THE DELANNOY TREE CATEGORY

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ABSTRACT. We consider an ordered counterpart to the arboreal categories constructed by Harman, Snowden, and Nekrasov. Our construction was inspired by the relationship of the Delannoy category of Harman, Snowden, and Snyder with the category $Rep(S_t)$ of Deligne. In the tree case, however, the relationship is more complicated, which we also study.

1. INTRODUCTION

In this paper, we describe a "Delannoy analogue" of the arboreal categories introduced by N. Harman, I. Nekrasov, and A. Snowden in [3]. We define a Fraïssé class $\mathfrak{T}^{\leq}$ of ordered planar trees, and consider the oligomorphic group of automorphisms of the universal real ordered tree $\mathscr{T}^{\leq}$ (its Fraïssé limit). We construct a regular measure, in the sense of [4], on the oligomorphic group $Aut(\mathscr{T}^{\leq})$ of order-preserving automorphisms of the tree valued in $\{\pm 1\}$. Thus, we obtain

Theorem 1. *The category $Rep(Aut(\mathscr{T}^{\leq}), \mu^{\leq})$ is a semisimple pre-Tannakian category over fields of every characteristic.*

The oligomorphic group $Aut(\mathscr{T}^{\leq})$ is a subgroup of the oligomorphic group corresponding the Fraïssé class $\mathfrak{T}$ of (unordered) trees considered in [3], but adding up the measures of the orbits of the Delannoy group contained in an orbit of [3], we do not obtain a measure. We obtain therefore an example of a measure on a oligomorphic subgroup not inducing a measure of a larger oligomorphic group.

On the other hand, the Fraïssé limit $\mathscr{T}^{\leq}$ is a countable totally ordered set with ($x < y \Rightarrow \exists z\ x < z < y$), and therefore $Aut(\mathscr{T}^{\leq})$ is the subgroup of the oligomorphic group $Aut(\leq)$ of permutations preserving the order.

It follows from the universal property of the Delannoy category [6], Theorem 4.9, that there is a tensor functor from the Delannoy category [5] to the arboreal Delannoy category:

$$\mathscr{D} \to Rep(Aut(\mathscr{T}^{\leq}), \mu^{\leq}).$$

However, we can also prove the following more precise statement:

Proposition 2. *Let C be an orbit of $Aut(\leq)$, and let*

$$C = C_1 \amalg \cdots \amalg C_k$$

where $C_1, \dots, C_k$ are orbits of $Aut(\mathscr{T}^{\leq})$. Then

$$\sum_{i=1}^{k} \mu^{\leq}(C_i) = \mu(C)$$

where μ denotes the regular Delannoy measure.

In other words, comparison with the Delannoy category behaves "correctly."

We also prove the following uniqueness result:

Theorem 3. *The measure $\mu^{\leq}$ on $Aut(\mathscr{T}^{\leq})$ is the only regular measure valued in a field of characteristic not 2.*

We also study isomorphism classes of the simple objects of the category $Rep(Aut(\mathscr{T}^{\leq}), \mu^{\leq})$. However, unlike the case of the Delannoy category [5] or [6], we are unable to determine the simple objects completely. Every ordered planar tree gives rise to an object. Following [1] (Remark 3.9 (ii)), we can consider its simple summands which occur via smaller trees. We call such summands *singular*. Among the non-singular (i.e. "top" summands) there is a collection of simple summands analogous to those occurring in the Delannoy category [5], which occurs with multiplicity one and which we call *regular*. However, even in the case of the tree with two leaves, we will see that there are 30 additional non-isomorphic simple summands of categorical dimensions 1, -1, which fall into neither of the above types. We call such summands *residual*. At the present time, we do not understand this well.

Nevertheless, we are able to prove the following

Theorem 4. *Every simple object in $Rep(Aut(\mathscr{T}^{\leq}), \mu^{\leq})$ has dimension ± 1.*

The present paper is organized as follows: In Section 2, we define the Fraïssé class of ordered planar trees. In Section 3, we define a regular measure $\mu^{\leq}$ on this class (proving Theorem 1). In Section 4, we prove uniqueness of the regular measure $\mu^{\leq}$ in characteristic not 2 (proving Theorem 3). In Section 5, we discuss the comparison of the measure $\mu^{\leq}$ with the unordered arboreal measures [3] and the regular Delannoy measure [5]. We prove Proposition 2. In Section 6, we discuss the regular simple objects of $Rep(Aut(\mathscr{T}^{\leq}), \mu^{\leq})$ and prove Theorem 4. In

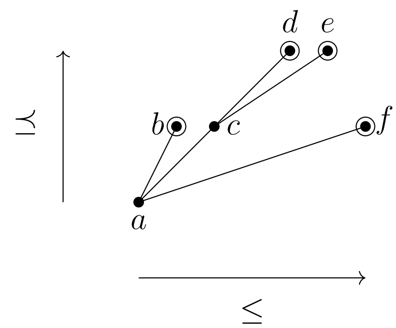


Figure 1. The ordered planar tree structure on vertices $\{a, b, c, d, e, f\}$ given by $a \leq b \leq c \leq d \leq e \leq f$ and the minimal partial ordering $\preceq$ such that $a \preceq b, c, f$ and $c \preceq d, e$. Leaves are circled.

