CS 201
Computational Physics
Assignment 6

Group 17
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Winter Semester 2017-18

System : Loaded horizontal spring-mass system within elastic limits, follows Hooke's law and has force constant k. Stable position x=0.
Goal: investigate the motion of a body attached to a spring.
Assumptions :

1. The spring is mass less.

2. No stress or strain gets applied on the spring.

1

1.1 Problem Statement

Harmonic oscillator with no damping factor.

1.2 Assumptions

Same as original problem statement.

1.3 Mathematical Model

In case of a harmonic oscillator, the equation of motion is given by :

$$\frac{d^2x}{dt^2} + kx = 0 \tag{1.3.1}$$

on further simplification :

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{1.3.2}$$

We can solve this using :

$$x_{i+1} = x_i + v_i dt \tag{1.3.3}$$

$$v_{i+1} = v_i - \frac{k}{m}x_{i+1}..(ApplyingEulerCorrection) \tag{1.3.4}$$

But since this causes errors due to errors being added up, we decide to use ODE.

1.4 Analytical Solution

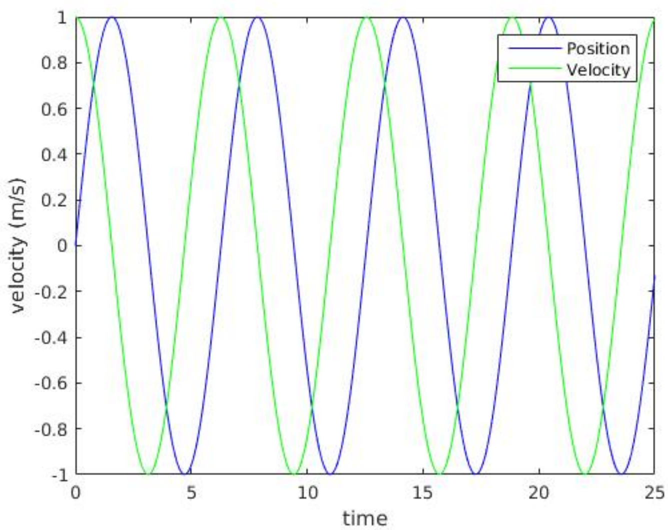
This equation is solved using :

$$x(t) = A * cos(\omega_o t + \phi) \tag{1.4.1}$$

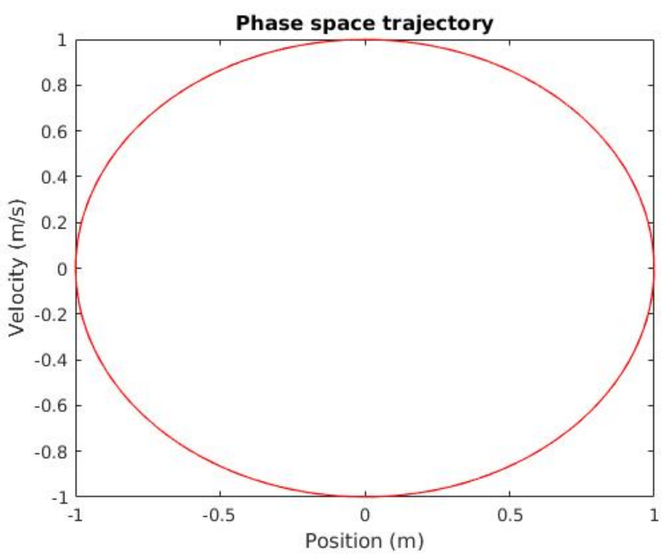
where, A is the maximum displacement of the block and

$$\omega_o = \sqrt{\frac{k}{m}} \tag{1.4.2}$$

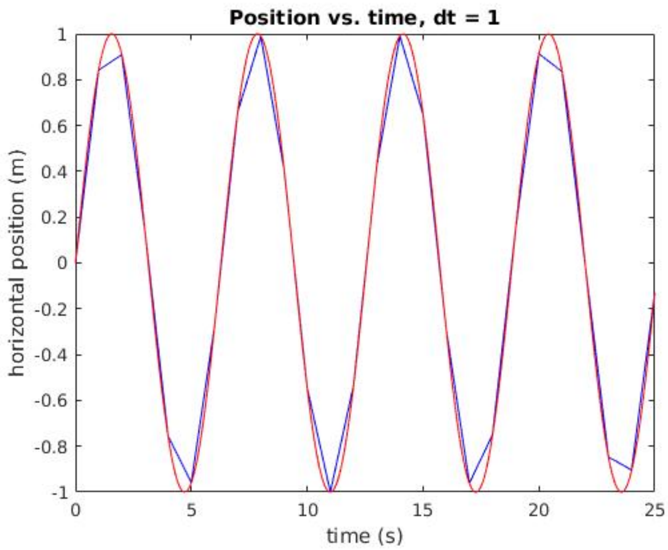
1.5 Results

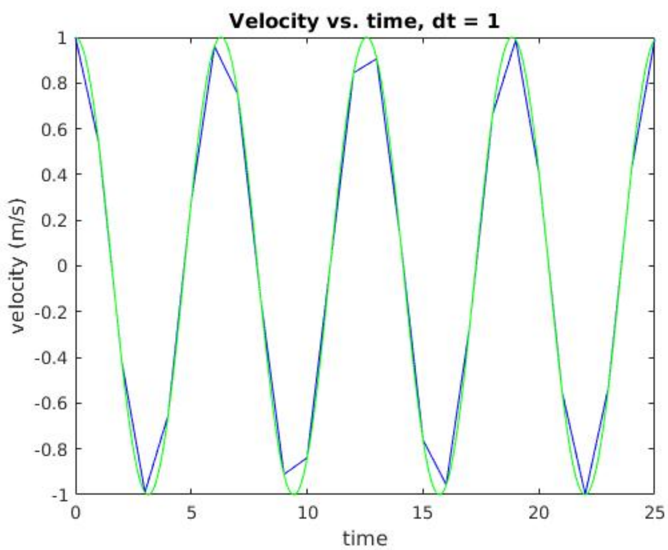


The position and velocity differ form each other in phase by $\frac{\pi}{2}$

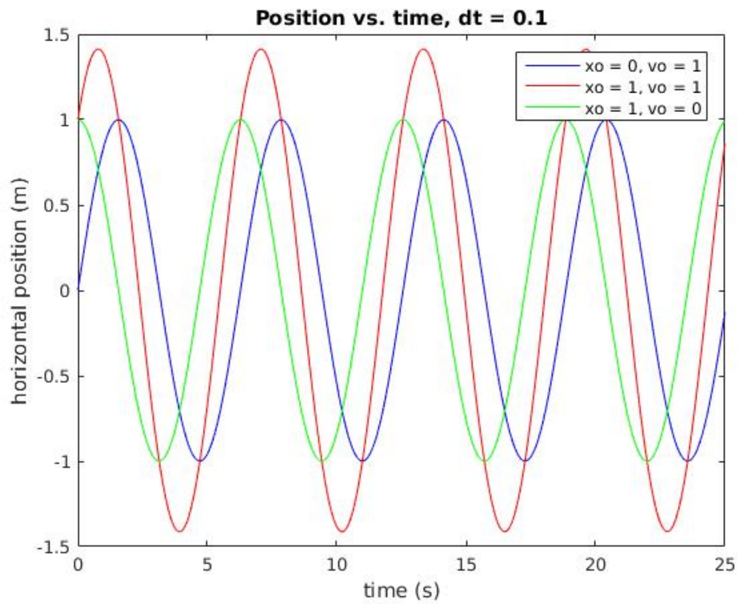


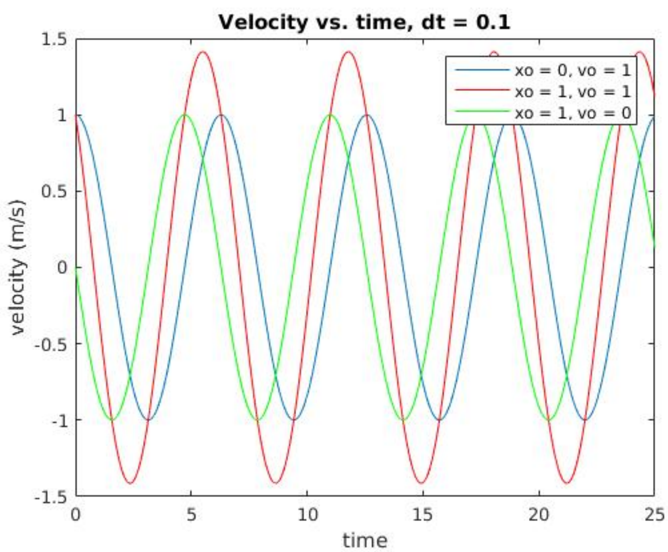
Effect of change of time step, comapared with trignometric functions :

