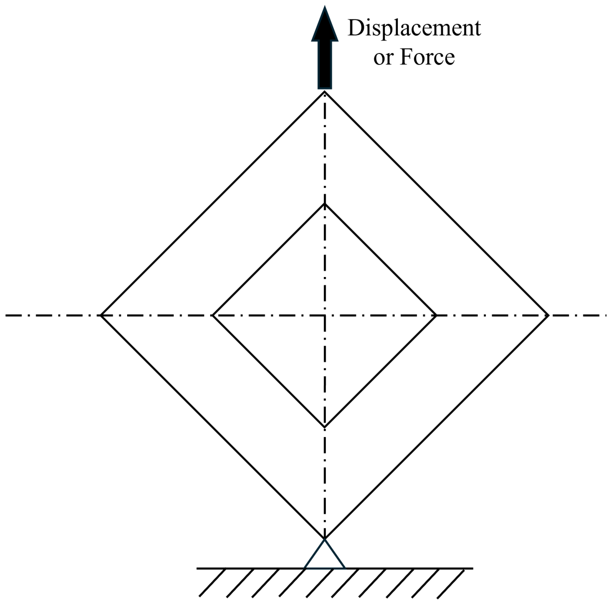
Computational Mechanics Final Project



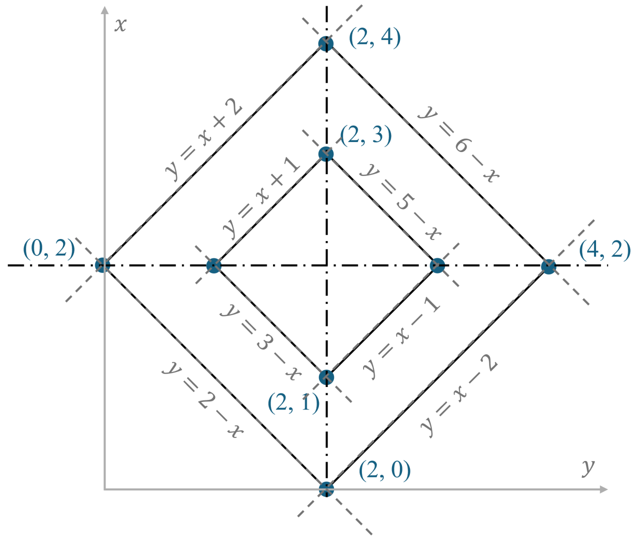
Outline

- Defining nodes and create elements
- Finding stiffness matrix of each element and total stiffness matrix of the material
- Compute the displacement of each node to obtain the deformed shape
- Compute the strain of each element
- Compute the stress of each element

In [ ]:
```
# Importing necessary library and package
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from scipy.spatial import Delaunay
from scipy import linalg
```

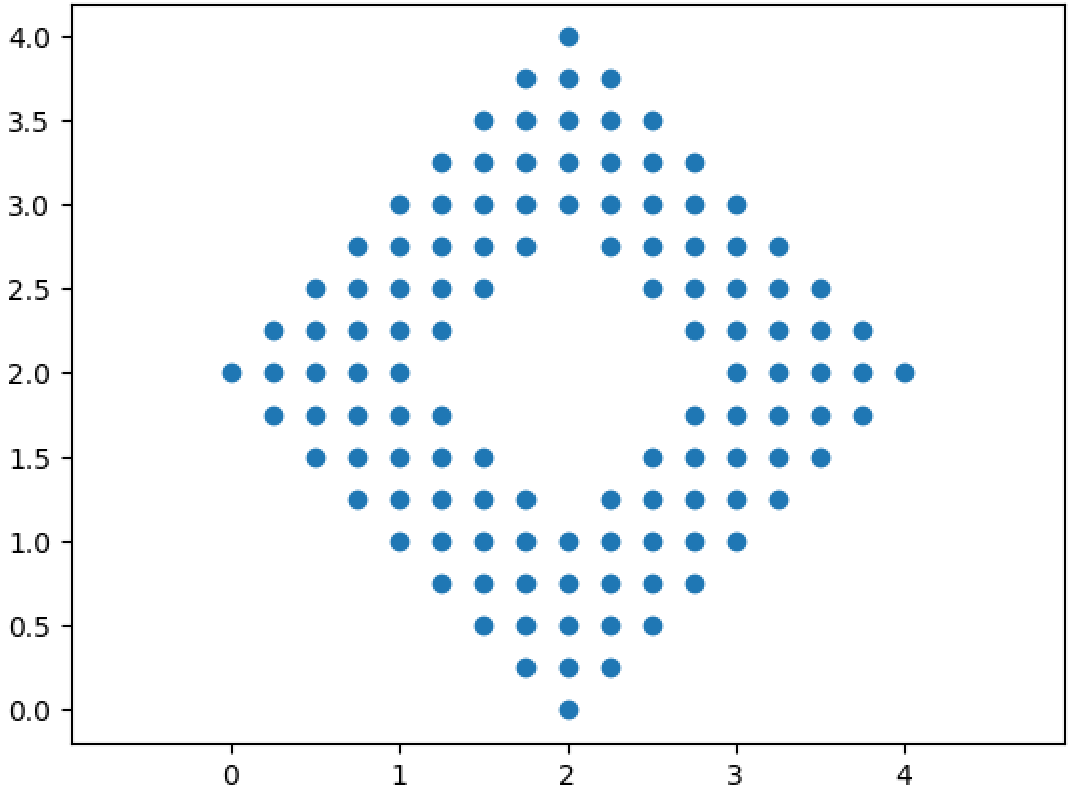
1. Defining nodes and create elements

Since each triangular element is formed by 3 nodes, we need firstly to create nodes. The coordinate (x, y) of the node will be appended to the list `node_list` which will later be converted to `nodes`, which is a numpy array. The size and coordinate of the sample are defined as follows:



