CSS 322 Reference sheets for Midterm Exam

$$
\begin{aligned}
(AB)^T &= B^T A^T, \\
(ABC)^T &= C^T B^T A^T,
\end{aligned}
$$

Theorem 1. *A square upper or lower triangular matrix is nonsingular if and only if all of its diagonal entries are nonzeros.*

Gaussian Elimination:

Eliminate a_{ik} by

$$
(\text{Row } i) := (\text{Row } i) - \left(\frac{a_{ik}}{a_{kk}}\right) \cdot (\text{Row } k).
$$

Call a_{kk} a **pivot** and (a_{ik}/a_{kk}) a **multiplier**.

Theorem 2. *Suppose you know all the row swaps performed by GEPP in advance. Then GEPP is equivalent to: first do all of swaps on A, then carry out plain GE on the permuted matrix.*

Properties of permutation matrices

1. Multiply a matrix on its left by a permutation matrix permutes its rows.
2. Multiply a matrix on its right by a permutation matrix permutes its columns.
3. $PP^T = I$, i.e. $P^{-1} = P^T$.

Theorem 3. *GEPP encounters a zero pivot if and only if A is singular.*

Sherman-Morrison formula:

$(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^T A^{-1} u)^{-1} v^T A^{-1}.$

Woodbury formula:

$$
(A - UV^T)^{-1} = A^{-1} + A^{-1}U(I - V^T A^{-1} U)^{-1} V^T A^{-1},
$$

where U and V are $n \times k$ matrices.

Rank-One Updating of Solution

Suppose we have the $P^T LU$ factorization of a matrix A. Solve $(A - uv^T)x = b$ efficiently by

- Solve $Az = u$ for z.
- Solve $Ay = b$ for y.
- Then $x = y + ((v^T y)/(1 - v^T z))z$.

The properties of norms

$\|\cdot\|$ is a norm if

1. $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$. $\|x\| = 0$ if and only if $x = 0$.
2. If α is a scalar,
$$\|\alpha x\| = |\alpha| \cdot \|x\|.$$
3. For all $x, y \in \mathbb{R}^n$,
$$\|x + y\| \leq \|x\| + \|y\|.$$

Some vector norm properties

- $$\|x\|_1 \geq \|x\|_2 \geq \|x\|_\infty.$$
- **Hölder inequality:**
$$\left|x^T y\right| \leq \|x\|_p \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$
- **"Cauchy-Swartz inequality":**
$$\left|x^T y\right| \leq \|x\|_2 \|y\|_2.$$
- Suppose $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix (which means $Q^T Q = QQ^T = I$ by definition).
$$\|Qx\|_2 = \|x\|_2.$$

Theorem 4 (The equivalence of norms)**.** *For any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$, there exists a pair of real numbers $0 < C_1 \leq C_2$ such that, for all x,*

$$
C_1 \|x\|_b \leq \|x\|_a \leq C_2 \|x\|_b.
$$

Theorem 5. *For any vector p-norm and induced matrix p-norm,*

$$\|Ax\|_p \le \|A\|_p \cdot \|x\|_p,$$

where $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$.

Theorem 6.

$$\|AB\|_p \le \|A\|_p \cdot \|B\|_p,$$

for any p-norm, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times l}$.

If $\hat{x} \in \mathbb{R}^n$ is an approximation to $x \in \mathbb{R}^n$, then the relative error is

$$\frac{\|\hat{x} - x\|}{\|x\|}.$$

Floating-point number system

$$\pm(d_0.d_1 d_2 \dots d_{p-1})_2 \times 2^E,$$

where p is the number of mantissa digits (precision), E is the (integer) exponent ($L \le E \le U$), d_i is either 0 or 1.

Unit roundoff, or **machine epsilon**, is the maximum **possible** relative error resulting from one scalar operation in floating-point arithmetic. Denoted as $\epsilon_{\texttt{mach}}$.

$$\epsilon_{\texttt{mach}} = 2^{1-p} \text{ or } 2^{-p}$$

depending on the roundoff method.

$$\begin{aligned} \text{cond}_p(A) &\ge 1. \\ \text{cond}_p(I) &= 1. \end{aligned}$$

If $\alpha \neq 0$ is a scalar, then

$$\text{cond}(A) = \text{cond}(\alpha A).$$

If D is a diagonal matrix, that is, $d_{ij} = 0$ for $i \neq j$, then

$$\text{cond}(D) = \frac{\max_i |d_{ii}|}{\min_i |d_{ii}|}.$$

Theorem 7 (Wilkinson (simplified))**.** *Suppose we solve $Ax = b$, $A \in \mathbb{R}^{n\times n}$ using GEPP and the four-steps procedure in presence of roundoff. The computed $\hat{x}$ satisfies*

$$(A + E)\hat{x} = b,$$

where E satisfies

$$\frac{\|E\|_\infty}{\|A\|_\infty} \le \rho n \epsilon_{mach} + O(\epsilon_{mach}^2).$$

In the (extremely rare) worst case, $\rho = 2^{n-1}$. In practice, however, it is usually observed that $\rho \approx 2$.

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is **positive definite** if

$$x^T A x > 0$$

for all nonzero $x \in \mathbb{R}^n$.

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is **positive semidefinite** if

$$x^T A x \ge 0$$

for all $x \in \mathbb{R}^n$.

Cholesky Factorization

for $k = 1$ to n
    $a_{kk} = \sqrt{a_{kk}}$
    for $i = k+1$ to n
        $a_{ik} = a_{ik}/a_{kk}$
    end
    for $j = k+1$ to n
        for $i = k+1$ to n
            $a_{ij} = a_{ij} - a_{ik} \cdot a_{jk}$
        end
    end
end

Solving tridiagonal linear systems

Let

$$A = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{bmatrix}.$$

Then the LU factorization of A are given by

$$L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ m_2 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & m_{n-1} & 1 & 0 \\ 0 & \cdots & 0 & m_n & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} d_1 & c_1 & 0 & \cdots & 0 \\ 0 & d_2 & c_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & d_{n-1} & c_{n-1} \\ 0 & \cdots & \cdots & 0 & d_n \end{bmatrix}.$$