ARBITRARILY HIGH GROWTH IN QUASI-PRE-TANNAKIAN CATEGORIES

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Abstract. The purpose of this note is to use a universal algebra formalism which we call a *T-algebra* to construct additive, locally finite categories with symmetric monoidal tensor product which are rigid and have arbitrarily high categorical growth. Our example solves a known open problem in the field, but leaves open the question as to whether there exist abelian categories with this property.

1. Introduction

Pre-Tannakian categories are abelian, k-linear (for a field k), locally finite symmetric monoidal categories which are rigid (i.e. have strong duality in the sense of [4]). Interest in these categories originated from the study of Tannakian categories, which were developed by P. Deligne and J. S. Milne in [3] in connection with constructing categories of motives. However, pre-Tannakian categories have evolved into a field of independent significance, due to their connections with representation theory. Background information about pre-Tannakian categories can be found for example in [5, 6, 7].

In this paper, we define and study a broader class of categories:

Definition 1. *A* quasi-pre-Tannakian (QPT) category *is an additive, locally finite category which has a symmetric monoidal tensor product and is rigid.*

We introduce a universal algebra type called a *T-algebra* which can be used to determine a QPT category by specifying the Hom-modules of the tensor powers of a certain generating object, with their equivariant, tensor, trace, and unital structure.

One important and longstanding question about tensor categories is the study of their *growth*. For a QPT category $\mathscr{C}$ which k-linear over a field k, we may define the *growth* of an object X as the sequence

$$rank(End_{\mathscr{C}}(X^{\otimes n})) \tag{1}$$

for positive integers n. It turns out that growth encodes deep information about the algebraic structure of a category. P. Deligne [2] proved that, over a field of characteristic 0, a pre-Tannakian category $\mathscr{C}$ generated by a single object X is equivalent (as a tensor category) to the category of finite-dimensional representations of an affine group superscheme if and only if it is of *moderate growth*, meaning that there exists a constant $c \in \mathbb{R}_{>0}$ such that $length(X^{\otimes n}) \leq e^{c \cdot n}$ for all n. This also implies an at most exponential upper bound on (1).

The first examples of semisimple QPT categories (which must therefore be abelian, and thus, in fact, pre-Tannakian) of superexponential growth were constructed by P. Deligne and J. S. Milne in 1982 in [3]. These categories can be thought of as free QPT categories on one object X (described by the free T-algebra on 0 generators). They can also be viewed as interpolated categories of representations of the general linear groups $\underline{Rep}(GL_t)$. They are semisimple after extending scalars to a field of characteristic 0, for t not a non-negative integer. They have growth $e^{n \cdot ln(n)}$. More recent literature on pre-Tannakian categories and their growth includes [1, 7, 8, 9, 10, 12], with the highest currently known growth attained in [11]. However, arbitrarily high growth for pre-Tannakian categories at this time remains unknown.

The purpose of this note is to construct a QPT category that attains arbitrarily high growth. Our main result is the following:

Theorem 2. *For any sequence $a_n \in \mathbb{R}$ for $n \in \mathbb{N} = \{1, 2, \dots\}$, there exists a quasi-pre-Tannakian category $\mathscr{C}$ with an object X such that*

$$rank(End_{\mathscr{C}}(X^{\otimes n})) \geq a_n.$$

The present note is organized as follows: In Section 2, we introduce the framework of T-algebras, which is our method of construction of the category in Theorem 2. In Sections 3, 4, and 5, respectively, we describe the construction of the T-algebra that gives this example and verify its consistency by explicitly describing its representation structure, product structure, and trace structure, respectively. In Section 6, we prove Theorem 2.

2. T-Algebras

Over a commutative ring R, an additive R-linear category with a symmetrical monoidal tensor product which is rigid and is generated by an object X is determined by the structure on $Hom(X^{\otimes m}, X^{\otimes n})$. The structure which determines such a category consists of data describing the properties of the traces and products:

Definition 3. *Define a* T-algebra *over a commutative ring R as a collection of the following data:*

(A) For all finite sets S, T, a choice of R-modules $V_{S,T}$ functorial with respect to isomorphisms of finite sets (covariantly on T, contravariantly on S). (In particular, $V_{S,T}$ is a $\Sigma_S \times \Sigma_T$-representation where Σ_S, Σ_T denote the symmetric groups of bijections on S, T, respectively.)