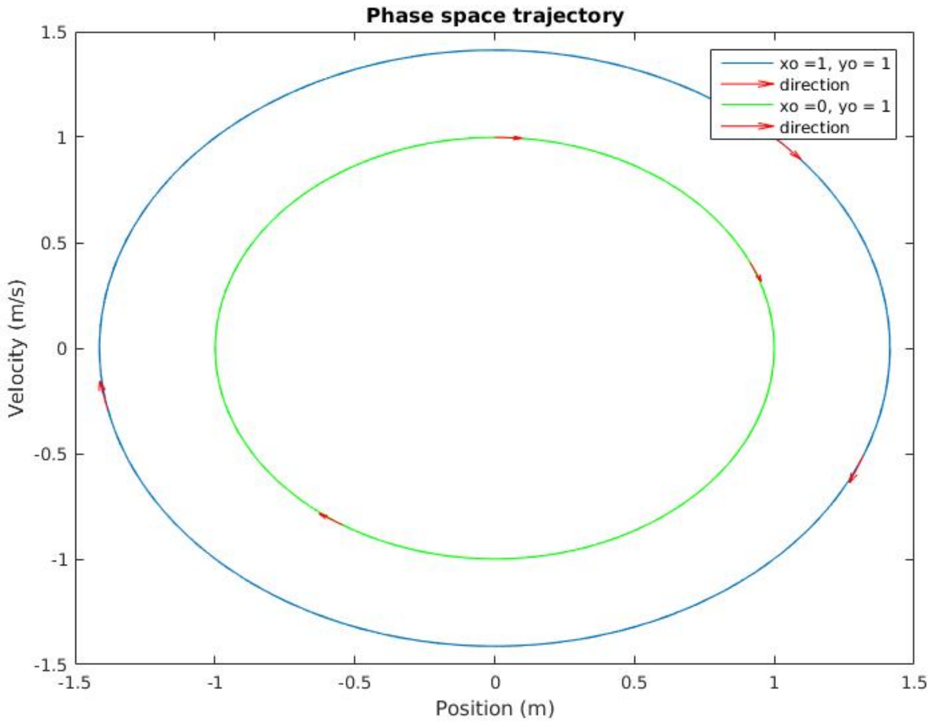




Variation of position and velocity with change in initial conditions:

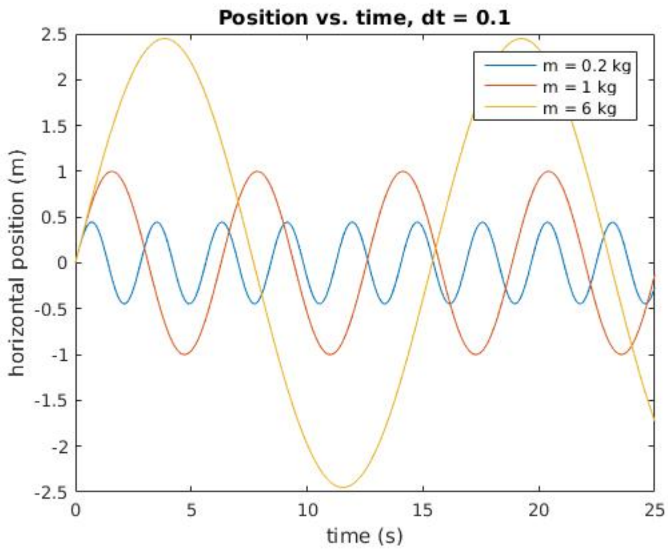

Position vs. time, dt = 0.1
xo = 0, vo = 1
xo = 1, vo = 1
xo = 1, vo = 0
horizontal position (m)
time (s)

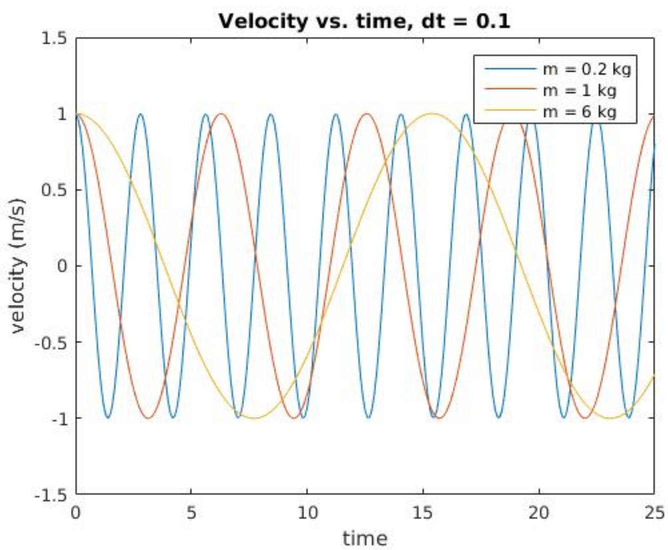

Velocity vs. time, dt = 0.1
xo = 0, vo = 1
xo = 1, vo = 1
xo = 1, vo = 0
velocity (m/s)
time

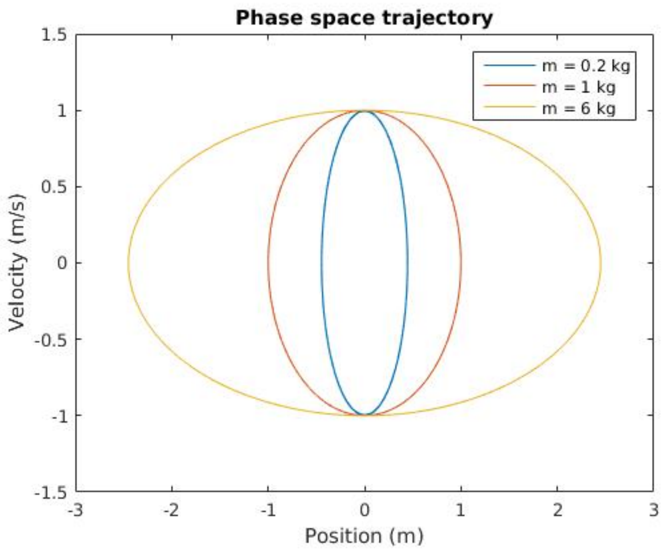


If the system has an initial displacement and initial velocity, the maximum amplitude is more, and energy of the system is more. If initial velocity is zero, the result shifts from initial displacement zero case by $\frac{\pi}{2}$ with same maximum amplitude.

