INTERPOLATION OF GENERAL AFFINE GROUPS AND SEMIDIRECT PRODUCTS OF SYMPLECTIC GROUPS WITH HEISENBERG GROUPS VIA REPRESENTATION STABILITY

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Abstract. Using methods of representation stability, we construct (non-semisimple) pre-Tannakian categories which could be interpreted as interpolation tensor categories of certain representations of groups of affine transformations. We also discuss an analogous construction for symplectic groups with their canonical actions on Heisenberg groups.

1. Introduction

The general linear group $GL_n(\mathbb{C})$ acts on the vector space $\mathbb{C}^n$. The resulting semidirect product is the general affine group $GA_n(\mathbb{C})$. P. Deligne and J. S. Milne [2, 3] defined pre-Tannakian categories, i.e. symmetric tensor (abelian) categories, linear over $\mathbb{C}$, with finite dimensional Hom-spaces, and such that all objects have strong dual, which can be interpreted as the categories of finite dimensional algebraic representations of $GL_c(\mathbb{C})$ for $c \notin \mathbb{Z}$.

Every finite dimensional algebraic representation W of $GL_n(\mathbb{C})$ also determines a representation $\widetilde{W}$ of $GA_n(\mathbb{C})$ where $\mathbb{C}^n$ acts trivially. However, such representations can have non-trivial extensions: For example, $GA_n(\mathbb{C})$ acts on $\mathbb{C}^{n+1}$ by

$$(A, (a_1, \dots, a_n)^T) \cdot (x_0, \dots, x_n)^T =$$

$$= \begin{pmatrix} x_0 \\ A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + x_0 \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{pmatrix} \tag{1}$$

which is an extension of the trivial representation by the representation corresponding to $\widetilde{W}$ where W is the vector representation of $GL_n(\mathbb{C})$.

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The purpose of this paper is to propose, in some sense, an "interpolation" of the category of finite extensions of the representations $\widetilde{W}$ of GA_c for $c \notin \mathbb{Z}$.

For the symplectic group $Sp_{2n}(\mathbb{C})$, the situation is analogous. The stabilizer of the vector $(1, 0, \dots, 0)^T \in \mathbb{C}^{n+1}$ in the standard representation of $Sp_{2n+2}(\mathbb{C})$ is a semidirect product of $Sp_{2n}(\mathbb{C})$ with the Heisenberg group

$$\mathbb{H}_{2n} = \{(b|a,\ c) \mid b \in \mathbb{C},\ a, c \in \mathbb{C}^n\}$$

with the product

$$(b|a,\ c) \cdot (y|x,\ z) = (b + y + a^T \cdot z - x^T \cdot c \mid a + x_1,\ c + z).$$

The embedding $\mathbb{H}_{2n} \subseteq Sp_{2n+2}(\mathbb{C})$ is given by

$$(b|a,\ c) \mapsto \left(\begin{array}{cc|cc} 1 & a^T & b & c^T \\ 0 & I & c & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -a & I \end{array}\right).$$

Again for a finite dimensional algebraic representation W of $Sp_{2n}(\mathbb{C})$, we can define a representation $\widetilde{W}$ of $Sp_{2n}(\mathbb{C}) \rtimes \mathbb{H}_{2n}$ by letting $\mathbb{H}_{2n}$ act trivially. Again, we have non-trivial extensions. For example, the basic representation V of $Sp_{2n+2}(\mathbb{C})$, considered as a representation of $Sp_{2n}(\mathbb{C}) \rtimes \mathbb{H}_{2n}$, has a composition series with trivial representation 1 at the top and bottom and $\widetilde{W}$ for the basic representation of $Sp_{2n}(\mathbb{C})$ in the middle.

In fact, we also have a non-trivial 2-dimensional representation of $\mathbb{H}_{2n}$ where $(b|a,\ c)$ acts by

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Again, we propose a pre-Tannakian interpolation of the category of finite extensions of the representations $\widetilde{W}$ of $Sp_c(\mathbb{C}) \rtimes \mathbb{H}_c$ for $c \notin \mathbb{Z}$.

What does this have to do with representation stability? The basic concept are *FI-modules*, which are functors from the category FI whose objects are finite sets (equivalently the sets $[n] = \{1, \dots, n\}$) and morphisms are injections to $\mathbb{C}$-vector spaces. The $\mathbb{C}$-linear abelian category FI-Mod of FI-modules (and natural transformations) has been studied extensively, see e.g. [1, 5, 6, 8, 11, 10, 13, 14, 15, 17, 18]. One passes to the quotient $FI\text{-Mod}^{gen}$ of FI-Mod by the Serre subcategory of *torsion modules* consisting of *torsion* elements (i.e those which are sent to 0 by one of the morphisms of FI). The category $FI\text{-Mod}^{gen}$ is a useful tool for capturing phenomena of representation stability [1].

Using Schur-Weyl duality, A. Snowden [19], Chapter 6, noted that the category of FI-modules corresponds to the category of polynomial GL_∞-equivariant $\mathbb{C}[x_1, x_2, \dots]$-modules. Thinking of these as "coherent sheaves on $\mathbb{A}^\infty$," he further observed that generic FI-modules correspond to "GL_∞-equivariant coherent sheaves" on $\mathbb{A}^\infty \smallsetminus \{0\}$ which correspond to representations of the stabilizer group of one point, i.e. GA_∞.

This was the motivation for our investigation, but to obtain the "interpolation" (in particular, to interpret the "dimension" c), the category of FI-modules has to be modified.

We begin by noting that there are *two* immediately visible tensor category structures on FI-Mod, $FI\text{-Mod}^{gen}$. One is the *level-wise structure*, i.e. for FI-modules M, N,

$$(M \otimes N)[n] = M[n] \otimes_{\mathbb{C}} N[n] \tag{2}$$

This structure was investigated for example in [1, 8, 11, 18]. The tensor structure $\otimes$ is most easily understood by the fact ([19], Exercise 5.29) that

$$M \mapsto \operatorname*{colim}_n M[n]$$

gives an equivalence between $FI\text{-Mod}^{gen}$ and smooth (finitely generated) representations of the countably infinite symmetric group Σ_∞, by which we mean the group of those permutations on $\mathbb{N} = \{1, 2, \dots\}$ that move only finitely many elements non-trivially, where *smooth* means that the stabilizer of any element contains a subgroup of the form $\Sigma_{\infty-n}$, which means the stabilizer of $[n]$. (In this note, we shall use Σ to denote a symmetric group, to avoid confusion with the notation for Specht modules.) Now smooth Σ_∞-representations have a tensor product (the tensor product of representations over $\mathbb{C}$) to which the tensor product in $FI\text{-Mod}^{gen}$ given by (2) corresponds.

While the resulting tensor category of smooth Σ_∞-representations is not rigid (i.e. does not have strong duality), it embeds naturally into any of the categories $\underline{Rep}(\Sigma_t)$ of P. Deligne [2], which are semisimple for $t \notin \mathbb{N}_0$. The embedding is defined as follows: Morphisms from $[m]$ to $[n]$ in $\underline{Rep}(\Sigma_t)$ can be identified with the free $\mathbb{C}$-vector spaces on *equivalence relations* on $[m] \amalg [n]$ (see [9] for more detail). Then morphisms of smooth Σ_∞-representations

$$V^{\otimes m} \to V^{\otimes n}$$

where $V = \mathbb{C}\mathbb{N}$ is the "standard" representations of Σ_∞ correspond to the free vector space on those equivalence relations whose every equivalence class has a non-empty intersection with $[m]$. Thus, for values $t \notin \mathbb{N}_0$, we obtain an embedding of FI-Mod^{gen} with the tensor product $\otimes$ into the semisimple pre-Tannakian category $\underline{Rep}(\Sigma_t)$.

Another, for our purposes more relevant, tensor category structure on FI-Mod, FI-Mod^{gen}, which can be called the *Day product* and which we will denote by $\boxtimes$, where for FI-modules M, N, the FI-module $M \boxtimes N$ is defined by taking the $FI \times FI$-module $(M_m \otimes N_n)_{m,n \in \mathbb{N}_0^2}$ and applying the left Kan extension along the functor

$$FI \times FI \to FI$$

given by disjoint union. The significance of this tensor structure is that under Schur-Weyl duality, it corresponds to $\otimes$ of representations.

The question we investigate in the present paper is fully embedding FI-Mod^{gen} into a pre-Tannakian category.

The basic idea is that we can easily embed the pseudo-abelian envelope of the category of (Day) tensor powers $X^{\boxtimes n}$ of the "basic" object into the category $(FI_c^{\pm})^{Op}$ whose objects are pairs $(m, n) \in \mathbb{N}_0^2$ and morphisms, for $(m, n), (p, q) \in \mathbb{N}_0^2$, are

$$Mor_{(FI_c^{\pm})^{Op}}((m, n), (p, q)) = \mathbb{C}Mor_{FI}([n] \amalg [p], [m] \amalg [q]).$$

When composing, one encounters "circles," which are replaced by multiplication by a given constant c. This is a variant (by replacing bijections with injections) of the category $\underline{Rep}(GL_c)$ considered by P. Deligne and J. S. Milne in [3], Subsections 1.26, 1.27, and later by P. Deligne in [2], Section 10. While this is a tensor embedding, the target category is not abelian. The main purpose of this paper is to show that one does in fact have a (non-semisimple) rigid pre-Tannakian category $FI_c^{\pm}$-Mod^{gen} of *generic* $FI_c^{\pm}$*-modules* into which FI-Mod^{gen} (with the Day product) embeds as a full tensor subcategory. In addition, we also identify the simple objects.

Theorem 1. *The simple objects $\mathscr{Y}_{\lambda,\mu}$ of the generic category of $FI_c^{\pm}$-modules are indexed by pairs of Young diagrams λ, μ. Further, the fusion rules of tensor products of these simple generic $FI_c^{\pm}$-modules exactly correspond to the fusion rules of tensor products of the simple objects $Y_{\lambda,\mu}$ in $\underline{Rep}(GL_{c-1})$.*

The symplectic story is largely, analogous, with a few notable differences. We begin with the symplectic (or twisted) analogue QB_c of the Brauer category, which, by P. Deligne [2], Section 9, has a semisimple representation category with self-dual simple objects Y_λ. There is a commutative algebra

$$\underline{\mathbb{C}} = Y_\emptyset \oplus Y_{(1)} \oplus Y_{(2)} \oplus \dots$$

in this tensor category, and we call its finite dimensional representations *QI_c-modules.* The suitably defined category QI_c-Mod^{gen} of *generic* QI_c-modules is our proposed interpolation category of $Sp_c \rtimes \mathbb{H}_c$-representations for $c \notin \mathbb{Z}$. The important difference is that since the simple objects are self-dual, there is no need for a "$\pm$"-construction.