Section 7, we discuss the complete decomposition of the tree with two leaves into simple summands, including the 30 residual summands.

2. The Fraïssé class of ordered trees

In this section, we will describe ordered planar trees using the formalism of Fraïssé structures [2].

Definition 5. *Define an* ordered planar tree *to consist of the data of a a set of vertices S with a well partial ordering $\preceq$ and a total ordering $\leq$ such that*

(1) For any $a, b \in S$,

$$a \preceq b \Rightarrow a \leq b \tag{1}$$

(2) For $a, b, c \in S$ such that $a \leq b \leq c$, we then have that

$$a \preceq c \Rightarrow a \preceq b \tag{2}$$

such that there is a unique minimal element of S with respect to $\preceq$, which we call the root*. A* leaf *is a maximal element with respect to $\preceq$. The set of leaves of an ordered planar tree $\mathcal{T}$ will be denoted by $\ell(\mathcal{T})$.*

We visualize this structure by drawing the minimal possible edges on S such that if $a \preceq b$, then there is a path connecting a and b. We draw this graph in a plane so that if $a \preceq b$, then a is lower than b, and if $a \leq b$, then a is to the left of b. Condition (2) then guarantees that none of these edges cross. An example is pictured in Figure 1.

Definition 6. *Call an ordered planar tree* reduced *if there do not exist distinct vertices $b \neq c$ such that $b \preceq c$ and for every $d \neq b \in S$,*

$$b \preceq d \ \Rightarrow\ c \preceq d. \tag{3}$$

The condition of being reduced is equivalent to requiring that every vertex which is not a leaf has degree ≥ 3 except the root, which has degree ≥ 2.

The structure of an ordered planar tree is not Fraïssé. However, it can be equivalently characterized by a Fraïssé structure on its totally ordered set X of leaves. The structure is characterized by two relations

$$R^+, R^- \subseteq \binom{X}{3} \tag{4}$$

where

(1) $\binom{X}{3} = R^+ \cup R^-$

(2) If $\{a < b < c\} \in R^\pm$ and $\{b < c < d\} \in R^\pm$, then $\{a < b < d\} \in R^\pm$ and $\{a < c < d\} \in R^\pm$.

(3) If $\{a < b < c\} \in R^-$, then
$$\{a < c < d\} \in R^\pm \ \Leftrightarrow\ \{a < b < d\} \in R^\pm.$$

(4) If $\{b < c < d\} \in R^+$, then
$$\{a < c < d\} \in R^\pm \ \Leftrightarrow\ \{a < b < d\} \in R^\pm.$$

To prove the equivalence of both structures, one first notes that it is obvious that the vertices of an ordered planar tree satisfy (1)-(4). On the other hand, for a relational structure satisfying (1)-(4), we recover the tree by letting the non-leaf vertices be written as $v = \inf_{\preceq}\{a, b\}$ for $a < b$ and setting

$$v_{a,c} = v_{a,b} \preceq v_{b,c} \text{ when } \{a < b < c\} \in R^-$$
$$v_{b,c} = v_{a,c} \preceq v_{a,b} \text{ when } \{a < b < c\} \in R^+$$

($\preceq$-equivalent vertices are identified). The axioms (1)-(4) imply consistency on 4 leaves, which is sufficient to conclude that we get an ordered planar tree.

Let $\mathfrak{T}^{\leq}$ denote the Fraïssé class of the sets of leaves of ordered trees.