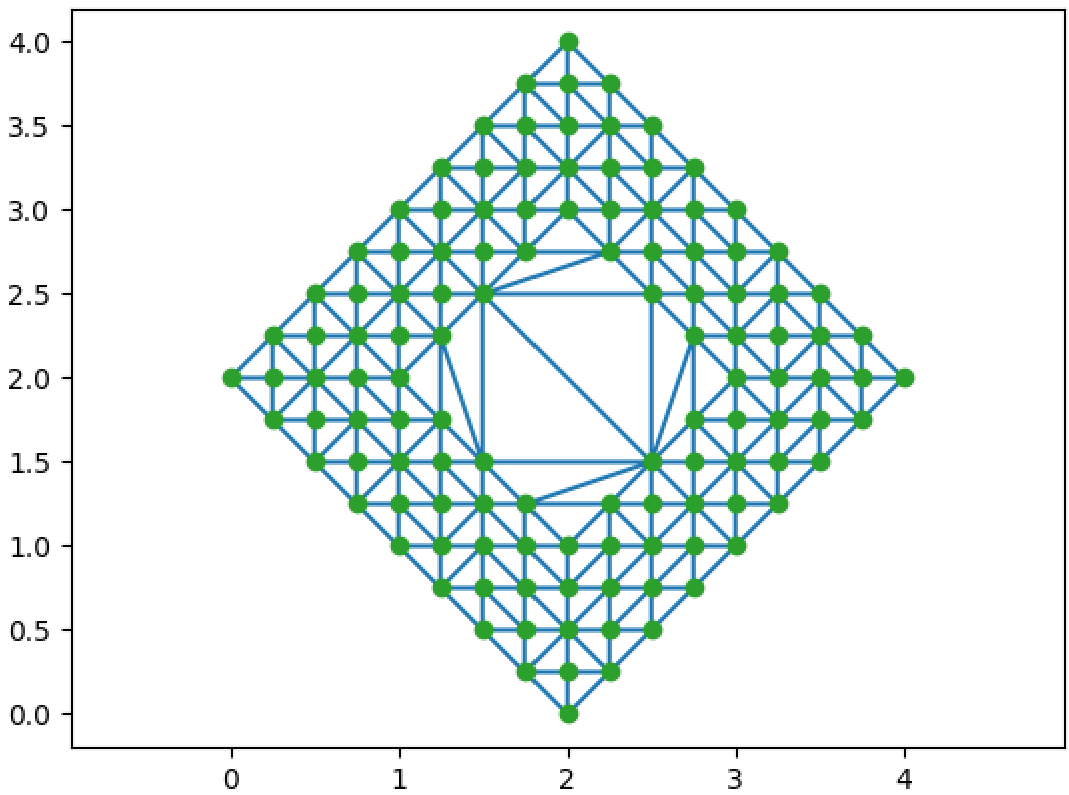
In [ ]:

```
#Defining nodes
RESOLUTION = 4 #input 1, 2 or 4
step = 1/RESOLUTION
y_cor = 4
node_list = []
while y_cor <= 4 and y_cor >= 2:
    x_cor = 0
    while x_cor >= 0 and x_cor <= 4:
        if (x_cor >= y_cor-2) and (x_cor <= 6-y_cor):
            if (x_cor <= y_cor-1) or (x_cor >= 5-y_cor):
                node_list.append([x_cor, y_cor])
        x_cor += step
    y_cor -= step

while y_cor < 2 and y_cor >= 0:
    x_cor = 0
    while x_cor >= 0 and x_cor <= 4:
        if (x_cor <= y_cor+2) and (x_cor >= 2-y_cor):
            if (x_cor >= y_cor+1) or (x_cor <= 3-y_cor):
                node_list.append([x_cor, y_cor])
        x_cor += step
    y_cor -= step


nodes = np.array(node_list)
plt.plot(nodes[:,0], nodes[:,1], 'o')
plt.axis('equal')
plt.show()
```



To create elements from node, we will utilize the Delaunay triangulation, which is the algorithm compute the triangular mesh from a given set of points. The Delaunay triangulation is already defined in `scipy.spatial` package, which we have already imported above.

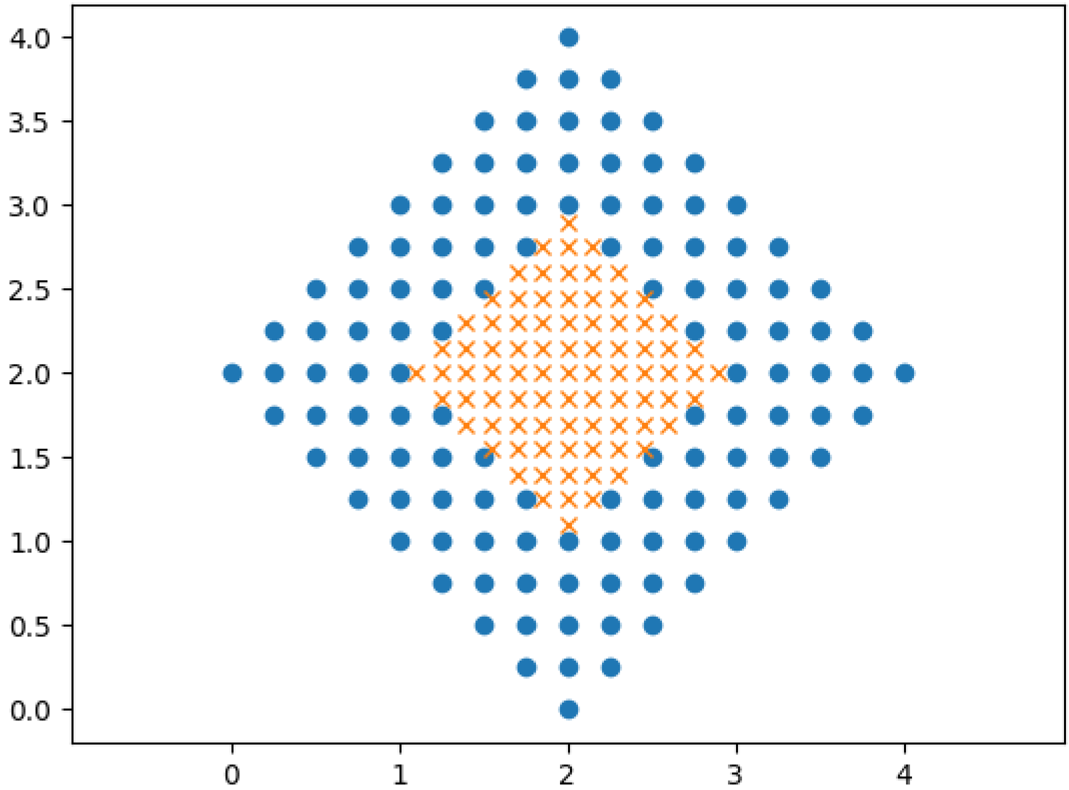
In [ ]:
```
tri = Delaunay(nodes)
plt.triplot(nodes[:,0], nodes[:,1], tri.simplices)
plt.plot(nodes[:,0], nodes[:,1], 'o')
plt.axis('equal')
plt.show()
```



Now, we have obtained the set of elements `tri`. However, the algorithm also created elements inside the hole that should be empty. So, we will first define nodes inside the hole in the `hole_node_list`. Then, we will find the elements that contain those points, using `tri.find_simplex(hole_nodes)` method, and remove them from `tri`. The final list of elements is called `element_list`