The symplectic variant of Theorem 1 is

Theorem 2. *The simple objects $\mathscr{Y}_\lambda$ of the generic category of $QI_c^\pm$-modules are indexed by Young diagrams λ. Further, the fusion rules of tensor products of these simple generic QI_c-modules exactly correspond to the fusion rules of tensor products of the simple objects Y_λ in $\underline{Rep}(Sp_{c-2})$, (i.e. the Newell-Littlewood rules).*

The present paper is organized as follows: In Section 2, we describe the category $FI_c^\pm$ and discuss the structure of $FI_c^\pm$-modules. The main goal of this section is to define generic $FI_c^\pm$-modules. In Section 3, we give the construction of the symmetric tensor structure of the Day product in the category of $FI_c^\pm$-modules and prove that it carries over to a symmetric tensor structure in generic $FI_c^\pm$-modules. In Section 4, we describe the simple generic $FI_c^\pm$-modules, show that the category of generic $FI_c^\pm$-modules has strong duality, and prove Theorem 1 as well as the exactness of $\boxtimes$ in generic $FI_c^\pm$-modules. In Section 5, we discuss the symplectic analogue.

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2. The Category of $FI_c^\pm$-Modules

Fix a value $c \in \mathbb{C}$. Let us consider the category $FI_c^\pm$ defined by taking

$$Obj(FI_c^\pm) = \mathbb{N}_0^2$$

and for pairs $(m,n),(p,q) \in Obj(FI_c^\pm)$, the morphisms

$$Mor_{FI_c^\pm}((m,n),(p,q)) = \mathbb{C}Mor_{FI}([m] \amalg [q], [p] \amalg [n]). \tag{3}$$

In particular, by an *expansion* in $FI_c^\pm$ we shall mean a morphism of the form (3) given by a disjoint union of an injection

$$[m] \hookrightarrow [p]$$

and an bijection

$$[q] \xrightarrow{\cong} [n].$$

Composition will depend on the fixed value of $c \in \mathbb{C}$. To describe the composition in $FI_c^{\pm}$, we first give a graphical representation of the morphisms

$$(m, n) \to (p, q)$$

freely generating (3) in $FI_c^{\pm}$, following the diagrammatic expressions of morphisms of the $\underline{Rep}(GL_c)$ described by P. Deligne in [2], Section 10.1:

On the left, we have m points labelled with the sign "+", and n points labelled with the sign "-," (representing the source of the morphism), and on the right, we have p and q points labelled "+" and "-," respectively (representing the target). The morphism then corresponds to a perfect pairing (or matching) of the disjoint union of the set of m points labelled "+" on the left, the set of q points labelled "-" on the right, a subset of the set of n points labelled "-" on the left, and a subset of the set of p points labelled "+" on the right, such that every point is paired with either a point of the same sign on the opposite side, or a point of opposite sign on the same side. For example, Figures 1 and 2 show an example of the graphical representation of a generating morphism $(2, 5) \to (4, 3)$. Figure 1 displays the injection, while Figure 2 shows the corresponding morphism in $FI_c^{\pm}$.

Composition in $FI_c^{\pm}$ is then described by defining it for free generating morphisms of (3) by placing two diagrams (as in Figure 2) next to each other, aligning the points corresponding to the intermediate pairs, and composing as in [2], Section 10.1: the lines and segments of the pairing are connected and a closed circuit is deleted and the obtained diagram is multiplied by c to form the final morphism of $FI_c^{\pm}$ forming the original morphisms' composition. We then may extend composition $\mathbb{C}$-linearly to define it for all composable morphisms $FI_c^{\pm}$.

It is formal to verify that $FI_c^{\pm}$ defined this way then indeed forms a category. We note that while $FI_c^{\pm}$, $(FI_c^{\pm})^{Op}$ pre-additive categories linear over $\mathbb{C}$, adding direct sums and taking a pseudoabelian envelope does not make an abelian category.

Definition 3. *We define $FI_c^{\pm}$-modules to be functors from $FI_c^{\pm}$ to vector spaces over $\mathbb{C}$*

$$FI_c^{\pm} \to \mathbb{C}\text{-}\mathit{Vect}. \tag{4}$$

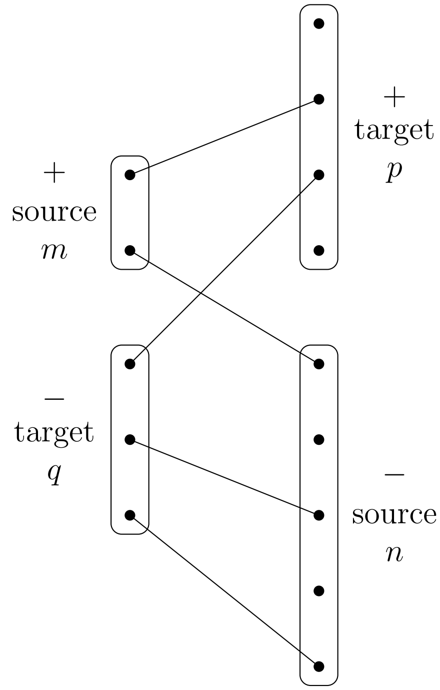


FIGURE 1. An injection $[m] \amalg [q] \to [p] \amalg [n]$, for $m = 2$, $n = 5$, $p = 4$, $q = 3$

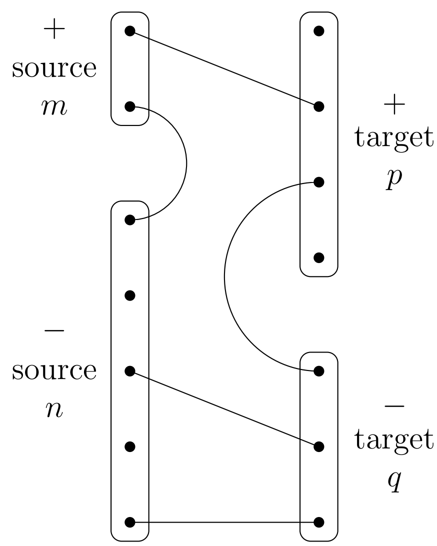


FIGURE 2. A generator of $Mor_{FI_c^{\pm}}((m, n), (p, q))$, for $m = 2$, $n = 5$, $p = 4$, $q = 3$

(Recall that, in this paper, we are restricting attention to *finitely generated* $FI_c^{\pm}$-modules.)

For a pair $(m,n) \in \mathbb{N}_0$, let us denote its endomorphism algebra in $FI_c^{\pm}$ by

$$\Sigma_{m,n}^c := End_{FI_c^{\pm}}((m,n)).$$

For $c \in \mathbb{C} \smallsetminus \mathbb{Z}$, the $\mathbb{C}$-algebra $\Sigma_{m,n}^c$ is semisimple (see [2], Section 10), and the irreducible $\Sigma_{m,n}^c$-representations are indexed by pairs of Young diagrams λ and μ such that $m-n = |\lambda| - |\mu|$. Write $\mathcal{Y}_{\lambda,\mu}(m,n)$ for the simple $\Sigma_{m,n}^c$-representation corresponding to λ, μ. This follows from the decomposition of the tensor of copies of the basic object and its dual

$$X^{\otimes m} \otimes (X^{\vee})^{\otimes n} \tag{5}$$

in $\underline{Rep}(GL_c)$ into simple objects $Y_{\lambda,\mu}$. The dimension of $\mathcal{Y}_{\lambda,\mu}(m,n)$ as a $\Sigma_{m,n}^c$-representation is the multiplicity of $Y_{\lambda,\mu}$ in (5):

$$dim(\mathcal{Y}_{\lambda,\mu}(m,n)) = dim(Hom_{\underline{Rep}(GL_c)}(Y_{\lambda,\mu}, X^{\otimes m} \otimes (X^{\vee})^{\otimes n}).$$

An $FI_c^{\pm}$-algebra F then determines, at each pair (m,n), a $\Sigma_{m,n}^c$-representation $F(m,n)$. Note that we have

$$dim(\Sigma_{m,n}^c) = |Mor_{FI}([m+n],[m+n])| = (m+n)!.$$

However, $\Sigma_{m,n}^c$ is not equal to the free representation of the symmetric group on $m+n$ elements, since its algebra structure is different and depends on c.

Similarly as for classical FI-modules, $FI_c^{\pm}$-modules form a category by taking morphisms between two functors of the form (4) to be natural transformations. This category, which we denote $FI_c^{\pm}$-Mod, is clearly a $\mathbb{C}$-additive category, by taking for $FI_c^{\pm}$-modules $F, G : FI_c^{\pm} \to \mathbb{C}$-Vect, for $(m,n) \in \mathbb{N}_0^2$

$$(F \oplus G)(m,n) = F(m,n) \oplus G(m,n)$$

(the proof that this definition forms a $FI_c^{\pm}$-module is analogous to the proof for the similar statement for FI-modules).