3. Measure

Let $\mathcal{T} \in \mathfrak{T}^{\leq}$ be an ordered with non-empty set of vertices S define the set of its *nodes* to be

$$n(\mathcal{T}) = S \smallsetminus \ell(\mathcal{T})$$

(i.e. the set of vertices which are not leaves). Then put

$$\mu^{\leq}(\mathcal{T}) = (-1)^{|n(\mathcal{T})|+1}. \tag{5}$$

In particular, the unique ordered tree $\{*\}$ with a single vertex has no nodes, so

$$\mu^{\leq}(\{*\}) = -1.$$

For the ordered tree where the set of vertices is empty, we take

$$\mu^{\leq}(\emptyset) = 1.$$

To prove the $\mu^{\leq}$ defines a measure on $\mathfrak{T}^{\leq}$, the only non-trivial property to check is the following (see Definition 6.4, [4]):

Proposition 7. *Consider all possible amalgamations $\mathcal{T}^i$, $i = 1, \dots n$ of reduced ordered planar trees $\mathcal{U}_1$, $\mathcal{U}_2$ along fixed ordered embeddings*

$$\iota_1 : \ell(\mathcal{V}) \hookrightarrow \ell(\mathcal{U}_1)$$

$$\iota_2 : \ell(\mathcal{V}) \hookrightarrow \ell(\mathcal{U}_2)$$

for some reduced ordered planar tree $\mathcal{V}$. Then

$$\frac{\mu^{\leq}(\mathcal{U}_1) \cdot \mu^{\leq}(\mathcal{U}_2)}{\mu^{\leq}(\mathcal{V})} = \sum_{i=1}^{n} \mu^{\leq}(\mathcal{T}^i) \tag{6}$$

Proof. Let $S_{\mathcal{U}_1}, S_{\mathcal{U}_2}$, and $S_{\mathcal{V}}$ be the sets of vertices of $\mathcal{U}_1, \mathcal{U}_2$, and $\mathcal{V}$. We first claim that we can reduce the statement to the case

$$|\ell(\mathcal{U}_1) \smallsetminus \ell(\mathcal{V})| = 1. \tag{7}$$

The reason is that, choosing $u \in \ell(\mathcal{U}_1) \smallsetminus \ell(\mathcal{V})$, we may first describe all the possible amalgamations of $\{u\} \cup \ell(\mathcal{V})$ with $\ell(\mathcal{U}_2)$ via $\ell(\mathcal{V})$, and then replace $\ell(\mathcal{V})$ by $\{u\} \cup \ell(\mathcal{V})$, and $\ell(\mathcal{U}_2)$ by $\{u\} \cup \ell(\mathcal{U}_2)$, thus reducing $|\ell(\mathcal{U}_1) \smallsetminus \ell(\mathcal{V})|$ by 1, which allows us to proceed by induction.

Now assuming (7), let $\{u\} = \ell(\mathcal{U}_1) \smallsetminus \ell(\mathcal{V})$. We shall first assume that

$$\ell(\mathcal{V}) \neq \emptyset.$$

Then consider the node

$$w = \sup_{\preceq}\{x \in n(\mathcal{U}_1) \mid x \preceq u\}.$$

Then we have several cases:

Case 1: w is not a node of $\mathcal{V}$. Then consider the vertices v_+, v_- of $\mathcal{V}$ such that

$$v_- \underset{\neq}{\preceq} w \underset{\neq}{\preceq} v_+ \in S_{\mathcal{U}_1},$$

$$v_- \preceq x \preceq v_+ \in S_{\mathcal{V}} \Rightarrow (x = v_- \text{ or } x = v_+).$$

Then let

$$\{z_1 \underset{\neq}{\preceq} \dots \underset{\neq}{\preceq} z_k\} = \{z \in S_{\mathcal{U}_2} \mid v_- \underset{\neq}{\preceq} z \underset{\neq}{\preceq} v_+\}. \tag{8}$$

Further, let

$$Q = \{x \in S_{\mathcal{U}_2} \mid \exists 1 \leq i \leq k \; z_i \preceq x\} \tag{9}$$

Now we claim that the sum of measures of the amalgamations considered is $\mu^{\leq}(\mathcal{U}_2)$ times an alternating sum of -1 and 1 starting and ending with -1. The beginning term is

$$v_- \underset{\neq}{\preceq} w \underset{\neq}{\preceq} z_1,$$

which has sign -1, due to w being an additional node. The next term is

$$w = z_1, \; u < x \text{ for all } z_1 \preceq x, \; x \in \ell(\mathcal{U}_2),$$

which has sign $+1$ due to there not being an additional node. The alternating sum then proceeds "clockwise" along the forest spanned by (9), with the last term being

$$z_k \underset{\neq}{\preceq} w \underset{\neq}{\preceq} v_+,$$

which again has sign -1. This case is illustrated in Figure 2, with $k = 1$, $|Q| = 2$. (Note: It is also possible for v_- not to exist. This case, however, proceeds the same way.)