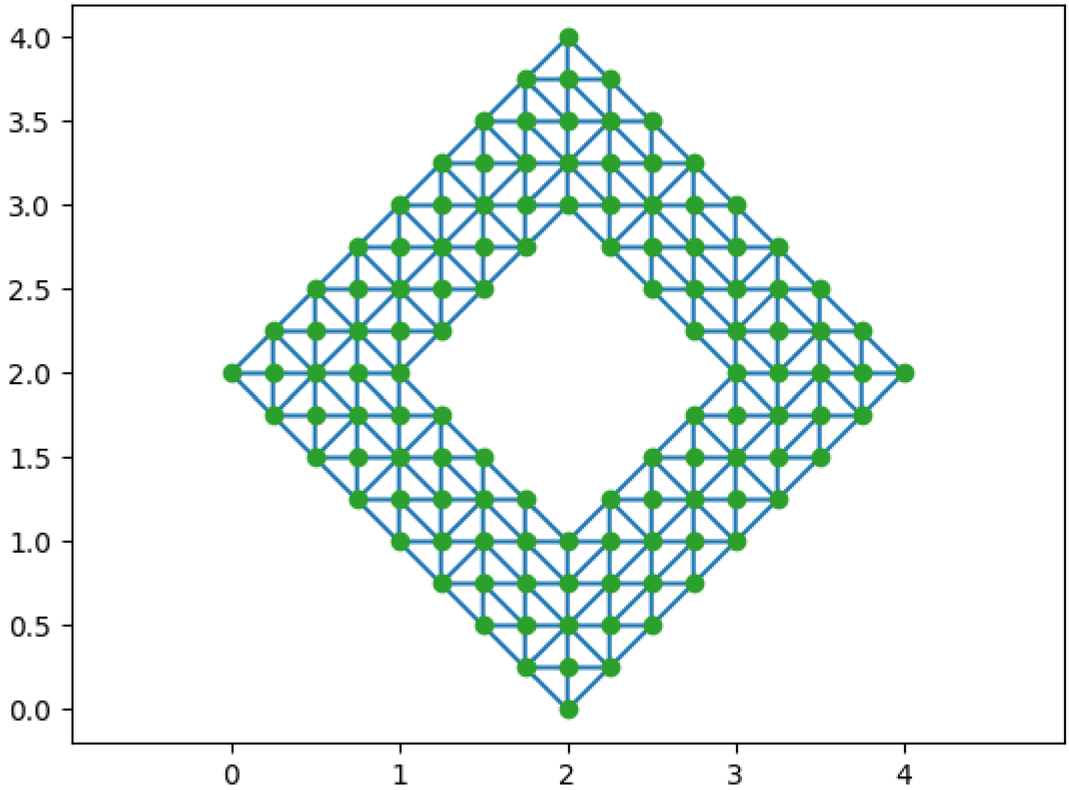
In [ ]:

```
hole_node_list = []
hole_step = 0.15
y_cor = 2.9
while y_cor <= 2.9 and y_cor > 2:
    x_cor = 1.1
    while x_cor >= 0 and x_cor <= 4:
        if (x_cor >= y_cor-1) and (x_cor <= 5-y_cor):
            hole_node_list.append([x_cor, y_cor])
        x_cor += hole_step
    y_cor -= hole_step

while y_cor > 1:
    x_cor = 1.1
    while x_cor >= 0 and x_cor <= 4:
        if (x_cor <= y_cor+1) and (x_cor >= 3-y_cor):
            hole_node_list.append([x_cor, y_cor])
        x_cor += hole_step
    y_cor -= hole_step


nodes = np.array(node_list)
hole_nodes = np.array(hole_node_list)
plt.plot(nodes[:,0], nodes[:,1], 'o')
plt.plot(hole_nodes[:,0], hole_nodes[:,1], 'x')
plt.axis('equal')
plt.show()
```



In [ ]:

```
# Find the elements which contain those points
hole_elements = list(set(tri.find_simplex(hole_nodes)))
hole_elements.sort()

# Remove elements that contain those points
element_list = np.delete(tri.simplices, hole_elements, 0)

plt.triplot(nodes[:,0], nodes[:,1], element_list)
plt.plot(nodes[:,0], nodes[:,1], 'o')
plt.axis('equal')
plt.show()
```



Here, we will show the format of the `element_list`, which each element contains 3 node indices. The coordinate of each index can be found in `nodes` array.

In [ ]:
```
# Display the mesh indices and coordinates
print("Example - Element list:\nindex   node indices")
for i, element in enumerate(element_list[:5]):
    print(f"  {i}        {element}")

print("\nExample - Nodes:\nindex   coordinate (x, y)")
for i, node in enumerate(nodes[:5]):
    print(f"  {i}         ({node[0]}, {node[1]})")
```

```
Example - Element list:
index   node indices
  0        [75 85 76]
  1        [86 85 95]
  2        [110 109 115]
  3        [114 118 115]
  4        [15  8 14]

Example - Nodes:
index   coordinate (x, y)
  0         (2.0, 4.0)
  1         (1.75, 3.75)
  2         (2.0, 3.75)
  3         (2.25, 3.75)
  4         (1.5, 3.5)
```

2. Finding stiffness matrix of each element and total stiffness matrix of the material

The stiffness matrix K_n of each element can be computed using the following formula:

$$K_n = a_n B_n^T D B_n$$

, where a_n is the area of the element, B matrix is the matrix that shows the realationship between the displacement and strain of each element, and D matrix is the Elastic modulus matrix.

Since D matrix is a constant for the material, only depending on Young's modulus (E) and poisson ratio (ν), we will first start by defining it. However, D matrix also varies in different conditions. In plain strain condition, D matrix is expressed by:

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

On the other hand, in plain stress condition, D matrix is expressed by:

$$D = \frac{E}{(1+\nu)(1-\nu)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Using the Aluminum alloy 1100, the Young's modulus is 69 GPa and the poisson ratio is 0.33

In [ ]:
```
v_ratio = 0.33
E = 69*(10**9)

# Defining Elastic modulus matrix (D)
def D_matrix_strain_cond(E, v_ratio):
    D = (E/((1+v_ratio)*(1-2*v_ratio)))*np.array([[1-v_ratio, v_ratio, 0],
                                                  [v_ratio, 1-v_ratio, 0],
                                                  [0, 0, (1-2*v_ratio)/2]])
    return D

def D_matrix_stress_cond(E, v_ratio):
    D = (E/((1+v_ratio)*(1-v_ratio)))*np.array([[1, v_ratio, 0],
                                                [v_ratio, 1, 0],
                                                [0, 0, (1-v_ratio)/2]])
    return D

# Compute D_matrix (can change the condition by changing the function)
D = D_matrix_stress_cond(E, v_ratio)
print("Elastic modulus matrix (D):\n", D)
```

```
Elastic modulus matrix (D):
 [[7.74323869e+10 2.55526877e+10 0.00000000e+00]
 [2.55526877e+10 7.74323869e+10 0.00000000e+00]
 [0.00000000e+00 0.00000000e+00 2.59398496e+10]]
```

Since each B matrix is specific to an element, we will define a function `B_matrix_and_area()` that compute B matrix and area (a_n) of each elements. Then, we will loop over every elements and put each element's B matrix and area into `B_list` and `area_list` according to its element index.