In the category of FI-modules, one often uses the representable objects $\underline{m}$. Analogously, for an element $(m,n) \in \mathbb{N}_0^2$, we may consider the corresponding *representable* $FI_c^{\pm}$*-module*

$$\underline{(m,n)} : FI_c^{\pm} \to \mathbb{C}\text{-Vect}$$

defined by sending a pair $(p,q) \in \mathbb{N}_0^2$ to the free $\mathbb{C}$-vector space

$$\underline{(m,n)}(p,q) = Mor_{FI_c^{\pm}}((m,n),(p,q)) = \mathbb{C}Mor_{FI}([m] \amalg [q], [p] \amalg [n]).$$

Following the definition of torsion FI-modules, we can make the following definition of *torsion $FI_c^{\pm}$-modules*:

Definition 4. *For an $FI_c^{\pm}$-module $F : FI_c^{\pm} \to \mathbb{C}$-Vect, for a pair $(m,n) \in \mathbb{N}_0^2$, an element $x \in F(m,n)$ is called* torsion *if there exists a pair $(p,q) \in \mathbb{N}_0^2$ and an expansion $f \in Mor_{FI_c^{\pm}}((m,n),(p,q))$ such that*

$$(F(f))(x) = 0.$$

At each pair $(m,n) \in \mathbb{N}_0^2$, the set of torsion elements of $F(m,n)$ forms a $\mathbb{C}$-subspace of $F(m,n)$, and sending (m,n) to this subspace defines a functor (4), and thus an $FI_c^{\pm}$-module, which we denote TF. We say that F is a torsion $FI_c^{\pm}$-module *if $TF = F$.*

We can take the category of torsion $FI_c^{\pm}$-modules, which we shall denote by $FI_c^{\pm}$-Mod^{tor} to be the full subcategory of $FI_c^{\pm}$-modules on these objects.

Proposition 5. *The category of torsion $FI_c^{\pm}$-modules $FI_c^{\pm}$-Mod^{tor} forms a Serre subcategory of $FI_c^{\pm}$-Mod.*

□

Therefore, we can define the category $FI_c^{\pm}$-Mod^{gen} of *generic $FI_c^{\pm}$-modules* as the Serre quotient of $FI_c^{\pm}$-Mod by $FI_c^{\pm}$-Mod^{tor}.

3. Symmetric Monoidal Structure - the "Day Product"

In this section, we will explicitly describe a symmetric tensor category structure on $FI_c^{\pm}$-modules which we call the *Day product*, defined as an analogue of the Day product on FI-Mod. For $FI_c^{\pm}$-modules

$$F, G : FI_c^{\pm} \to \mathbb{C}\text{-Vect},$$

we can first define the functor

$$F\overline{\boxtimes}G : FI_c^{\pm} \times FI_c^{\pm} \to \mathbb{C}\text{-Vect}$$

(defined by taking $F\overline{\boxtimes}G((m,n),(p,q)) = F(m,n)\otimes_{\mathbb{C}} G(p,q))$. We then define their Day product $F \boxtimes G$ as the left Kan extension

$$\begin{tikzcd} FI_c^{\pm} \times FI_c^{\pm} \arrow[r, "F\overline{\boxtimes}G"] \arrow[d, "\amalg"'] & \mathbb{C}\text{-Vect} \\ FI_c^{\pm} \arrow[ur, dashed, "F\boxtimes G"'] & \end{tikzcd}$$

where the vertical map

$$\amalg : FI_c^{\pm} \times FI_c^{\pm} \to FI_c^{\pm}$$

is defined by taking coordinate-wise addition

$$(m,n) \amalg (p,q) = (m+p, n+q).$$

Recall that, for $x \in \mathbb{N}_0^2$, $F \boxtimes G(x)$ is defined as the coequalizer of two functors

$$\phi, \psi : \bigoplus_{\alpha:a\to a',\beta:b\to b',\theta:a'+b'\to x} F(a)\otimes_{\mathbb{C}} G(b) \to \bigoplus_{\theta:a+b\to x} F(a)\otimes_{\mathbb{C}} G(b)$$

(all sums are over choices of morphisms in $FI_c^{\pm}$), where ϕ is defined by sending a direct summand $F(a)\otimes_{\mathbb{C}} G(b)$ of the source corresponding to morphisms $\alpha : a \to a', \beta : b \to b', \theta : a'+b' \to x$ in $FI_c^{\pm}$ to the summand $F(a')\otimes_{\mathbb{C}} G(b')$ of the target corresponding to θ by the linear map $F(a')\otimes G(b')$, and ψ is defined by sending such a summand of the source to the summand $F(a)\otimes G(b)$ corresponding to $\theta\circ(\alpha+\beta)$ by the identity.

This construction parallels the "Day product" of FI-modules already mentioned in the Introduction. In fact, the reader may use the above paragraphs as a review of the Day product of FI-modules by replacing $FI_c^{\pm}$ by $\mathbb{C}FI$. We also note that by using the characterization of FI-modules as modules over the twisted commutative algebra $\underline{\mathbb{C}}$ (see [16] and [19], Exercise 2.8), the Day product of FI-modules M, N can also be described as

$$M \circledast_{\underline{\mathbb{C}}} N.$$

We then have a functor

$$\Phi : FI\text{-Mod} \to FI_c^{\pm}\text{-Mod}$$

given by left Kan extension along the inclusion

$$\mathbb{C}FI \hookrightarrow FI_c^{\pm},$$

which is then automatically a tensor functor with respect to the Day product $\boxtimes$.

Note that for an FI-module M, we have in fact

$$\Phi(M)(p,q) = \begin{cases} 0 & \text{if } p < q \\ M_{p-q} \otimes_{\mathbb{C}\Sigma_{p-q}} Hom_{FI_c^{\pm}}((p-q,0),(p,q)) & \text{if } p \geq q \end{cases}$$

which implies:

Proposition 6. *The functor*

$$\Phi FI\text{-}Mod \to FI_c^{\pm}\text{-}Mod$$

is exact.

□

Every FI-module is a quotient of a (finite) direct sum of representable objects and, moreover,

$$\underline{m} \boxtimes \underline{n} \cong \underline{m+n}. \tag{6}$$

Analogously, we have the following facts in $FI_c^{\pm}$-modules:

Proposition 7. *For pairs $a_0, b_0 \in \mathbb{N}_0^2$, we have*

$$\underline{a_0} \boxtimes \underline{b_0} \cong \underline{(a_0 + b_0)}.$$

□

Proposition 8. *Every (finitely generated) $FI_c^{\pm}$-module is, by definition, a quotient of a direct sum of (finitely many) objects of the form $\underline{(m_i, n_i)}$.*

□

Note that, by Propositions 7 and 8, $\underline{(0,0)}$ is the unit of the symmetric monoidal structure on $FI_c^{\pm}$-modules.

Denote by $Tor_i^{\boxtimes}$, the ith left derived functor of $\boxtimes$ (in FI-Mod or $FI_c^{\pm}$-Mod). (Note that both abelian categories clearly have enough projectives.)

Lemma 9. *Let M, N be FI-modules (resp. $FI_c^{\pm}$-modules) where M is torsion. Then for every $i \geq 0$, $Tor_i^{\boxtimes}(M,N)$ is torsion.*

Proof. By considering a projective resolution of N, it suffices to prove the statement for $i = 0$. For $i = 0$, the statement follows from the fact that taking the Day product of an expansion and an identity morphism still gives an expansion.

□

By Lemma 9, the Day tensor product $\boxtimes$ passes to a tensor structure on FI-Mod^{gen}, $FI_c^{\pm}$-Mod^{gen} and Φ induces a $\boxtimes$-tensor functor

$$\Phi^{gen} : FI\text{-Mod}^{gen} \to FI_c^{\pm}\text{-Mod}^{gen},$$

which is exact by Proposition 6.

In the category FI-Mod, we can further decompose

$$\underline{m} = \bigoplus_{|\lambda|=m} P_\lambda$$

(see [19]) where P_λ is the subobject generated by the Specht module S_λ as a Σ_m-representation (called the *principal projective*). Denoting, for a Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, the Young diagram

$$\overline{\lambda} = (\lambda_1, \lambda_1, \lambda_2, \dots, \lambda_k) \tag{7}$$

one defines the *Spechtral module* $\mathscr{S}_\lambda$ as the submodule of P_λ generated by the Specht module $S_{\overline{\lambda}}$ (see the paragraph of [19] before Proposition 2.7). One then has a short exact sequence on FI-Mod^{gen}

$$0 \to \mathscr{S}_\lambda \to P_\lambda \to Q \to 0 \tag{8}$$

where Q is an extension of Spechtral modules $\mathscr{S}_{\lambda'}$ with $|\lambda'| < |\lambda|$.

A. Snowden ([19], Proposition 5.10) proved that $\mathscr{S}_\lambda$ are all the non-isomorphic simple objects of FI-Mod^{gen}. The Spechtral FI-modules satisfy the Littlewood-Richardson rule with respect to the Day product in FI-Mod^{gen}:

Proposition 10. *For Young diagrams*

$$\mathscr{S}_{\lambda_1} \boxtimes \mathscr{S}_{\lambda_2} \cong \bigoplus_{\lambda} \kappa_\lambda^{\lambda_1,\lambda_2} \mathscr{S}_\lambda$$

where $\kappa_\lambda^{\lambda_1,\lambda_2}$ denote the Littlewood-Richardson numbers.

Proof. By the Pieri rule, this statement is equivalent to the statement that

$$(S_\lambda \otimes_{LR} S_\mu)^{\Sigma_n} = \bigoplus_{m+\ell=n} S_\lambda^{\Sigma_m} \otimes_{LR} S_\mu^{\Sigma_\ell}$$

(as $\Sigma_{|\lambda|+|\mu|-n}$-representation, the superscripts denoting fixed points). Since we have assumed that the ground field is $\mathbb{C}$, we can dualize to obtain an equivalent statement involving cofixed points, which holds for all representations V, W (instead of simple S_λ, S_μ), since

$$\begin{gathered} Ind_{\Sigma_N}^{\Sigma_M \times \Sigma_L}((V \otimes W) \otimes_{\mathbb{C}\Sigma_n} \mathbb{C}) = \\ \bigoplus_{m+\ell=n} Ind_{\Sigma_{N-n}}^{\Sigma_{M-m} \times \Sigma_{L-\ell}}((V \otimes_{\mathbb{C}\Sigma_m} \mathbb{C}) \otimes (W \otimes_{\mathbb{C}\Sigma_\ell} \mathbb{C})) \end{gathered} \tag{9}$$

for all L, M, N with $N = L + M$. This statement follows from the fact that, for a bijection

$$f : [M] \amalg [L] \to N,$$

the bijection obtained by restricting away from elements that f sends to a certain choice of n elements, is exactly equivalent to restricting f away from some m elements of $[M]$ and some ℓ elements of $[L]$ for some choice of m, ℓ with $m + \ell = n$.