Variation with m and k:

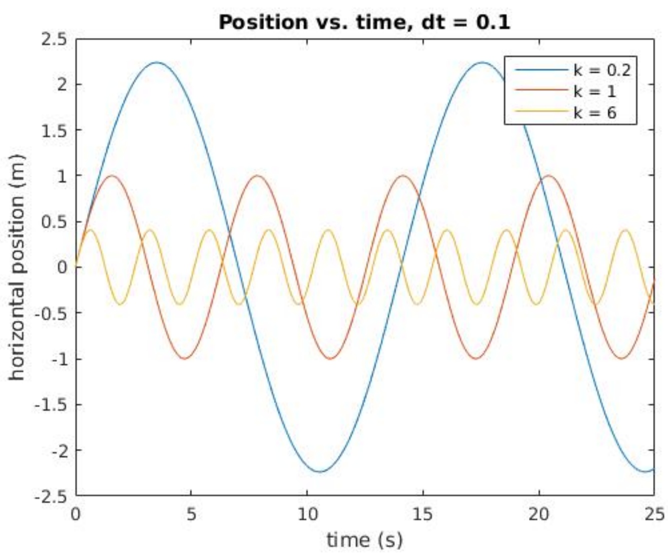

Position vs. time, dt = 0.1
horizontal position (m)
time (s)
m = 0.2 kg
m = 1 kg
m = 6 kg
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
5
10
15
20
25

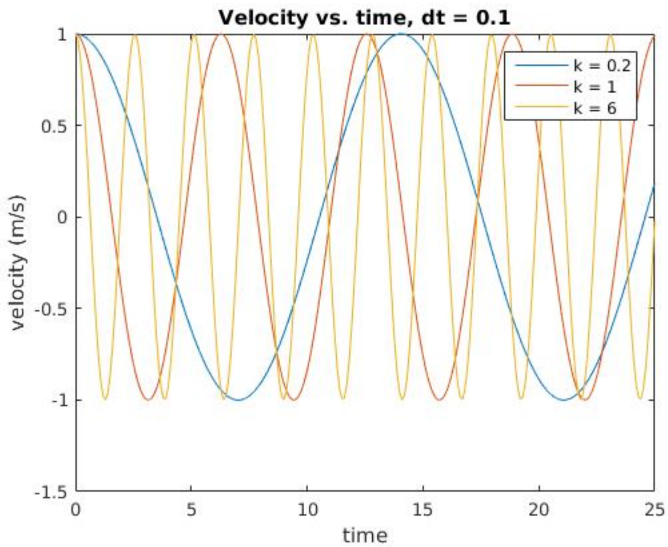

Velocity vs. time, dt = 0.1
velocity (m/s)
time
m = 0.2 kg
m = 1 kg
m = 6 kg
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25

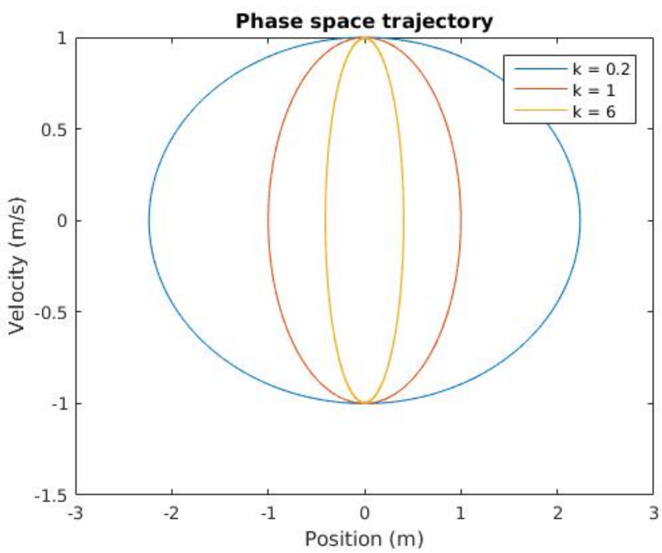


As mass increases, the maximum amplitude increases and hence energy of the system increases, and the frequency of oscillations decreases.

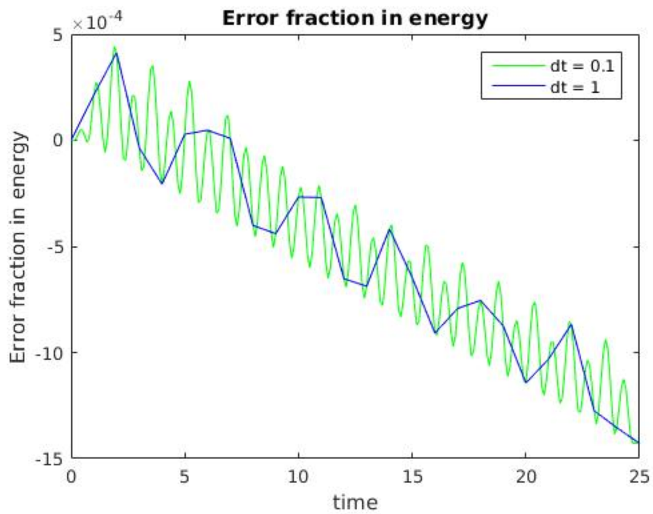
Variation with k:







As k increases, the maximum amplitude decreases and hence energy of the system decreases, and the frequency of oscillations increases.



2

2.1 Problem Statement

Damped Oscillator with c as damping coefficient.

2.2 Assumptions

Same as main statement.

2.3 Mathematical Model

We have the equation :

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{2.3.1}$$

Hence we will use the foll equations :

$$v_{t+1} = -(\frac{c}{m}v_t + kx_t)dt \tag{2.3.2}$$

$$x_{t+1} = x_t + v_t dt \tag{2.3.3}$$

2.4 Analytical Solution

we have,

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{2.4.1}$$

let

$$\omega = \sqrt{\frac{k}{m}} \tag{2.4.2}$$

$$\beta = \frac{c}{2m} \tag{2.4.3}$$

We have solution of the form,

$$x = Be^{rt} \tag{2.4.4}$$

On putting x in the differential equation :

$$r^2 + 2\beta r + \omega^2 = 0 \tag{2.4.5}$$

So the solution is :

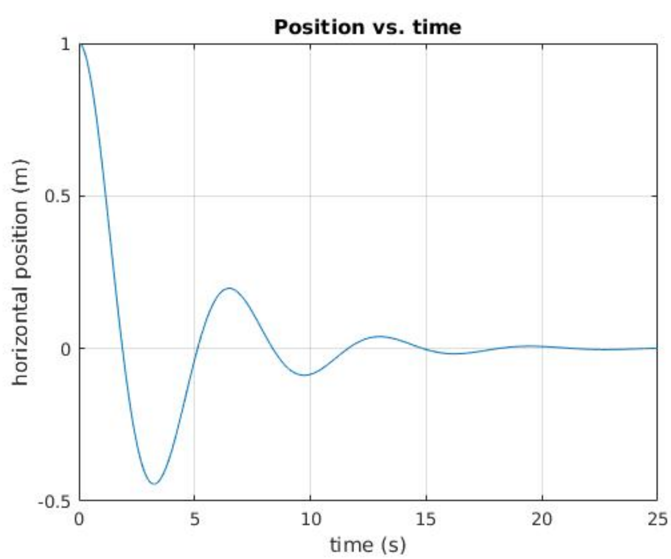
$$r = -\beta + \sqrt{\beta^2 - \omega^2} \tag{2.4.6}$$

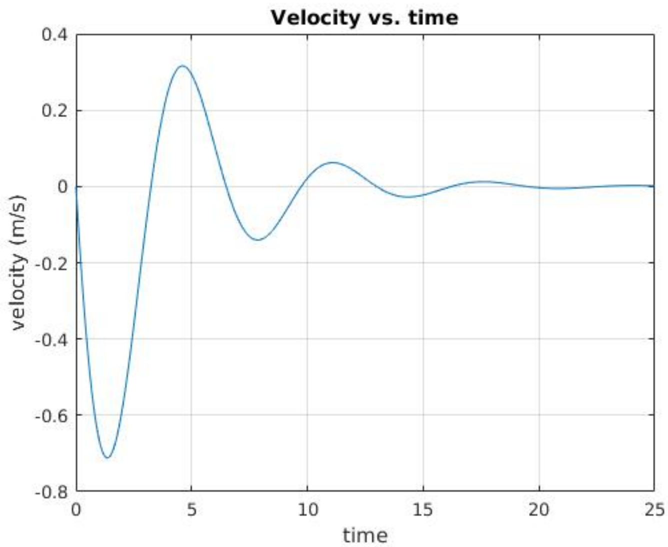
So there are three solutions :