The formula of B matrix is shown as follows:

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

, where

$$\frac{\partial N_1}{\partial x} = \frac{y_2 - y_3}{2a}, \quad \frac{\partial N_2}{\partial x} = \frac{y_3 - y_1}{2a}, \quad \frac{\partial N_3}{\partial x} = \frac{y_1 - y_2}{2a}$$

$$\frac{\partial N_1}{\partial y} = \frac{x_3 - x_2}{2a}, \quad \frac{\partial N_2}{\partial y} = \frac{x_1 - x_3}{2a}, \quad \frac{\partial N_3}{\partial y} = \frac{x_2 - x_1}{2a}$$

, and

$$a = \frac{1}{2} det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

In [ ]:

```
# Defining a function to compute B matrix
def B_matrix_and_area(element_index, nodes_coordinate):
    # mesh_node is the array containing 3 node indices = element in mesh
    # nodes_coordinate is the array containing coordinate of each node indix = n
    node1, node2, node3 = element_list[element_index]

    node1_x, node1_y = nodes_coordinate[node1]
    node2_x, node2_y = nodes_coordinate[node2]
    node3_x, node3_y = nodes_coordinate[node3]

    area = 0.5*np.linalg.det(np.array([[1, node1_x, node1_y],
                                       [1, node2_x, node2_y],
                                       [1, node3_x, node3_y]]))

    dN1_dx = (node2_y - node3_y)/(2*area)
    dN2_dx = (node3_y - node1_y)/(2*area)
    dN3_dx = (node1_y - node2_y)/(2*area)
    dN1_dy = (node3_x - node2_x)/(2*area)
    dN2_dy = (node1_x - node3_x)/(2*area)
    dN3_dy = (node2_x - node1_x)/(2*area)

    B_matrix = np.array([[dN1_dx, 0, dN2_dx, 0, dN3_dx, 0],
                         [0, dN1_dy, 0, dN2_dy, 0, dN3_dy],
                         [dN1_dy, dN1_dx, dN2_dy, dN2_dx, dN3_dy, dN3_dx]])
    return (B_matrix, area)

# Compute B matrix of each element and put them into B_list
B_list = []
area_list = []

for i in range(len(element_list)):
    B_tuple = B_matrix_and_area(i, nodes)
    B_list.append(B_tuple[0])
    area_list.append(B_tuple[1])

print("The B matrix of element 0:\n", B_list[0], "\n")
print("The area of element 0:\n", area_list[0])
```

```
The B matrix of element 0:
 [[-4.  0.  0.  0.  4.  0.]
 [ 0.  0.  0. -4.  0.  4.]
 [ 0. -4. -4.  0.  4.  4.]]

The area of element 0:
 0.03125
```

Now, we will define a function `K_matrix()`, which will input area and B matrix of each element and compute K_n matrix. Then, similar to before, we will loop over every element, compute K_n and put it in `K_list` according to the element index.

In [ ]:

```
# Defining a function to compute B matrix
def K_matrix(area, B_matrix, D_matrix):
    K = area*B_matrix.transpose().dot((D_matrix.dot(B_matrix)))
    return K

# Computing K matrix for each element and put them into K_list
K_list = []
for i in range(len(element_list)):
    K_list.append(K_matrix(area_list[i], B_list[i], D))

print("The K matrix of element 0:\n", K_list[0])
```

```
The K matrix of element 0:
 [[ 3.87161935e+10  0.00000000e+00  0.00000000e+00  1.27763438e+10
  -3.87161935e+10 -1.27763438e+10]
 [ 0.00000000e+00  1.29699248e+10  1.29699248e+10  0.00000000e+00
  -1.29699248e+10 -1.29699248e+10]
 [ 0.00000000e+00  1.29699248e+10  1.29699248e+10  0.00000000e+00
  -1.29699248e+10 -1.29699248e+10]
 [ 1.27763438e+10  0.00000000e+00  0.00000000e+00  3.87161935e+10
  -1.27763438e+10 -3.87161935e+10]
 [-3.87161935e+10 -1.29699248e+10 -1.29699248e+10 -1.27763438e+10
   5.16861183e+10  2.57462687e+10]
 [-1.27763438e+10 -1.29699248e+10 -1.29699248e+10 -3.87161935e+10
   2.57462687e+10  5.16861183e+10]]
```

Now that we obtained the K matrix for each element, we will define `K_total_matrix()` function to superimpose every K matrix according to their node to compute the total stiffness matrix (K_t). To do that, first, we need to increase the dimension of each K matrix to [2n(node) $\times$ 2n(node)]. The increased dimensionality marix is defined as `Ke`. Then, we will loop over `Ke` and performed element-wise addition to superimpose them.

In [ ]:

```
# Combine all K matrices to create the total stiffness matrix
def K_total_matrix(K_list, element_list, total_nodes):
    #test_output = []
    K_t = np.zeros((total_nodes*2, total_nodes*2))

    for i in range(len(K_list)):
        K_e = np.zeros((total_nodes*2, total_nodes*2))
        element = element_list[i]

        for row in range(6):
            row_node = element[row//2]
            for col in range(6):
                col_node = element[col//2]
```

```
                if (row%2==0) and (col%2==0):
                    K_e[row_node*2][col_node*2] = K_list[i][row][col]
                elif (row%2==0) and (col%2==1):
                    K_e[row_node*2][(col_node*2)+1] = K_list[i][row][col]
                elif (row%2==1) and (col%2==0):
                    K_e[(row_node*2)+1][col_node*2] = K_list[i][row][col]
                elif (row%2==1) and (col%2==1):
                    K_e[(row_node*2)+1][(col_node*2)+1] = K_list[i][row][col]
        #test_output.append(K_e)
        K_t += K_e
    return K_t

# Compute the total stiffness matrix
K_t = K_total_matrix(K_list, element_list, len(nodes))


# save the dataframe as a csv file
K_t_DF = pd.DataFrame(K_t)
K_t_DF.to_csv("Total_stiffness_matrix.csv")
```

3. Compute the displacement of each node to obtain the deformed shape

Now that we obtained total stiffness matrix `K_t`, we can compute the displacement vector u, by using the following relation:

$$K_t u = f$$

Since our geometry is symmetric to the axis $x = 2$, and the force will only be applied in the vertical direction at the symmetric axis, the node at the symmetric axis will have displacement in x-direction equals to zero (excluding the node that the force is applied, `node_for_pulling`). Moreover, the base node, which is fixed, will have the displacement in y-direction equals to zero as well. Because of that, some entries in u vector will equal to 0, and they need to be excluded from the K_t matrix before computing the u vector.

