MATH 425 8/31/2022

Note Title 8/31/2022

Example: Suppose I flip a fair coin 4 times. What is the probability that the number of times H come up is even?

Solution: 16 possibilities

1st flip		2nd		3rd		4th
H / T	$\times$	H / T	$\times$	H / T	$\times$	H / T

$$2 \cdot 2 \cdot 2 \cdot 2 = 16$$

each of the outcomes has probability $\frac{1}{16}$

(equal by symmetry)

H

even # of times

0 times $\frac{1}{16}$

2 times

(which tosses?)

2 out of 4 $\binom{4}{2}$ $\frac{6}{16}$

6

12, 13, 14, 23, 24, 34

4 times $\frac{1}{16}$

Answer: $\frac{1}{16} + \frac{6}{16} + \frac{1}{16} = \frac{8}{16} = \boxed{\frac{1}{2}}$

Follow-up: What is the probability that in 4 flips, H come up an even # of times and T come up an odd number of times?

$\boxed{0}$.

Counting methods in probability theory

Basic counting principle: Counting ordered n-tuples $(s_1, s_2, \dots s_n)$ where $s_i \in S_i$ $(S_1, \dots S_n)$ are sets. Suppose the set S_i has N_i elements: $|S_i| = N_i$.

number of elements

Number of n-tuples $(s_1, \ldots s_n) = N_1 \cdot \ldots \cdot N_n$

$$= \prod_{i=1}^{n} N_i .$$

The set of all such n-tuples is denoted by

$$S_1 \times S_2 \times \ldots \times S_n = \prod_{i=1}^{n} S_i$$

(Cartesian product).

René Descartes

Examples: Suppose I cast a standard dice and then flip a coin 2 times. What is the probability that the number that comes up on the dice plus the number of times H come up on the coin is 4?

Solution: Sample space

$S = \{1, \ldots, 6\} \times \{H, T\} \times \{H, T\}$ ← equally likely outcomes

$$|S| = 6 \cdot 2 \cdot 2 = 24$$

$$E = \{(4, T, T), (3, H, T), (3, T, H), (2, H, H)\}$$

$$|E| = 4$$

$$P(E) = \frac{|E|}{|S|} = \frac{4}{24} = \boxed{\frac{1}{6}}$$

Permutations

Example: How many times can we arrange 3 different books on a bookshelf?

Moby Dick M

Ross : Prob. R

Calculus 3 C

Solution: Pick one book and put it on the left : 3 choices

Pick another book to put next to it on the right: 2 choices

last book only 1 choice

By basic counting principle, # of choices is $3 \cdot 2 \cdot 1 = \boxed{6}$

M R C

M C R

R M C

R C M

C M R

C R M

Arranging n different books on a shelf:

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 =: n! \quad (n \text{ factorial})$$

choices

These choices are called <u>permutations</u>.

$$\|$$

arranging the numbers $\{1, 2, \cdots n\}$ in a row (in any order)

<u>Example</u>: Suppose in a lottery, I am asked to arrange the numbers 1, 2, 3, 4, 5 in an order and I win if the order matches an order

chosen at the draw. What are the chances of winning?

Solution: Sample space S: permutations of $\{1,2,3,4,5\}$

$$|S| = 5! = 120$$

Equally likely outcomes. Chance of winning:

$$\boxed{\frac{1}{120}}$$

Example: Suppose a study plan requires purchasing 4 particular books in math, 3 in physics and 2 in chemistry. Suppose I buy all of them at once and arrange them on the shelf so each subject is together. Suppose a student in room next to mine also did the same thing independently. What is the probability that we will have the same arrangement?

Solution:

First choose the order of subjects: $3!$ choices

Then choose order of books within each subject: $4! \cdot 3! \cdot 2!$

$$|S| = 3! \cdot 4! \cdot 3! \cdot 2!$$

sample space.

Answer: $\boxed{3! \cdot 4! \cdot 3! \cdot 2!}$

An exam answer (no calculators)

answers can be expressions, unless specified

$3! = 6$

$4! = 24$

$$\frac{1}{6 \cdot 24 \cdot 6 \cdot 2} = \frac{1}{72 \cdot 24} = \boxed{\frac{1}{1728}}$$

$$\begin{array}{r} 72 \\ \times\ 24 \\ \hline 288 \\ 144 \\ \hline 1728 \end{array}$$

HW answer

$\doteq 0.000578$

Example: How many different "words" (including nonsensical ones) can we make using all the letters of the word PEPPER?

Solution: $6!$ ← overcounts by swapping the same letters

P	$3!$
E	$2!$
R	$1!$

Answer: $\frac{6!}{3!\,2!\cdot 1!} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot 4\cdot 5\cdot 6}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot \cancel{1}\cdot 2\cdot \cancel{1}} = \boxed{60}$

HW ③ How many different words (including nonsensical ones) can I make using all the letters of the word REFEREE?

④ Suppose I cast a standard die and flip a fair coin three times. What is the probability that the number of times H come up is equal to the number that comes up on the die?