CS – 201

Computational Physics

Assignment 8

Group 17

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Winter 2017-18

(a) Hypothetical solar system

Assumptions:

- Orbits of the planets are assumed to be circular instead of elliptical. In reality, the orbit of planets is elliptical with sun at one focal point.

- Initially, eccentricity of the path is taken 0.02.

- Motion is assumed to be planar.

- Characteristic time scale is taken as 1 year (3.2*10^7 s) and length scale as 1 AU (1.5*10^11 m). Hence velocity of earth is 2*pi.

- Centripetal force of the planet balances the gravitational force between planet and Sun.

- Sun’s mass is sufficiently large to neglect motion of the sun.

Mathematical Model:

$$
\begin{aligned}
v_{x,i+1} &= v_{x,i} - \frac{4\pi^2 x_i}{r_i^3}\Delta t \\
x_{i+1} &= x_i + v_{x,i+1}\Delta t \\
v_{y,i+1} &= v_{y,i} - \frac{4\pi^2 y_i}{r_i^3}\Delta t \\
y_{i+1} &= y_i + v_{y,i+1}\Delta t\,,
\end{aligned}
$$

Computational Model:

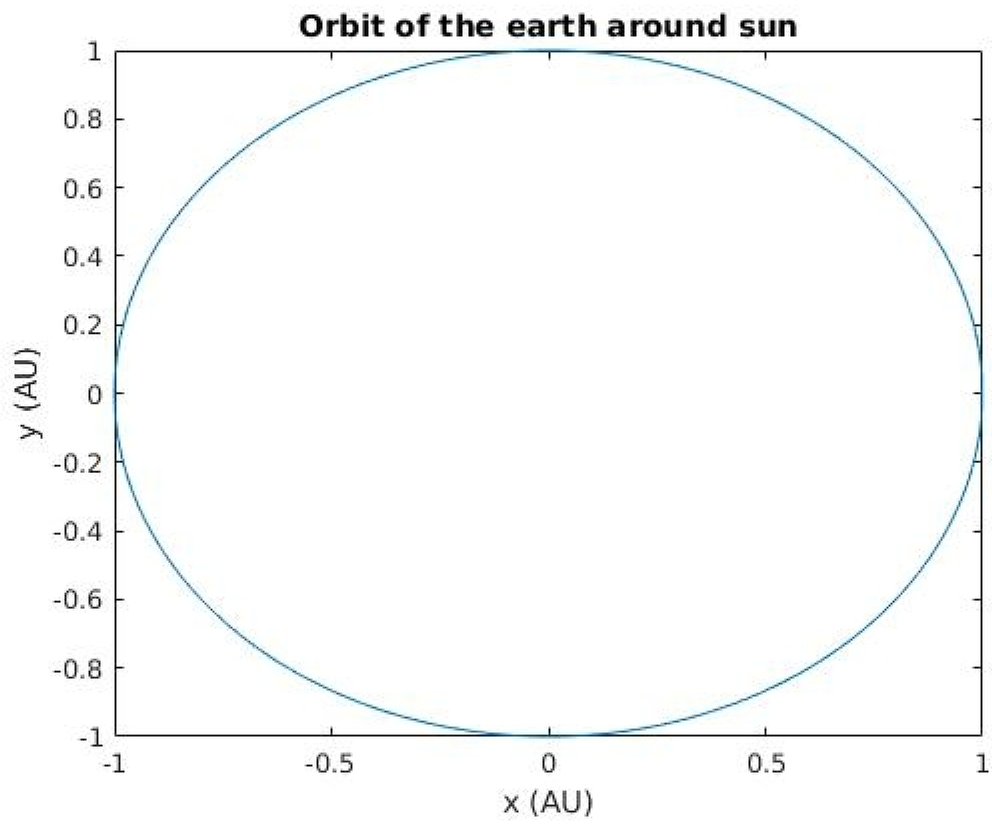
Euler-Cromer method is used to compute this problem as using normal Euler method would not work since errors will keep accumulating with each iteration and the total energy of the planet will keep on increasing. This is similar to the problem faced in a driven pendulum.

At each time step *i,* calculate the position (x,y) and the velocity (v_x ,v_y) for time step *i*+1 using Euler-Cromer method.