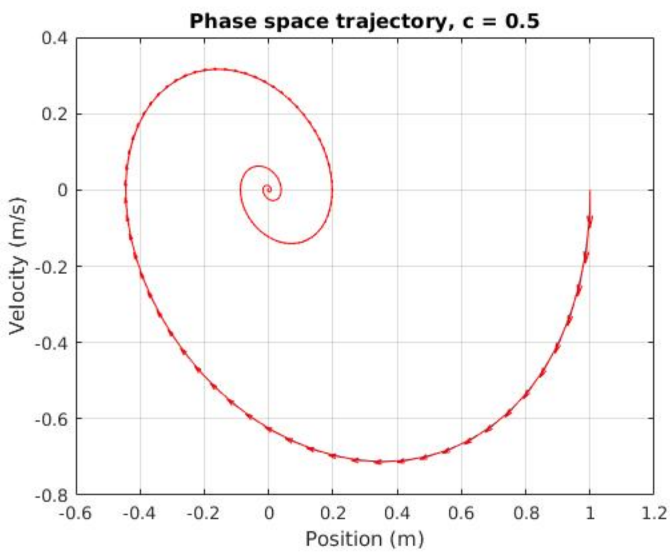
1. $\beta > \omega$, Overdamped
2. $\beta = \omega$, Critically damped
3. $\beta < \omega$, Underdamped

2.5 Results

For damping coefficient $c = 0.5$ -

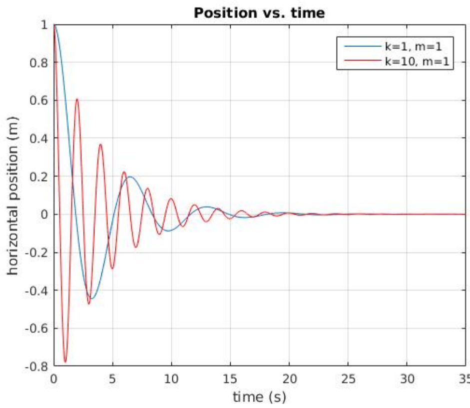


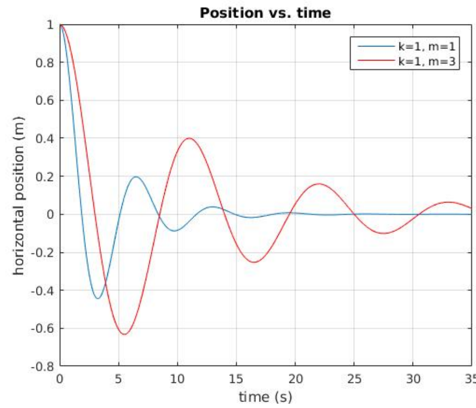




As seen from the graph, the phase space curve reaches the initial position, $x = 0, y = 0$ in the end.

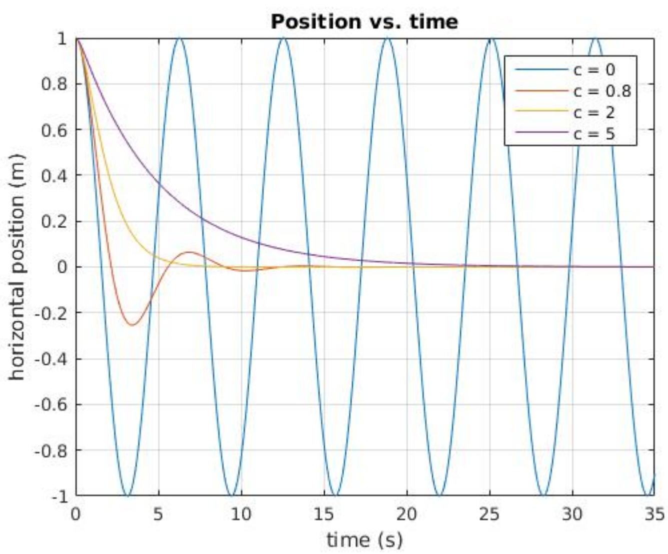
Different times for reaching equilibrium position -

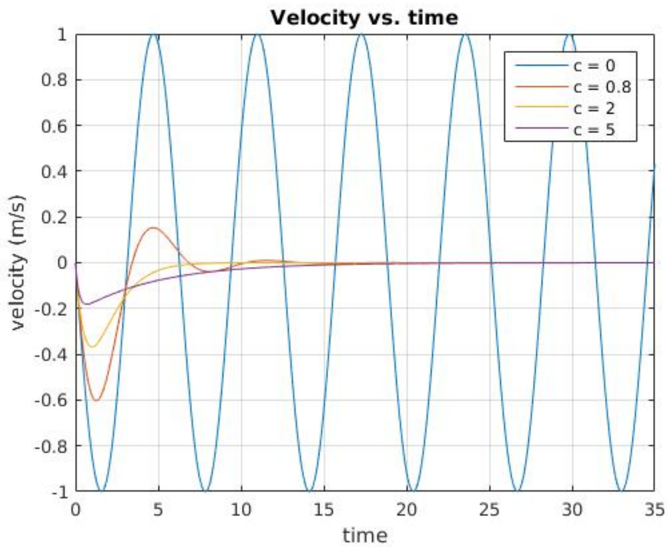


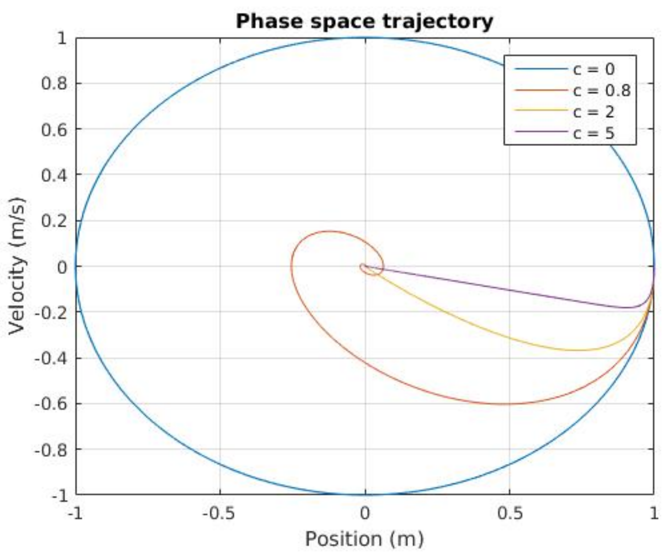


On varying the values of k and m in our code, the time taken to reach equilibrium increased as k and m increased, as seen above. The time taken is more heavily influenced by change in mass, as compared to change in k.

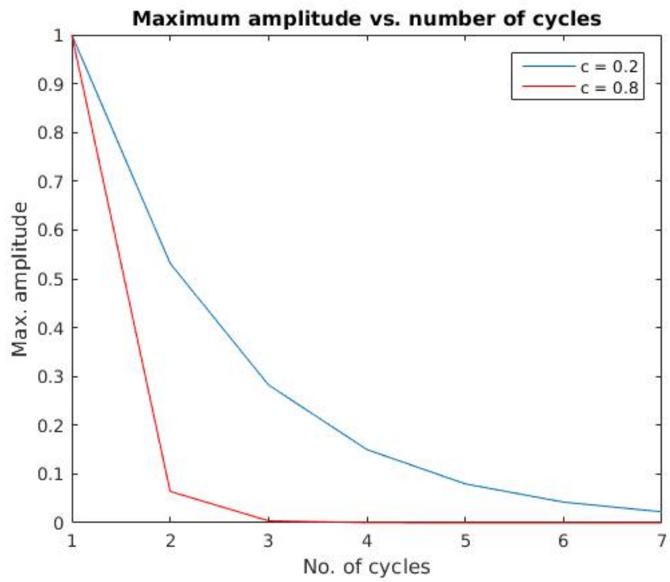
Variation with value of c (Damping constant) -







As seen from the graphs, at critical damping (c = 2) as shown by the yellow curve, the amplitude does not oscillate and reaches equilibrium the fastest.
The blue line indicates no damping.
The red curve indicates underdamped system. The amplitude oscillates initially then dies down to equilibrium.
The violet curve indicates overdamped system.