In [ ]:

```
# Find nodes that in the middle (u_x = 0)
node_for_pullling = 0
row_to_crop = []
for i in range(len(nodes)):
    if nodes[i][0] == 2:
        if nodes[i][1] == 0:            # Condition for Based node
            row_to_crop.append(i*2)
            row_to_crop.append(i*2+1)
        elif nodes[i][1] == 4:
            node_for_pullling = i
        else:
            row_to_crop.append(i*2)     # Appending the node that displacement i
print("Rows that should be cropped:\n",row_to_crop)

row_to_crop.sort(reverse=True)
K_t_crop = K_t
for i in row_to_crop:
    K_t_crop = np.delete(K_t_crop, i, 0)
    K_t_crop = np.delete(K_t_crop, i, 1)
```

```
# Computing u vector
# Defining force vector
f_vector = np.zeros((len(nodes)*2-len(row_to_crop),1))
f_vector[1][0] = 5*(10**9)
K_t_crop_inv = linalg.inv(K_t_crop)
u_vector = K_t_crop_inv.dot(f_vector)
```

```
Rows that should be cropped:
 [4, 12, 24, 40, 198, 214, 226, 234, 238, 239]
```

Since the matrix is cropped before the computation, the resulted u vector will have less dimension than the actual displacement vector. The following code is meant to append the value of `u_vector` into `displacement` vector, according to their node index.

In [ ]:

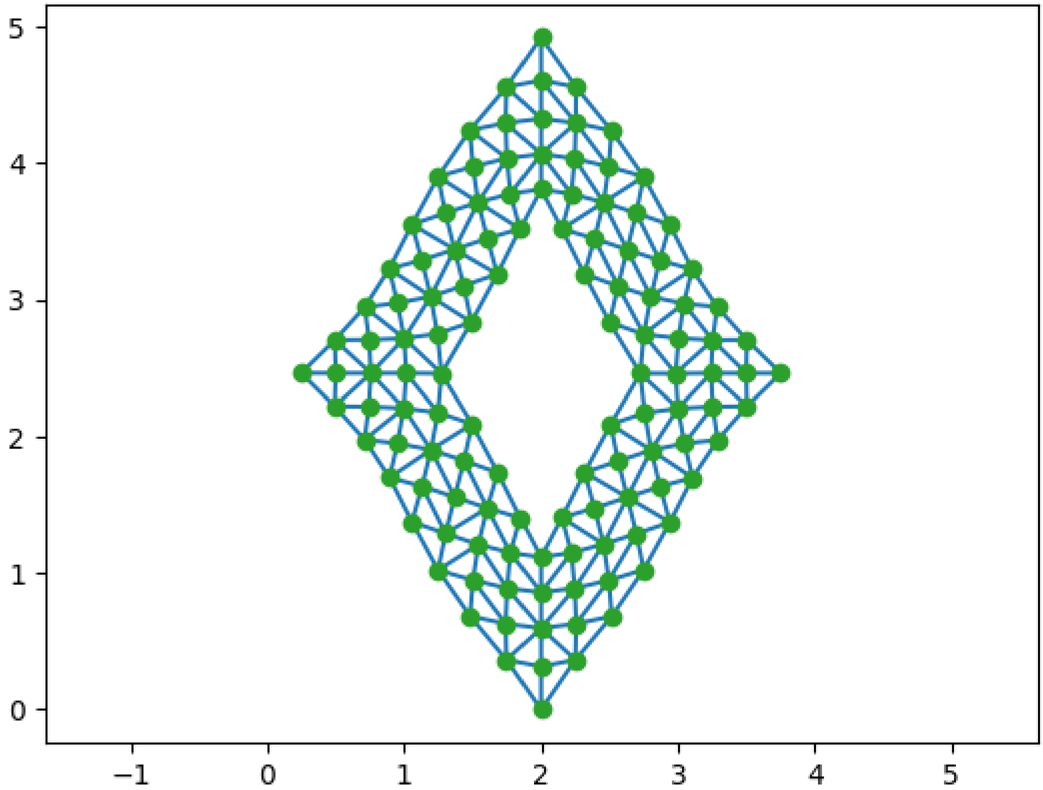
```
displacement = np.zeros((len(nodes),2))
k = 0
w = 0
for i in range(len(displacement)):
    for j in range(2):
        if k not in row_to_crop:
            displacement[i][j] = u_vector[w]
            w += 1
        k += 1

print("Example - Displacement:")
print("node            (x-displacement, y-displacement)")
for i, dis in enumerate(displacement[:5]):
    print(f" {i}         ({dis[0]}, {dis[1]})")
```

```
Example - Displacement:
node            (x-displacement, y-displacement)
 0         (-0.0002533324598887061, 0.9178856417999617)
 1         (-0.0004390814950629667, 0.8050387259378183)
 2         (0.0, 0.8541581695205898)
 3         (0.004682079042777487, 0.8055453908575957)
 4         (-0.017752131951369956, 0.7387984862948522)
```

After we obtained the `displacement` vector, we can obtain the new coordinates by adding the displacement into the old coordinate and plot them.