Comment: *We use "S, T" here to suggest "source" and "target." To avoid confusion, we write* T*-algebra with roman* T.

(B) For all finite sets $S' \subseteq S$, $T' \subseteq T$, for every choice of bijection

$$b : S' \to T',$$

a $\Sigma_{S'} \times \Sigma_{S \smallsetminus S'} \times \Sigma_{T \smallsetminus T'} \cong \Sigma_{T'} \times \Sigma_{S \smallsetminus S'} \times \Sigma_{T \smallsetminus T'}$-equivariant map

$$\sigma_b : V_{S,T} \to V_{S \smallsetminus S', T \smallsetminus T'}$$

(where we consider

$$\Sigma_{S'} \times \Sigma_{S \smallsetminus S'} \times \Sigma_{T \smallsetminus T'} \cong \Sigma_{T'} \times \Sigma_{S \smallsetminus S'} \times \Sigma_{T \smallsetminus T'} \subseteq \Sigma_S \times \Sigma_T$$

by embedding $\Sigma_{S'}$, $\Sigma_{T'}$ diagonally) such that, for finite sets $S'' \subseteq S$, $T'' \subseteq T$ disjoint from S' and T' with a bijection

$$b' : S'' \to T'',$$

(2)
$$\sigma_{b \amalg b'} = \sigma_b \circ \sigma_{b'} = \sigma_{b'} \circ \sigma_b,$$

as maps over

$$\Sigma_{S'} \times \Sigma_{S''} \times \Sigma_{S \smallsetminus (S' \amalg S'')} \times \Sigma_{T \smallsetminus (T' \amalg T'')} \subset \Sigma_{S' \amalg S''} \times \Sigma_{S \smallsetminus (S' \amalg S'')} \times \Sigma_{T \smallsetminus (T' \amalg T'')}.$$

(C) For finite sets S_1, T_1, S_2, T_2, a tensor product map

$$\pi : V_{S_1,T_1} \otimes V_{S_2,T_2} \to V_{S_1 \amalg S_2, T_1 \amalg T_2}$$

which is equivariant with respect to

$$(\Sigma_{S_1} \times \Sigma_{T_1}) \times (\Sigma_{S_2} \times \Sigma_{T_2}) \subset \Sigma_{S_1 \amalg S_2} \times \Sigma_{T_1 \amalg T_2}.$$

We will typically denote $x\pi y := \pi(x \otimes y)$ for $x \in V_{S_1,T_1}$, $y \in V_{S_2,T_2}$. We require that π is compatible with all partial traces. More specifically, for $S' \subseteq S_1 \amalg S_2$, $T' \subseteq T_1 \amalg T_2$ with a bijection

$$b : S' \to T'$$

the diagram

$$\begin{tikzcd}
V_{S_1,T_1} \otimes V_{S_2,T_2} \arrow[r, "\pi"] \arrow[d, "\sigma_{b|_{S_1}} \otimes \sigma_{b|_{S_2}}"'] & V_{S_1 \amalg S_2, T_1 \amalg T_2} \arrow[d, "\sigma_b"] \\
V_{S_1 \smallsetminus S', T_1 \smallsetminus T'} \otimes V_{S_2 \smallsetminus S', T_2 \smallsetminus T'} \arrow[r, "\pi"] & V_{(S_1 \amalg S_2) \smallsetminus S', (T_1 \amalg T_2) \smallsetminus T'}
\end{tikzcd}$$

(where in the lower left corner, we use the convention that

$$S \smallsetminus R = S \smallsetminus (R \cap S)$$

if we may not have $R \subseteq S$*).*

We also require that π *is commutative, associative, unital in the obvious sense with a unit* $R \to V_{\emptyset,\emptyset}$*. For example, commutativity means commutativity of the diagram*

$$\begin{tikzcd}
V_{S_1 \amalg T_1} \otimes V_{S_2 \amalg T_2} \arrow[r, "\pi"] \arrow[d, "\tau"'] & V_{S \amalg T} \arrow[d, "Id"] \\
V_{S_2 \amalg T_2} \otimes V_{S_1 \amalg T_1} \arrow[r, "\pi"] & V_{S \amalg T}
\end{tikzcd}$$

where τ *denotes the switch of tensor factors.*

(D) An element $\iota \in V_{\{1\},\{1\}}$ *such that for all* $x \in V_{\{1\},\{1\}}$

$$\sigma_{Id_{\{1\}}}((12)(x\pi\iota)) = \sigma_{Id_{\{1\}}}((12)(\iota\pi x)) = x$$