Case 2: w is a node $\mathcal{V}$. If this case, let $v_<$ (resp. $v_>$) be the $\leq$-largest (resp. $\leq$-least) leaf of $\mathcal{V}$ with $v_< < u$ (resp. $v_> > u$) in $\mathcal{U}_1$. We let

$$Q = \{x \in S_{\mathcal{U}_2} \mid v_< < x < v_>\}.$$

This time, the sum of measures of the amalgamation is $\mu^{\leq}(\mathcal{U}_2)$ times an alternating sum of 1 and -1, starting an ending with 1, "moving clockwise" around the tree spanned by Q. The first term is

$$u < x \;\text{ for every } x \in Q$$

and the last term is

$$u > x \;\text{ for every } x \in Q.$$

(Note: there is also the possibility of $v_<$ or $v_>$ not existing. However, the conclusion is the same.)

It remains to consider the case when $S_{\mathcal{V}} = \emptyset$. In this case, however, we have $\mu^{\leq}(\mathcal{U}_2)$ times an alternating sum moving clockwise around $\mathcal{U}_2$, starting and ending with -1, corresponding to the cases when w is the root of the amalgamation of degree 2. (Note that then $|S_{\mathcal{U}_1}| = |\ell(\mathcal{U}_1)| = 1$, so the $\mu^{\leq}(\mathcal{U}_1) = -1$.)

□

Recall that a measure μ on the orbits of an oligomorphic group is called *regular* if it always non-zero, see [4]. By definition, for every ordered tree $\mathcal{T}$, its measure $\mu^{\leq}(\mathcal{T}) = \pm 1$, and is therefore regular over any base field. Thus, Theorem 1 follows by Theorem of [4].

4. Uniqueness of Measure

The purpose of this section is to prove Theorem 3.

Proof of Theorem 3. Suppose ν is a regular measure on $\mathfrak{T}^{\leq}$. We begin by introducing the following notation (see Figure 3): Let $\mathcal{X}_1$ be the unique ordered planar tree with a single leaf. Let $\mathcal{X}_2$ be the unique ordered planar tree with two leaves.

Let $\mathcal{X}_n$ for $n \geq 3$ be the ordered planar tree on a set of n leaves $\{x_1 < \cdots < x_n\}$ such that, in terms of the Fraïssé structure on the set of leaves, all $\{x_i < x_j < x_k\}$ for $i < j < k$ are elements of R^+ and R^-

$$R^+ = R^- = \binom{\{x_1 < \cdots < x_n\}}{3}.$$

In other words, graphically, $\mathcal{X}_n$ has n leaves and a single node.

Let $\mathcal{A}$ be the ordered planar tree with tree leaves $\{x < y < z\}$ such that, in the Fraïssé structure

$$R^+ = \{\{x < y < z\}\} \text{ and } R^- = \emptyset.$$

Let $\mathcal{B}$ be the ordered planar tree with tree leaves $\{x < y < z\}$ such that, in the Fraïssé structure