- Calculate the distance r_i from the sun,

$$r_i = (x_{i2}+y_{i2})^{1/2}$$

- Compute $v_{x,i+1}$ and $v_{y,i+1}$ using the above formulae



For all planets other than Mercury and Pluto, using the equation,

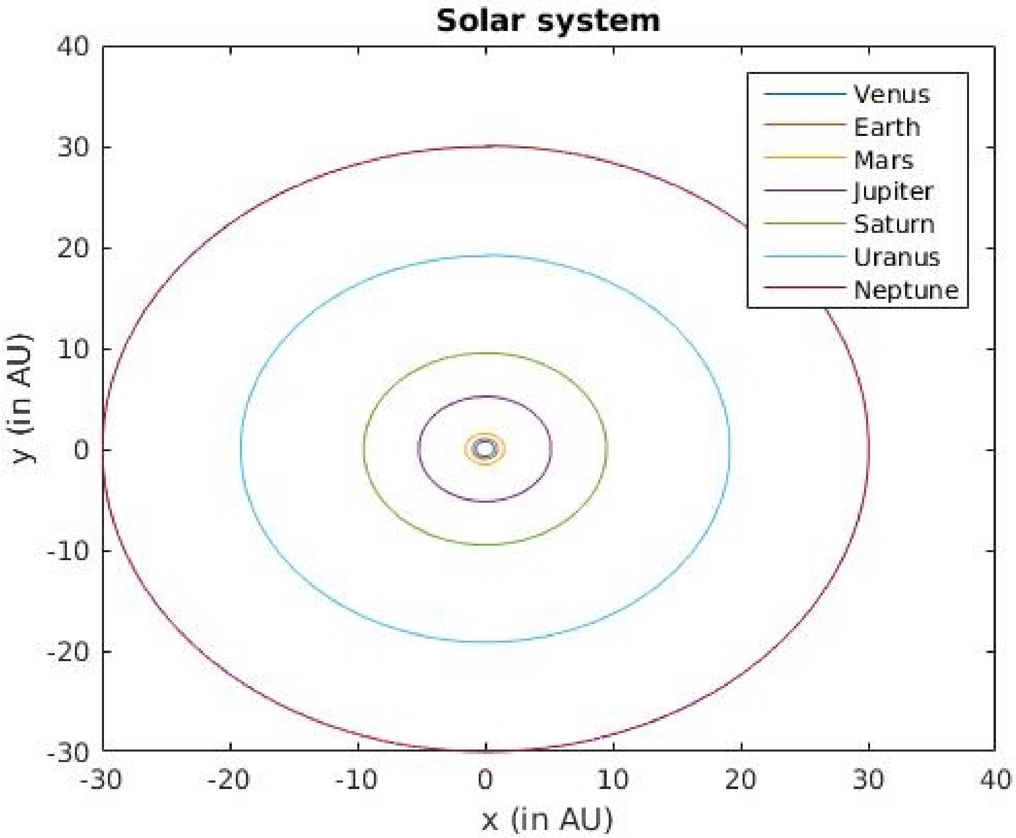
$mv^2/R = Gmm/R$,

$v\_init = (GM/R)^{1/2}$

$a_x = -Gm_s x/R$;
$a_y = -Gm_s y/R$;

We calculate the time period by checking when the planet crosses its original position again.

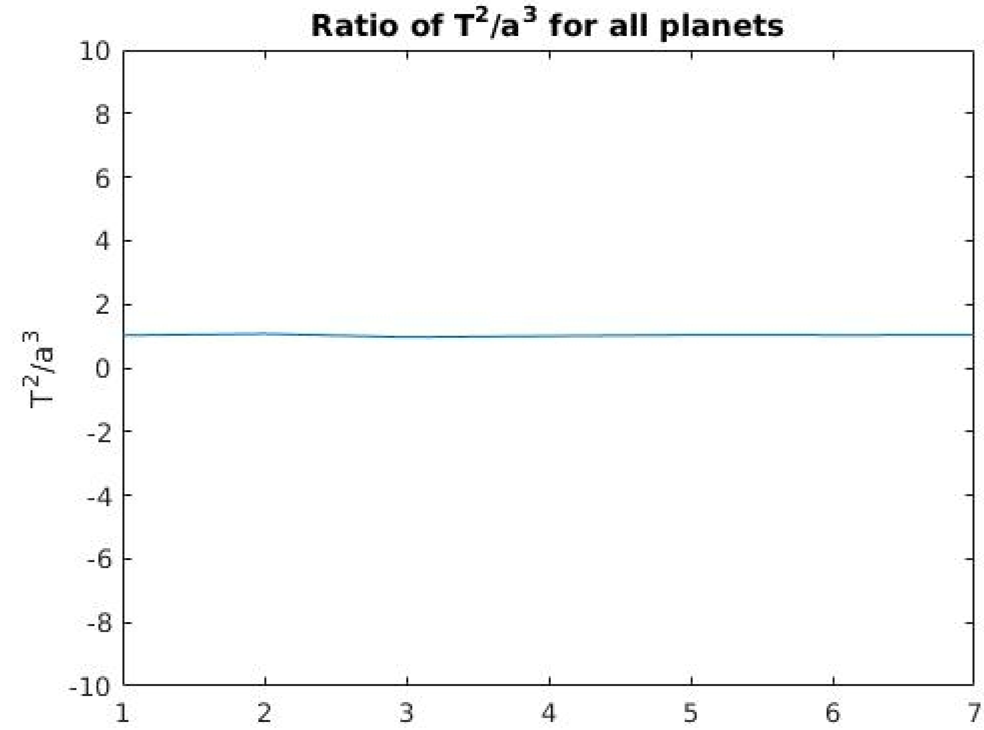
Planet	**Mass (kg)**	**Orbit (distance in AU)**	**Time Period (years) actual**	**Time Period (years) calculated**
Mercury				
Venus	4.87 e24	0.72	0.62	0.6204
Earth	5.97 e24	1	1	1.0349
Mars	0.64 e24	1.52	1.88	1.8498
Jupiter	1898 e24	5.20	12	11.8791
Saturn	568 e24	9.54	29	29.9111
Uranus	86.8 e24	19.18	84	85.2862
Neptune	102 e24	30.06	165	167.3133



Using the above method to calculate the timeperiod, we can confirm Keplar’s 3rd law by calculating the ratio T^2/a^3 for each planet :

