MATH 592 3/8/2024

Topics for exam:

- Universal covering of a CW-complex
- Free group F, homomorphism $h: F \twoheadrightarrow G$ find free generators and rank of $\mathrm{Ker}(h)$ (Cayley graph)
- Free generators of a subgroup G of a free group F generated by given elements (casting out $\rightsquigarrow$ covering corresponding to G)
  - regular covering
- Categories of coverings (based, unbased) vs. category of π_1-sets
  - all coverings of a given degree (the level of HW)

Homology of spaces X (X Hausdorff)

Different approaches. (analog: theory of integration)

$\pi_n(\mathbb{Z}X)$

another path to homology

Singular homology

chain complex

$$X \xrightarrow{?} CX \xrightarrow{H_n} H_n X$$

? higher-dimensional analogs of paths

$[0,1] \xrightarrow{\omega} X$

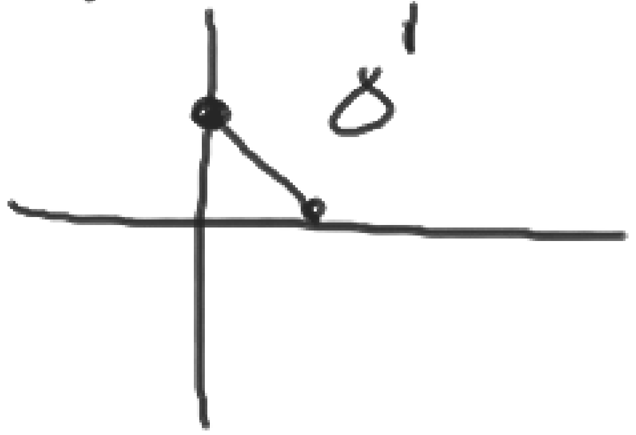
? higher-dim. analogue?

cubes? Massey: Singular homology theory

The standard simplex:

$$\Delta^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1 \,,\; \underbrace{x_i \geq 0}_{\text{purely aesthetic}} \right\}$$

historic notation: $[x_0, \dots, x_n]$ ← barycentric coordinates.



Defining the chain complex CX:

$$\cdots \to C_nX \xrightarrow{d} C_{n-1}X \xrightarrow{d} \cdots \xrightarrow{d} C_0X \to 0 \to 0 \to \cdots$$

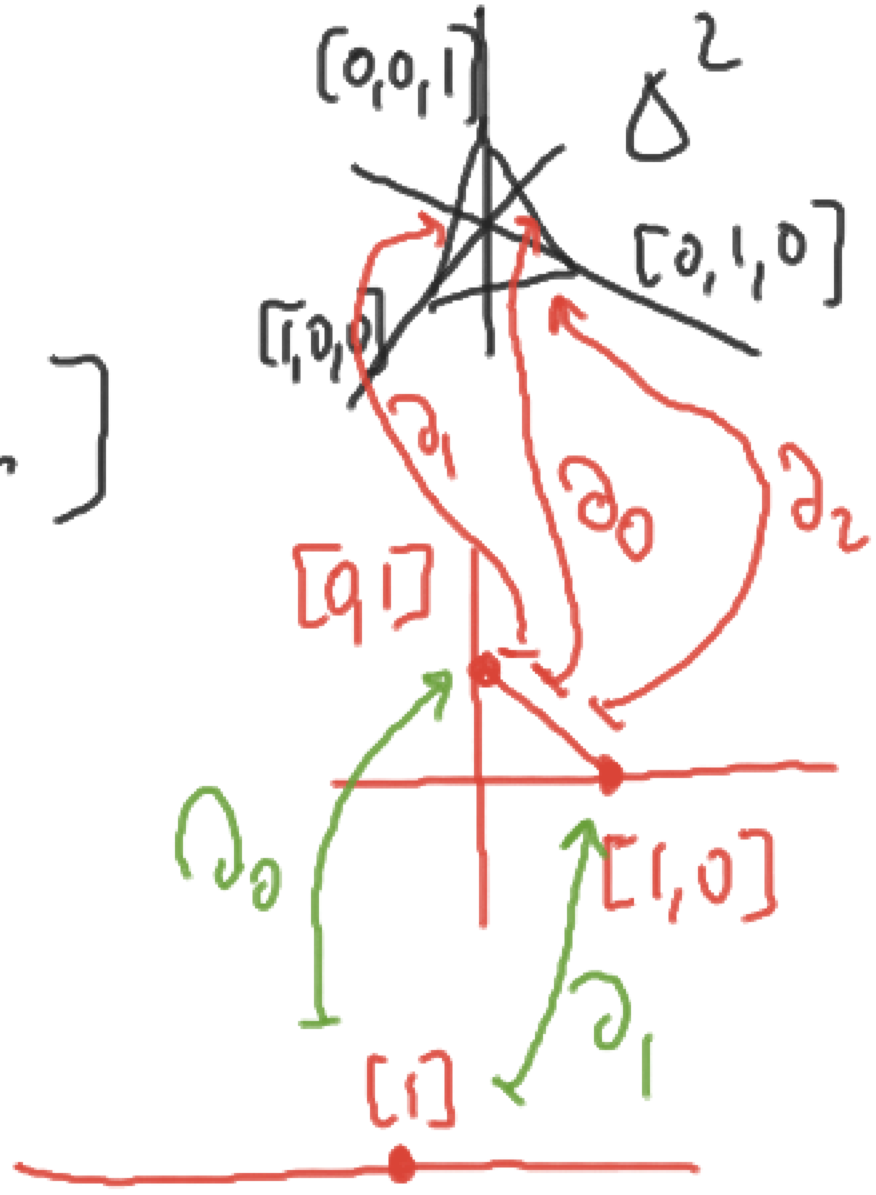
$$C_nX = \underset{\text{free abelian group}}{\mathbb{Z}} \{ \underbrace{\sigma : \Delta^n \to X}_{\text{singular simplices}} \text{ continuous} \}$$