(considering $\{1,2\} \cong \{1\} \amalg \{1\}$*).*

Remark: In particular, the definition implies that $V_{S,T}$ gives a lax symmetric monoidal functors from the product of two copies of the category of finite sets and bijections (where the monoidal structure is the disjoint union) to R-modules (where the monoidal structure is the tensor product).

Note that by the assumption (2) for partial traces, all trace maps can be constructed as a composition of, for all choices of $i \in S$, $j \in T$, the maps

$$\sigma_{i,j} := \sigma_{\{i\}\to\{j\}} : V_{S,T} \to V_{S \smallsetminus \{i\}, T \smallsetminus \{j\}}$$

(where the partial trace map is with respect to the only bijection

$$\{i\} \to \{j\},$$

sending i to j). Call these the *elementary partial trace maps* in a T-algebra.

Definition 4. *Call a* T*-algebra V a* graded T-algebra *if, for all finite sets S, T with $|S| \neq |T|$, we have*

$$V_{S,T} = 0.$$

We say that an additive R-linear category with a symmetric monoidal tensor product which is rigid and is generated by an object X is graded *if for all n, m, k, ℓ such that $m + \ell \neq n + k$,*

$$Hom(X^{\otimes m} \otimes (X^{\vee})^{\otimes k}, X^{\otimes n} \otimes (X^{\vee})^{\otimes \ell}) = 0).$$

Comment 5. In the graded case, V is determined by a sequence of $\Sigma_n \times \Sigma_n$-representations $V_n := V_{\{1,\dots,n\},\{1,\dots,n\}}$, along with partial trace maps, for all $k \leq n$,

$$\sigma_k := \sigma_{Id_{\{1,\dots,k\}}} : V_n \to V_{\{k+1,\dots,n\},\{k+1,\dots n\}} \cong V_{n-k},$$

where the second isomorphism in the composition corresponds to the order-preserving bijection

$$\{k+1, \dots, n\} \to \{1, \dots, n-k\},$$

and product maps

$$\pi : V_n \otimes V_m \to V_{\{1,\dots,n\} \amalg \{1,\dots,m\}} \cong V_{n+m}$$

where the second isomorphism in the composition corresponds to the bijection which is the disjoint union of $Id_{\{1,\dots,n\}}$ with the order-preserving bijection

$$\{1, \dots, m\} \to \{n+1, \dots, n+m\},$$

giving a bijection

$$\{1, \dots, n\} \amalg \{1, \dots, m\} \to \{1, \dots, n+m\}$$

(note that the required consistency conditions and diagrams for partial trace and products will contain conjugation by certain permutations to be equivalent to the conditions in Definition 3).

Proposition 6. *Given a* T*-algebra V, there exists a rigid R-linear category $\mathcal{C}(V)$ with a symmetric monoidal tensor product such that for a certain $X \in Obj(V)$,*

$$Obj(\mathcal{C}(V)) = \{X^{\otimes m} \otimes (X^{\vee})^{\otimes n} | m, n \in \mathbb{N}_0\}$$

$$Mor(X^{\otimes m_1} \otimes (X^{\vee})^{\otimes n_1}, X^{\otimes m_2} \otimes (X^{\vee})^{\otimes n_2}) =$$

$$V_{\{1,\dots m_1+n_2\},\{1,\dots m_2+n_1\}}.$$

The category $\mathcal{C}(V)$ is graded when the T*-algebra V is graded.*

Proof. The defining axioms of a T-algebra are translations of the corresponding categorical ones.

□

In the remaining sections, we shall construct a graded T-algebra $\mathbb{T}$ such that $\mathbb{T}_0 = \mathbb{C}$ and $dim(\mathbb{T}_n)$ is finite but grows faster than any given function in n. This gives a QPT category of arbitrarily high growth.