The above plot shows an exponential decay of the amplitude, over each passing cycle. As the damping coefficient comes nearer to critical damping, the amplitude reaches zero in lesser number of cycles. (the effect of damping the system is increasing).

3

3.1 Problem Statement

Spring Mass system with friction.

3.2 Assumptions

Same as main problem statement.

3.3 Mathematical Model

In this case we have :

For oscillator going from left to right :

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x - c \tag{3.3.1}$$

and for right to left :

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x + c \tag{3.3.2}$$

So we use :
    if v_t is not equal to zero :

$$v_{i+1} = -\frac{k}{m} x \Delta t + sgn(v_t) \Delta t + v_t \tag{3.3.3}$$

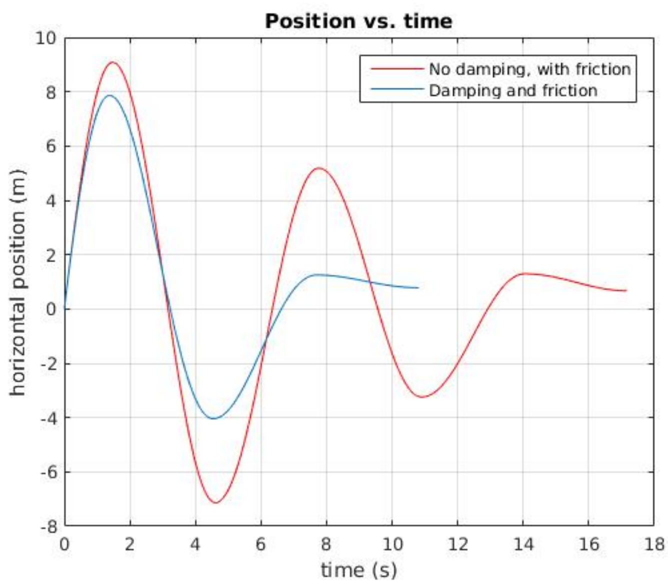
here

$$sgn(x) = \frac{|x|}{x} \tag{3.3.4}$$

if $v_t = 0$

$$v_{i+1} = -\frac{k}{m} x \Delta t - sgn(v_{t-1}) \Delta t + v_t \tag{3.3.5}$$

3.4 Results



4

4.1 Problem Statement

Forced oscillator

4.2 Assumptions

Same as original question.

4.3 Mathematical Model

We have :

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f \tag{4.3.1}$$

here $f = f_o cos(\omega t + \phi)$
So we use :

$$v_{t+1} = (\frac{f}{m} - (\frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x))dt + v_t \tag{4.3.2}$$

But as the error will get added up, we use an ODE.

4.4 Analytical Solution

We have :

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f \tag{4.4.1}$$

On solving, we get :

$$A^2 = \frac{f_o}{(\omega^2 - \omega_o^2)^2 + 4\beta^2\omega^2} \tag{4.4.2}$$

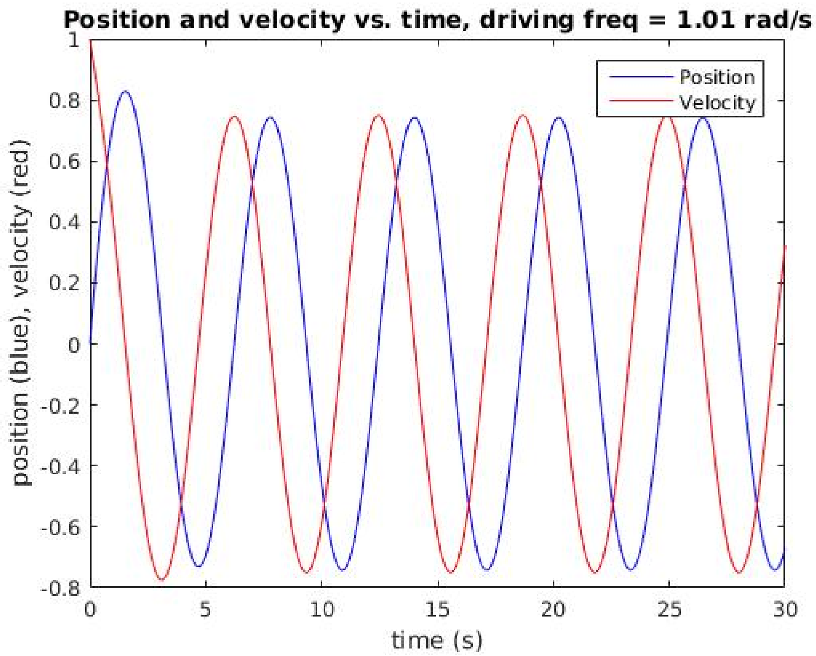
here ω_o is the natural frequency of the oscillator.

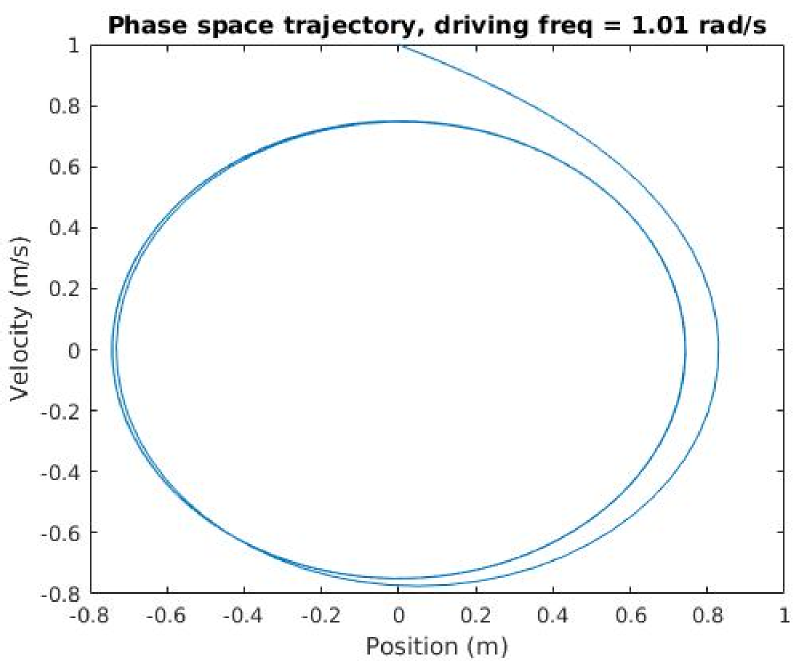
$$\delta = tan^{-1}\frac{2\beta\omega}{\omega^2 - \omega_o^2} \tag{4.4.3}$$

