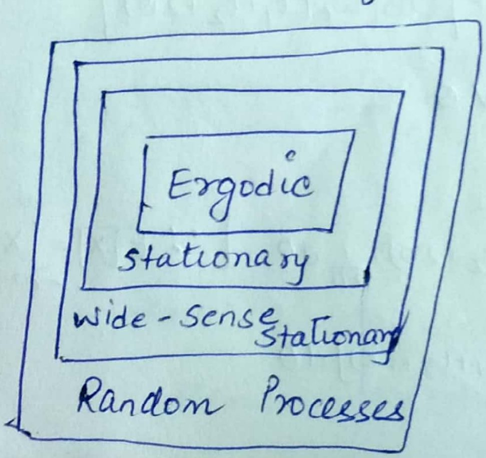
Ergodicity of a Process :-

When all the ensemble averages equal the corresponding time averages, the process is called an ergodic process.

i.e. Time average = Ensemble average

& Time averaged Autocorrelation function = Auto correlation function of system

Ensemble average = Time Average



Time average can't be a function of time so, ensemble average is also independent of time

∴ Ergodic process is stationary process, but the converse is not true.

Que:- Show that the random process

$$X(t) = A\cos(\omega_c t + \theta)$$

where θ is a random variable uniformly distributed in the range $[0, 2\pi)$ is a WSS process.

Solⁿ:- For a WSS process, show that

① Ensemble mean of sample function amplitudes at any time t is same.

② Autocorrelation $R_Z(t_1, t_2) = R_Z(t_2 - t_1)$

So, Ensemble mean is $m_X(t)$

$$m_X(t) = \int_{-\infty}^{\infty} x f_X(x,t)\,dx$$

As θ is uniformly distributed

$$f_X(x) = f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

Here $X = x(t) = A\cos(\omega_c t + \theta)$

and $f_X(x, t) = f_\theta(\theta) = \frac{1}{2\pi}$

So, $m_X(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} A\cos(\omega_c t + \theta)\,d\theta$

$$= \frac{A}{2\pi}\left[\sin(\omega_c t + \theta)\right]_0^{2\pi}$$

$$= 0$$

So, ensemble mean of sample function is zero i.e. independent of t.

Autocorrelation function $R_Z(t_1, t_2)$ is

$$R_Z(t_1, t_2) = E[X(t_1)\cdot X(t_2)]$$

$$= E[A\cos(\omega_c t_1 + \theta)\cdot A\cos(\omega_c t_2 + \theta)]$$

$$= E[A^2\cos(\omega_c t_1 + \theta)\cdot \cos(\omega_c t_2 + \theta)]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$R_Z(t_1, t_2) = E\left[\frac{A^2}{2}\{\cos[\omega_c(t_1 - t_2)] + \cos[\omega_c(t_1 + t_2 + 2\theta)]\}\right]$$

or $R_Z(t_1, t_2) = \underbrace{\frac{A^2}{2}E[\cos[\omega_c(t_1 - t_2)]]}_{\text{does not contain the RV } \theta} + \frac{A^2}{2}E[\cos[\omega_c(t_1 + t_2 + 2\theta)]]$

So, $E[\cos(\omega_c(t_1 - t_2))] = \cos[\omega_c(t_1 - t_2)]$

$$E[\cos(\omega_c(t_1 + t_2 + 2\theta))] = \int_{-\infty}^{\infty} \cos(\omega_c(t_1 + t_2 + 2\theta))\cdot\frac{1}{2\pi}\,d\theta \quad \left[\because E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx\right]$$

$$= \int_0^{2\pi} \frac{1}{2\pi}\cdot\cos[\omega_c(t_1 + t_2 + 2\theta)]\,d\theta$$

So, $R_Z(t_1,t_2) = \frac{A^2}{2}\left[\cos(\omega_c(t_1-t_2))\right] + 0$

$$R_Z(t_1,t_2) = \frac{A^2}{2}\left[\cos(\omega_c(t_1-t_2))\right]$$

Que:- A Random process is expressed as

$$X(t) = A\cos(\omega t + \theta)$$

where ω & θ are constant and A is a random variable. Determine whether $X(t)$ is WSS.

