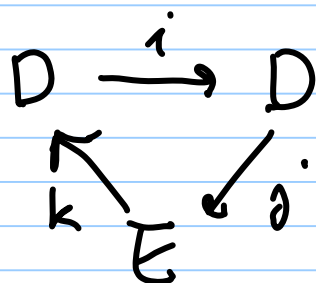


Spectral sequences

Where does a spectral sequence come from?



D, E abelian
groups

(typically, bi-graded)
 $\mathbb{Z} \times \mathbb{Z}$

exact: $\text{Ker } j = \text{Im } i$

$\text{Ker } k = \text{Im } j$, $\text{Ker } i = \text{Im } k$.

= Exact couple

Philosophy: Try to get information on D
from knowing E .

The kind of information we are entitled to
is precisely a spectral sequence.

Example: X (W-complex, E generalised
homology theory)

LGS



$$\begin{array}{ccc} E_{p+q}(X_{p-1}) & \xrightarrow{i} & E_{p+q}(X_p) \\ \downarrow j & & \downarrow j \\ E_{p+q-1}(X_{p-1}) & & E_{p+q-1}(X_p) \end{array} \left\{ \begin{array}{ccc} D & \xrightarrow{i} & D \\ \uparrow k & & \downarrow j^r \end{array} \right.$$

$$\begin{array}{ccc} & \nearrow k & \\ & E_{p+q}(X_p, X_{p-1}) & \nwarrow \varepsilon \\ & & \text{adding up over } p, q \end{array}$$

The bigrading: $E'_{pq} = E_{p+q}(X_p, X_{p-1})$

$$\begin{array}{ccc} E & \xrightarrow{(1, -1)} & E \\ & \searrow i & \nearrow i \\ (-1, 0) & \xrightarrow{k} & D & \xrightarrow{i} & (0, 0) \end{array}$$

$$\begin{pmatrix} E_q & I_p \end{pmatrix}$$

$$O'_{pq} = E_{p+q}(X_p)$$

degree of map:
what does it
add to p, q ?

Single grading $n = p + q$

total degree
 p filtration degree
 q complementary degree

In terms of the single grading:

$$\begin{array}{ccc} D & \xrightarrow{0} & D \\ & \nwarrow k \quad \nearrow j \circ & \\ & E & \end{array}$$

Massey observed that we have, given an exact couple, a derived exact couple

given:

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & \nwarrow k \quad \nearrow j \circ & \\ & E & \end{array}$$

$$\left| \begin{array}{ccc} D' & \xrightarrow{i'} & D' \\ & \nwarrow k' \quad \nearrow j' \circ & \\ & E' & \end{array} \right.$$

$$D' = \text{Im } i$$

$$E' = \text{Ker } d / \text{Im } d$$

where $d = j \circ k$

$$i' \Rightarrow i \mid \mathcal{O}'$$

$$k'([i]) = [k(x)] \leftarrow$$

$$dd = \underbrace{j \circ k \circ j \circ k}_0 = 0$$

$$\text{supposing } x \in \text{Ker}(d) \\ = \text{Ker}(jk)$$

$$jk(x) = 0$$

$$k(x) \in \text{Im}(i).$$

What about if $x \in \text{Im}(d)$?

$$x = jk(y)$$

$$k(x) = kjk(y) = 0.$$

$i'_! k'$ induced
by i, k

$$j' = j \circ i^{-1}$$

$$\begin{array}{ccc} D & \xrightarrow{j} & D \\ \nwarrow k & & \searrow j \\ & \mathbb{G} & \end{array}$$

$$j'(i(x)) = [j(x)]$$

$$D' = \text{Im } i$$

Need to show

$$j \circ \underbrace{k}_{=0} \circ j(x) = 0$$

$$\text{but } k_j(x) = 0 \quad \text{OK.}$$

$$\text{Also need: If } i(x) = 0 \Rightarrow [j(x)] = 0$$

$$x = k(y) \quad j(x) = \underbrace{j \circ k}_{=d}(y) = d(y).$$

(HW 2:) show that

$$\begin{array}{ccc} D' & \xrightarrow{i'} & D \\ \nwarrow k' & & \searrow j' \\ & \mathbb{G} & \end{array}$$

is an exact couple.

This is called the derived exact couple.