$\square$

We can deduce the following

Proposition 11. *The functor $\boxtimes$ in FI-Mod^{gen} is exact in each variable.*

Proof. The principal projectives P_λ are flat with respect to the Day product $\boxtimes$ in FI-Mod^{gen}. Therefore, we can define its left derived functors, which we will denote by Tor_i^{gen}.

Now

$$Tor_i^{gen}(\mathscr{S}_\lambda, M) = 0 \text{ for } i > 0$$

for every $M \in Obj(FI\text{-Mod}^{gen})$ follows by induction on $|\lambda|$ from the short exact sequence (8). The long exact sequence in Tor_i^{gen} implies the statement of the Proposition.

$\square$

We will prove analogous statements about the category $FI_c^{\pm}$-Mod^{gen} in the next Section.

4. Simple Generic $FI_c^{\pm}$-Modules and Strong Duality

Proposition 12. *The objects $\underline{(m,n)}$ are strongly dualizable in $FI_c^{\pm}$-Mod^{gen}, and one has*

$$\underline{(m,n)}^\vee = \underline{(n,m)}.$$

Proof. The unit and counit

$$\eta : \underline{(0,0)} \to \underline{(m,n)} \boxtimes \underline{(n,m)} \cong \underline{(m+n, n+m)}$$

$$\epsilon : \underline{(m,n)} \boxtimes \underline{(n,m)} \cong \underline{(m+n, n+m)} \to \underline{(0,0)}$$

are represented by

$$\sigma : [m] \amalg [n] \to [n] \amalg [m]$$

where σ is the shuffle switching $[m]$ and $[n]$, see Figure 3.

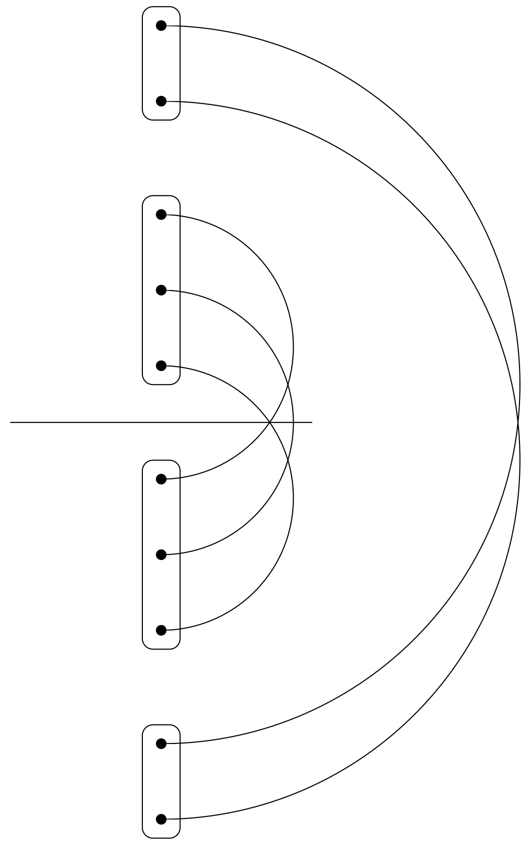


Figure 3. The (co)unit of duality of $\underline{(m,n)}^\vee = \underline{(n,m)}$ for $m=2$, $n=3$

One of the triangle identities is represented by Figure 4 below, the other is symmetrical.

□

For every pair of Young diagrams

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i),$$

$$\mu = (\mu_1, \mu_2, \dots, \mu_j),$$

define the *principal projective* generic $FI_c^\pm$-module $P_{\lambda,\mu}$ as the sub-$FI_c^\pm$-module of $\underline{(|\lambda|, |\mu|)}$ generated (in $FI_c^\pm$-Mod^{gen}) by $\mathcal{Y}_{\lambda,\mu}(|\lambda|, |\mu|)$.

Proposition 13. *For a pair* $(m,n) \in \mathbb{N}_0$

$$\underline{(m,n)} = \bigoplus_{|\lambda|=m, |\mu|=n} P_{\lambda,\mu}$$

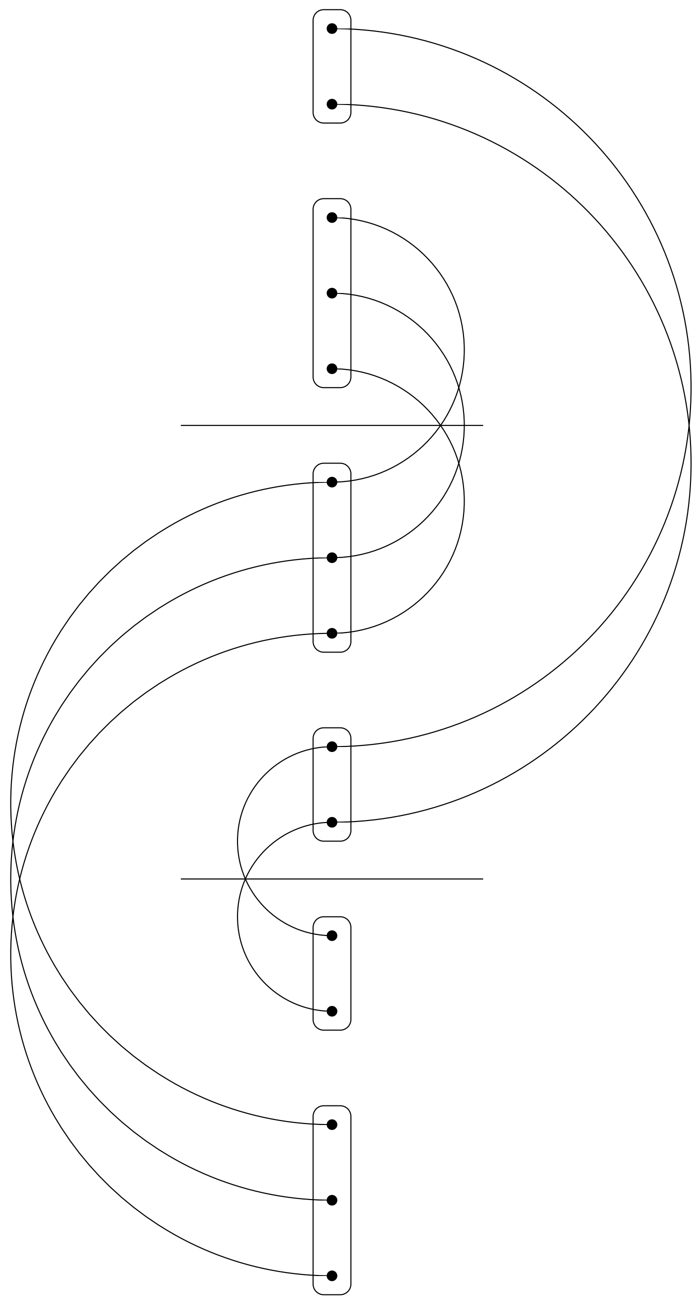


FIGURE 4. Triangle identity for $\underline{(m,n)}^{\vee} = \underline{(n,m)}$ for $m = 2$, $n = 3$

where the direct sum runs over all Young diagrams λ, μ with $|\lambda| = m$, $|\mu| = n$.

$\square$

For $N \in \mathbb{N}_0$, denote

$$\lambda_N^+ = (N - |\lambda|, \lambda_1, \lambda_2, \dots, \lambda_i). \tag{10}$$

(Note that the Young diagram $\overline{\lambda}$ defined in (7) is $\lambda^+_{|\lambda|+\lambda_1}$.)

Let us now define the $FI_c^{\pm}$-module $\mathscr{Y}_{\lambda,\mu}$ as the quotient of the sub-$FI_c^{\pm}$-module of $P_{\lambda,\mu}$ generated (in $FI_c^{\pm}$-Mod^{gen}) by

$$\mathcal{Y}_{\lambda^+_{|\lambda|+\lambda_1},\mu}(|\lambda| + \lambda_1, |\mu|) \tag{11}$$

by the submodule generated by all $\mathcal{Y}_{?,\mu'}(|?|, |\mu'|)$ with $|\mu'| < |\mu|$.

This is analogous to the definition of the "Spechtral FI-module" $\mathscr{S}_\lambda$ corresponding to a Young diagram λ as the submodule generated by $S_{\lambda^+_{|\lambda|+\lambda_1}}$ of the principal projective FI-module P_λ in [19] before Proposition 2.7. The reason we need to consider subquotients is the duality, which we discuss in more detail below. In our present setting, Proposition 2.7 of [19] has the following analogue:

Proposition 14. *For a pair $(m, n) \in \mathbb{N}_0^2$ with $m \geq |\lambda| + \lambda_1$, $n \geq |\mu|$, we have*

$$\mathscr{Y}_{\lambda,\mu}(m, n) = \mathcal{Y}_{\lambda^+_m,\mu}(m, n). \tag{12}$$

Proof. Let $m \geq |\lambda| + \lambda_1$, $n \geq |\mu|$. By the Pieri rule, $P_{\lambda,\mu}(m, n)$ then contains a copy of

$$\mathcal{Y}_{\lambda^+_{m-n+|\mu|},\mu}(m, n) \tag{13}$$

along with $\Sigma^c_{m,n}$-representations of the form

$$\mathcal{Y}_{\lambda',\mu'}(m, n)$$

where $\lambda' = \overline{\lambda}^+_{m'}$, with $|\overline{\lambda}| < |\lambda|$, or $|\mu'| < |\mu|$. Thus, by the structure maps of $FI_c^{\pm}$, (11) can only map to (13), as desired.

$\square$

In particular, in generic $FI_c^{\pm}$-modules, we have short exact sequences

$$0 \to \mathscr{Y}_{(1),\emptyset} \to \underline{(1, 0)} \to \underline{(0, 0)} \to 0 \tag{14}$$

$$0 \to \underline{(0, 0)} \to \underline{(0, 1)} \to \mathscr{Y}_{\emptyset,(1)} \to 0. \tag{15}$$

We also note that by (14), using the long exact sequence for Tor^{gen}, $\mathscr{Y}_{(1),\emptyset}$ is flat.

Lemma 15. *In a tensor abelian category, if we have short exact sequences*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{f} C \longrightarrow 0 \tag{16}$$

$$0 \longrightarrow C' \xrightarrow{f'} B' \longrightarrow A' \longrightarrow 0 \tag{17}$$

where f' is strongly dual to f and the tensor product of any of the exact sequences (16), (17) with any term of the other is exact, then A' is strongly dual to A.