$$R^+ = \emptyset \text{ and } R^- = \{\{x < y < z\}\}.$$

Now let us write

$$t := \nu(\mathcal{X}_1).$$

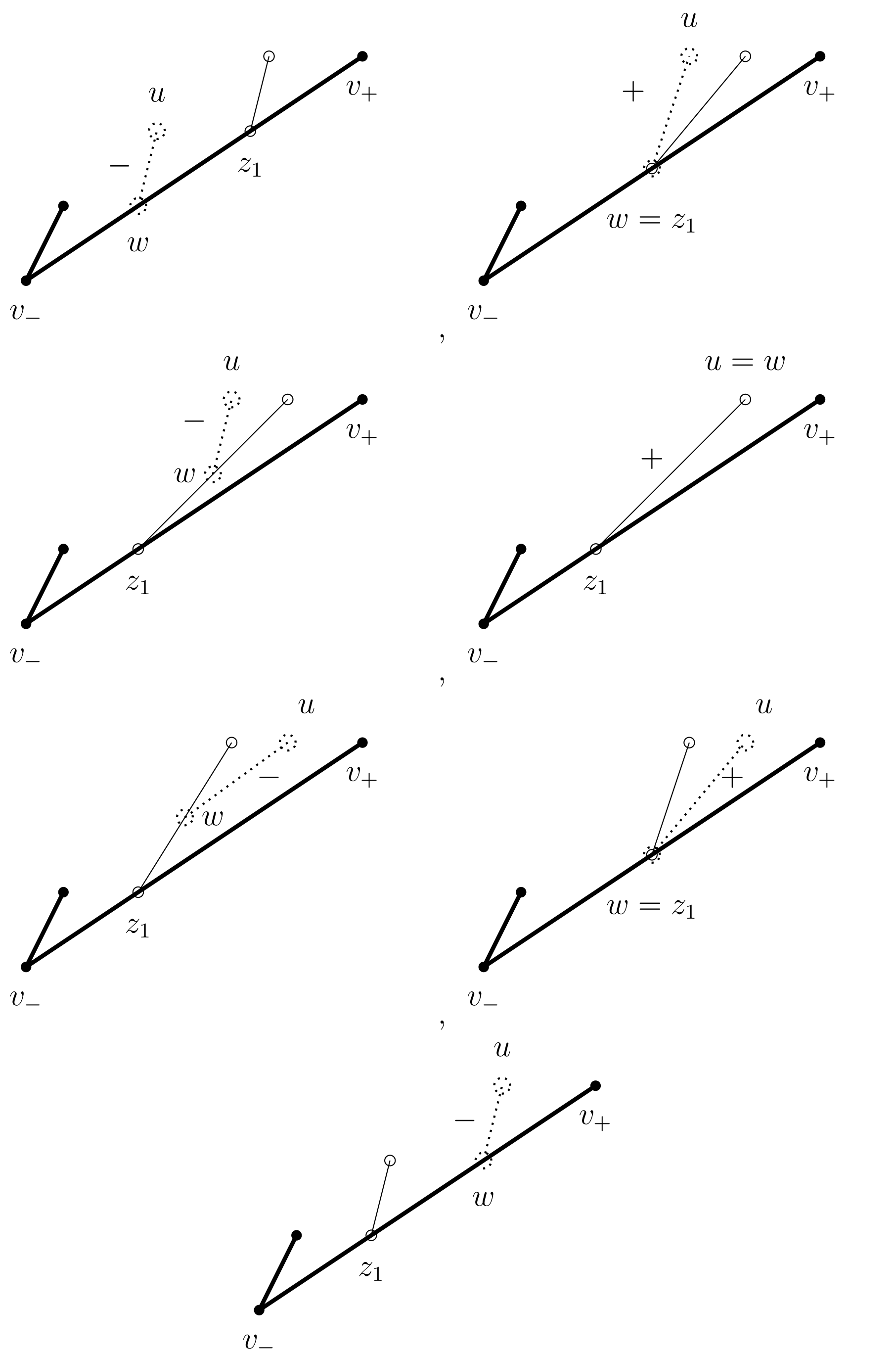


Figure 2. An example of all possible amalgamations $\mathcal{T}^i$ in Case 1, where $\mathcal{V}$ is the bold tree, $\mathcal{U}_1$ is the minimal tree containing $\mathcal{V}$ and the dotted node, branch, and leaf, and $\mathcal{U}_2$ is the minimal tree containing $\mathcal{V}$ and the thin node, branch, and leaf. The sign notes $\mu^{\leq}(\mathcal{T}^i) = \pm\mu^{\leq}(\mathcal{U}_2)$.

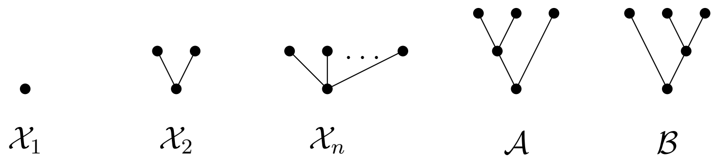


FIGURE 3. The ordered planar trees $\mathcal{X}_n$, $\mathcal{A}$, $\mathcal{B}$. (For the sake of graphical space, we do not tilt the diagram so that the total ordering on all vertices is from left to right. We take the leaves to be ordered with respect to $\leq$ from left to right; the orderings $\preceq$, $\leq$ are then determined.)

Let us apply (6) to $\mathcal{U}_1 = \mathcal{X}_1$, $\mathcal{U}_2 = \mathcal{X}_1$, $\mathcal{V} = \emptyset$. There are two possible amalgamations with two leaves, corresponding to choosing that the leaf in $\mathcal{U}_1$ is (strictly) less than or greater than the leaf in $\mathcal{U}_2$, both of which are isomorphic to $\mathcal{X}_2$. There is one possible amalgamation with one leaf, corresponding to identifying the leaves in $\mathcal{U}_1$ and $\mathcal{U}_2$, which is isomorphic to $\mathcal{X}_1$. Therefore, (6) gives that

$$\nu(\mathcal{X}_1)^2 = 2 \cdot \nu(\mathcal{X}_2) + \nu(\mathcal{X}_1),$$

thus giving

$$\nu(\mathcal{X}_2) = \frac{t^2 - t}{2}. \tag{10}$$

Now let us consider amalgamations with $\mathcal{U}_1 = \mathcal{X}_2$, $\mathcal{U}_2 = \mathcal{X}_2$, $\mathcal{V} = \mathcal{X}_1$.

$$\ell(\mathcal{U}_1) = \{x_1 < y_1\},\ \ell(\mathcal{U}_2) = \{x_2 < y_2\},\ \ell(\mathcal{V}) = \{z\}. \tag{11}$$