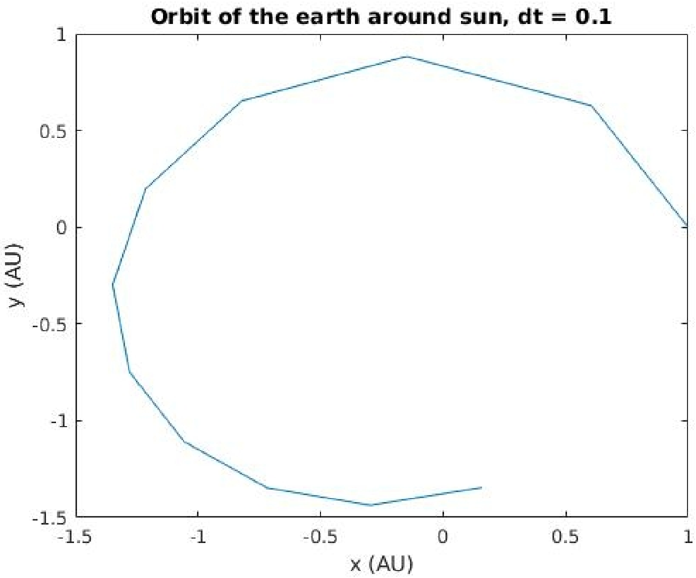
Confirmation of the Kepler’s Third Law

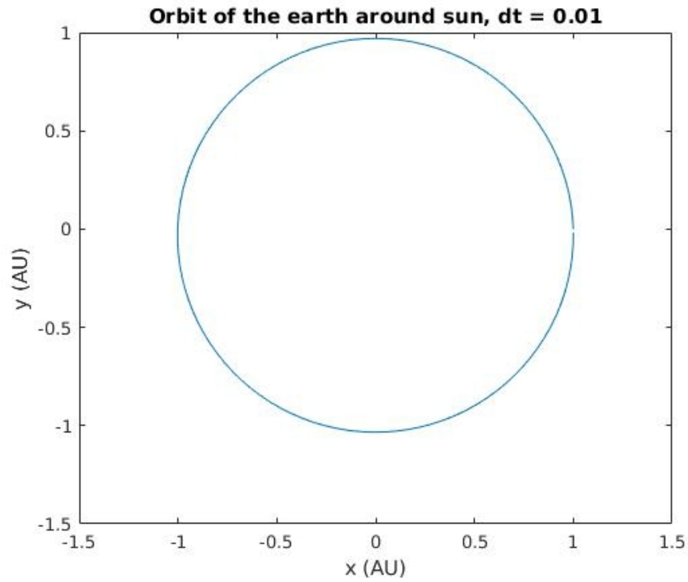
Planet	**T^2/a^3 (calculated)**
Venus	1.0178
Earth	1.0674
Mars	0.9632
Jupiter	0.9982
Saturn	1.0309
Uranus	1.0255
Neptune	1.0270



The table consists of values calculated of T^2/a^3 for all planets except Mercury and Pluto. All the orbits are nearly circular, so a is nearly equal to the orbital radius mentioned in table 1. According to Kepler's third law, T^2/a^3 should be constant (1 for Astronomical units). The given values are, hence, in accordance with Kepler's Third Law of Planetary Motion.

For different time steps:





As seen from the above graphs, for larger time steps, the orbit is not perfectly circular and is incomplete.

2) Orbits:

Assumptions:

- Motion is assumed to be happening in 2D-plane.

- Initial velocity magnitude - this value must be less than or greater than that of a
  circular orbit, up to the limiting value of the velocity of a parabolic orbit, where
  the kinetic energy of motion is equal to the gravitational energy of gravity

  KE=PE ==> $mv^2/2 = GMm/R$ ==> v (parabolic) = $(2GM/R)^{1/2}$

- Sun's mass is sufficiently large to neglect motion of the sun.

- We assume that the Sun is stationary at one focus of the elliptical orbit.

The direction of the gravitational force between planet and the sun is along the line joining the two bodies.

Velocity and Acceleration of the planet:

For a Circular Orbit:

- **Direction of velocity:** Tangential
- **Magnitude of the velocity:** $v=\sqrt{(GMr)}$
- **Direction of acceleration:** Radial
- **Magnitude of the acceleration:** $a = GM/r_2$

For Elliptical Orbit: The velocity will no longer be constant but will change with radius from earth. Kepler's Laws still control the motion.

- **Direction of velocity:** Tangential

- **Magnitude of the velocity:** changes at every instant

- **Direction of acceleration:** Radial

- **Magnitude of the acceleration:** changes at every instant

Circular Orbits –

On absence of external torque, the planet undergoes uniform circular motion if its **velocity is $\sqrt{(GM_{sun}/R)}$**. We can see that in the plot of the potential energy vs. time. We notice that Kinetic Energy remains a constant. The observation can be made for the potential energy because the object remains at a constant distance from the center of mass i.e. Sun.

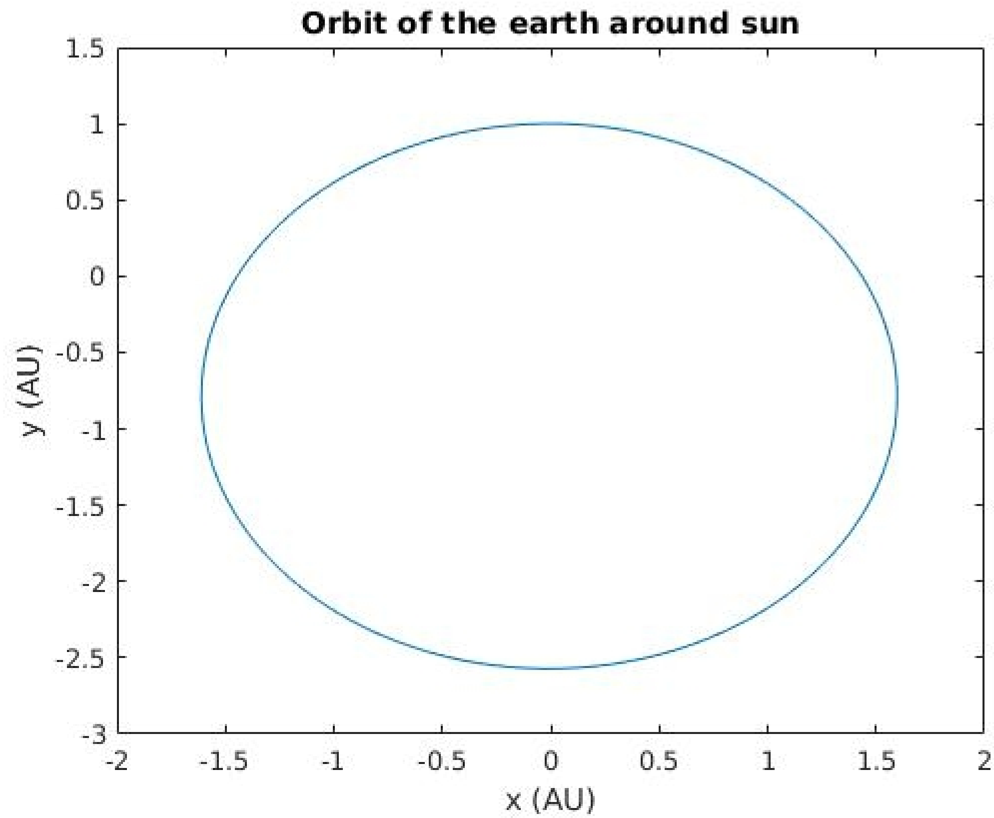
As KE and PE are both constants, we can conclude that the Total energy is also a constant on absence of an external torque.