Proof. The unit of duality $\eta : 1 \to A \otimes A'$ is defined by the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & C \otimes C' \arrow[d] & \\
 & 1 \arrow[r] \arrow[urr] \arrow[d, dashed, "\eta"] & B \otimes B' \arrow[r] \arrow[d] & C \otimes B' \arrow[d] & \\
0 \arrow[r] & A \otimes A' \arrow[r] & B \otimes A' \arrow[r] & C \otimes A' \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

The counit of adjunction is symmetrical. One triangle identity follows from the commuting diagram

$$\begin{tikzcd}
 & A \arrow[r, "1\otimes\eta"] \arrow[dl] \arrow[d, "1\otimes\eta"] & A \otimes A' \otimes A \arrow[d] \arrow[dr, "\epsilon\otimes 1"] & \\
B \arrow[dr, "1\otimes\eta"] & A \otimes B' \otimes B \arrow[r] \arrow[d] & A \otimes A' \otimes B \arrow[d, "\epsilon\otimes 1"] & A \arrow[dl] \\
 & B \otimes B' \otimes B \arrow[r, "\epsilon\otimes 1"] & B &
\end{tikzcd}$$

and one of the triangle identity for B and B'. The other is symmetrical. $\square$

By applying Lemma 15 to the short exact sequences (14) and (15), we obtain the following

Corollary 16. *$\mathscr{Y}_{(1),\emptyset}$ and $\mathscr{Y}_{\emptyset,(1)}$ are strongly dual in $FI_c^{\pm}$-Mod^{gen}.*

□

In fact, we have

Lemma 17. *For all Young diagrams λ, $\mathscr{Y}_{\lambda,\emptyset}$, $\mathscr{Y}_{\emptyset,\lambda}$ of $FI_c^{\pm}$-Mod^{gen} are strongly dual to each other (and hence flat).*

Proof. Tensoring with a strongly dualizable object A in a symmetric tensor category is exact because $A\otimes ?$ is both a left and a right adjoint. We can then obtain the claim by applying Lemma 15 to the short exact sequence of the form

$$0 \to \mathscr{Y}_{\lambda,\emptyset} \to P_{\lambda,\emptyset} \to Q \to 0$$

where Q is an extension of $\mathscr{Y}_{\lambda',\emptyset}$ with $|\lambda'| < |\lambda|$, going by induction on $|\lambda|$ (using (8)), using the exactness of the symmetric monoidal functor Φ^{gen}.

□

We moreover have the following

Theorem 18. *The generic $FI_c^{\pm}$-modules $\mathscr{Y}_{\lambda,\mu}$ are exactly the simple objects of $FI_c^{\pm}$-Mod^{gen}, and every finitely generated generic $FI_c^{\pm}$-module has a composition series with associated graded pieces isomorphic to a $\mathscr{Y}_{\lambda,\mu}$ for some λ, μ.*

Proof. As objects of $FI_c^{\pm}$-Mod^{gen}, the $\mathscr{Y}_{\lambda,\mu}$ must be simple, since at each pair (m, n), the corresponding $\Sigma^c_{m,n}$-representation is simple. We shall follow a proof analogous to the well-know analogue for classical generic FI-modules given, for example in [19].

To prove the claim that every finitely generated generic $FI_c^{\pm}$-module has a finite length filtration whose associated graded pieces are of the form $\mathscr{Y}_{\lambda,\mu}$, we will follow the terminology of [19].

For a Young diagram $\lambda = (\lambda_1, \dots, \lambda_n)$, recall that its *amplitude* is defined to be the sum $\lambda_2 + \lambda_3 + \cdots + \lambda_n$. For pairs $(m, n) \in \mathbb{N}_0^2$, we consider the amplitude of a $\Sigma^c_{m,n}$-representation V to be the maximum amplitude of Young diagrams λ such that the simple $\mathcal{Y}_{\lambda,\mu}(m, n)$ $\Sigma^c_{m,n}$-representation is a summand of V. We additionally define its *coamplitude* to be the maximal total number of boxes $|\mu|$ of Young diagrams μ such that the simple $\mathcal{Y}_{\lambda,\mu}(m, n)$ $\Sigma^c_{m,n}$-representation is a summand of

V. For an $FI_c^{\pm}$-module, we can then define its amplitude, resp. coamplitude, as the supremum of the amplitudes, resp. coamplitudes, of the $\Sigma^c_{m,n}$-representations they give at each pair $(m,n) \in \mathbb{N}_0^2$. Similarly as for FI-modules, every finitely generated $FI_c^{\pm}$-module has a finite amplitude. Since at each degree, the $\Sigma^c_{m,n}$-representations $\mathcal{Y}_{\lambda,\mu}(m,n)$ are defined only for Young diagrams λ, μ such that

$$|\lambda| - |\mu| = m - n,$$

we therefore know that the coamplitude of every finitely generated $FI_c^{\pm}$-module is also finite.

This implies that for every finitely generated generic $FI_c^{\pm}$-module M there exists a finite length filtration

$$0 = F_0M \subset F_1M \subset F_2M \subset \cdots \subset F_{N-1}M \subset F_NM = M \tag{18}$$

such that for every $1 \leq n \leq N$, the $F_nM/F_{n-1}M$ has constant amplitude and co-amplitude. Similarly as in [19], one then argues that $F_nM/F_{n-1}M$ is an extension of $\mathscr{Y}_{\lambda,\mu}$ with given $|\lambda|$, $|\mu|$, and torsion $FI_c^{\pm}$-modules.

Refining the filtration, we obtain a filtration in $FI_c^{\pm}$-Mod^{gen} where the associated graded pieces are of the form $\mathscr{Y}_{\lambda,\mu}$, as desired.

$\square$

Our next goal is to describe the "fusion rules" for the generic $FI_c^{\pm}$-modules $\mathscr{Y}_{\lambda,\mu}$, and prove their strong dualizability. The fusion rules of Spechtral FI-modules with respect to the Day product $\boxtimes$ are described in Proposition 10.

The analogous statement for $FI_c^{\pm}$-modules, completing the proof of Theorem 1, is the following

Theorem 19. *The generic simple $FI_c^{\pm}$-modules $\mathscr{Y}_{\lambda,\mu}$ are strongly dualizable with strong dual $\mathscr{Y}_{\mu,\lambda}$. In fact, there is a tensor functor*

$$\Xi : \underline{Rep}(GL_{c-1}) \to FI_c^{\pm}\text{-}Mod^{gen}$$

(with respect to the Day product on the right hand side) such that

$$\Xi(Y_{\lambda,\mu}) = \mathscr{Y}_{\lambda,\mu}.$$

Consequently, the category $FI_c^{\pm}$-Mod^{gen} of finitely generated generic $FI_c^{\pm}$-modules has strong duality and $\boxtimes$ is exact in each variable.

Proof. By Corollary 16, $\mathscr{Y}_{(1),\emptyset}$ is strongly dualizable and has dimension $c-1$ (by additivity of dimensions in short exact sequences of strongly dualizable objects), and its strong dual is $\mathscr{Y}_{\emptyset,(1)}$. This already defines a tensor functor

$$\Xi : \underline{Rep}(GL_{c-1}) \to FI_c^{\pm}\text{-Mod}^{gen}, \tag{19}$$

which maps

$$\Xi(X) = \mathscr{Y}_{(1),\emptyset}$$

$$\Xi(X^\vee) = \mathscr{Y}_{\emptyset,(1)}.$$

Moreover, from the case of FI-Mod^{gen} (Proposition 10), we already know that $\mathscr{Y}_{\lambda_1,\emptyset}$, $\mathscr{Y}_{\lambda_2,\emptyset}$ fuse according to the Littlewood-Richardson rule (and in particular are strongly dualizable). So what remains to prove is that the image of $Y_{\lambda,\mu}$ under (19) is $\mathscr{Y}_{\lambda,\mu}$.

To show that, note that $Y_{\lambda,\emptyset} \otimes Y_{\emptyset,\mu}$ contains $Y_{\lambda,\mu}$ plus summands of the form $Y_{\lambda',\mu'}$, $|\lambda'| < |\lambda|$, $|\mu'| < |\mu|$. Thus, if we can show that $\mathscr{Y}_{\lambda,\emptyset} \boxtimes \mathscr{Y}_{\emptyset,\mu}$ has, in each degree, the exact same summands as

$$\bigoplus_{i=1}^{N} \mathscr{Y}_{\lambda_i,\mu_i} \tag{20}$$

where

$$Y_{\lambda,\emptyset} \otimes Y_{\emptyset,\mu} = \bigoplus_{i=1}^{N} Y_{\lambda_i,\mu_i},$$

then we can argue by induction. Now (20) can be seen by induction from studying

$$\mathscr{Y}_{\lambda,\mu} \boxtimes P_{(1),\emptyset}. \tag{21}$$

Indeed, one sees, by definition, that in degree $(m+1,n)$, (21) has the $\Sigma^c_{m+1,n}$-representation

$$\bigoplus_{i=1}^{N} \mathcal{Y}_{\overline{\lambda^+},\overline{\mu}}(m+1,n), \tag{22}$$

where $\overline{\lambda^+}$ and $\overline{\mu}$ are obtained from λ^+ and μ (where $\mathcal{Y}_{\lambda^+,\mu}(m,n)$ is the summand of $\mathscr{Y}_{\lambda,\mu}$ in degree (m,n)) by the Pieri rule, adding one square to some row to λ^+ or subtracting a square from some row of μ, while still obtaining a Young diagram.

The summands where we add a square to the first row of λ^+ match a copy of $\mathscr{Y}_{\lambda,\mu}$ again, while the others match the precise copies of

$$\bigoplus_{i=1}^{N'} \mathscr{Y}_{\lambda'_i,\mu'_i}$$

where

$$Y_{\lambda,\mu} \otimes Y_{(1),\emptyset} = \bigoplus_{i=1}^{N'} Y_{\lambda'_i,\mu'_i}$$

(using, again, the Pieri rule). This is, of course, the right number, by exactness and the short exact sequence (14).

Since the $\mathscr{Y}_{\lambda,\mu}$ are strongly dualizable, they are flat, and hence $\boxtimes$ is exact on $FI_c^{\pm}$-Mod^{gen} using Theorem 18 and the long exact sequence in Tor^{gen}.