3. Representation Structure

Write $[[n]] = \{1, \dots, n\} \times \{0, 1\}$, $[n]_\epsilon = \{1, \dots, n\} \times \{\epsilon\}$, for $\epsilon = 0, 1$.

Definition 7. *Fix a sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$. Define $\mathbb{T}_n$ as the free $\mathbb{C}$-vector space on the set $\mathbb{S}_n$ of choices of the following data:*

(1) For all $1 \leq k \leq n$, a subset
$$\mathscr{T}(k) \subseteq \mathscr{P}([[n]]),$$
letting $\mathscr{P}$ denote the power set.

(2) For all $1 \leq k \leq n$, a function
$$\chi : \mathscr{T}(k) \to \{1, \dots, n_k\}.$$

(3) Subsets $W_0 \subseteq [n]_0$ and $W_1 \subseteq [n]_1$ with
$$|W_0| = |W_1|,$$
and a bijection
$$\beta : W_0 \to W_1$$

(4) A bijection $b : Z_0 \to Z_1$ where
$$Z_\epsilon := [n]_\epsilon \smallsetminus \left(W_\epsilon \cup \bigcup_{k=1}^{n} \bigcup_{T \in \mathscr{T}(k)} T \right)$$

satisfying the following conditions:

(1) For all $1 \leq k \leq n$, for each $T \in \mathscr{T}(k)$, for both $\epsilon = 0, 1$,
$$|T \cap [n]_\epsilon| = k.$$

(2) For all $1 \leq k \leq n$, for all distinct $T \in \mathscr{T}(k)$, $T' \in \mathscr{T}(\ell)$,
$$T \cap T' = \emptyset$$
and for $\epsilon = 0, 1$,
$$T \cap W_\epsilon = \emptyset.$$
(Note that conditions (1), (2) imply
$$|Z_0| = |Z_1|.)$$

Remark: In a more condensed notation, the definition can be restated as follows: For a subset $S \subseteq [[n]]$ and $\epsilon \in \{0, 1\}$, write

$$S_\epsilon = S \cap [n]_\epsilon.$$

Write

$$P_n := \{S \subseteq [[n]] \mid |S_0| = |S_1|\}.$$

The data and conditions of Definition 7 can be equivalently stated as:

(1) A set partition

$$W \amalg Z \amalg \coprod \mathscr{T}(1) \amalg \cdots \amalg \coprod \mathscr{T}(n) = [[n]]$$

(2) Bijections

$$\beta : W_0 \to W_1$$
$$b : Z_0 \to Z_1$$

(3) For all $1 \leq k \leq n$, functions

$$\chi : \mathscr{T}(k) \to \{1, \ldots, n_k\}$$

such that

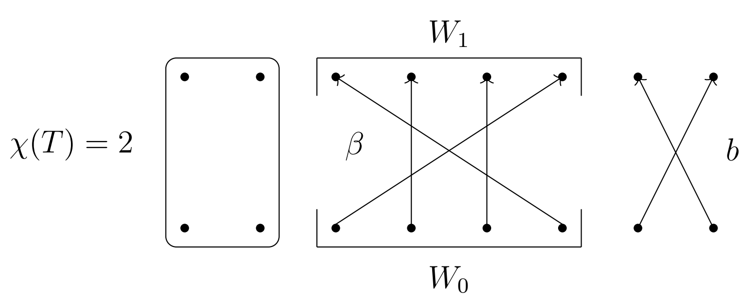
(1) $\{W\}, \{Z\}, \mathscr{T}(1), \ldots, \mathscr{T}(n) \subseteq P_n$

(2) For all $1 \leq k \leq n$, for all $S \in \mathscr{T}(k)$, $|S| = 2k$.

For example, the following diagram is a visualization of an element of $\mathbb{S}_8$ corresponding to taking $T = \{(1, 0), (2, 0), (1, 1), (2, 1)\}$,

$$\mathscr{T}(2) = \{T\}, \ \chi(T) = 2,$$

$\mathscr{T}(k) = \emptyset$ for all $k \neq 2$, and $W_\epsilon = \{3, 4, 5, 6\} \times \{\epsilon\}$:



The $\mathbb{C}$-vector space $\mathbb{T}_n$ also has a $\Sigma_n \times \Sigma_n$-action induced by the $\Sigma_n \times \Sigma_n$-action on $[[n]]$ given by letting the first and second factors act on $[n]_0$ and $[n]_1$, respectively.