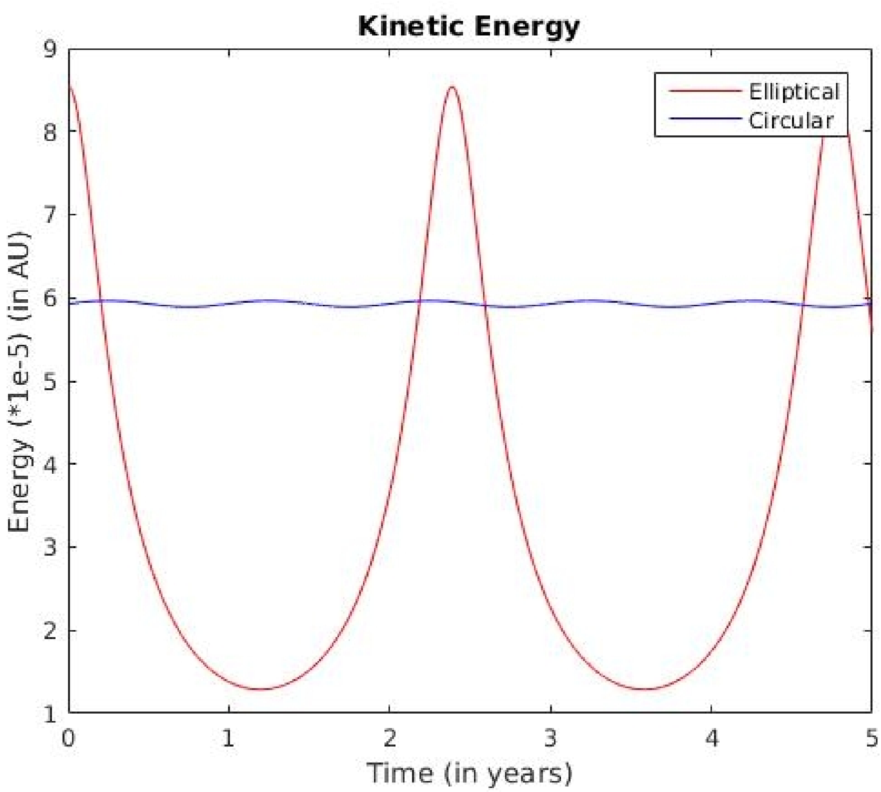
Elliptical Orbits -

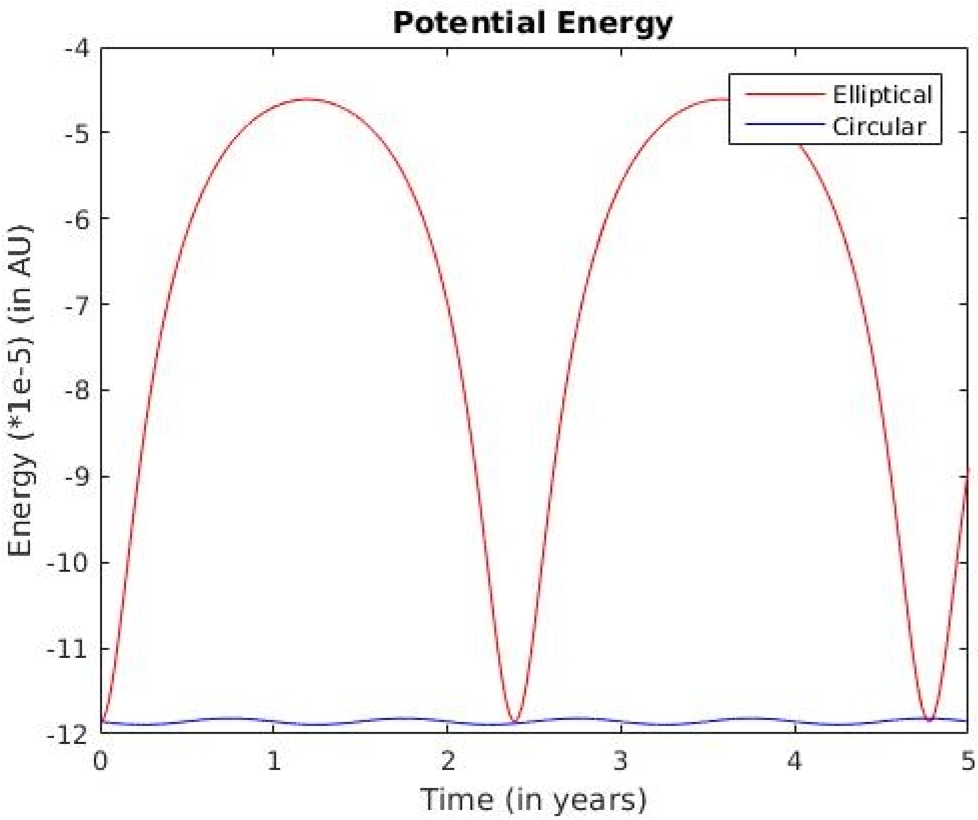
For a lower velocity or a slightly higher velocity, we see that the planet undergoes an elliptical motion, with the Sun lying at one of the foci of the elliptical orbit. Unlike in the case of the circular orbit, the velocity in this case constantly changes. Thus, the Kinetic energy also varies in a periodic manner. But at the same time whenever the velocity increases, the radius of orbit decreases, thereby lowering the Potential energy i.e. making it more negative. Therefore, when KE increases, PE decreases accordingly so that the value of total energy remains constant.