First let us consider amalgamations along embeddings

$$\begin{aligned} \iota_1 : \ell(\mathcal{V}) &\to \ell(\mathcal{U}_1) \\ z &\mapsto y_1 \\ \iota_2 : \ell(\mathcal{V}) &\to \ell(\mathcal{U}_2) \\ z &\mapsto x_2 \end{aligned} \tag{12}$$

There are three possible amalgamations of (11) along (12), each of which has exactly three leaves $\{x_1 < z < y_2\}$, which can either be taken to be in $R^+ \smallsetminus (R^+ \cap R^-)$, $R^+ \cap R^-$, or $R^- \smallsetminus (R^+ \cap R^-)$. These amalgamations are isomorphic to $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}_3$, respectively (see Figure 4). Thus, (6) gives

$$\frac{\nu(\mathcal{X}_2)^2}{\nu(\mathcal{X}_1)} = \nu(\mathcal{A}) + \nu(\mathcal{X}_3) + \nu(\mathcal{B}). \tag{13}$$

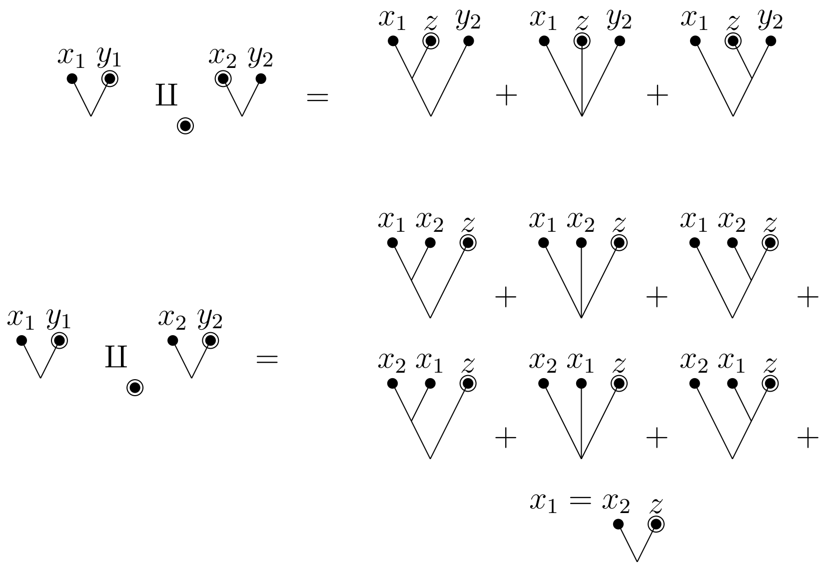


FIGURE 4. Amalgamations of (11) over (12) and (14), respectively. The leaves identified by the embeddings ι_1, ι_2.

We may also consider the amalgamation of (11) along embeddings

$$\begin{aligned} \iota_1 : \ell(\mathcal{V}) &\to \ell(\mathcal{U}_1) \\ z &\mapsto y_1 \\ \iota_2 : \ell(\mathcal{V}) &\to \ell(\mathcal{U}_2) \\ z &\mapsto y_2 \end{aligned} \tag{14}$$

There are three amalgamations with ordered set of leaves $\{x_1 < x_2 < z\}$ and three (isomorphic) amalgamations with ordered set of leaves $\{x_2 < x_1 < z\}$, again by taking the triplet of leaves to be in $R^+ \smallsetminus (R^+ \cap R^-)$, $R^+ \cap R^-$, or $R^- \smallsetminus (R^+ \cap R^-)$, (giving ordered planar trees isomorphic to $\mathcal{A}$, $\mathcal{X}_3$, and $\mathcal{B}$, respectively). Finally, there is one amalgamation with two leaves $\{x_1 = x = 2 < z\}$ coming from further identifying x_1 and x_2, which is isomorphic to $\mathcal{X}_2$. (See Figure 4.) Therefore, we also obtain the following relation:

$$\frac{\nu(\mathcal{X}_2)^2}{\nu(\mathcal{X}_1)} = 2(\nu(\mathcal{A}) + \nu(\mathcal{X}_3) + \nu(\mathcal{B})) + \nu(\mathcal{X}_2). \tag{15}$$

Combining this with (13), this gives

$$\frac{\nu(\mathcal{X}_2)^2}{\nu(\mathcal{X}_1)} = 2\frac{\nu(\mathcal{X}_2)^2}{\nu(\mathcal{X}_1)} + \nu(\mathcal{X}_1).$$

Applying (10), this reduces to give

$$t(t-1)(t+1) = 0,$$

i.e. $t = 0, 1$, or -1. Since ν is assumed to be regular, $t \neq 0$. If t were 1, the $\nu(\mathcal{X}_2) = 0$ by (10). Hence,

$$t = -1,$$

and therefore $\nu(\mathcal{X}_2) = 1$.

