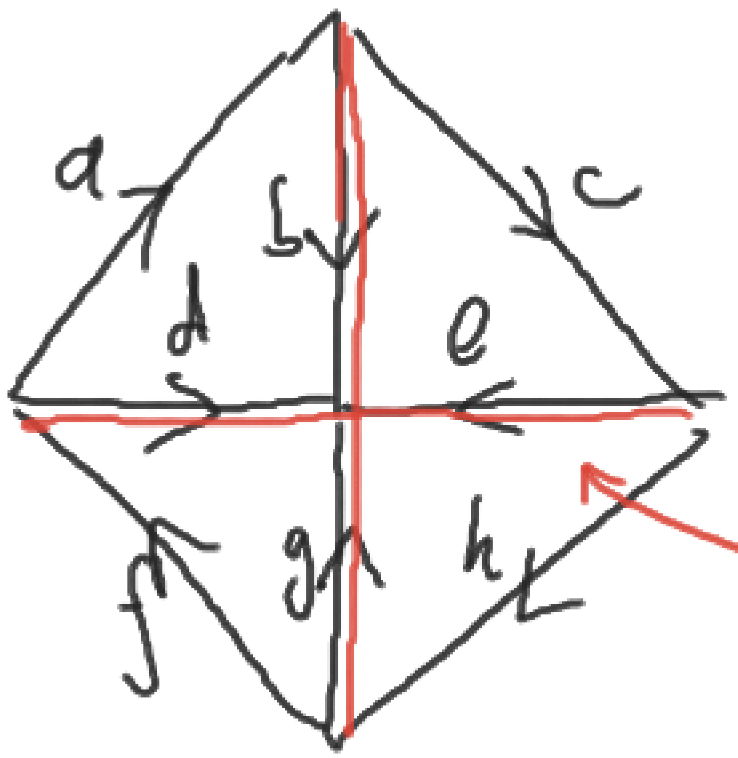
MATH 592 4/19/2024

Review

①



$= X_1$

$\tilde{H}_*(X) \cong H_*(X, T)$

T spanning tree - ignore edges in it

$X = X_2$ is formed by attaching 2-cells via the cycles

$9a + 9c + 9h + 9f$ $\alpha : (achf)^9$

$-6a$ $\beta : (a^{-1}db^{-1})^6$

0 $\gamma : (b^{-1}ceg^{-1}h^{-1}c^{-1}be^{-1}hg)^2$

Calculate $H_* X$

$H_0 X = \mathbb{Z}$

work with $\tilde{H}$

factor out a spanning tree

$\tilde{C}^{cell}(X)$: $\overset{\text{dim.}}{\text{degree}}$ 2 1

$$\mathbb{Z}^3 \xrightarrow{d} \mathbb{Z}^4$$

$$\begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline a & 9 & -6 & 0 \\ c & 9 & 0 & 0 \\ f & 9 & 0 & 0 \\ h & 9 & 0 & 0 \end{array}$$

↖ rank = 2

(over $\mathbb{Q}$ or $\mathbb{R}$: number of pivots)

$$\begin{array}{c} \tilde{H}_2 X \\ \| \\ H_2 X = \mathbb{Z} \end{array}$$

rank $\tilde{H}_1 X$ = rank $H_1 X$ =

= 2

$H_1 X = \mathbb{Z} \oplus \mathbb{Z} \oplus$ torsion

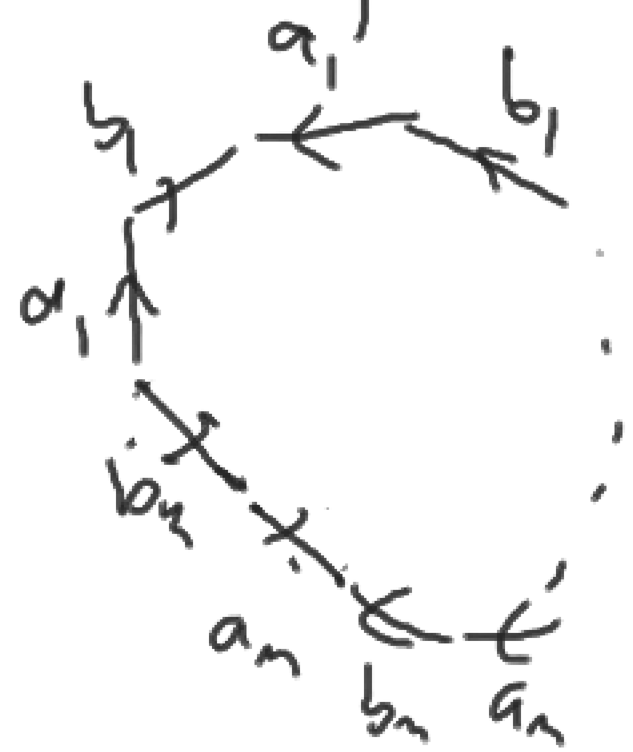
$$1 \circlearrowright \begin{pmatrix} 9 & 9 & 9 & 9 \\ -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \text{\textcircled{3}} & \overset{-3}{9} & \overset{-3}{9} & \overset{-3}{9} \\ -6 & 0 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \sim 2\downarrow \begin{pmatrix} 3 & 0 & 0 & 0 \\ -6 & 18 & 18 & 18 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 18 & 18 & 18 \end{pmatrix}$$