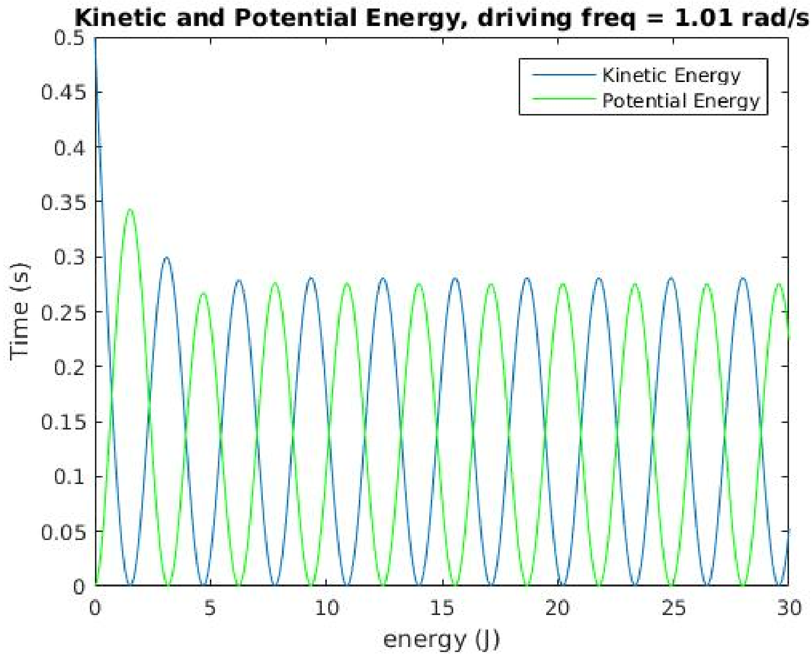
$$x = Ae^{i(\omega t - \delta)} \tag{4.4.4}$$

4.5 Results

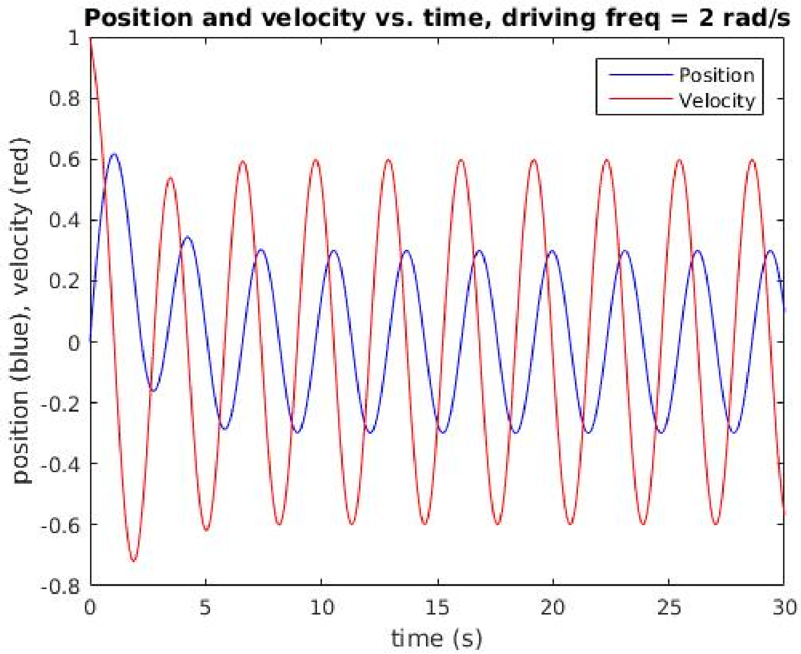
Initial conditions taken: $x_o = 0, v_o = 1, m = 1, k = 1$

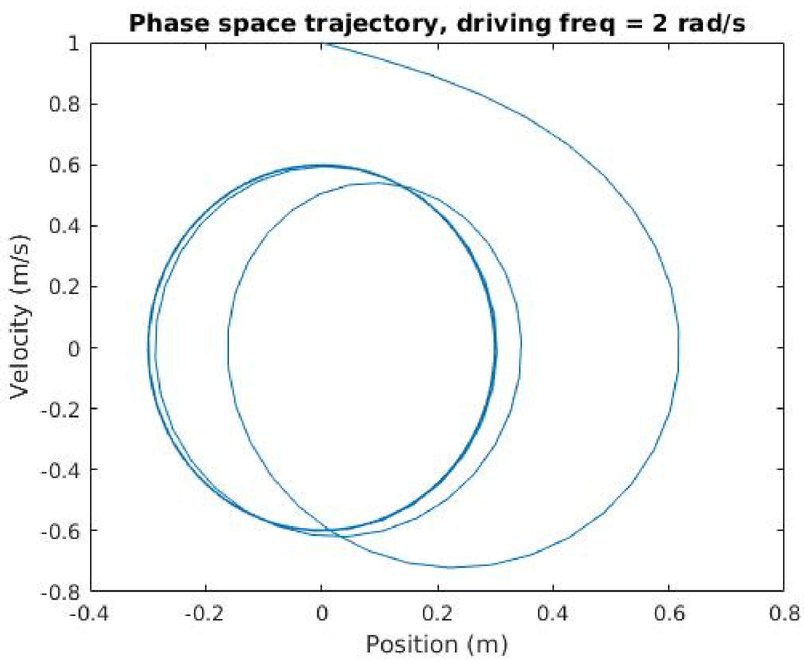


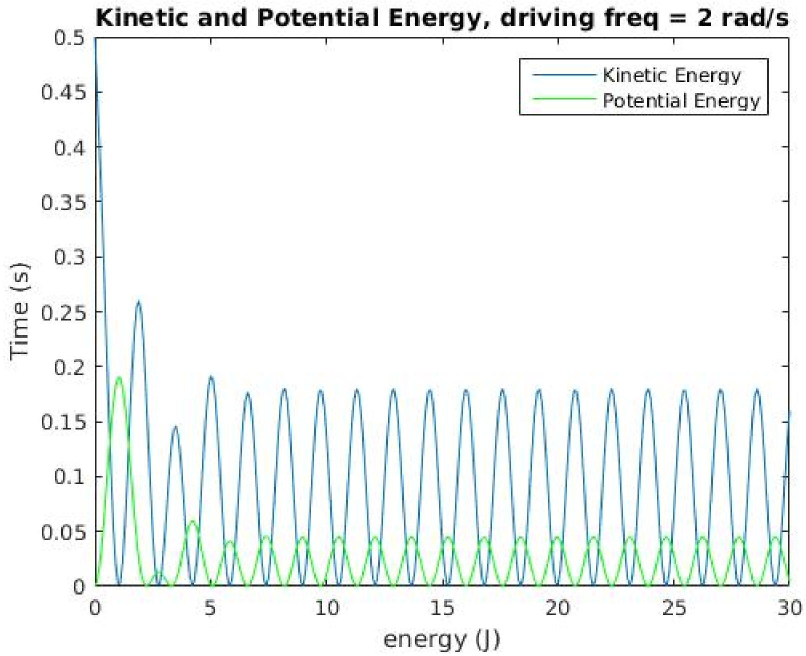




The system initially has a transient solution and then starts oscillating in a steady state. The maximum velocity reduces from the initial value and the first maximum of the displacement is more than the remaining ones. The maximums of the KE and PE also decrease.

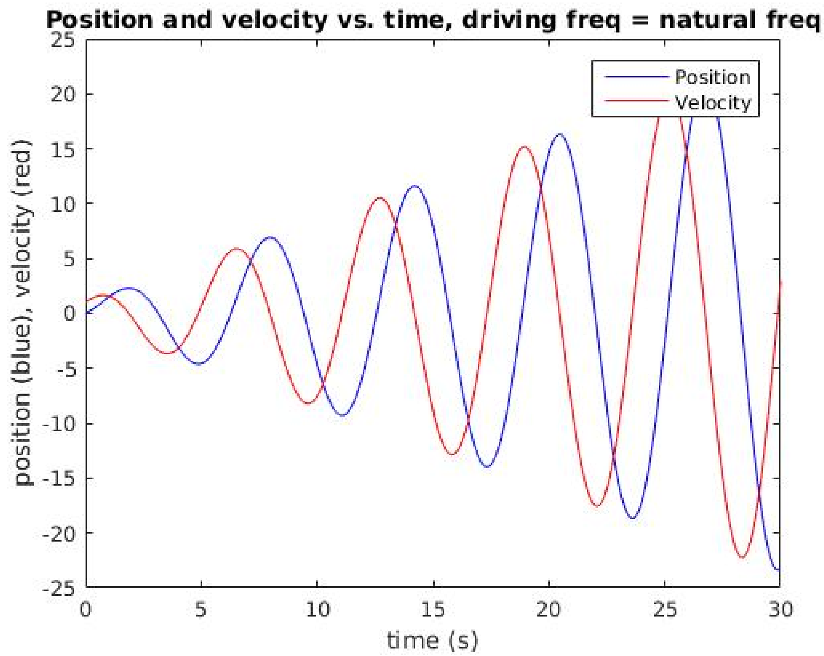

Position and velocity vs. time, driving freq = 2 rad/s
Position
Velocity
position (blue), velocity (red)
time (s)