4. Product Structure

For all n, m, we further give a homomorphism over $(\Sigma_n)^2 \times (\Sigma_m)^2 \subseteq (\Sigma_{n+m})^2$ mapping

$$\pi_{n,m} : \mathbb{T}_n \times \mathbb{T}_m \to \mathbb{T}_{n+m}. \tag{3}$$

In diagrams, we will take this operation to be placing diagram side by side, i.e. using disjoint union. More precisely, let us fix elements

$$\Phi = (\mathscr{T}(1), \dots, \mathscr{T}(n), \chi, \beta, W_0, W_1, b) \in \mathbb{T}_n$$
$$\Phi' = (\mathscr{T}'(1), \dots, \mathscr{T}'(m), \chi', \beta', W_0', W_1', b') \in \mathbb{T}_m$$

Take, then,

$$\widetilde{\mathscr{T}(k)} = \mathscr{T}(k) \amalg \mathscr{T}'(k)$$

(taking undefined sets to be empty and identifying

$$\{1, \dots, n\} \amalg \{1, \dots, m\} \cong \{1, \dots, n+m\}$$

by sending $j \mapsto j+n$ for $j \in \{1, \dots, m\}$),

$$\widetilde{\chi} = \chi \amalg \chi' : \widetilde{\mathscr{T}(k)} \to \{1, \dots, n_k\},$$

for $\epsilon = 0, 1$

$$\widetilde{W_\epsilon} = W_\epsilon \amalg W_\epsilon',$$

$\widetilde{\beta} = \beta \amalg \beta'$ and $\widetilde{b} = b \amalg b'$.

Definition 8. *Define the product operation (3) by putting*

$$\pi_{n,m}((\Phi, \Phi')) = (\widetilde{\mathscr{T}(1)}, \dots, \widetilde{\mathscr{T}(n)}, \widetilde{\chi}, \widetilde{\beta}, \widetilde{W_0}, \widetilde{W_1}, \widetilde{b})$$

and extending linearly.

5. Trace Structure

Fix a constant

$$c = tr(\iota) \in \mathbb{C} \smallsetminus \mathbb{Z} \tag{4}$$

(recall Definition 3, part (D)). To give the sequence of $\Sigma_n \times \Sigma_n$-representations $(\mathbb{T}_n)$ the structure of a graded T-algebra, we must also describe partial trace maps. We define $\Sigma_{n-i} \times \Sigma_{n-i} \times \Sigma_i$-equivariant maps

$$tr_\sigma : \mathbb{T}_n \to \mathbb{T}_{n-i} \tag{5}$$

(embedding $\Sigma_{n-i} \times \Sigma_{n-i} \times \Sigma_i$ diagonally into

$$\Sigma_{n-i} \times \Sigma_{n-i} \times \Sigma_i \times \Sigma_i \subseteq \Sigma_n \times \Sigma_n$$

for the left hand side) after being given a bijection σ between two i-element subsets of $[n]_0$ and $[n]_1$.

Suppose we are given two such subsets $R_0 \subseteq [n]_0$, $R_1 \subseteq [n]_1$ with

$$|R_0| = |R_1| = i$$

and a bijection

$$\sigma : R_0 \to R_1.$$

(This is also sometimes referred to as partial bijection from $[n]_0$ to $[n]_1$.)

Extending Comment 5, our convention is to use the order-preserving bijections

$$[n-i]_\epsilon \to [n]_\epsilon \smallsetminus R_\epsilon \tag{6}$$

for the definition of tr_σ.

Consider the (say, unordered) graph Γ with vertices $[[n]]$ and edges $\{i, \sigma(i)\}$ for $i \in R_0$, and $\{j, b(j)\}$ for $j \in Z_0$. The vertices of Γ have degree ≤ 2, so components can be individual vertices, (connected) cycles, or paths. First of all, we eliminate all (connected) cycles and replace each with a factor c (recall (4)). Let s be the number of such cycles.

Paths from $[n]_0$ to $[n]_1$ can be identified with the data of subsets $\widehat{R_0} \subseteq [n]_0$, $\widehat{R_1} \subseteq [n]_1$ and a bijection $\widehat{\sigma} : \widehat{R_0} \to \widehat{R_1}$.