$\square$

5. The Symplectic Case

Consider the $\mathbb{C}$-linear *twisted Brauer category* QB_c, defined as follows: Take

$$Obj(QB_c) = \mathbb{N}_0$$

and, for $m, n \in \mathbb{N}_0$, take the space of morphisms $Hom_{QB_c}(m, n)$ from m to n to be the quotient of the free $\mathbb{C}$-vector space on the data

$$(m, n, \phi : S \to T, \phi_S, \phi_T) \tag{23}$$

where $S \subseteq [m]$, $T \subseteq [n]$ are subsets, and

$$\phi : S \xrightarrow{\cong} T$$

is a bijection, and there are some decompositions

$$[m] \smallsetminus S = S_1 \amalg S_2, \quad [n] \smallsetminus T = T_1 \amalg T_2$$

and

$$\phi_S : S_1 \xrightarrow{\cong} S_2$$

$$\phi_T : T_1 \xrightarrow{\cong} T_2$$

are bijections, by the following relations: The data (23) is equivalent to

$$-(m, n, \phi : S \to T, \phi'_S, \phi_T)$$

where, if for some $x \in S_1$, we have $\phi_S(x) = y$, we take

$$S'_1 = (S_1 \smallsetminus \{x\}) \amalg \{y\}$$

$$S'_2 = (S_2 \smallsetminus \{y\}) \amalg \{x\}$$

and the new bijection

$$\phi'_S : S'_1 \to S'_2$$

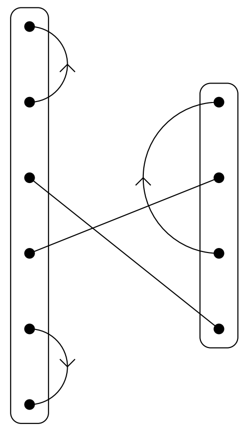


FIGURE 5. An example of a generator of $Mor_{QB_c}(m,n)$, for $m = 6$, $n = 4$

is defined by putting

$$\phi'_S(y) = x$$
$$\phi'_S(z) = \phi_S(z) \text{ for } z \in S_1 \smallsetminus \{x\} = S'_1 \smallsetminus \{y\},$$

and we have a similar relation

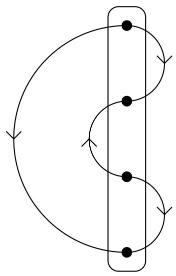
$$(m, n, \phi : S \to T, \phi_S, \phi_T) \sim -(m, n, \phi : S \to T, \phi_S, \phi'_T)$$

where ϕ'_T is, in an analogous way, ϕ_T after switching an element of the original T_1 and T_2.

For example, we have, for n even, the dimension (as a $\mathbb{C}$-vector space) of $Hom_{QB_c}(0,n)$ is equal to the number of pairings on $[n]$, which can be calculated as the product of odd positive integers less than n:

$$dim(Hom_{QB_c}(0,n)) = (n-1) \cdot (n-3) \ldots 3 \cdot 1 = \frac{n!}{2^{\frac{n}{2}} \cdot (\frac{n}{2})!} \tag{24}$$

Note that the data of a generator of $Hom_{QB_c}(m,n)$ (for m, $n \in \mathbb{N}_0$ such that $m+n$ is even) described above can be represented graphically as follows: For a given choice of the data (23), we draw a column of m dots on the left and n dots on the right, connecting dots in S to dots in T according to ϕ. We then draw arrows from elements of S_1 (resp. T_1) to elements of S_2 (resp. T_2) according to ϕ'_S (resp. ϕ'_T). For example, see Figure 5. In this graphical representation, again, note that switching the direction of an arrow is the same as multiplying the diagram by -1.



$$= c \in \mathbb{C} = Mor_{QB_c}(0,0)$$

FIGURE 6. An example of the composition of an element of $Hom_{QB_c}(0,4)$ and an element of $Hom_{QB_c}(4,0)$

Composition in QB_c, for generating elements of $Hom_{QB_c}(\ell, m)$ and $Hom_{QB_c}(m,n)$, can then be defined by placing their corresponding diagrams side by side, identifying both sets of dots corresponding to $[m]$, and composing the connections and pairings. If a loop arises, we reverse arrows until all arrows have compatible directions, delete the loop, and multiply the resulting generator by c and -1 to the number of arrows we reversed. For example, Figure 6 shows the composition of a generating element of $Hom_{QB_c}(0,4)$ with a generating element of $Hom_{QB_c}(4,0)$ which is c (identifying $\mathbb{C}$ with $Hom_{QB_c}(0,0)$ by identifying $1 \in \mathbb{C}$ with the generating element of $Hom_{QB_c}(0,0)$ corresponding to the empty diagram), since there is a single loop an no sign, since two arrows can be reversed to resolve the disagreements in the arrows' directions.

Definition 20. *We define a QB_c-module to be a (finitely generated) functor from QB_c to the category of $\mathbb{C}$-vector spaces*

$$QB_c \to \mathbb{C}\text{-}Vect. \tag{25}$$

Denote by QB_c-Mod the category with objects QB_c-modules and morphisms natural transformations.

Comment: Recall that we may also consider the *non-twisted Brauer category B_c*, which is defined by taking the same objects

$$Obj(B_c) = \mathbb{N}_0$$

and (using the above notation) taking the space of morphisms from a $m \in \mathbb{N}_0$ to an $n \in \mathbb{N}_0$ to be the quotient of the free $\mathbb{C}$-vector space on the data (23) by the relations

$$(m, n, \phi : S \to T, \phi_S, \phi_T) \sim (m, n, \phi : S \to T, \phi'_S, \phi_T) \sim$$
$$\sim (m, n, \phi : S \to T, \phi_S, \phi'_T)$$

for ϕ'_S (resp. ϕ'_T) obtained from ϕ_S (resp. ϕ_T) by switching an element of S_1 and S_2 (resp. T_1 and T_2), as defined above.

The generating morphism of $Hom_{B_c}(m, n)$ corresponding to the data (23) can, again, be expressed diagrammatically by drawing a column of m points on the left representing $[m]$, and a column of n points on the right representing $[n]$, and connecting every point with the one it is paired with (we now do not give the connections between elements of the same column a direction). Composition in B_c is then defined as the $\mathbb{C}$-linear extension of composition of diagrams performed exactly as in the Brauer algebra. The non-twisted Brauer category appears in the description of the interpolation of the orthogonal group $Rep_0(O_c)$ given in [2], Section 9, where the objects of the category are tensor powers of a basic object X and

$$Hom_{Rep_0(O_c)}(X_0^{\otimes m}, X_0^{\otimes n}) = Hom_{B_c}(m, n).$$

Analogously as in Definition 20, we can define B_c*-modules* as (finitely generated) functors

$$B_c \to \mathbb{C}\text{-Vect}.$$

We, in fact, have an equivalence of categories

$$QB_c\text{-Mod} \boxtimes \text{sVect} \cong B_{-c}\text{-Mod} \boxtimes sVect$$

(the operator $\boxtimes$ used as in, for example, [4]) defined by sending

$$M \boxtimes V \mapsto M \boxtimes (V \otimes \mathbb{C}_{(0,1)}).$$

In particular, considering the Young diagram (n), we will have that, as a $\mathbb{C}$-vector space,

$$Y_{(n)}(n) = \mathbb{C}, \tag{26}$$

on which the generators of $End_{QB_c}(n)$ without any pairings of elements in the same copy of $[n]$ (which form a copy of the symmetric group $\Sigma_n \subseteq End_{QB_c}(n)$) act trivially, while any generator of $End_{QB_c}(n)$ involving a pairing of two elements of the same copy of $[n]$ (i.e. whose corresponding diagram has a connection between points in the same column) acts by 0.

One has

$$Y_{(n)}(k) = Y_{(n)}(n) \otimes_{End_{QB_c}(n)} Hom_{QB_c}(n, k). \tag{27}$$

In particular, $Y_{(n)}(k) = 0$ unless $k \geq n$ and $k - n$ is even.

Now the category of QB_c-modules has a symmetric tensor category structure which we call the *Day product*, described, for QB_c-modules

$$F, G : QB_c \to \mathbb{C}\text{-Vect}$$

as the left Kan extension

$$\begin{tikzcd} QB_c \times QB_c \arrow[r, "F\overline{\otimes}G"] \arrow[d, "\amalg"'] & \mathbb{C}\text{-Vect} \\ QB_c \arrow[ur, dashed, "F\otimes G"'] & \end{tikzcd}$$

where the vertical arrow is the functor

$$\amalg : QB_c \times QB_c \to QB_c$$
$$(m, n) \mapsto m + n$$

and the horrizontal arrow is the functor

$$F\overline{\otimes}G : QB_c \times QB_c \to \mathbb{C}\text{-Vect}$$
$$(m, n) \mapsto F(m) \otimes_{\mathbb{C}} N(n).$$

Definition 21. *We define a QB_c-algebra to be an algebra with respect to the Day product in the category of QB_c-modules, meaning a QB_c-module F with a given multiplication morphism (in the category QB_c-Mod)*

$$F \otimes F \to F$$

(satisfying associativity).

Note that by (27), we have a natural commutative associative unital pairing

$$Y_{(m)} \otimes Y_{(n)} \to Y_{(m+n)}. \tag{28}$$

Thus, we have a QB_c-algebra

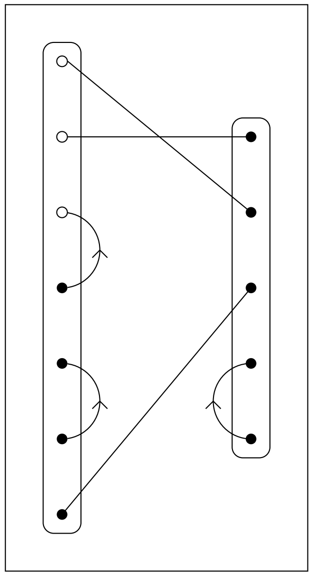
$$\underline{\mathbb{C}} = Y_{\emptyset} \oplus Y_{(1)} \oplus Y_{(2)} \oplus \dots. \tag{29}$$

To mirror the classical case of FI-modules, we begin by defining QI_c-modules in the following way:

Definition 22. *Define a QI_c-module M to be a module over the QB_c-algebra $\underline{\mathbb{C}}$, i.e. we are given the data of a multiplication morphism (in the category of QB_c-modules)*

$$\underline{\mathbb{C}} \otimes M \to M$$

with the usual axioms.