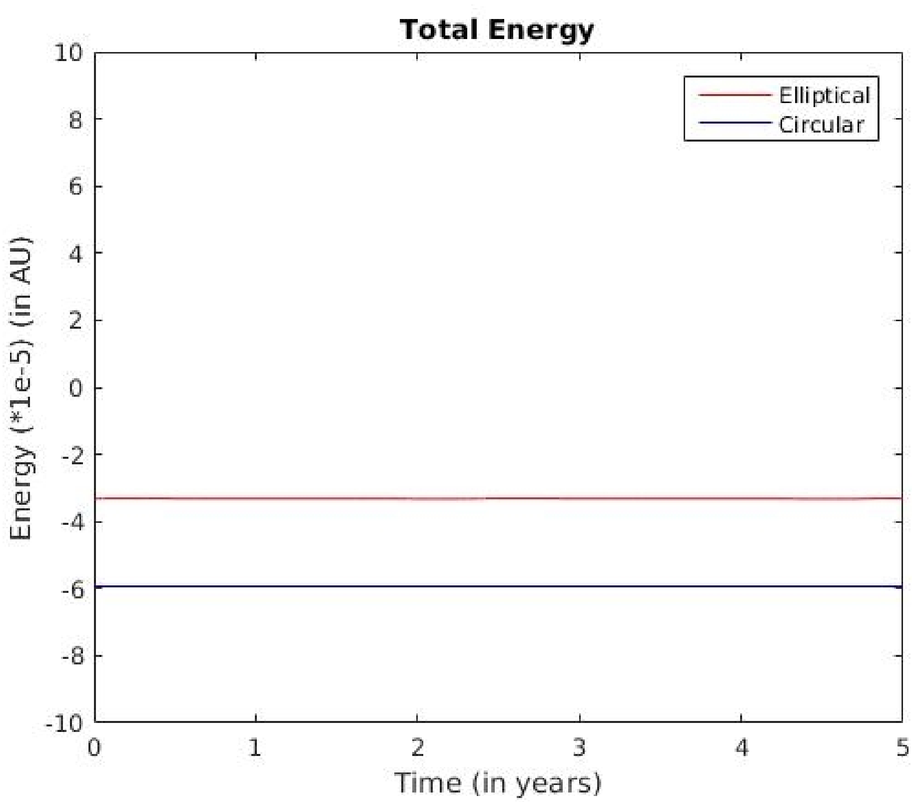
Therefore, we see that both the orbits follow the law of conservation of energy.

For v_init = 1.2 * v_init$_{circular}$ gives us the following elliptical orbit for the Earth around the Sun,






Potential Energy
Elliptical
Circular
Energy (*1e-5) (in AU)
-4
-5
-6
-7
-8
-9
-10
-11
-12
0
1
2
3
4
5
Time (in years)


Total Energy
Elliptical
Circular
Energy (*1e-5) (in AU)
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
Time (in years)

According to Kepler's Second Law, the radius vector sweeps equal area in equal intervals of time. Hence, we can infer from this that when the planet is nearer to the Sun, its velocity is less than when it is farther. Also, from Kepler's Second Law,

$$dA/dt = l2/2m$$

For small displacement, $A=rdx$ where r is the radius at that point.

$$\therefore rdx/dt = l_2/2m$$

$$\therefore dx/dt = l2/2mr$$

$$\therefore v = l_2/2mr$$

$$\therefore v \propto 1/r$$

This equation is in accordance with our claim that the velocity of the planet is inversely proportional to its distance from Sun.

In case of circular orbit, planet's distance from the Sun remains constant and hence its velocity also remains constant.

During equal time intervals, the radius vector from the sun to a planet sweeps out equal areas. What does this tell you about the angular momentum of the planets? What does this tell you about the motion of the planets/ planet's orbit?

Extending the above argument, let the position of the planet at any instant be r1 and velocity be v1. Let the position and velocity is r2 and v2 respectively at another instant of time. Then, from the previous relation between position and velocity,