A path from $[n]_\epsilon$ to $[n]_\epsilon$ ends with a σ-edge on one side and a b-edge on the other side. Thus, from these paths, we can extract sets $\overline{R_\epsilon} \subseteq [n]_\epsilon$, $\overline{R_\epsilon} \cap \widehat{R_\epsilon} = \emptyset$ and injections

$$\rho_\epsilon : \overline{R_\epsilon} \to [n]_\epsilon \smallsetminus \widehat{R_\epsilon}$$

which send the σ-end of the path to the b-end.

Definition 9. *Call an element of $\mathbb{S}_n$, i.e. a collection of data*

$$\Phi = (\mathscr{T}(1), \dots, \mathscr{T}(n), \chi, \beta, W_0, W_1, b),$$

matchable with respect to σ *if for $x \in W_0 \cap \widehat{R_0}$, $y \in W_1 \cap \widehat{R_1}$, $\widehat{\sigma}(x) = y$ implies $\beta(x) = y$, and for all $T \in \mathscr{T}(k)$ one of the following is true:*

1. *There exists $T' \neq T \in \mathscr{T}(k)$ such that $T \cap \widehat{R_0} \neq \emptyset$ or $T \cap \widehat{R_1} \neq \emptyset$,*

$$\widehat{\sigma}(T \cap \widehat{R_0}) \subseteq (T' \cap \widehat{R_1})$$

$$\widehat{\sigma}^{-1}(T \cap \widehat{R_1}) \subseteq (T' \cap \widehat{R_0}).$$

   Note that the conditions imply that the above formulae must also then be true for T and T' switched and that T' is unique.

2. *We have $T \cap \widehat{R_0} = \emptyset$ and $T \cap \widehat{R_1} = \emptyset$.*

If $\Phi \in \mathbb{S}_n$ is not matchable with respect to σ, put

$$tr_\sigma(\Phi) = 0.$$

We shall now define $tr_\sigma(\Phi)$ in the case when $\Phi \in \mathbb{S}_n$ is matchable.

Let $\widehat{W_\epsilon}$ be obtained from W_ϵ by deleting any source (resp. target) elements of $\widehat{\sigma}$ and replacing $x \in W_\epsilon \cap \overline{R_\epsilon}$ by $\widehat{x} = \rho_\epsilon(x)$ and define $\widehat{\beta}$ by taking β and replacing an element x of its source (resp. target) by $\widehat{X}$ when applicable. Similarly, for each $T \in \mathscr{T}(k)$, let $\widehat{T}$ be obtained by replacing each $x \in T \cap \overline{R_\epsilon}$ by $\rho_\epsilon(x)$.

Replace each $T \in \mathscr{T}(k)$ satisfying Case 2 of Definition 9 by $\widehat{T}$. Let $\widetilde{\mathscr{T}(k)}$ be the set of all such $\widehat{T}$, and put $\widetilde{\chi}(\widehat{T}) = \chi(T)$.

Now let $\widehat{\mathscr{T}(k)}$ be the set of all unordered pairs $\{T, T'\} \subseteq \mathscr{T}(k)$ satisfying Case 1 of Definition 9. For such a pair $\{T, T'\}$, define

$$\begin{aligned} \beta_{\{T,T'\}} = q(\textstyle\sum(\gamma : (\widehat{T} \cap [n]_0) \smallsetminus \widehat{R_0} \xrightarrow{\cong} (\widehat{T'} \cap [n]_1) \smallsetminus \widehat{R_1}))\cdot \\ (\textstyle\sum(\gamma' : (\widehat{T'} \cap [n]_0) \smallsetminus \widehat{R_0} \xrightarrow{\cong} (\widehat{T} \cap [n]_1) \smallsetminus \widehat{R_1})). \end{aligned} \tag{7}$$

(In (7), we consider a bijection as a "product" of its pairs, the product is distributive with respect to sums). Define, also,

$$W_\epsilon^{\{T,T'\}} = ((\widehat{T} \cup \widehat{T'}) \cap [n]_\epsilon) \smallsetminus \widehat{R_\epsilon}.$$