$$\sim \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 18 & 0 & 0 \end{pmatrix}$$

$$\boxed{H_2 X = \mathbb{Z} \,,\ H_1 X = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/18 \,,\ H_0 X = \mathbb{Z}}$$

Reviewing homology of compact surfaces:

$X = \underbrace{T \# \cdots \# T}_{n}$:



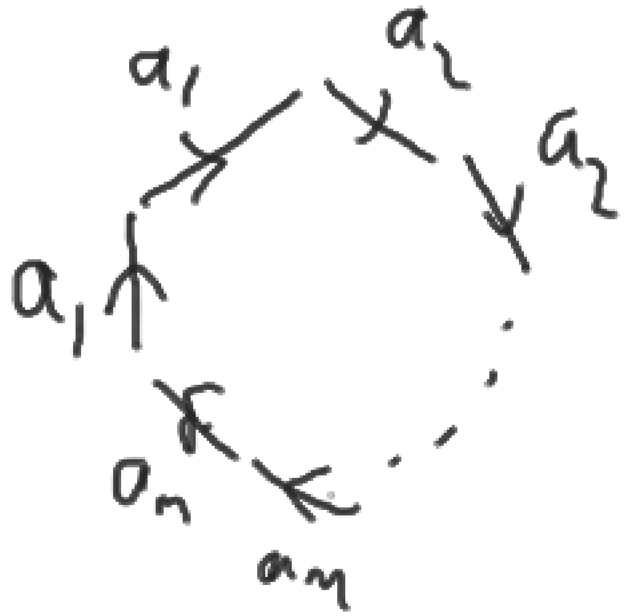
$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2n}$$

$H_2 X = \mathbb{Z}$

$H_1 X = \mathbb{Z}^{2n}$

$H_0 X = \mathbb{Z}$

$X = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}$



$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}} \mathbb{Z}^n$$

$H_2 X = 0$

$H_1 X = \mathbb{Z}/2 \underbrace{\oplus \oplus \oplus}_{n-1} \mathbb{Z}$

$H_0 X = \mathbb{Z}$

② Find the homology of the CW-cx X given as a pushout

$$\begin{tikzcd} U\cap V & \mathbb{C}P^1 \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\ulcorner"] & \mathbb{C}P^n & V \\ & \mathbb{C}P^n & & \\ & U & & X = U \cup V \end{tikzcd}$$

Mayer-Vietoris

	$H_* U\cap V$		$H_* U \oplus H_* V$	$H_* U \cup V$
$2n$	$\vdots$		$\vdots$	$\mathbb{Z}\oplus\mathbb{Z}$
$\vdots$				$\vdots$
				$\mathbb{Z}\oplus\mathbb{Z}$
6				
4	0		$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$
3	0			
2	$\mathbb{Z}$	$\xrightarrow{(i)}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$
1	0			
0	$\mathbb{Z}$	$\xrightarrow{(i)}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$

$H_k \mathbb{C}P^n = \mathbb{Z}$ $0 \leq k \leq 2n$ even

0 else

$H_k \mathbb{R}P^n = \mathbb{Z}$ if $k=0$ or n odd

$\mathbb{Z}/2$ if $0<k<n$ odd

0 else

$H_k X = \mathbb{Z}$ if $k=0,2$

$\mathbb{Z}\oplus\mathbb{Z}$ if $4 \leq k \leq 2n$ even

0 else

Example ㉖ : $U \cap V$

$$\begin{tikzcd} S^1 \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\cap|"] & \mathbb{R}P^2 \; U \\ \mathbb{R}P^2 \; V & \end{tikzcd}$$

generate $\pi_1 \mathbb{R}P^2 = \mathbb{Z}/2$

Pushout X. ? $H_* X$

Work reduced Mayer - Vietoris

$$\boxed{\begin{array}{l} H_0 X = \mathbb{Z} \\ H_1 X = \mathbb{Z}/2 \\ H_2 X = \mathbb{Z} \end{array}}$$

$$\begin{tikzcd} 2 & & 0 \arrow[r] & \mathbb{Z} \arrow[dll, bend right] \\ 1 & \mathbb{Z} \arrow[r, "\begin{pmatrix}1\\1\end{pmatrix}"] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \\ 0 & & & \\ \text{deg} & \tilde{H}_*(U \cap V) & \tilde{H}_*(U) \oplus \tilde{H}_* V & \tilde{H}_*(\underbrace{U \cup V}_{X}) \end{tikzcd}$$

