

MATH 695

10/31/2022

Cell chain ex. in an abelian category: $C = \text{colim } C_{(n)}$

$$C_{(-1)} \rightarrow C_{(0)} \rightarrow C_{(1)} \rightarrow \dots$$

mapping
cone

$$C_{(-1)} = 0 \text{ (cell pair: } C_{(-1)} \text{ is anything)}, C_{(n+1)} = C \varphi_{(n)}$$

$$\varphi_{(n)} : P_{(n)} \rightarrow C_{(n)}$$

where $P_{(n)}$ is a chain complex of projective objects with 0 differential.

(notation of a co-cell chain ex. vs dual)

Theorem: All chain complexes in an abelian category are co-local in h.c-chain. Co-all chain complexes are local in h.c-chain.

Proof: One approach is exactly analogous to the case of games:

HGLP. \square

Note: it works more generally. For example, a Differential Graded Algebra is a chain αA of R -modules (R comm. ring) with a multiplication $A \otimes_R A \rightarrow A$ which is associative and unital. The co-local part of the Theorem is true for the category of $^{(H)}DG$ -modules

over A . Cell objects : $\varphi_{(n)} : P_{(n)} \rightarrow C_{(n)} \quad (K, May \dots)$
 $\uparrow \quad C_{(n+1)} = C \varphi_{(n)} \quad \uparrow$

$A \otimes_R M \rightarrow M$

associative
unital.

\oplus free DG- A -modules

Existence of (ω) -localisation: If \mathcal{A} is an abelian category

with enough projectives and coproducts then for every

$\forall M \text{ obj of } \mathcal{A} \exists P \rightarrow M$
 $P \text{ proj.}$

chain complex C in \mathcal{A} -chain there exists $g: C' \xrightarrow{\sim} C$ quasiisomorphism
 with C' cell. (dual for enough injectives & products \rightarrow localisation by ω -cell).

Proof (localization): construct

$$C'_{(n)} \xrightarrow{\gamma_{(n)}} C$$

↑ epimorphism after applying homology
 $H_i(\gamma_{(n)})$ is an epimorphism $\forall i$.

Choose $P_{(n)} \xrightarrow{\varphi_{(n)}} C'_{(n)}$

where $P_{(n)}$ is projective, \odot diff.

$H_i(\varphi_{(n)})$ is epimorphism to
 $\text{Ker } H_i(\gamma_{(n)})$

Set $C'_{(n+1)} = C \varphi_{(n)}$.

$C' = \text{colim}(C_{(n)}),$

□

Example: Projection resolutions of an object of \mathcal{A} are cell chain complexes.
 (Injective resolutions are co-cell chain complexes.)

$$\dots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0 \rightarrow 0 \dots \quad \underline{C_i \text{ projective}}$$

is always a cell chain complex.

$$C_{(n)} := (C_n \rightarrow \dots \rightarrow C_0)$$

$$P_{(n)} := P_{n+1} \oplus P_{(n)}$$

$$\varphi_{(n)} \left\{ \begin{array}{l} 0 \rightarrow C_{n+1} \rightarrow 0 \rightarrow 0 \dots \quad 0 \rightarrow 0 \\ \quad \downarrow \quad \downarrow \\ 0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \dots \rightarrow C_0 \rightarrow 0 \end{array} \right.$$

$$C\varphi_{(n)} \subseteq 0 \rightarrow C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \dots \rightarrow C_0 \rightarrow 0$$

A cell approximation of an object is achieved by a projective resolution (exists if \uparrow there are enough projectives).

(duals, for co-cell).

thought of as a chain c. in degree 0.

Where do Tor and Ext come from?

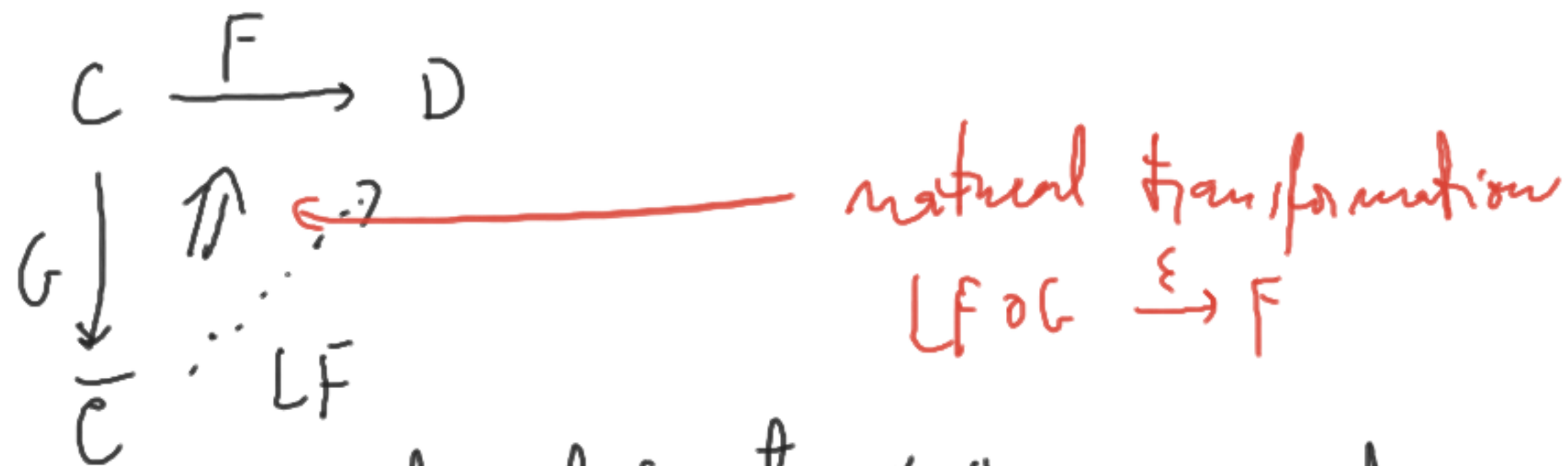
There is a notion of a derived functor. Most generally, a functor