$$r1v1=r2v2$$

$$\therefore m r_1 v_1 = m r_2 v_2$$

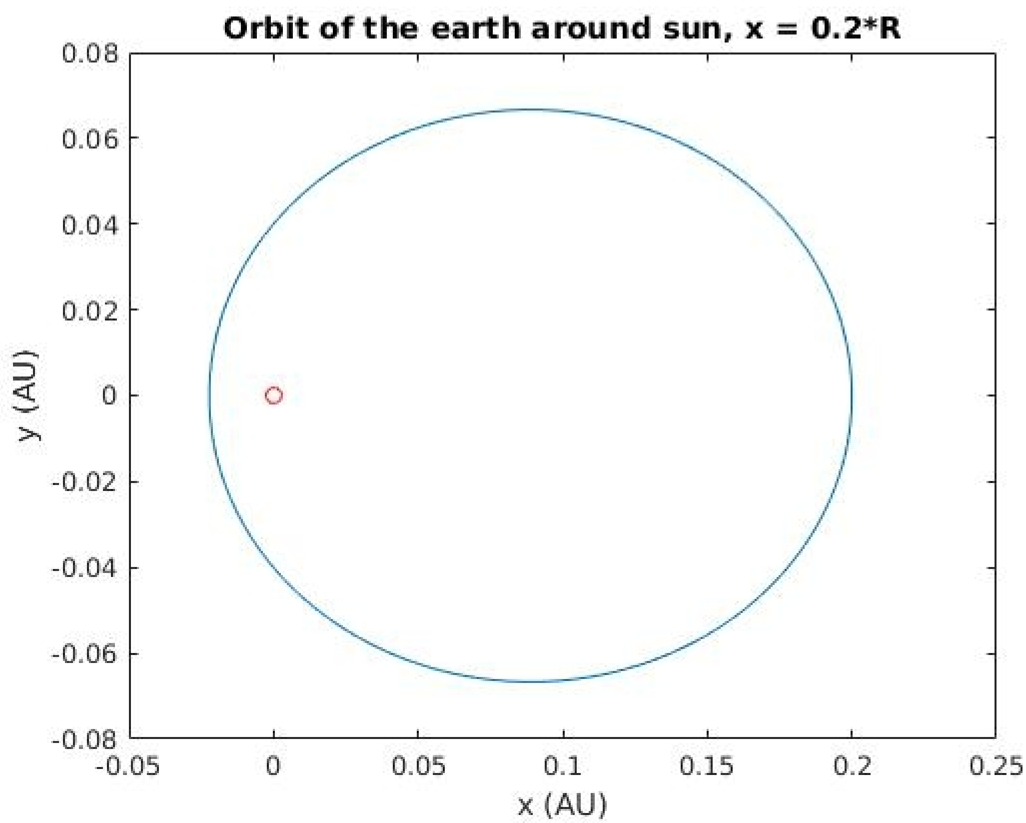
$$\therefore l_1 = l_2$$

∴ ***Angular momentum of the planet remains constant*.**

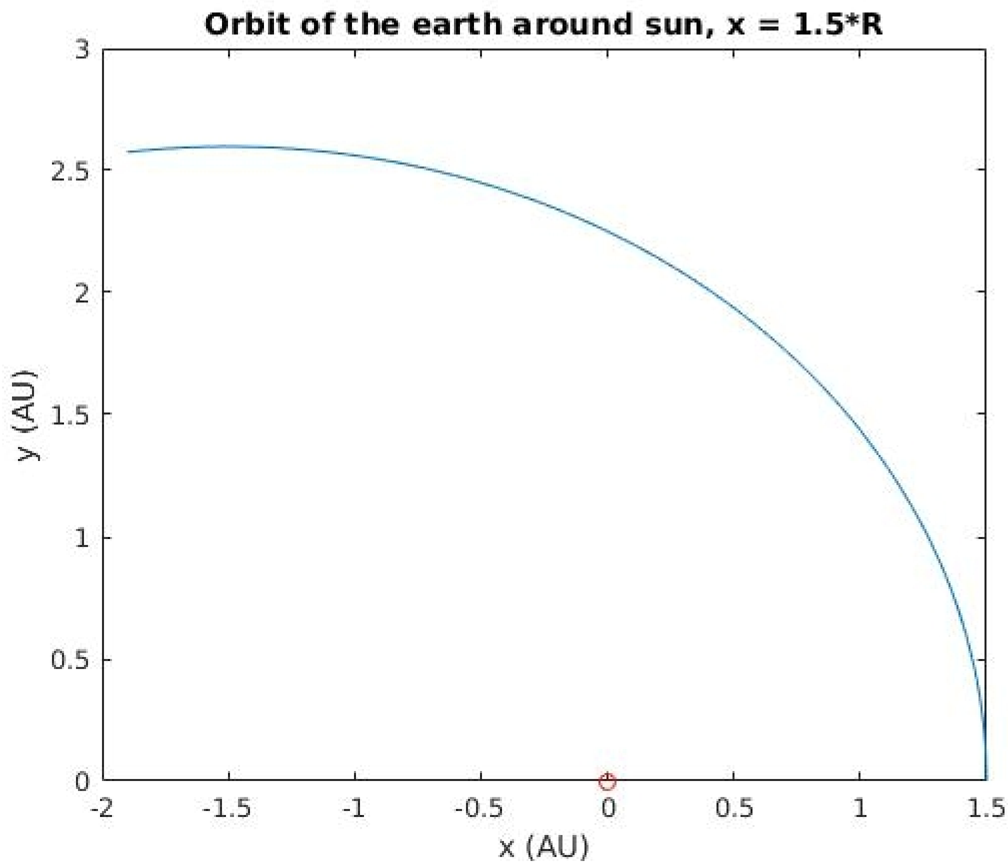
Here, the net force acting on the planet is the gravitation force and since the gravitation force and position vector are always in the same direction, the net torque due to gravitational force is zero and hence the angular momentum remains conserved which is in accordance with our observations.

■ Summarizing the observations:

● What happens to the orbit when x gets really small?