Claim 1. *The measures of the tree with three leaves must be*

$$\nu(\mathcal{A}) = \nu(\mathcal{B}) = -1,\ \nu(\mathcal{X}_3) = 1. \tag{16}$$

Proof of Claim 1. Let us consider amalgamations of $\mathcal{U}_1 = \mathcal{A}$, $\mathcal{U}_2 = \mathcal{A}$, $\mathcal{V} = \mathcal{X}_2$. Writing

$$\ell(\mathcal{U}_1) = \{a_1 < b_1 < c_1\},\ \ell(\mathcal{U}_2) = \{a_2 < b_2 < c_2\}, \ell(\mathcal{V}) = \{t < u\},$$

consider embeddings

$$\begin{aligned} \iota_1 : \ell(\mathcal{V}) &\to \ell(\mathcal{U}_1) \\ t &\mapsto a_1 \\ u &\mapsto b_1 \end{aligned}$$

$$\begin{aligned} \iota_2 : \ell(\mathcal{V}) &\to \ell(\mathcal{U}_2) \\ t &\mapsto a_2 \\ u &\mapsto b_2. \end{aligned}$$

There are three possible amalgamations with ordered set of leaves $\{t < u < c_1 < c_2\}$, and three (isomorphic) possible amalgamations with ordered set of leaves $\{t < u < c_2 < c_1\}$. The measures of these ordered planar trees can be reduced to a product of $\nu(\mathcal{A})$ and $\nu(\mathcal{A})$, $\nu(\mathcal{X}_3)$, or $\nu(\mathcal{B})$, respectively (by Claim 2 below, taking x_i and x_j as shown in Figure 5).

There is a single amalgamation with three leaves $\{t < u < c_1 = c_2\}$, which is isomorphic to $\mathcal{A}$. Thus, (6) gives

$$\nu(\mathcal{A})^2 = 2 \cdot \nu(\mathcal{A}) \cdot (\nu(\mathcal{A}) + \nu(\mathcal{X}_3) + \nu(\mathcal{B})) + \nu(\mathcal{A}).$$

Combing this with (13), we conclude that $\nu(\mathcal{A}) = -1$. By symmetry, $\nu(\mathcal{B}) = -1$. Finally, $\nu(\mathcal{X}_3) = 1$ follows from (13). □

This fixes the measure of all ordered planar trees with ≤ 3 leaves. Let $\mathcal{T}$ be an ordered planar tree with

$$\ell(\mathcal{T}) = \{x_1 < \cdots < x_n\},\ n \geq 4. \tag{17}$$

Let $1 \leq i < j \leq n$.

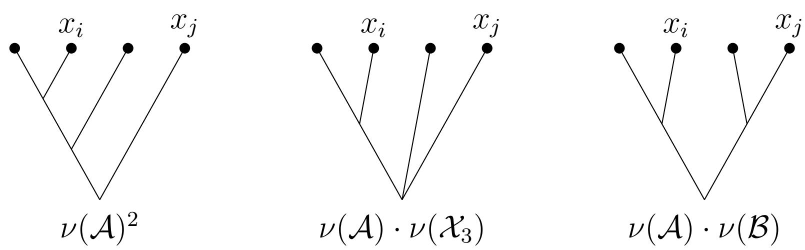


FIGURE 5. Choice of leaves "x_i, x_j" which satisfy Condition 1 and, applying Claim 2, (20), compute each tree's measure as the indicated product.

We introduce the following

Condition 1: *$j - i \geq 2$ and, letting $\mathcal{V}$ be the restriction of $\mathcal{T}$'s ordered planar tree structure to leaves $\ell(\mathcal{T}) \smallsetminus \{x_i, x_j\}$, there does not exist a pair of vertices a, b of $\mathcal{V}$ such that*

$$b \underset{\neq}{\preceq} a, \quad \textit{and (for } c \in S_{\mathcal{V}},\ c \underset{\neq}{\preceq} a \Rightarrow c \preceq b) \tag{18}$$

(when (18) holds, we say that $\{b \preceq a\}$ forms an edge *of $\mathcal{V}$) and such that the nodes*

$$y := \max{}_{\preceq}\{y \in S_{\mathcal{T}} \mid y \underset{\neq}{\preceq} x_i\}$$

$$z := \max{}_{\preceq}\{z \in S_{\mathcal{T}} \mid z \underset{\neq}{\preceq} x_j\}$$

both satisfy

$$b \underset{\neq}{\preceq} y \underset{\neq}{\preceq} a \textit{ and } b \underset{\neq}{\preceq} z \underset{\neq}{\preceq} a \tag{19}$$

in $\mathcal{T}$.