Now let

$$\widetilde{W_\epsilon} = \widehat{W_\epsilon} \cup \bigcup_{k=1}^{n-i} \bigcup_{\{T,T\} \in \widehat{\mathscr{T}(k)}} W_\epsilon^{\{T,T'\}}$$

and

$$\widetilde{\beta} = \widehat{\beta} \cdot \prod_{k=1}^{n-i} \prod_{\{T,T'\} \in \widehat{\mathscr{T}(k)}} \beta_{\{T,T'\}}.$$

Finally, let $\widetilde{b}$ be the restriction of $\widehat{\sigma}$ to

$$[n]_0 \smallsetminus \left(\bigcup_{k=1}^{n-i} \bigcup_{T \in \widetilde{\mathscr{T}(k)}} T \cup \widetilde{W_0} \right) \to [n]_1 \smallsetminus \left(\bigcup_{k=1}^{n-i} \bigcup_{T \in \widetilde{\mathscr{T}(k)}} T \cup \widetilde{W_1} \right).$$

Definition 10. *When Φ is matchable, define the partial trace operator (5) by letting*

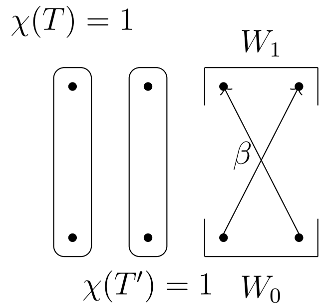
$$tr_\sigma(\Phi) = c^s \widetilde{\Phi}$$

where

$$\widetilde{\Phi} = (\widetilde{\mathscr{T}(1)}, \dots, \widetilde{\mathscr{T}(n-i)}, \widetilde{\chi}, \widetilde{\beta}, \widetilde{W_0}, \widetilde{W_1}, \widetilde{b}),$$

using the idenitification (6), and extending linearly.

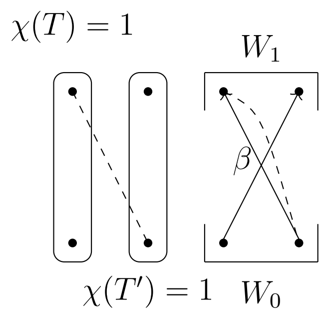
For example, the element Φ of $\mathbb{T}_4$ corresponding to the diagram



is matchable with respect to

$$\sigma : \{2,4\} \times \{0\} \to \{1,3\} \times \{1\}$$

given by $\sigma((2,0)) = (1,1)$, $\sigma((4,0)) = (3,1)$ (represented by the dotted lines):



Then to find $tr_\sigma(\Phi)$ we delete the elements of the T's and W's connected by σ. In this case, Z_ϵ, $\overline{R_\epsilon}$ and $\widehat{R_\epsilon}$ are all empty. None of the $\widetilde{\mathscr{T}(k)}$'s will be non-empty, and all remaining points will belong to the new $\widetilde{W_\epsilon}$'s. There is only one unordered pair of sets $\{T, T'\} \in \widehat{\mathscr{T}(k)}$, and

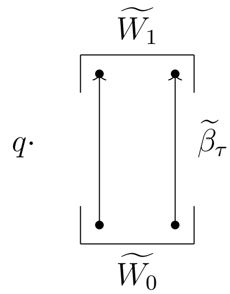
$$W_0^{\{T,T'\}} = \{(1,0)\}$$

$$W_1^{\{T,T'\}} = \{(2,1)\},$$

with

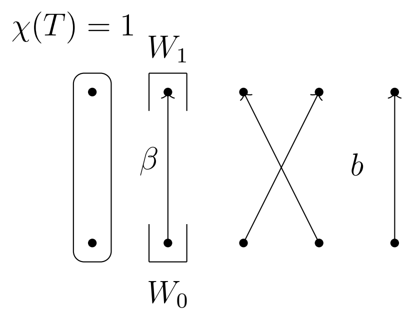
$$\beta_{\{T,T'\}} = q \cdot ((1,0) \mapsto (2,1))$$

Thus, $tr_\sigma(\Phi)$ can be visualized as



(the top row of points representing $\{(2,1),(4,1)\}$ and the bottom row of points representing $\{(1,0),(3,0)\}$).