In single grading the degrees are

$$\begin{array}{c|c}
 \begin{array}{ccc}
 D & \xrightarrow{0} & D \\
 \nwarrow k & & \nearrow j \\
 \epsilon & & 0
 \end{array}
 &
 \begin{array}{ccc}
 D' & \xrightarrow{0} & D' \\
 \nwarrow k' & & \nearrow j' \\
 \epsilon' & & 0
 \end{array}
 \end{array}$$

i', k' induced
by i, k
" $j' = j \circ i^{-1}$ "

Bogomolov is derived ^{exact} couple

$$\begin{array}{c}
 D \xrightarrow[i]{(1,-1)} D \\
 \begin{array}{c} \nearrow k \\ \nwarrow j \end{array} \\
 (-1,0) \quad (0,0) \\
 E
 \end{array}
 \quad \Bigg| \quad
 \begin{array}{c}
 D' \xrightarrow[i']{(1,-1)} D' \\
 \begin{array}{c} \nearrow k' \\ \nwarrow j' \end{array} \\
 (-1,0) \quad (-1,1) \\
 E'
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{c}
 D'' \xrightarrow[i'']{(1,-1)} D'' \\
 \begin{array}{c} \nearrow k'' \\ \nwarrow j'' \end{array} \\
 (-1,0) \quad (-2,2) \quad \dots
 \end{array}$$

Convention

||

$$\begin{array}{c}
 D^1 \xrightarrow[i^1]{(1,-1)} D^1 \\
 \begin{array}{c} \nearrow k^1 \\ \nwarrow j^1 \end{array} \\
 (-1,0) \quad (0,0) \\
 E^1
 \end{array}$$

$$d^1 = j^1 \circ k^1$$

has deg $(-1,0)$

$$\begin{array}{c}
 D^2 \xrightarrow[i^2]{(1,-1)} D^2 \\
 \begin{array}{c} \nearrow k^2 \\ \nwarrow j^2 \end{array} \\
 (-1,0) \quad (-1,1) \\
 E^2
 \end{array}$$

$$d^2 = j^2 \circ k^2$$

has degree $(-2,1)$

$$\begin{array}{c}
 D^3 \xrightarrow[i^3]{(1,-1)} D^3 \\
 \begin{array}{c} \nearrow k^3 \\ \nwarrow j^3 \end{array} \\
 (-1,0) \quad (-2,2) \\
 E^3
 \end{array}$$

$$d^3 = j^3 \circ k^3$$

degree $(-3,2)$

$$\begin{array}{ccc}
 D^r & \xrightarrow{i^r} & D^r \\
 \nwarrow & \nearrow & \\
 (-1, 0) & E^r & (1-r, r-1)
 \end{array}$$

$$d^r = j^r \circ k^r$$

has degree
 $(-r, r-1)$

$$E^{r+1} = H(E^r, d^r)$$

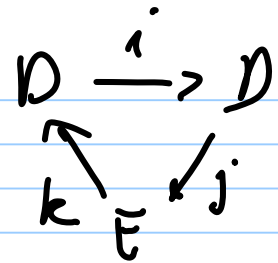
a homological
 spectral sequence!

We also need a direct description of
the $(r-1)$ st derived exact couple

$$\begin{array}{ccc} D^r & \rightarrow & D^r \\ & \nwarrow \quad \nearrow & \\ & E^r & \end{array}$$

The start is an indirect description of E^r in the
alternator
derived exact couple, avoiding d :

$$E' = k^{-1}(\text{Im } i) / j(\text{Ker } i)$$



$$\text{Why } \underbrace{j(\text{Ker } i) \subseteq k^{-1} \text{Im } i}_{\text{14}}$$

$$k j(\text{Ker } i) \subseteq \text{Im } i \quad \text{OK.}$$

$$\begin{array}{c} \parallel \\ 0 \end{array}$$

$$\text{Why } k^{-1}(\text{Im } i) / j(\text{Ker } i) \cong \text{Ker}(j \circ k) / \text{Im}(j \circ k)$$

$$k^{-1}(\text{Im } i) = k^{-1}(\text{Ker } j) = \text{Ker}(j \circ k)$$

$$j(\text{Ker } i) = j(\text{Im } k) = \text{Im}(j \circ k) \quad \checkmark$$

This generalizes. By induction, if we denote the n th derived exact couple by

$$\begin{array}{ccc} D^{(n)} & \xrightarrow{i^{(n)}} & D^{(n)} \\ & \swarrow k^{(n)} \quad \nwarrow j^{(n)} & \\ & E^{(n)} & \end{array}$$

$$D^{(n)} = \text{Im}(\underbrace{i \circ \dots \circ i}_n)$$

$$E^{(n)} = k^{-1} \text{Im}(\underbrace{i \circ \dots \circ i}_n)$$

$$\gamma \cdot \text{Ker}(\underbrace{i_0 \dots i_r}_{m}).$$

In the other notation $\bar{E}^r = \bar{E}^{(r-1)}$, $D^r = D^{(r-1)}$:

$$D^r = \text{Im}(\underbrace{i_0 \dots i_r}_{r+1})$$

$$\left[\bar{E}^r = h^{-1} \text{Im}(\underbrace{i_0 \dots i_r}_{r+1}) / \gamma \cdot \text{Ker}(\underbrace{i_0 \dots i_r}_{r+1}). \right]$$