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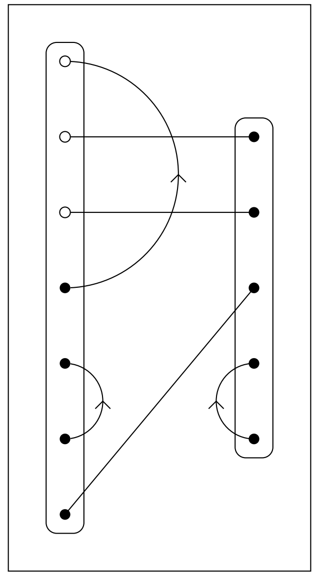
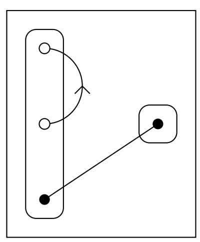


$\in Hom_{QI_c}(4,5)$



$= 0 \in Hom_{QI_c}(3,1)$

Figure 7. Examples of morphisms in QI_c

By explicitly considering the construction of the Day product $\underline{\mathbb{C}} \otimes M$, we can give a more a more elementary diagrammatic description of QI_c-modules.

Let QI_c be the category defined as follows: The objects are

$$Obj(QI_c) = \mathbb{N}_0$$

and for $m, n \in \mathbb{N}_0$, define the space of morphisms in QI_c from m to n to be, as a $\mathbb{C}$-vector space,

$$Hom_{QI_c}(m,n) = \bigoplus_{k \geq 0} Hom_{QB_c}(m+k, n) \otimes_{End_{QB_c}(k)} \mathbb{C} \tag{30}$$

where $\mathbb{C} = Y_{(k)}(k)$ as in (26).

A generator (30) is graphically described by drawing a column of m black dots under a column of k white dots on the left and n left dots

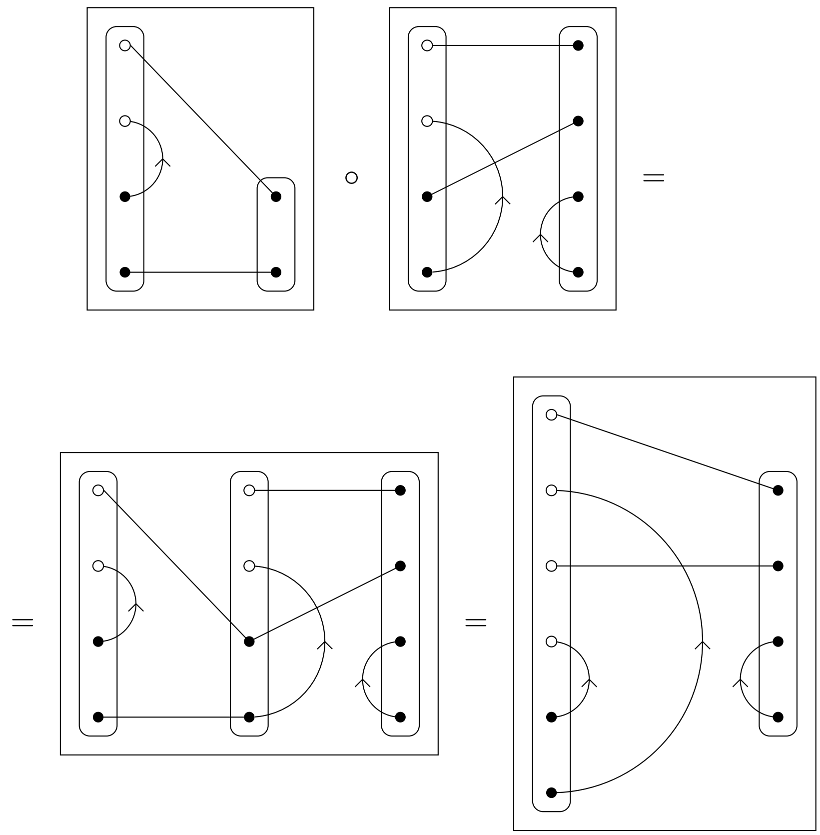


Figure 8. An example of a composition a generator $Hom_{QI_c}(2,2)$ and a generator of $Hom_{QI_c}(2,4)$

on the right, connected by a diagram representing a QB_c-morphism. The identifications mean that the order of white dots does not matter and that if we connect two white dots by a self-arrow, we get 0 (see Figure 7).

To compose two such diagrams, we compose the same way as in QB_c, and then group the white dots together in the top of the column corresponding to the source (see Figure 8).

Given the category QI_c, it is then natural to define a QI_c-module as a functor from QI_c to $\mathbb{C}$-vector spaces. These definitions are, in fact equivalent.

Proposition 23. *The category of QI_c-modules (as defined in Definition 22) is equivalent to the category of functors*

$$QI_c \to \mathbb{C}\text{-}Vect$$

and natural transformations.

$\square$

Again, there is a notion of a Day product on QI_c-modules, which can be interpreted equivalently both as left Kan extensions over diagrams similarly as in our definition of the Day product of QB_c-modules with QB_c replaced by QI_c, or as the QB_c-module Day product over $\underline{\mathbb{C}}$ (considering QI_c-modules as objects of QB_c-Mod which are modules over $\underline{\mathbb{C}}$).

Definition 24. *For $m \in \mathbb{N}_0 = Obj(QI_c)$, define the corresponding* representable QI_c-module

$$\underline{\underline{m}} : QI_c \to \mathbb{C}\text{-}Vect$$

by taking, for $n \in \mathbb{N}_0 = Obj(QI_c)$,

$$\underline{\underline{m}}(n) = Hom_{QI_c}(m, n)$$

(with the action of morphisms being given by composition).

Analogously to Proposition 7, we have

Proposition 25. *For $m, n \in \mathbb{N}_0 = Obj(QI_c)$, we have that*

$$\underline{\underline{m}} \otimes \underline{\underline{n}} \cong \underline{\underline{m+n}}.$$

$\square$

Example: We have $\underline{\underline{0}} = \underline{\mathbb{C}}$. Omitting the white dots in the generators' graphical representations, $\underline{\underline{0}}(m)$ for $m \in \mathbb{N}_0 = Obj(QI_c)$ can be described as free on generators of the form described in Figure 9, up to permutation and taking all arrows to be pointed upwards.

In particular, considering $m \in \mathbb{N}_0 = Obj(QI_c)$, $dim(\underline{\underline{0}}(m))$ is the number of possible pairings on a subset of $[m]$, which is (recalling the

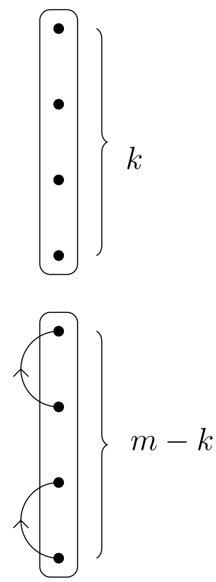


FIGURE 9. The generators of $\underline{\underline{0}}$ corresponding to generators $Hom_{QI_c}(0, m)$ with k white dots for $k \leq m$ such that $m - k$ is even (where the indicated k dots correspond to the subset of $[m]$ which was connected to white dots in the source)

computation (24))

$$\begin{aligned} &\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2i} \cdot (2i-1) \cdot (2i-3) \cdot \dots \cdot 3 \cdot 1 \\ &\qquad = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2i} \cdot \frac{(2i)!}{2^i \cdot (i)!}. \end{aligned} \tag{31}$$

Now let us consider $\underline{\underline{1}}$. The free generators of $\underline{\underline{1}}(m)$ are, up to permutation, of the form described in Figure 10. Considering

$$m \in \mathbb{N}_0 = Obj(QI_c),$$

we then have that

$$dim(\underline{\underline{1}}(m)) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2i} \cdot \frac{(2i)!}{2^i \cdot (i)!} + m \cdot \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2i} \frac{(2i)!}{2^i \cdot (i)!}$$

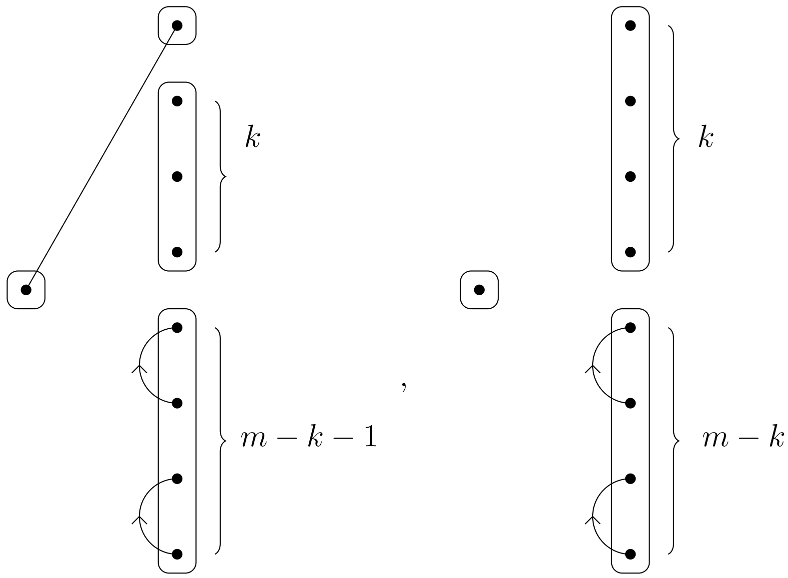


FIGURE 10. The generators of $\underline{\underline{1}}(m)$ on the left correspond to generators of $Hom_{QI_c}(1, m)$ with k white dots for $k+1 \leq m$ such that $m-k-1$ is even (where the indicated k dots correspond to the subset of $[m]$ which was connected to white dots in the source). The generators of $\underline{\underline{1}}(m)$ on the right correspond to generators of $Hom_{QI_c}(1, m)$ with $k+1$ white dots for $k \leq m$ such that $m-k$ is even where one of the white dots is connected to the point in the source.

