

Vector bundles and characteristic classes

(Milnor - Stasheff: Characteristic classes)

Vector bundle on X : open cover U_i .

$$U_i \times \mathbb{C}^n$$

$$\text{or } U_i \times \mathbb{R}^n$$

$$U_i \cap U_j \xrightarrow[\text{continuous}]{h_{ij}} GL_n(\mathbb{C}) \quad \text{or } GL_n(\mathbb{R})$$

$$h_{jk} \circ h_{ij} = h_{ik}$$

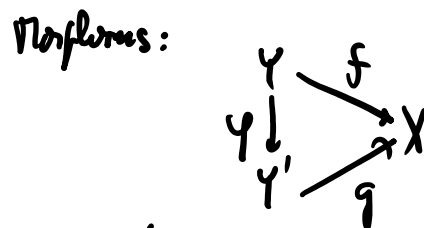
- Details needed: passing to a refinement of the open cover.

- Isomorphism: $\varphi_i: U_i \rightarrow GL_n(\mathbb{C}) \dots h_{ij} \varphi_i = \varphi_j h_{ij}$

$$H_{\text{int.}}^1(X; GL_n(\mathbb{C})) \leftarrow \text{classifies vector bundles over } \mathbb{C} \text{ or } \mathbb{R}$$

$\nwarrow GL_n(\mathbb{R})$

Another approach to vector bundles: Category Top_X : topological spaces $/ X$.
 Objects: $Y \xrightarrow{f} X$
 Continuous



Abelian group in Top_X . \rightsquigarrow \mathbb{C} -vector space $/ X$. (= in Top_X).
 \mathbb{R} -

Vector bundle: \mathbb{C} -vector space $/ X$ which, when pulled back to U_i , becomes $\cong \mathbb{C}^n \times U_i \rightarrow U_i$.
 \mathbb{R}^n

Classification of vector-bundles using homotopy classes: consider

$$\left. \begin{array}{l} \text{Define } BGL_n(\mathbb{C}) \simeq BU_n \\ BGL_n(\mathbb{R}) \simeq BO_n. \end{array} \right\} \begin{array}{l} n\text{-dim. vector subspaces in } \mathbb{C}^{\oplus \infty} \\ \mathbb{R}^{\oplus \infty} \end{array}$$

We have a tautological (= universal) n -bundle

$$\gamma_{\mathbb{C}}^n \text{ on } BU_n$$

eq. $\gamma_{\mathbb{R}}^n \text{ on } BO_n$

"The fiber over a point $V \in BU_n$
is V " $V \subset \mathbb{C}^{\oplus \infty}$

$$p: E \rightarrow BU_n$$

analogously for \mathbb{R}

$$\{(x, V) \mid V \in BU_n, x \in V\}$$

Theorem: If X is paracompact, we have a bijection

$$\{ \cong \text{ classes of } \text{vector } n\text{-bundles on } X \} \cong [X, BU_n]$$

unbased homotopy classes of maps

$$f^*(\gamma_{\mathbb{C}}^n)$$

$$\longleftarrow f: X \rightarrow BU_n$$

□

Characteristic classes: E be a generalized cohomology theory.

$X \simeq CW$ -complex.

ξ is a real n -vector bundle on X
complex

$\mapsto \alpha(\xi) \in E^i(X)$ only depends on \cong class of ξ

E^i :

characteristic class α on real n -bundles is.

The only axiom: Functoriality: $f: Y \rightarrow X$ continuous map, ξ is an n -bundle on X :

$$\begin{array}{ccc} \alpha(f^*\xi) & = & f^*(\alpha(\xi)) \\ \cap & & \cap \\ E^i(Y) & \xleftarrow{f^*} & E^i(X) \end{array}$$

The cases we will discuss at this point: real vector bundles: $E = H(\cdot; \mathbb{Z}/2)$

Stiefel-Whitney classes $H\mathbb{Z}/2$

complex vector bundles: $E = H(\cdot; \mathbb{Z})$

Chern classes

$H\mathbb{Z}$

$E = H\mathbb{Z}[\frac{1}{2}]$ Pontryagin classes

oriented n -bundles

Euler class $H^n(\cdot; \mathbb{Z})$

Computing characteristic classes:

Most fundamental case:
Chern classes - via Schubert calculus

To characterise a characteristic class α , all we need is $\alpha(\gamma_{\mathbb{C}}^n)$
 $\alpha(\gamma_{\mathbb{R}}^n)$
 ξ any bundle (say, complex) on X

$$\alpha(\xi) = f^*(\alpha(\gamma_{\mathbb{C}}^n))$$

$$f: X \rightarrow BU_n$$

$$\xi \cong f^*(\gamma_{\mathbb{C}}^n)$$

\therefore Calculating E - characteristic classes of real & vector bundles (complex)

is equivalent to calculating $E^+(BO_n)$

$$E^+(BU_n).$$

$H\mathbb{Z}/2$

$H\mathbb{Z}$

CW - (co)homology: We will give BU_n (w.p. BO_n)
 a structure of a CW-complex.

We can say something even more basic:

$$BU_1 = \mathbb{C}P^\infty \quad \swarrow \text{all lines in } \mathbb{C}^{\oplus \infty}$$

$$BO_1 = \mathbb{R}P^\infty \quad \swarrow \text{lines in } \mathbb{R}^{\oplus \infty}$$

$$U_1 = S^1$$

$$O_1 = \mathbb{Z}/2$$

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[t]$$

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[z]$$

dimension: 1
dimension: 2

Poincaré duality for $\mathbb{R}P^n$ $\mathbb{C}P^n$

First thing next time: An easy guess of what might, most naively, happen for $n > 1$.