Sol^n:- $m_X(t) = E[X(t)] = E[A\cos(\omega t+\theta)]$

$$= \int_{-\infty}^{\infty} A\cos(\omega t+\theta) f_\theta(\theta)\,d\theta$$

Since θ is constant

$$m_X(t) = \cos(\omega t+\theta)\cdot E[A]$$

So, $m_X(t)$ is not constant unless $E[A]=0$.

(ii) $R_{XX}(t,t+\tau) = E[X(t)\cdot X(t+\tau)]$

$$= \int_{-\infty}^{\infty} A^2\cos(\omega t+\theta)\cos[\omega(t+\tau)+\theta]\cdot f_\theta(\theta)\,d\theta$$

$$= \int_{-\infty}^{\infty} \frac{A^2}{2}\left[\cos\omega\tau + \cos(2\omega t+2\theta+\omega\tau)\right] f_\theta(\theta)\,d\theta$$

Since θ is constant

$$R_{XX}(t,t+\tau) = \frac{1}{2}\left[\cos\omega\tau + \cos(2\omega t+2\theta+\omega\tau)\right]E[A^2]$$

$\Rightarrow$ Auto correlation is not a function of time difference τ only

Since $m_X(t)$ is not constant and since the auto correlation is not a function of time difference τ only, the given process is not WSS

Que:- Consider a Random process $X(t)$ is given by

$$X(t) = A\cos(\omega t+\theta)$$

where A and ω are constants and θ is RV over $(-\pi,\pi)$. Show that $X(t)$ is ergodic in both the mean and autocorrelation.

Sol^n:- Time average

$$\langle X(t)\rangle = \lim_{T\to\infty}\int_{-T/2}^{T/2} A\cos(\omega t+\theta)\,dt$$

$$= \frac{A}{T_0}\int_{-T_0/2}^{T_0/2} \cos(\omega t+\theta)\,dt$$

where $T_0 = \frac{2\pi}{\omega}$ since $\omega T_0 = 2\pi$

Hence $\langle X(t)\rangle = 0$

Time averaged autocorrelation is

$$\bar{R}_{xx}(\tau) = \langle X(t) \cdot X(t+\tau) \rangle$$

$$= \lim_{T\to\infty} \int_{-T/2}^{T/2} x(t) \cdot x(t+\tau)\, dt = \frac{A^2}{T_0} \int_{-T_0/2}^{T_0/2} \cos(\omega t + \theta) \cdot \cos[\omega(t+\tau) + \theta]\, dt$$

or $$\bar{R}_{xx}(\tau) = \frac{A^2}{2T_0} \int_{-T_0/2}^{T_0/2} [\cos(2\omega t + \omega\tau + 2\theta) + \cos\omega\tau]\, dt$$

$$= \frac{A^2}{2T_0} \int_{-T_0/2}^{T_0/2} \cos[2\omega t + \omega t + 2\theta]\, dt + \frac{A^2}{2T_0} \int_{-T_0/2}^{T_0/2} \cos(\omega\tau)\, dt$$

$$\bar{R}_{xx}(\tau) = \frac{A^2}{2T_0} \times 0 + \frac{A^2}{2T_0} \cos\omega(\tau) \cdot [T]_{-T_0/2}^{T_0/2}$$

$$\therefore \bar{R}_{xx}(\tau) = \frac{A^2}{2} \cos(\omega\tau)$$

Here, $m_x(t) = \langle x(t) \rangle = 0$

and $R_{xx}(\tau) = \bar{R}_{xx}(\tau)$

Hence the process is ergodic process.

Que:- If the Random process is WSS, then prove that

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

Soln:- $R_{xx}(-\tau) = E[X(t) \cdot X(t+\tau)]$

Substituting $(t+\tau) = t'$

$\Rightarrow t = t' - \tau$

$$R_{xx}(-\tau) = E[X(t'-\tau) X t']$$

$$= E[X(t') X(t'-\tau)]$$

R.H.S represents autocorrelation of $R_{xx}(\tau)$

$$R_{xx}(\tau) = R_{xx}(-\tau)$$

Que- If the Random Process $X(t)$ is WSS, then prove that

$$|R_{xx}(\tau)| \leq R_{xx}(0)$$

Soln:- $E\{[X(t) \pm X(t+\tau)]^2\} \geq 0$

because mean square value will always be non-negative

$$E[X^2(t)] \pm 2E[X(t) \cdot X(t+\tau)] + E[X^2(t+\tau)] \geq 0$$

$$R_{xx}(0) \pm 2R_{xx}(\tau) + R_{xx}(0) \geq 0$$

$$2R_{xx}(0) \geq |2R_{xx}(\tau)|$$

So, $R_{xx}(0) \geq |R_{xx}(\tau)|$

Mean and Variance of Poisson Distribution

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{where } \lambda = \text{mean} = np \text{ where } n \to \infty, \ p \to 0$$