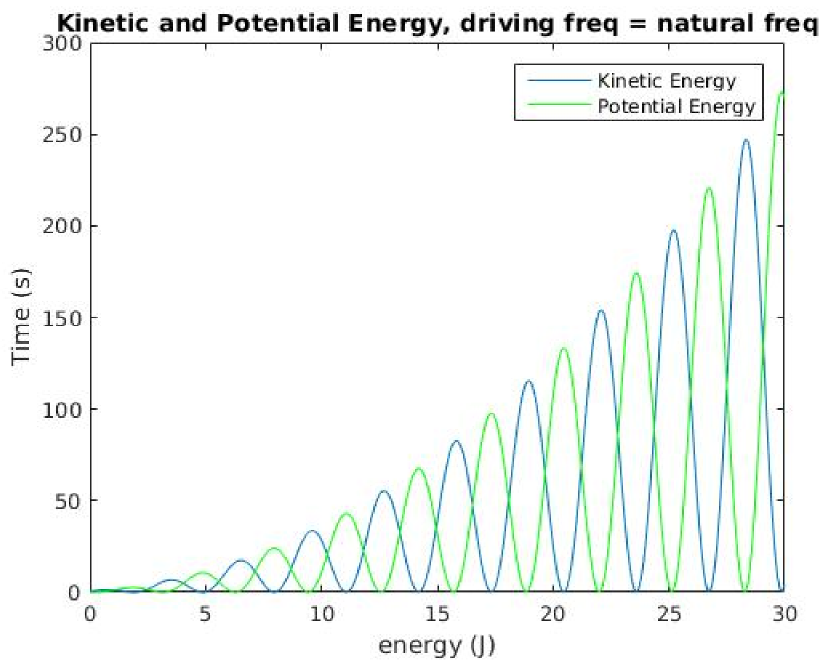

Phase space trajectory, driving freq = 2 rad/s
Velocity (m/s)
Position (m)

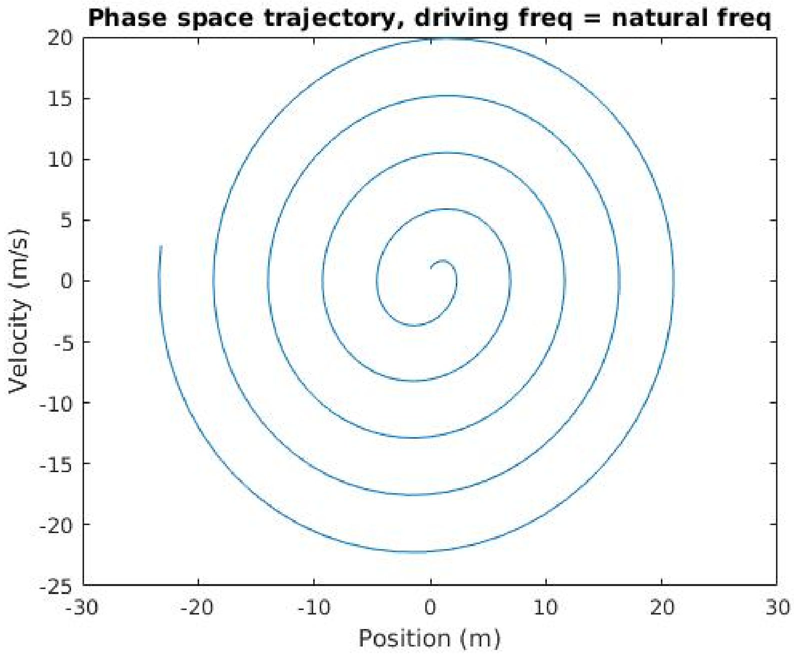


If the driving frequency is increased, the system oscillates more number of times before reaching steady state. The potential energy of the system decreases, as compared to the kinetic energy.

$\omega = \omega_o$:

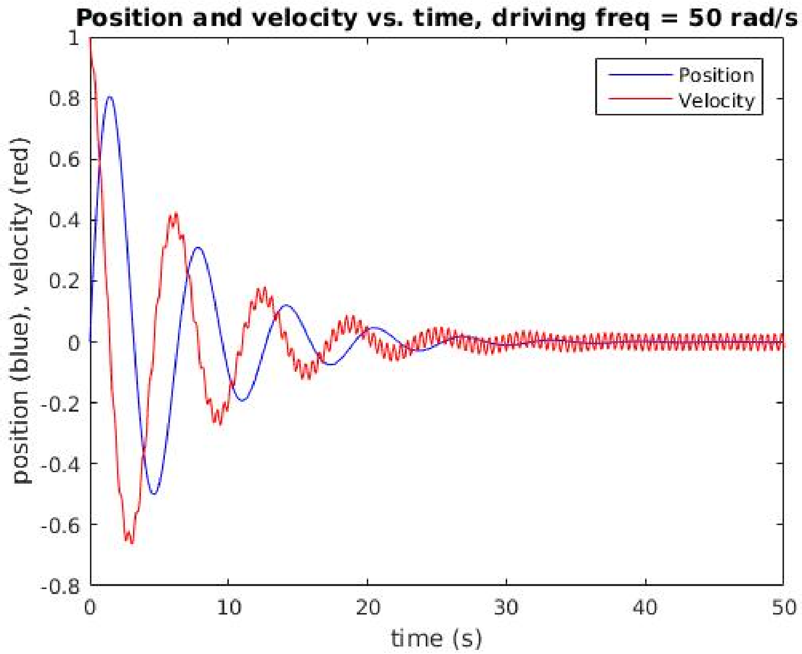

Position and velocity vs. time, driving freq = natural freq
Position
Velocity
position (blue), velocity (red)
time (s)

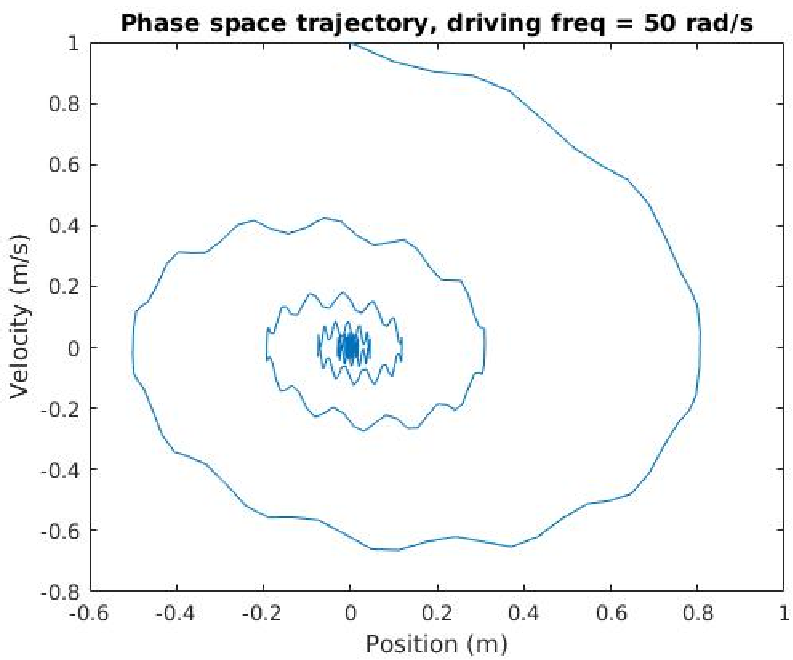

Kinetic and Potential Energy, driving freq = natural freq
Kinetic Energy
Potential Energy
Time (s)
energy (J)