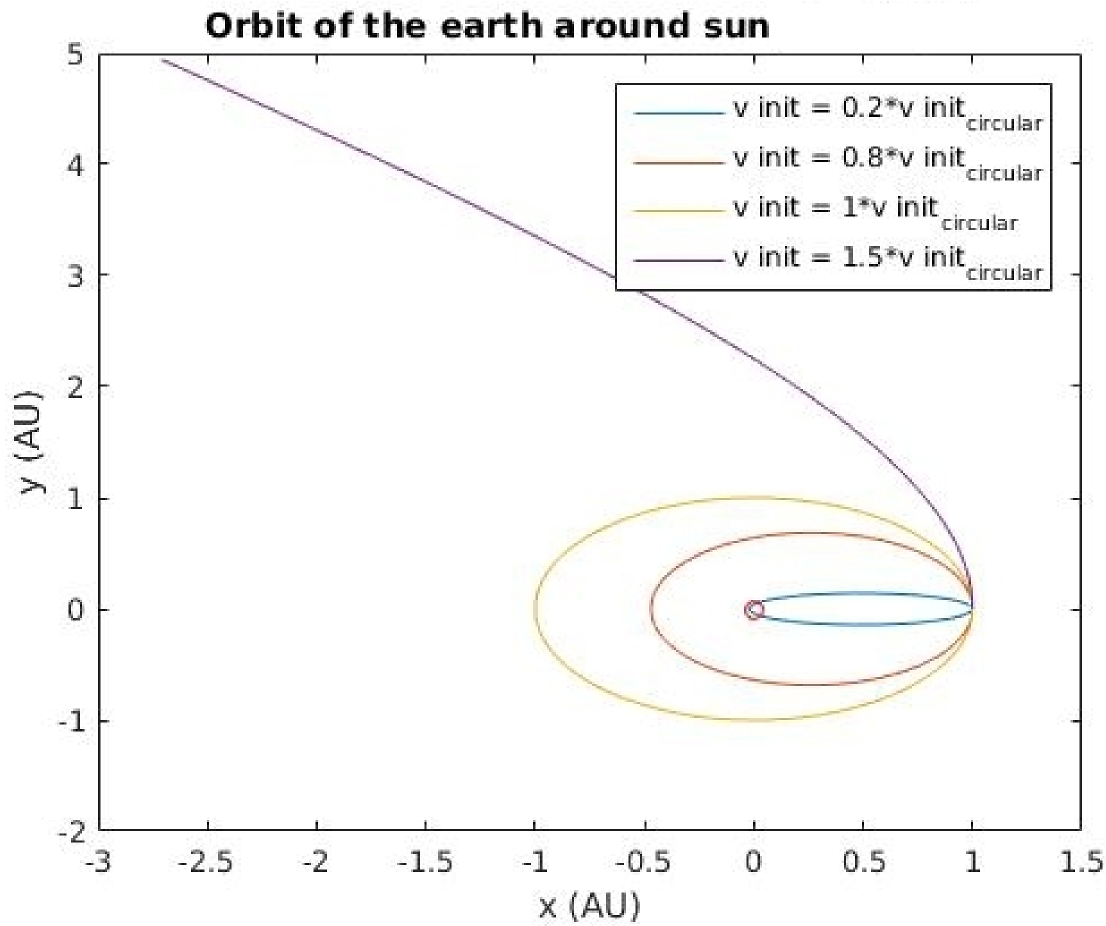


● What happens to the orbit when x gets really large?



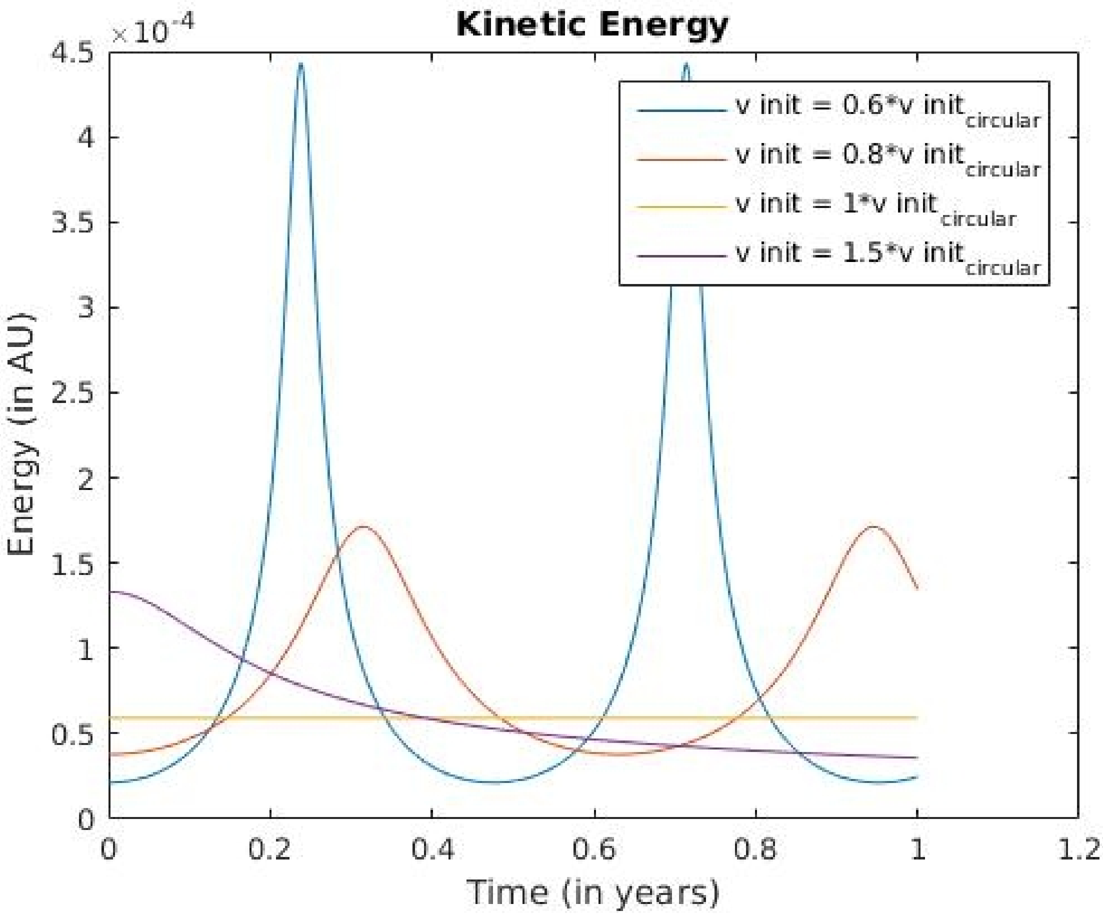
As seen in the figure above, if x is very large, time taken to complete one revolution will become very high. Also, the orbit becomes more elliptical (for above image, eccentricity ≅ 0.8771)