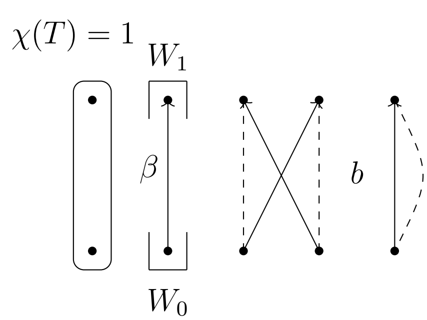
For another example, consider the generator Φ of $\mathbb{T}_5$ corresponding to the diagram



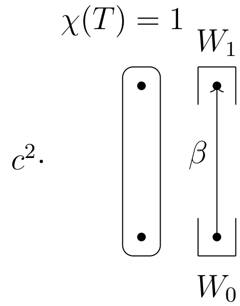
is matchable with respect to

$$\sigma : \{3,4,5\} \times \{0\} \to \{3,4,5\} \times \{1\}$$
$$(i,0) \mapsto (i,1),$$

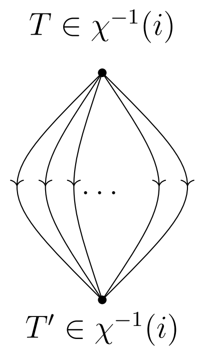
which is represented by dotted lines in the following diagram:



Then $tr_\sigma(\Phi)$ is



Remark 11. The motivation of this definition comes from making traces of diagrams of the form



equal to q, and "introducing no other non-zero traces." The formalism of the set W is introduced to eliminate negligible elements that would arise from different values of i.

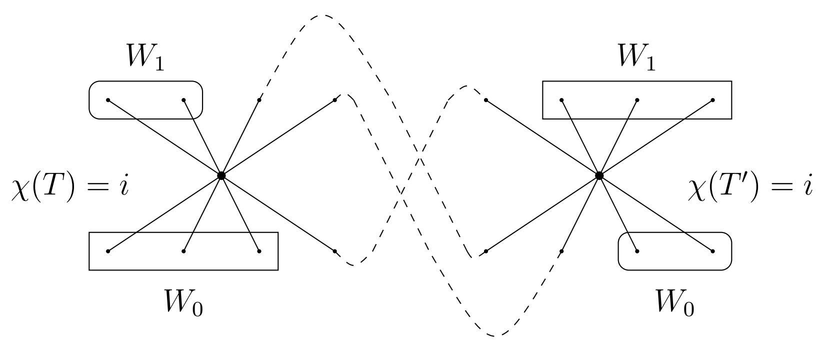
6. Proof of the Main Result

Proposition 12. *Definitions 7, 8, and 10 describe the structure of a* T-*algebra on* $\mathbb{T}$.

Proof. Axiom (A) is immediate from Definition 7. The product and permutations of the element ι required in Axiom (D) are given by the bijections b. Axiom (D) follows from the discussion prior to Definition 9 similarly as in [1], Chapter 10. Axiom (C) is also immediate from Definitions 8 and 10.

The non-trivial property of our construction is transitivity of traces.

To prove this property, the key point is to realize how we are implementing the picture of Remark 11 in the set-up of Definition 10. We can represent this graphically as follows:



The only non-trivial traces between elements of the sets $\mathscr{T}(i)$ are for $T, T' \in \mathscr{T}(i)$, $\chi(T) = \chi(T')$ (see Definition 7) is to match bijectively a chosen subset of elements of $T \cap [n]_\epsilon$ to $T' \cap [n]_{1-\epsilon}$. The remaining elements of $(T \cup T') \cap [n]_\epsilon$ will be added to W_ϵ where the bijection β will be extended by all permutations matching the remaining elements of $T' \cap [n]_{1-\epsilon}$. Transitivity then follows from the fact that this case of the trace can only be followed by tracing W_0 with W_1 along the bijection β, with the understanding that elements can be moved around using Axiom (D), analogously to the dimension of [1], Chapter 10.

□

Proof of Theorem 2. Consider the $\mathbb{C}$-linear category $\mathcal{C}(\mathbb{T})$ (defined in Proposition 6) and take its additive and pseudo-abelian envelope ([1], Subsections 1.7, 1.8). This gives a QPT category by the definition of $\mathbb{T}$ and Proposition 12. All that remains to prove is that the lower bound on growth, which follows from the fact that the growth of the basic object X is bounded below by the sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$, which was arbitrary.

□

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