MATH 592 3/15/2024

X topological space.

Define <u>singular chain complex</u> CX:

$$C_n X := \mathbb{Z} S_n X \qquad S_n X := \{\sigma : \Delta^n \to X \text{ continuous}\}$$

$$d_n : C_n X \to C_{n-1} X \quad : \quad d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \partial_i$$

$$H_n X := H_n(CX)$$

Functors

$C : \mathrm{Top} \to \mathrm{Chain}$

$H_n : \mathrm{Chain} \to \mathrm{Ab}$

$H_n : \mathrm{Top} \to \mathrm{Ab}$

HW 1 Compute $H_0 X$ for every space X.

Pair is the category of pairs of topological spaces (Hausdorff)

$\mathrm{Obj}(\mathbf{Pair})$: (X, Y), $Y \subseteq X$ subspace topology

$$\mathrm{Mor}_{\mathbf{Pair}}\big((X_1, Y_1), (X_2, Y_2)\big) := \{ f : X_1 \to X_2 \mid f \text{ continuous}, f(Y_1) \subseteq Y_2 \}.$$

The singular chain complex and homology of a pair:

$$C(X, Y) = CX / CY$$

$$\left(\begin{matrix} S_n Y & \subseteq & S_n X \\ C_n Y & \subseteq & C_n X \end{matrix} \right)$$

$$H_n(X, Y) := H_n C(X, Y).$$

$$0 \to CY \to CX \to C(X, Y) \to 0$$

Note: $C_n(X,Y)$ is a free abelian group.

HW ② If $T \subseteq S$ then $\mathbb{Z}S / \mathbb{Z}T \cong \mathbb{Z}(S \setminus T)$.

Functors: $C : \mathrm{Pair} \to \mathrm{Chain}$

$H_n : \mathrm{Pair} \to \mathrm{Ab}$.

A homotopy of pairs $f, g : (X_1, Y_1) \to (X_2, Y_2) \in \mathrm{Mor}(\mathrm{Pair})$

$h : f \simeq g$ means $h : (X_1 \times [0,1], Y_1 \times [0,1]) \to (X_2, Y_2)$

$\forall x \in X_1 \quad h_0(x) = f(x)$

$\qquad\qquad h_1(x) = g(x)$

equivalence relation

$f \simeq g$ means $\exists h : f \simeq g$ $\qquad$ hPair : Category, $\mathrm{Obj}\, h(\mathrm{Pair}) = \mathrm{Obj}\, \mathrm{Pair}$

Mor (h Pair) = homotopy classes of morphisms in Pair.

Inclusion functor $\quad \mathrm{Top} \xrightarrow{\subseteq} \mathrm{Pair}$

$$X \longmapsto (X, \emptyset).$$

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "\subseteq"] \arrow[dr, "H_n"'] & \mathrm{Pair} \arrow[d, "H_n"] \\ & \mathrm{Ab} \end{tikzcd}$$

<u>Note</u>: $Z \subseteq Y \subseteq X$ subspace topology

$$0 \to C(Y,Z) \to C(X,Z) \to C(X,Y) \to 0$$

(by the 9-lemma = Noether isomorphism theorem)

Eilenberg - Steenrod axioms (make H_n calculable for CW-complexes)

① $H_n : hTop \to Ab$, $H_n : hPair \to Ab$ are functors. (Homotopy axiom)

② (Exactness axiom): For a pair (X,Y), we have a long exact sequence
$(i : Y \xrightarrow{\subseteq} X,\ j : (X,\emptyset) \xrightarrow{\subseteq} (X,Y))$

$$\cdots \to H_n(Y) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,Y) \xrightarrow{\partial} H_{n-1}(Y) \xrightarrow{i_*} \cdots$$