$F: C \rightarrow D$ can be derived along any functor $G: C \rightarrow \bar{C}$.

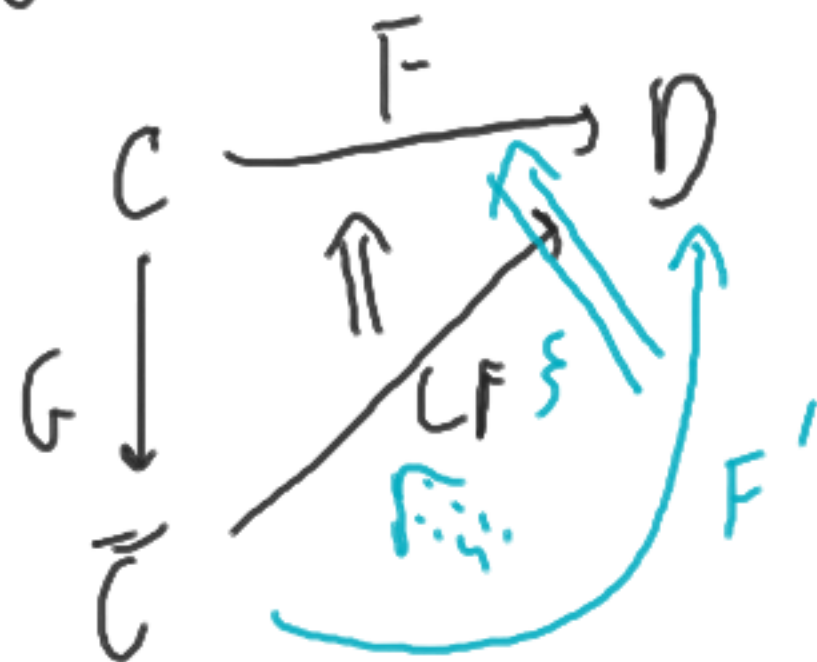
we are interested in

$$\bar{F}: C \rightarrow D\bar{C}$$

derived category
a class of equivalences.



LF is called the left derived functor if it is universal with respect to this property.



$$\begin{array}{ccc}
 F' & \xrightarrow{\xi} & F \\
 \exists! F' \xrightarrow{\kappa} LF & & \\
 \xi = \xi \circ (\kappa \circ G) & &
 \end{array}$$

Dually, RF :

(the right Kan extension)

$$\begin{array}{ccc}
 C & \xrightarrow{F} & D \\
 G \downarrow \eta \Downarrow & & \\
 \bar{C} & \xrightarrow{RF} &
 \end{array}$$

(ω ?) universal with
this property.

Theorem: If C has co-localisation with respect to a class \mathcal{B}
 Then the left derived functor of any functor
 $F: C \rightarrow D$ (with respect to $\Phi: C \rightarrow D_{\mathcal{B}}(C)$)
 is naturally isomorphic to

$$LF(X) = F(X').$$

(dual for ω -localisation). \square

$$\begin{array}{l}
 \gamma_X: X' \xrightarrow{\sim} X \\
 X' \in \mathcal{B}
 \end{array}$$

Example: $\text{Hom} : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{Ab}$

\nearrow
 \nearrow
functor of two variables

RHom with respect to the co-variable
or both variables

but in \mathcal{A}^{op} it gets confusing

\nwarrow cell in $\mathcal{A}^{\text{op}} = \text{cell in } \mathcal{A}$

In particular, for objects $M, N \in \text{Obj}(\mathcal{A})$, one defines

$$\text{Ext}_{\mathcal{A}}^n(M, N) := H^n(\text{RHom}(M, N))$$

Depending on whether it has enough projectives or injectives (or both), we can

① find a proj. resolution C of M

$$\text{Ext}_A^n(M, N) = H^n(\text{Hom}_A(C, N))$$

② find an injective resolution D of N

$$\text{Ext}_A^n(M, N) = H^n(\text{Hom}_A(M, D))$$

same
answers!

HW ① Let $\Lambda[x] = \langle 1, x \rangle$, $x^2 = 0$. Consider the following

chain α of free $\Lambda[x]$ -^{free module} modules.

$$C: \dots \rightarrow \Lambda[x] \xrightarrow{x} \Lambda[x] \xrightarrow{x} \Lambda[x] \xrightarrow{x} \dots$$

$$1 \mapsto x$$

$$x \mapsto 0$$

Prove that C is not cell.