Mean :- $E[X] = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

$$= e^{-\lambda}\lambda \cdot \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

Taylor's expansion

$$e^{\lambda} = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \cdots$$

$$= \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{\lambda}$$

So $\cancel{e^{-\lambda}}\lambda \cdot \cancel{e^{\lambda}}$

$E[X] = \lambda$

Variance :- $\sigma^2 = E[X^2] - E[X]^2$

$$E[X^2] = \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{(x-1)!} = e^{-\lambda} \sum_{y=0}^{\infty} \frac{(y+1)\lambda^{y+1}}{y!}$$

$$= e^{-\lambda} \sum_{y=0}^{\infty} \frac{y\lambda^{y+1}}{y!} + e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y+1}}{y!}$$

$$= \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{y \cdot \lambda^y}{y!} + e^{-\lambda}\lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

$$= \lambda e^{-\lambda}[\lambda \cdot e^{\lambda}] + e^{-\lambda}\lambda[e^{\lambda}]$$

$$= \lambda^2 + \lambda$$

variance $= \lambda^2 + \lambda - [\lambda^2] = \lambda$

So, mean = variance for a Poisson Distribution

Que:- Let X and Y be two jointly continuous RV with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find c

b) Find $P(0 \le x \le \frac{1}{2}, 0 \le y \le \frac{1}{2})$

Sol^n $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$

$$\int_0^1\int_0^1 x+cy^2\,dx\,dy$$

$$\int_0^1 \left[\frac{x^2}{2}+cy^2x\right]_0^1 dy = \int_0^1 \left[\frac{1}{2}+cy^2\right]_0^1 dy$$

$$= \left[\frac{1}{2}y+\frac{cy^3}{3}\right]_0^1 = \frac{1}{2}+\frac{c}{3}$$

$$\frac{1}{2}+\frac{c}{3}=1$$

$$\frac{c}{3}=1-\frac{1}{2}$$

$$\frac{c}{3}=\frac{1}{2} \quad \boxed{c=\frac{3}{2}}$$

(b) $P(0\le X\le \frac{1}{2}, 0\le Y\le\frac{1}{2}) = \int_0^{1/2}\int_0^{1/2}(x+\frac{3}{2}y^2)\,dx\,dy$

$$= \int_0^{1/2}\left[\frac{1}{2}x^2+\frac{3}{2}y^2x\right]_0^{1/2} dy$$

$$\int_0^{1/2} \frac{1}{8}+\frac{3}{4}y^2\,dy = \left[\frac{1}{8}y+\frac{3}{4}\cdot\frac{y^3}{3}\right]_0^{1/2} = \frac{1}{16}+\frac{1}{4}\cdot\frac{1}{8} = \frac{3}{32}$$

So, $\boxed{P(0\le X\le\frac{1}{2}, 0\le Y\le\frac{1}{2}) = 3/32}$

Q:- Find marginal densities $f_X(x)$ & $f_Y(y)$ in the above question

a) for $f_X(x)$ $\quad f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^1 (x+\frac{3}{2}y^2)\,dy = [xy+\frac{1}{2}y^3]_0^1$

$$= x+\frac{1}{2}$$

$$f_X(x)=\begin{cases} x+\frac{1}{2} & 0\le x\le 1 \\ 0 & \text{otherwise}\end{cases}$$

IIely for $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

$$= \int_0^1 x+\frac{3}{2}y^2\,dx$$

$$= \left[\frac{1}{2}x^2+\frac{3}{2}y^2x\right]_0^1$$

$$= \frac{3}{2}y^2+\frac{1}{2}$$

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2+\frac{1}{2} & 0\le y\le 1 \\ & \text{otherwise}\end{cases}$$