Note also: $Z \subseteq Y \subseteq X$ $\quad i : (Y,Z) \subseteq (X,Z)$ $\quad j : (X,Z) \to (X,Y)$

$$\cdots \to H_n(Y,Z) \xrightarrow{i_*} H_n(X,Z) \xrightarrow{j_*} H_n(X,Y) \xrightarrow{\partial} H_{n-1}(Y,Z) \xrightarrow{i_*} \cdots$$

where ∂ is natural. $(X_1,Y_1) \xrightarrow{f} (X_2,Y_2)$

$$\begin{tikzcd} H_n(X_1,Y_1) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(Y_1) \arrow[d, "f_*"] \\ H_n(X_2,Y_2) \arrow[r, "\partial"] & H_{n-1}(Y_2) \end{tikzcd}$$

similarly in relative case

③ (Excision axiom): $Z \subseteq Y \subseteq X$ subspace topologies

Closure of Z in X $\subseteq$ interior of Y in X

$i : (X \setminus Z, Y \setminus Z) \xrightarrow{\subseteq} (X, Y)$ induces an isomorphism in homology:

$$H_n(X \setminus Z, Y \setminus Z) \underset{\cong}{\xrightarrow{i_*}} H_n(X, Y).$$

This lets us calculate $H_n X$ X CW-cx.

④ (Dimension axiom): $H_0(*) = \mathbb{Z}$

$H_n(*) = 0 \quad n \neq 0.$

But if we omit this axiom, we get $\boxed{\text{Generalised homology}}$

main subject of algebraic topology

$$\left(C_n(*) = \mathbb{Z} \quad \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \underset{\substack{\uparrow \\ \deg 0}}{\mathbb{Z}} \to 0 \to 0 \to 0 \cdots \right)$$

(5) (co) limit axioms: I = indexing set, X_i spaces, $i \in I$

$$X_i \xrightarrow{\iota_i} \coprod_{i \in I} X_i$$

The coproduct of abelian groups: $\bigoplus_{i \in I} A_i = \{ (a_i)_{i \in I} \mid \exists F \subseteq I \text{ finite}$
$a_i = 0 \quad i \notin F \}$

Product : $\prod_{i \in I} A_i = \{ (a_i)_{i \in I} \}$ Cartesian product

HW (3) Prove this characterisation of coproduct and product in Ab.

$$\bigoplus_{i \in I} H_n(X_i) \xrightarrow[\cong]{\bigoplus (\iota_i)_*} H_n\Big(\coprod_{i \in I} X_i\Big)$$ is an isomorphism

$X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots\cdots$ $\qquad X = \bigcup_{i \in \mathbb{N}} X_i$ colimit topology

subspace topologies

$\iota_i : X_i \xrightarrow[\subseteq]{\cong} X$ inclusions

$$\boxed{\operatorname{colim}_i H_n X_i \xrightarrow{\cong} H_n X \text{ is an isomorphism}}$$

colimit map

We already proved the exactness axiom (purely algebraic).
The axioms we need to prove are the <u>homotopy axiom</u> and the <u>excision axiom</u>.

<u>Chain homotopy</u>. If $f, g : C \to D$ are morphisms in Chain (chain maps)
a <u>chain homotopy</u> $h : f \simeq g$ is a sequence of homomorphisms

$$h_n : C_n \to D_{n+1}, \quad \text{for all } n \in \mathbb{Z}$$

such that

$$\boxed{dh + hd = f - g}$$

$$d_{n+1} h_n + h_{n-1} d_n = f_n - g_n$$

$$\begin{tikzcd}
 & D_{n+1} \arrow[d, "d_{n+1}"] \\
C_n \arrow[ur, "h_n"] \arrow[r, shift left, "f_n"] \arrow[r, shift right, "g_n"'] \arrow[d, "d_n"'] & D_n \\
C_{n-1} \arrow[ur, "h_{n-1}"'] &
\end{tikzcd}$$

HW ④ Prove that if $h: f \simeq g$ is a chain homotopy then f, g induce the same homomorphism on H_n for all n.

$\exists h: f \simeq g$ write $f \simeq g$ equivalence relation preserves composition, we have a category h Chain. (Chain complexes and homotopy classes of chain maps.)

homotopy on a singular simplex

the (triangulated) prism is the chain homotopy