← S_nX = all singular simplices on X

Some useful maps between standard simplices:

$$\partial_i : \Delta^n \longrightarrow \Delta^{n+1}, \quad i=0,\dots,n$$

$$[x_0,\dots,x_n] \longmapsto [x_0,\dots,x_{i-1},0,x_i,\dots x_n]$$



$$\begin{array}{ccc} d: C_n X & \longrightarrow & C_{n-1} X \\ \| & & \| \\ \mathbb{Z} S_n X & & \mathbb{Z} S_{n-1} X \end{array}$$

universal property: $\boxed{d(\sigma:\Delta^n \longrightarrow X) = \sum_{i=0}^{n} (-1)^i \sigma\circ\partial_i}$

$$\sigma\circ\partial_i : \Delta^{n-1} \xrightarrow{\partial_i} \Delta^n \xrightarrow{\sigma} X$$

(*) $\mathbb{Z}S = \{a : S \longrightarrow \mathbb{Z} \mid \exists F \subset S$ finite $\forall x \in S \setminus F \quad a(x) = 0\}$

"$\sum a(s) \cdot s$" finite sums only

$S \ni s \longmapsto 1_s$

$1_s(s) = 1$

$1_s(t) = 0 \quad t \in S \setminus \{s\}$

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "\text{any map}"'] & \mathbb{Z}S \arrow[d, dashed, "\exists!\ \text{homomorphism}"] \\ & G \end{tikzcd}$$

abelian group

Free abelian group

$S \longmapsto \mathbb{Z}S$

is left adjoint to the forgetful functor

$Ab \longrightarrow Set$

HW (7) Prove that the definition (*) of the free abelian group on S satisfies the universal property.

Lemma: In ⊞, we have $d \circ d = 0$.

Proof:

$$d \circ d(\sigma) = d\left(\sum_{i=0}^{n} (-1)^i \sigma \circ \partial_i\right) =$$

$\sigma : \Delta^n \to X$

$$= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j \sigma \circ \partial_i \circ \partial_j$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} \sigma \circ \partial_i \circ \partial_j = 0 \qquad \square$$

cancels in pairs

$\bar{i} = j+1$

$\bar{j} = i$

$$\Delta^{n-2} \xrightarrow{\partial_j} \Delta^{n-1} \xrightarrow{\partial_i} \Delta^n$$

Assume $i < j$

$$\partial_i \circ \partial_j = \partial_{j+1} \circ \partial_i$$

$\{(i,j) \mid i \in \{0, \dots, n-1\}, j \in \{0, \dots, n\}, i < j\}$

$i = \bar{j}$, $j = \bar{i} - 1$ ← inverse

bijection $(i,j) \mapsto (\bar{i}, \bar{j})$

$\{(\bar{i}, \bar{j}) \mid \bar{i} \in \{0, \dots, n-1\}, \bar{j} \in \{0, \dots, n\}, \bar{i} \geq \bar{j}\}$