③ Lefschetz Thm. / Euler characteristic (X = finite CW-complex)

③a For which n does there exist a continuous map $f: \mathbb{R}P^n \to \mathbb{R}P^n$ with no fixed point?

We can ignore torsion in $H_* \mathbb{R}P^n$.

n even

$$\begin{tikzcd} 0 & 0 \\ \mathbb{Z} \arrow[r, "\mathrm{Id}"'] & \mathbb{Z} \end{tikzcd}$$

$\boxed{\operatorname{tr} f = 1}$

Does not exist for n even.

$H_0 X = \mathbb{Z}\pi_0 X$ functorially n odd $\mathbb{Z} \xrightarrow{f_*} \mathbb{Z}$ could be anything ??

(-1 => no conclusion
$\operatorname{tr} f_* = 0$)

$n = 0$ $\mathbb{Z} \xrightarrow{Id} \mathbb{Z}$ (

n odd:

$$\begin{tikzcd} \mathbb{Z}/k \arrow[r, phantom, "\text{acts freely on}"] & S^{2k+1} \\ S^1 \arrow[u, phantom, "\cap"] \arrow[r] & \mathbb{C}^{k+1} \arrow[u, phantom, "\cap"] \end{tikzcd}$$

$$\lambda(x_0, \dots x_k) = (\lambda x_0, \dots \lambda x_k)$$

Free action of $\mathbb{Z}/4$ on S^n
n odd

=> free action of $\mathbb{Z}/2$ on $\mathbb{RP}^n = S^n/\mathbb{Z}/2$

generator no fixed points

$\boxed{\text{YES if exist for } n \text{ odd}}$

3b Is $T \# T$ a 3-fold cover of a CW-complex?

↑

X

$$\chi(T\#T) = -2$$

$$\underbrace{\chi(T\#T)}_{-2} = 3\,\chi(X)$$

X

No.

$$\chi(\#_n T) = 2-2n$$

$$\chi(\#_k \mathbb{R}P^2) = 2-k$$

(3c) Attach a 3-cell to S^2 by a map of degree 4 to form a CW-complex X. Can X have a free $\mathbb{Z}/3$-action?

deg:	3	2	1	0
$C_*^{cell} X$	$\mathbb{Z} \xrightarrow[4]{}$	$\mathbb{Z}$	0	$\mathbb{Z}$
$H_* X$		$\mathbb{Z}/4$	0	$\mathbb{Z}$

torsion can be ignored

has to induce Id in dg = 0

$\alpha \in \mathbb{Z}/3$ generator has no fixed points

$$\underset{X}{L(\alpha_*)} = 1$$

$\boxed{\text{Does not exist}}$

④ topological manifold means without boundary on this exam

(4a) For what values n is $X = \mathbb{C}^n / z \sim iz$ a topological manifold?

? $x \in X$

$$H_k(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = \text{dimension} \\ 0 & \text{else} \end{cases}$$

LES in reduced homology

$$H_k(X, X \setminus \{x\}) \cong \tilde{H}_{k-1}(X \setminus \{x\})$$

$X \simeq *$

$(h_t(x) = tx$
$t \in [0,1])$

Suspect point: $0 = (0, \cdots 0)$

$\tilde{H}_{k-1}(X \underset{\mathbb{Z}}{\setminus} \{0\})$ $\leftarrow$ $2n$

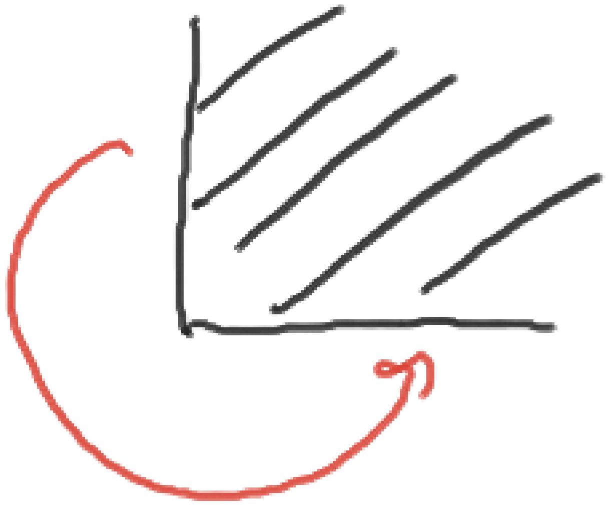
$k =$		possible
$\vdots$	$\mathbb{Z}/4$	
2	$\mathbb{Z}/4$	$\mathbb{Z}$
1		
0		
	$n > 1$ ✗	$n = 1$