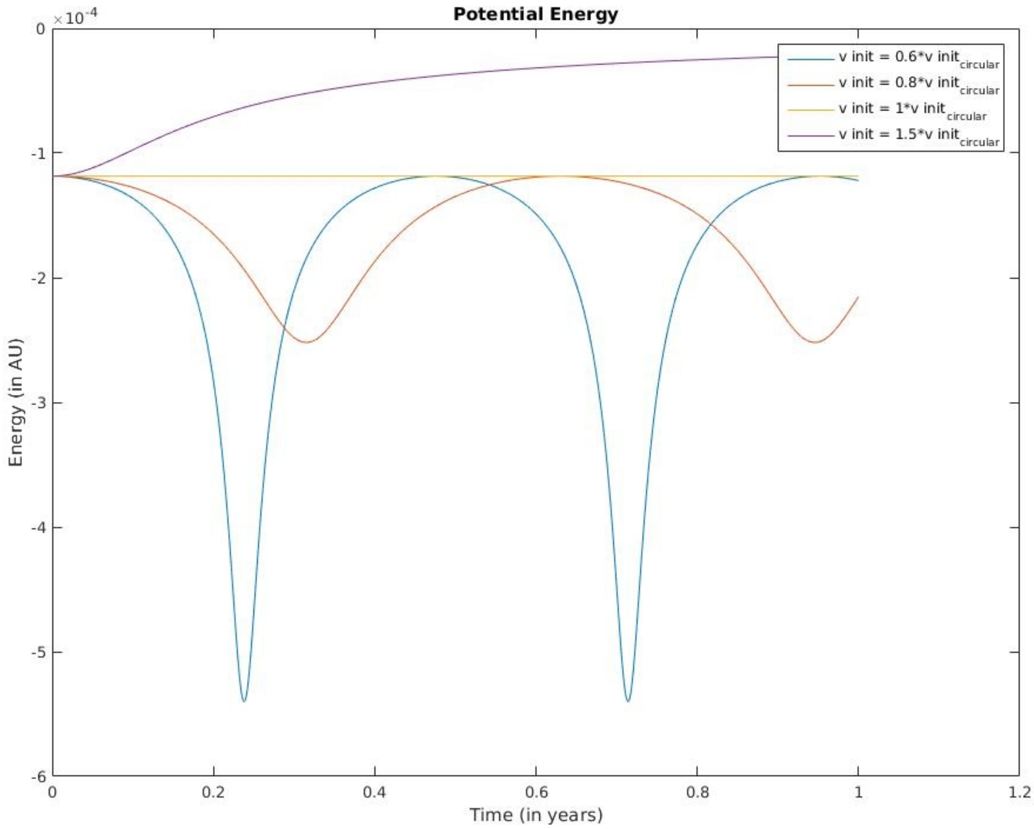
Now, as you vary the initial velocities of the planets, how do the orbital trajectories change?

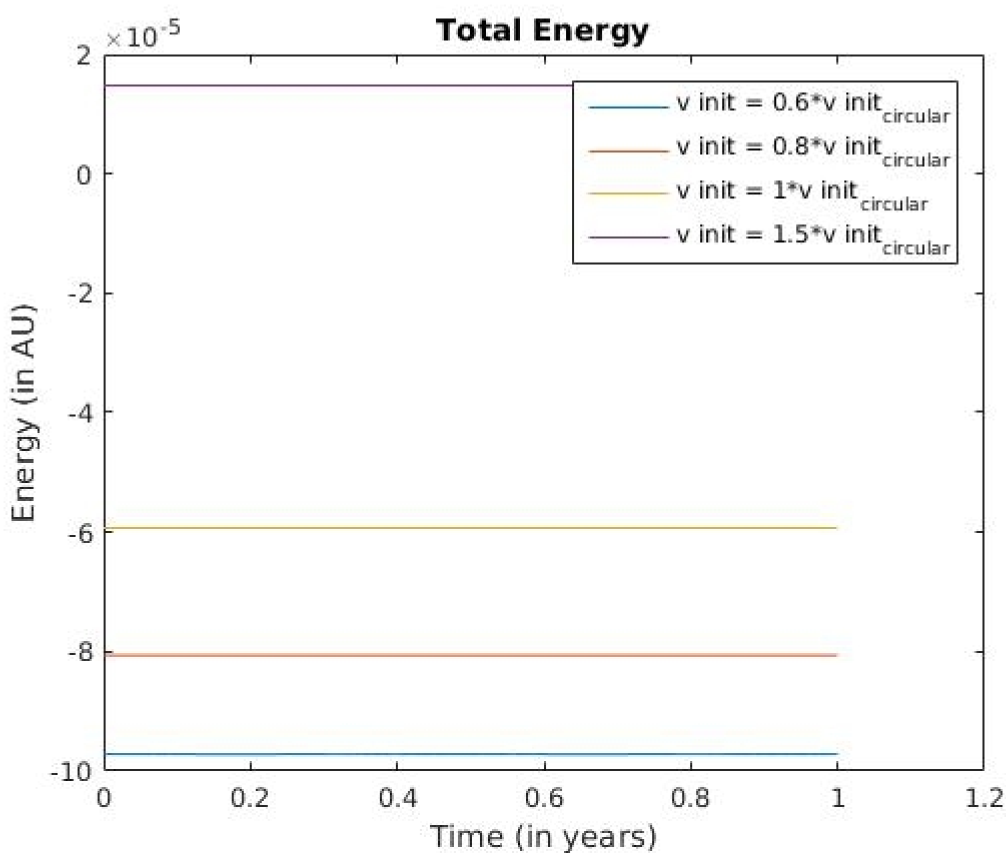


For the above graph, the initial velocity is very less. Hence, we can see that the planet will spiral into the Sun since it doesn't have the required energy to continue its motion.

How do the values for total energy change when the type of orbit is changed?




Potential Energy
×10^-4
v init = 0.6*v init_circular
v init = 0.8*v init_circular
v init = 1*v init_circular
v init = 1.5*v init_circular
Energy (in AU)
Time (in years)


Total Energy
×10^-5
v init = 0.6*v init_circular
v init = 0.8*v init_circular
v init = 1*v init_circular
v init = 1.5*v init_circular
Energy (in AU)
Time (in years)

From all the above graphs, we can see that the energy of the planet is the highest when the orbit is circular and when we change the orbit to elliptical (by decreasing the velocity), the total energy of the planet decreases.