In [ ]:

```
new_nodes = nodes + displacement

plt.triplot(new_nodes[:,0], new_nodes[:,1], element_list)
plt.plot(new_nodes[:,0], new_nodes[:,1], 'o')
plt.axis('equal')
plt.show()
```



4. Compute the strain of each element

The strain of each element can be computed using the following relation:

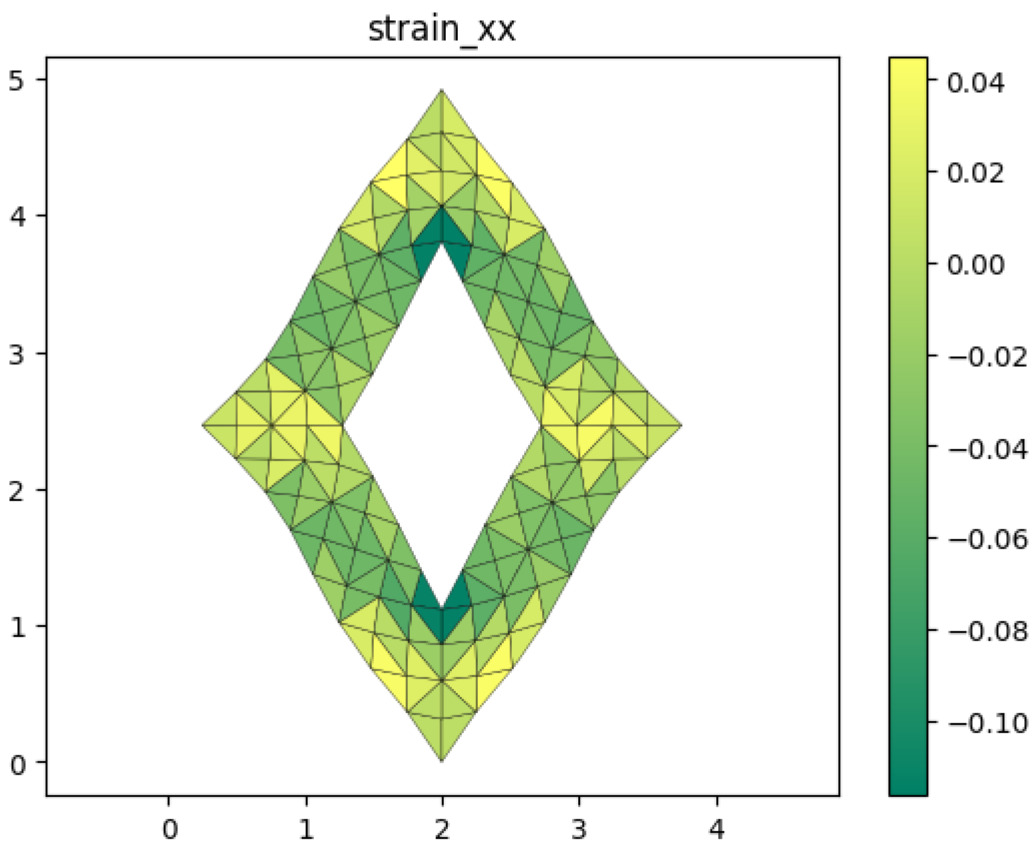
$$\varepsilon = Bd$$

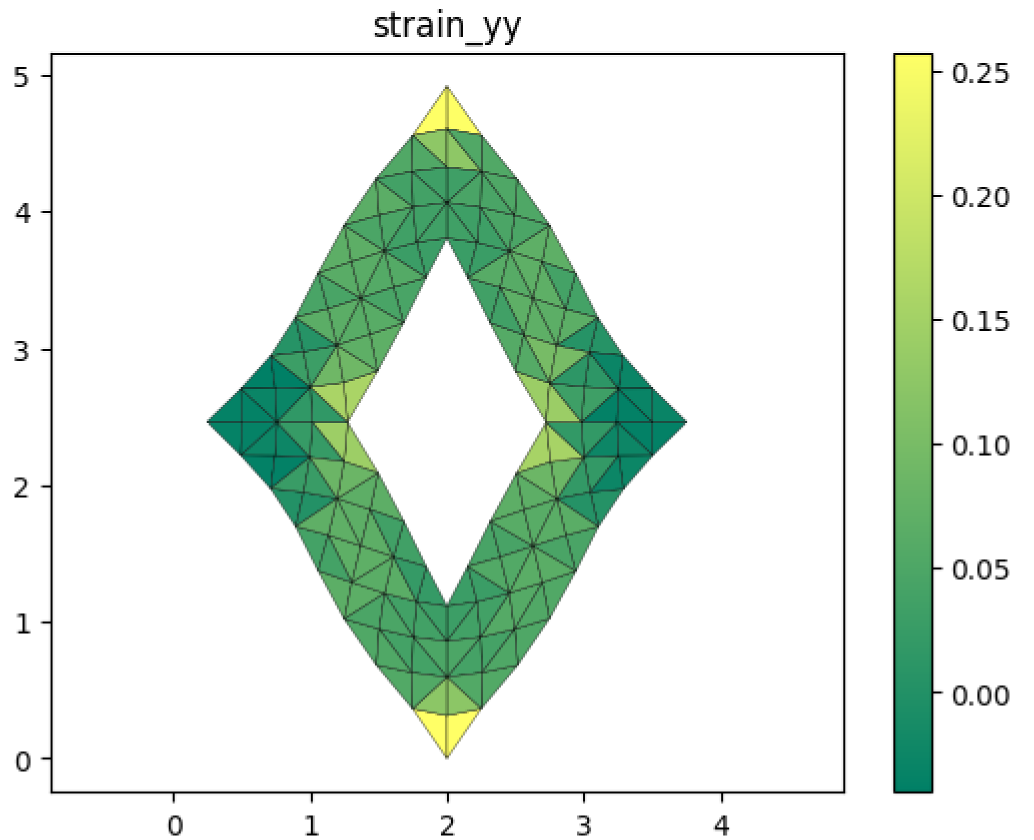
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{Bmatrix}$$