$X \setminus \{0\} \simeq S^{2n-1}/\mathbb{Z}/4$

$\uparrow$ free action by complex unit scalar mult.

homology	0	1	2	3			$2n-1$
	$\mathbb{Z}$	$\mathbb{Z}/4$	0	$\mathbb{Z}/4$	$\cdots$	$\mathbb{Z}/4$ 0	$\mathbb{Z}$

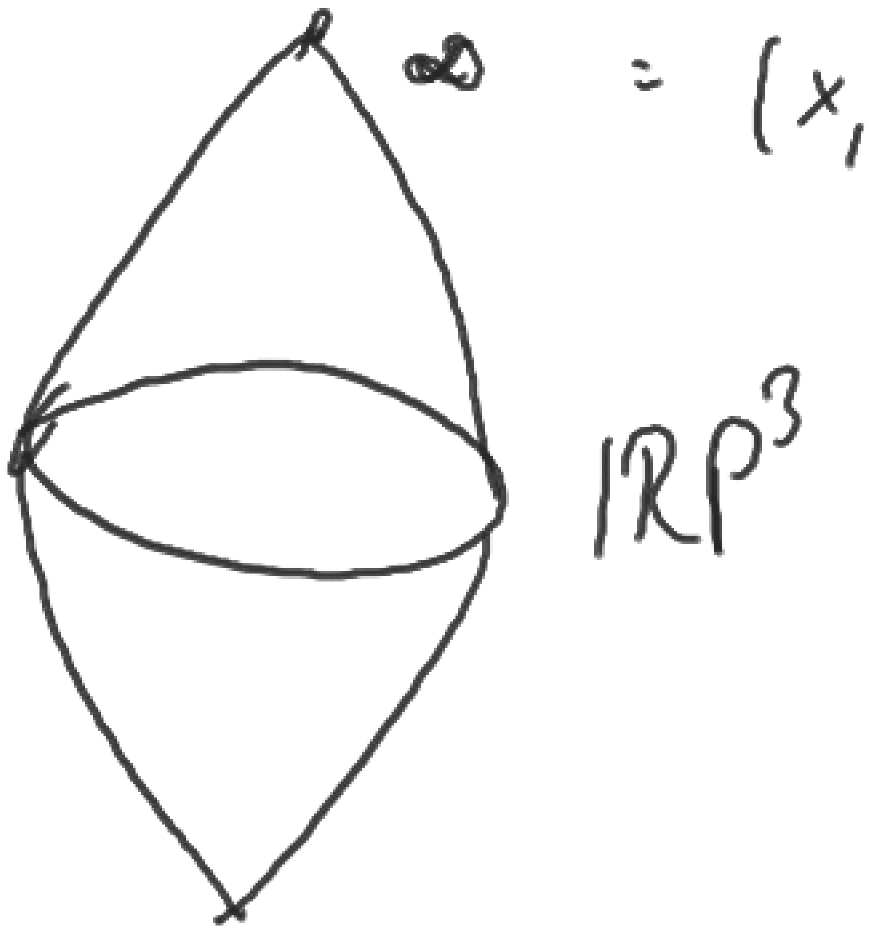
$\mathbb{C}/x \sim ix \cong \mathbb{C}$



Answer: Yes $n = 0, 1$

No $n \neq 1$

(4b) Is the unreduced suspension X of $\mathbb{RP}^3$ a topological manifold



$\infty = (x,0) / \ldots$

Please do teaching evaluations!

$X \setminus \{\infty\} \simeq *$

$H_k(X, X \setminus \{\infty\}) \cong \tilde{H}_k(X)$

$\mathbb{Z}$

0

X → $\mathbb{Z}/2$

0

0

Answer: NO

$\rightarrow \tilde{H}_k(S\mathbb{RP}^3) \cong \tilde{H}_{k-1}(\mathbb{RP}^3) \leftarrow$

deg	
3	$\mathbb{Z}$
2	0
1	$\mathbb{Z}/2$
0	0

Good luck!