Similarly as for FI-modules, we have a notion of *torsion* QI_c-modules. An *expansion* is a generating morphism in $Hom_{QI_c}(m, n)$ given by a bijection

$$[m+k] \to [n] \in Hom_{QB_c}(m+k, n).$$

Definition 26. *We say that an QI_c-module M is* torsion *if for every $m \in \mathbb{N}_0$, for every $x \in M(m)$, there exists an $n \in \mathbb{N}_0$ and an expansion $f \in Hom_{QI_c}(m, n)$ such that*

$$M(f)(x) = 0 \in M(n).$$

Denote the full subcategory of torsion QI_c-modules by QI_c-Mod^{tor}.

Similarly as for FI-modules, QI_c-Mod^{tor} forms a Serre subcategory of QI_c-Mod. Therefore we can make the following

Definition 27. *Define* generic QI_c-modules *to be the quotient category of QI_c-Mod by QI_c-Mod^{tor}.*

Similarly as in Lemma 9 in the case of FI-modules and $FI_c^{\pm}$-modules, if M is a torsion QI_c-module and N is any QI_c-module, then $M \otimes N$ is torsion and thus (by considering a free resolution of N), $Tor_i^{\otimes}(M, N)$ is torsion for all $i \geq 0$. Consequently, $\otimes$ passes to a symmetric monoidal tensor product on QI_c-Mod^{gen}.

Example: Recalling formula (29), the commutative QB_c-algebra $\underline{\mathbb{C}}$ has an ideal

$$I = Y_{(1)} \oplus Y_{(2)} \oplus Y_{(3)} \oplus \dots .$$

The quotient $\underline{\mathbb{C}}/I$ is torsion, and thus, the inclusion

$$\iota : I \xrightarrow{\subseteq} \underline{\mathbb{C}}$$

is an isomorphism in QI_c-Mod^{gen}, and hence, so is

$$\iota^n : I^n \xrightarrow{\subseteq} \underline{\mathbb{C}}$$

for every $n \geq 1$. (Note that

$$I^n = Y_{(n)} \oplus Y_{(n+1)} \oplus Y_{(n+2)} \oplus \dots .)$$

Also, note that the generators pictured on the right of Figure 10 give an inclusion

$$\kappa : \underline{\underline{0}} \hookrightarrow \underline{\underline{1}} \tag{32}$$

and thus a short exact sequence in QI_c-Mod^{gen}

$$0 \longrightarrow \underline{\underline{0}} \xrightarrow{\kappa} \underline{\underline{1}} \longrightarrow Q \longrightarrow 0. \tag{33}$$

On the other hand, we have a surjection

$$Q \twoheadrightarrow I \tag{34}$$

by sending the generator on the left of Figure 10 to the generators of Figure 9 obtained by erasing the line (with k replaced by $k+1$).

By the observation in the above Example, in QI_c-Mod^{gen}, corresponds to an epimorphism

$$Q \twoheadrightarrow \underline{\underline{0}}. \tag{35}$$

Thus, we have a short exact sequence in QI_c-Mod^{gen} of the form

$$0 \to \mathscr{Y}_{(1)} \to Q \to \underline{\underline{0}} \to 0. \tag{36}$$

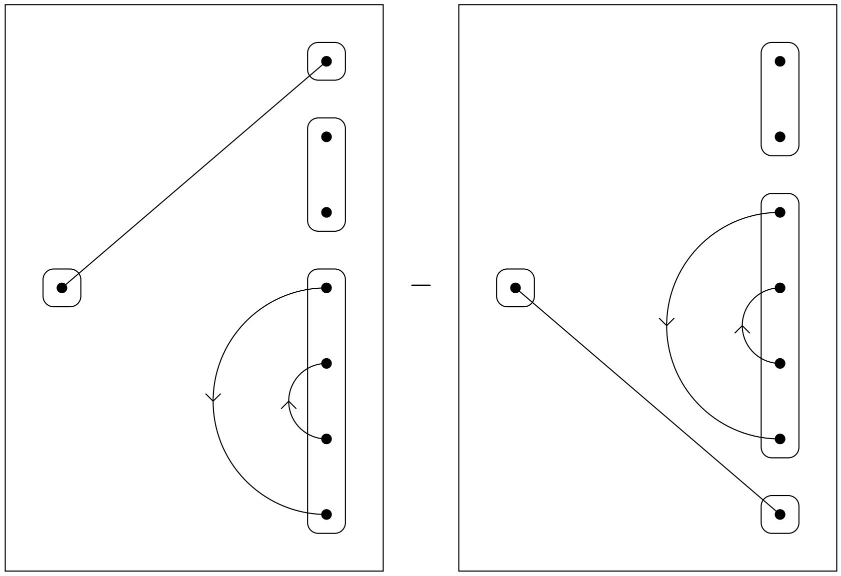


Figure 11. An example of a generator of $\mathscr{Y}_{(1)}(7)$, considering $\mathscr{Y}_{(1)}$ as the kernel of (35) (where in both terms we omit the connections between two white dots in the source with the two separated white dots in the target, similarly as for the left generator in Figure 10)

The $\mathbb{C}$-vector spaces $\mathscr{Y}_{(1)}(n)$ can also be described directly, consisting of linear combinations of generators on the left of Figure 10 whose coefficients, for each fixed configuration of ordered self-arrows, add up to 0 (see Figure 11).

Next, note that in QB_c-Mod, the representable object determined by $2 \in Obj(QB_c)$ contains $Y_\emptyset$ as a unique direct summand. Using Proposition 25, QI_c-Mod^{gen}, (given by self-arrows), this gives morphisms

$$\underline{\underline{0}} \xrightarrow{\eta} \underline{\underline{1}} \otimes \underline{\underline{1}}, \qquad \underline{\underline{1}} \otimes \underline{\underline{1}} \xrightarrow{\epsilon} \underline{\underline{0}},$$

which makes $\underline{\underline{1}}$ its own dual, η being anti-symmetric.

Proposition 28. *In QI_c-Mod^{gen}, $\underline{\underline{m}}$ is strongly dualizable and*

$$\underline{\underline{m}}^\vee \cong \underline{\underline{m}}.$$

Proof. The above observation proves the statement for $m = 1$. For general m, it then follows from Proposition 25.

□

Observing also that the morphism (35) composed with the second morphism (33) is the dual of κ, one can use Lemma 15 to conclude the following

Lemma 29. *In QI_c-Mod^{gen}, $\mathscr{Y}_{(1)}$ is strongly dualizable and one has*
$$\mathscr{Y}_{(1)}^{\vee} \cong \mathscr{Y}_{(1)}.$$
□

Note that additivity of dimensions gives
$$dim(\mathscr{Y}_{(1)}) = dim(V) - 2 \cdot dim(1) = c - 2.$$

By [2], Chapter 9, we have
$$Y_\lambda \otimes Y_\mu = \bigoplus_\nu N_{\lambda,\mu,\nu} \cdot Y_\nu$$
where $N_{\lambda,\mu,\nu}$ are the Newell-Littlewood numbers [12].

Now, for a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, let us further define QI_c-modules $\mathscr{Y}_\lambda$ by
$$\mathscr{Y}_\lambda = \bigoplus_{m \geq |\lambda| + \lambda_1} Y_{\lambda_m^+}, \tag{37}$$
recalling the notation (10), where the QI_c-module is determined by composition with the diagrammatic description of the generators of every $Y_{\lambda_m^+}(n)$, as in the above Example. Again, because of the composition rules described in Figure 8, since there is no QI_c-morphism that composes with a diagram with n free dots to give a diagram with fewer than n free dots. Thus, (37) defines a QI_c-module.

In particular, we have
$$1 = \mathscr{Y}_\emptyset.$$

For a given n, one can now form the Serre subcategory
$$QI_c\text{-Mod}(n) \subseteq QI_c\text{-Mod}$$
consisting of those QI_c-modules, which, as QB_c-modules,a re sums of Y_λ where the amplitude $|\lambda| - \lambda_1$ is $\leq m$. Considering the successive images of a QI_c-module in the quotient category
$$QI_c\text{-Mod}/QI_c\text{-Mod}(n),$$
one obtains the following analogue of Theorem 18 and Proposition 5.10 of [19]:

Theorem 30. *The $\mathscr{Y}_\lambda$ for Young diagrams form a complete system of non-isomorphic simple objects in QI_c-Mod^{gen}.*

$\square$

Theorem 31. *There is a tensor functor*

$$\Xi^q : \underline{Rep}(Sp_{c-2}) \to QI_c\text{-}Mod^{gen}$$

such that

$$\Xi^q(Y_\lambda) = \mathscr{Y}_\lambda. \tag{38}$$

In particular, the simple objects $\mathscr{Y}_\lambda$ are self-dual, and the category QI_c-Mod^{gen} of generic QI_c-modules has strong duality.

Proof. Analogous to the proof of Theorem 19. By Lemma 15 and the universality of $\underline{Rep}(Sp_t)$ with respect to $\mathbb{C}$-linear categories with an associative, commutative, unital tensor product such that $End(1) = \mathbb{C}$, with a self-dual object where the unit

$$\eta : \underline{\underline{0}} \to \underline{\underline{1}} \otimes \underline{\underline{1}}$$

is antisymmetric, (which follows from Proposition 9.4 of [2]), we obtain the tensor functor Ξ^q.

Thus, all that remains to prove is (38). This follows again from identifying the correct QB_c-simple summands of $\mathscr{Y}_\lambda \otimes \underline{\underline{1}}$, similarly as in the proof of Theorem 15.

Specifically, calculating

$$\mathscr{Y}_\lambda \otimes \underline{\underline{1}},$$

by the Newell-Littlewood analogue of the Pieri rule applied to the QB_c-summand of $\mathscr{Y}_\lambda$, contains summands $Y_{\lambda'}$ where λ' is obtained from λ_n^+ by adding or subtracting one square. Adding or subtracting a square from the first row gives terms that belong to one of two copies of $\mathscr{Y}_\lambda$, which correspond to the two copies of the unit $\underline{\underline{0}}$ in $\underline{\underline{1}}$. Adding or subtracting a square from another row corresponds to applying the Newell-Littlewood variant of the Pieri rule to λ, as needed.

$\square$

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