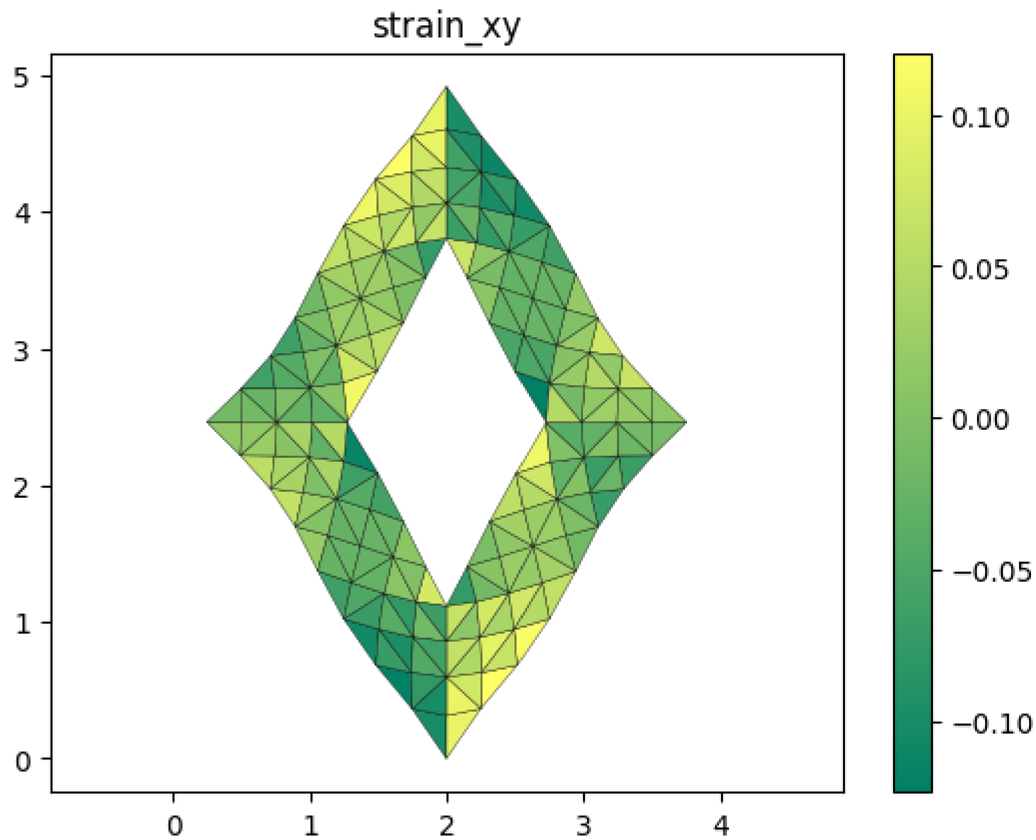
To compute that, we will define a `strain_vector()` function, which compute each element strain vector (ε). Then, loop over every element to compute every element's strain vector, and plot them using color gradient to show the distribution of strain.

In [ ]:
```
# Defining the function to compute strain vector
def strain_vector(B_matrix, element_index, displacement):
    node1, node2, node3 = element_list[element_index]
    disp = np.array([[displacement[node1][0]],
                     [displacement[node1][1]],
                     [displacement[node2][0]],
                     [displacement[node2][1]],
                     [displacement[node3][0]],
                     [displacement[node3][1]]])

    strain = B_matrix.dot(disp)
    return strain

# Loopoing over every element to obtain every element's strain vector
```

```
strain_list = []
for i in range(len(element_list)):
    strain_list.append(strain_vector(B_list[i], i, displacement))

# Plotting strain
colormap = "summer"

strain_xx_list = np.array([strain_list[i][0][0] for i in range(len(element_list)
plt.tripcolor(new_nodes[:,0], new_nodes[:,1], element_list, facecolors=strain_xx
plt.gca().set_aspect('equal')
plt.axis('equal')
plt.colorbar()
plt.title("strain_xx")
plt.show()

strain_yy_list = np.array([strain_list[i][1][0] for i in range(len(element_list)
plt.tripcolor(new_nodes[:,0], new_nodes[:,1], element_list, facecolors=strain_yy
plt.gca().set_aspect('equal')
plt.axis('equal')
plt.colorbar()
plt.title("strain_yy")
plt.show()

strain_xy_list = np.array([strain_list[i][2][0] for i in range(len(element_list)
plt.tripcolor(new_nodes[:,0], new_nodes[:,1], element_list, facecolors=strain_xy
plt.gca().set_aspect('equal')
plt.axis('equal')
plt.colorbar()
plt.title("strain_xy")
plt.show()
```







5. Compute the strain of each element

The stress of each element can be computed using the following relation:

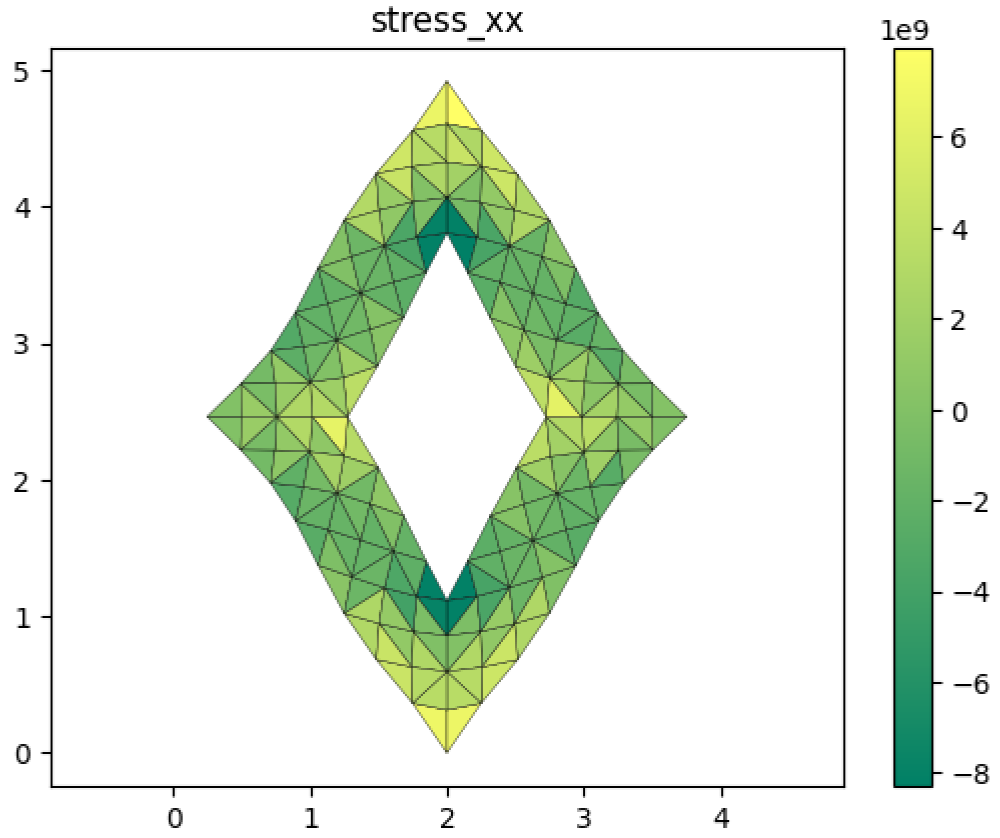
$$\sigma = D\varepsilon$$

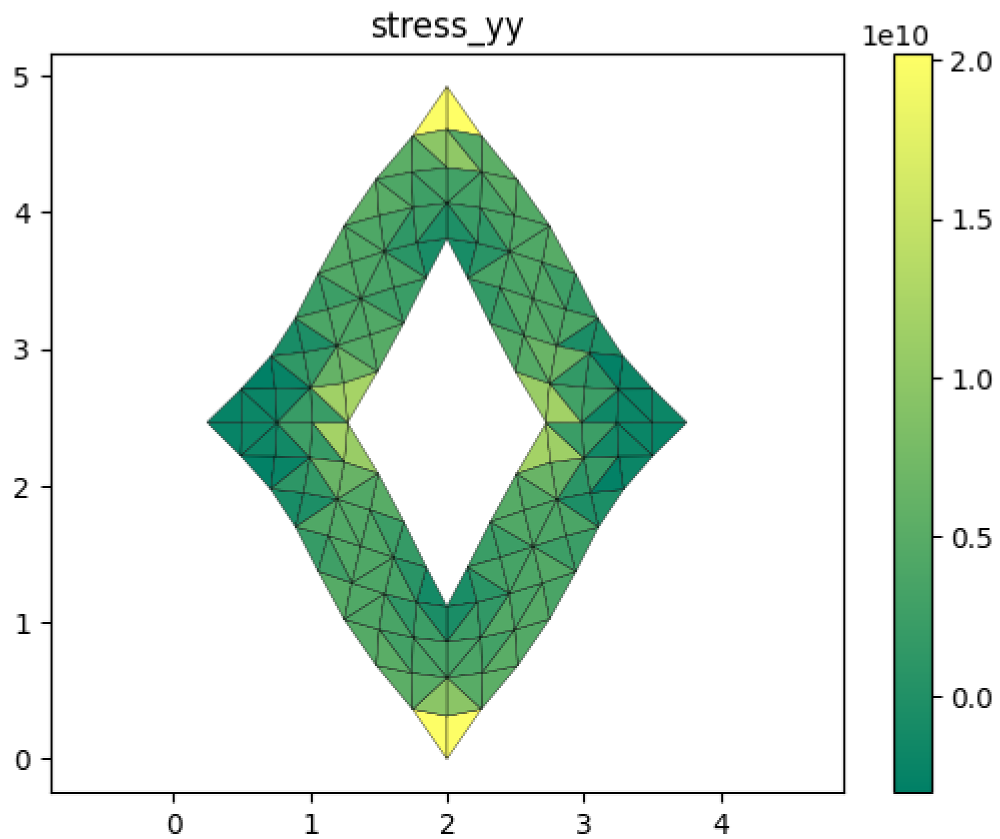
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = D \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

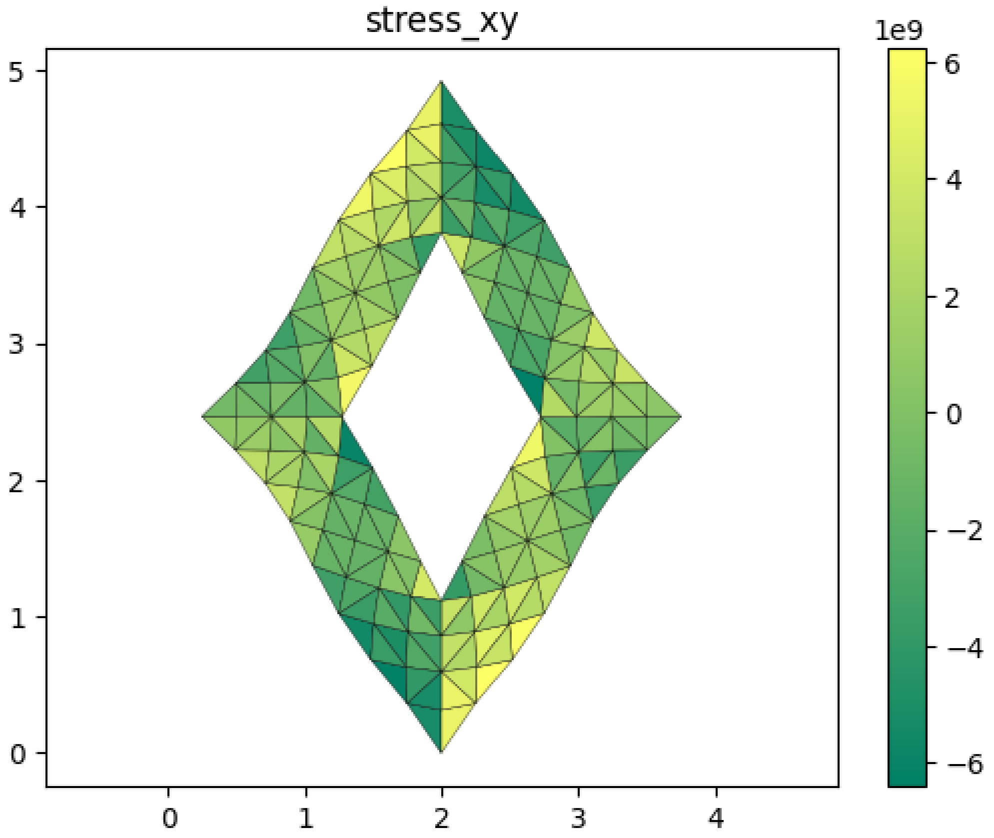
To compute that, we will define a `stress_vector()` function, which compute each element strain vector (σ). Then, loop over every element to compute every element's stress vector, and plot them using color gradient to show the distribution of stress.

In [ ]:

```
# Defining the function to compute stress vector
def stress_vector(D_matrix, strain_vector):
    stress = D_matrix.dot(strain_vector)
    return stress

# Loopoing over every element to obtain every element's stress vector
stress_list = []
for i in range(len(element_list)):
    stress_list.append(stress_vector(D, strain_list[i]))

# Plotting stress
colormap = "summer"

stress_xx_list = np.array([stress_list[i][0][0] for i in range(len(element_list)
plt.tripcolor(new_nodes[:,0], new_nodes[:,1], element_list, facecolors=stress_xx
plt.gca().set_aspect('equal')
plt.axis('equal')
plt.colorbar()
plt.title("stress_xx")
plt.show()

stress_yy_list = np.array([stress_list[i][1][0] for i in range(len(element_list)
plt.tripcolor(new_nodes[:,0], new_nodes[:,1], element_list, facecolors=stress_yy
plt.gca().set_aspect('equal')
plt.axis('equal')
plt.colorbar()
plt.title("stress_yy")
plt.show()

stress_xy_list = np.array([stress_list[i][2][0] for i in range(len(element_list)
plt.tripcolor(new_nodes[:,0], new_nodes[:,1], element_list, facecolors=stress_xy
plt.gca().set_aspect('equal')
plt.axis('equal')
plt.colorbar()
plt.title("stress_xy")
plt.show()
```


stress_xx
1e9
6
4
2
0
−2
−4
−6
−8
0
1
2
3
4
5


stress_yy
1e10
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5



stress_xy
1e9
6
4
2
0
−2
−4
−6
5
4
3
2
1
0
0
1
2
3
4