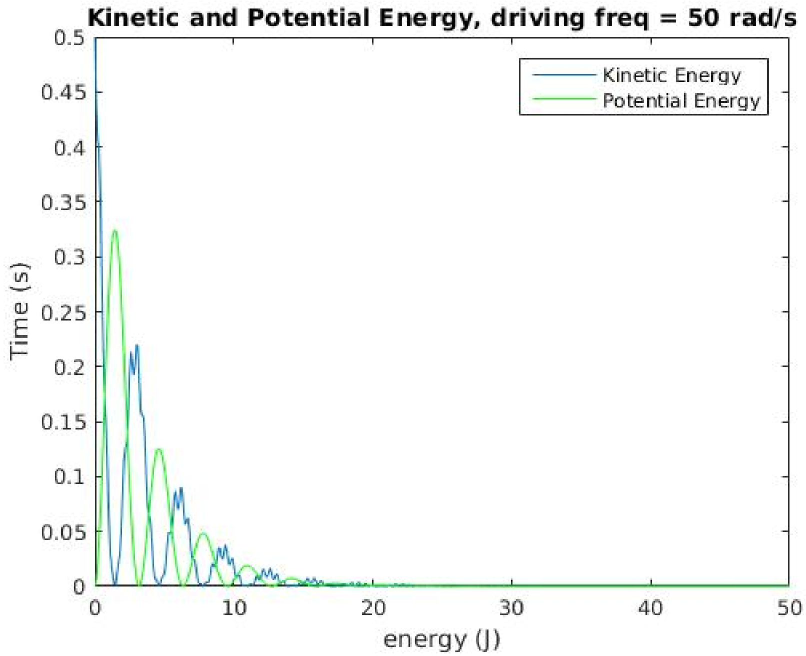


At driving frequency equal to the natural frequency of the system, the amplitude of position and velocity keeps increasing with each cycle. The total energy of the system increases. The phase space curve spirals out. This is the case of resonance.

$\omega >> \omega_o$:

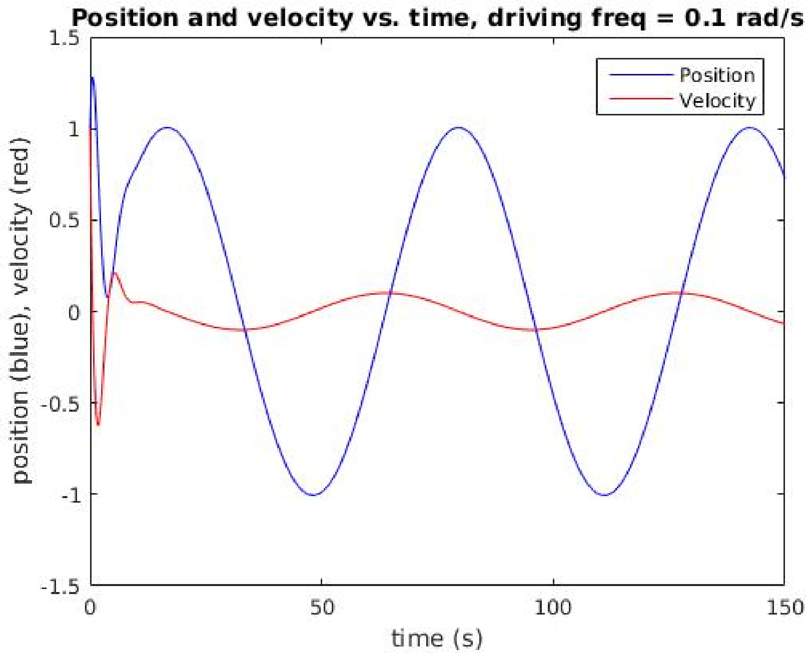

Position and velocity vs. time, driving freq = 50 rad/s
Position
Velocity
position (blue), velocity (red)
time (s)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
10
20
30
40
50

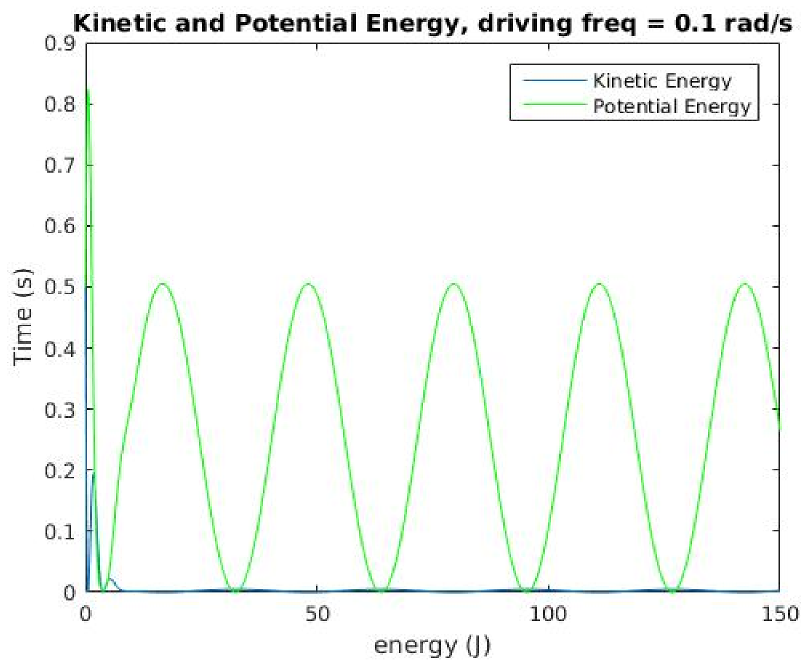

Phase space trajectory, driving freq = 50 rad/s
Velocity (m/s)
Position (m)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1



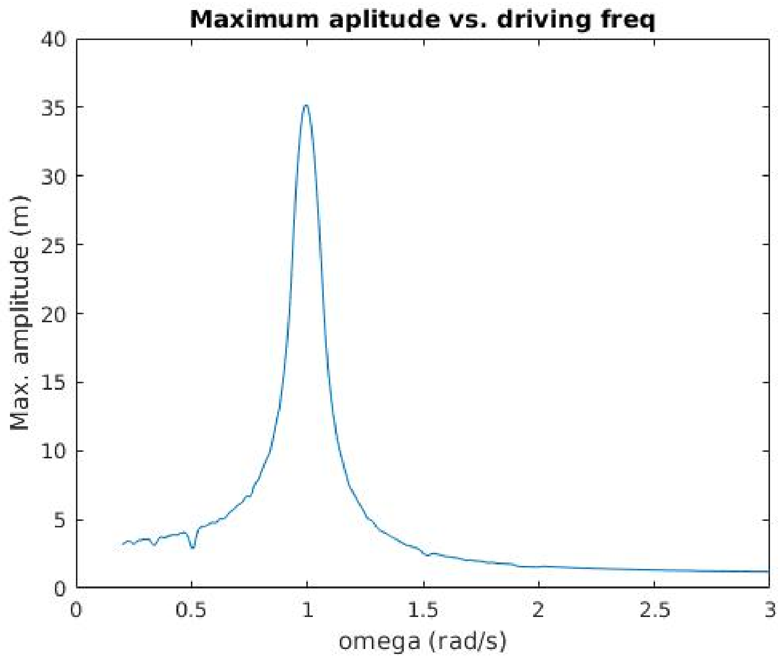
At very high driving frequency, the system initially oscillates with its natural frequency also, but the influence of the large driving freq. is more and it starts steadily oscillating with the driving frequency after some time. After a large amount of time, the energy of the system dies out.

$\omega << \omega_o$:





At driving freq much smaller than natural freq, the external force dampens the motion of the system. The system oscillates with the same amplitude but, very very slowly (the velocity is very small).



The graph of maximum amplitude vs. driving frequency has a peak at driving freq = natural freq. If the time period of stimulation tended to infinity, the max amplitude would also tend to infinity.