Claim 2. *Suppose there are $i < j$ such that Condition 1 holds. Let $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{V}$ be the restrictions of $\mathcal{T}$'s ordered planar tree structure to sets of leaves*

$$\ell(\mathcal{U}_1) := \ell(\mathcal{T}) \smallsetminus \{x_i\}, \ell(\mathcal{U}_2) := \ell(\mathcal{T}) \smallsetminus \{x_j\},$$
$$\ell(\mathcal{V}) := \ell(\mathcal{T}) \smallsetminus \{x_i, x_j\}$$

Then the only amalgamation of $\mathcal{U}_1$, $\mathcal{U}_2$ along the inclusions $\ell(\mathcal{V}) \subseteq \ell(\mathcal{U}_1), \ell(\mathcal{U}_2)$ is $\mathcal{T}$ itself. Consequently,

$$\nu(\mathcal{T}) = \frac{\nu(\mathcal{U}_1) \cdot \nu(\mathcal{U}_2)}{\nu(\mathcal{V})}. \tag{20}$$

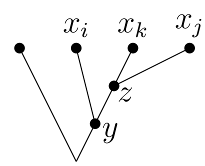


FIGURE 6. An example of a choice of $x_i < x_j$ failing Condition 1. In amalgamations of $\mathcal{U}_2$ and $\mathcal{U}_1$ over $\mathcal{V}$, there is ambiguity of whether $y \underset{\neq}{\preceq} z$, $y = z$, or $z \underset{\neq}{\preceq} y$, leading to multiple possible amalgamations.

Proof of Claim 2. Suppose $\mathcal{R}$ is such an amalgamation of $\mathcal{U}_1$, $\mathcal{U}_2$ over $\mathcal{V}$. First note that since $j - i \geq 2$, there exists a leaf $x_k \in \ell(\mathcal{V})$ such that $x_i < x_j < x_k$, so x_i cannot be identified with x_j in $\mathcal{R}$ and the order $x_i < x_j$ is fixed. Also, note that there exists an x_k for $k \neq j$ such that $y = \inf_{\preceq}(x_i, x_k)$ (otherwise, if $y = \inf_{\preceq}(x_i, x_j)$ only for x_j, Condition 1 fails). Therefore, y is a vertex $y \in S_{\mathcal{U}_2}$. Similarly, z is a vertex $z \in S_{\mathcal{U}_1}$.

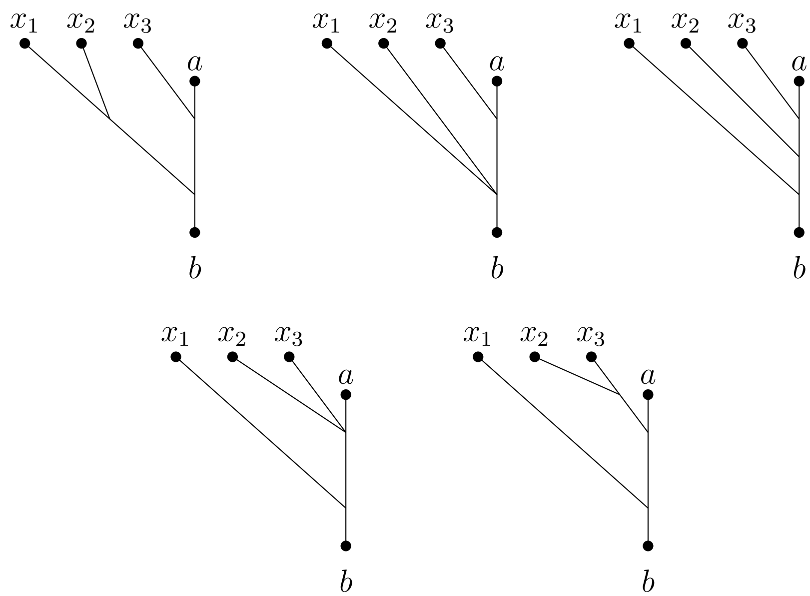
Now by Condition 1, y and z are on different edges of the tree $\mathcal{V}$, and therefore there is no ambiguity on how y and z attach in $\mathcal{R}$, making $\mathcal{T}$ the only possible amalgamation. Figure 6 exhibits why Condition 1 is necessary.

□

Figure 5 shows three examples of Claim 2.

Claim 3. *Let $\mathcal{T}$ be an ordered planar tree with (17). Then there exist $1 \leq i < j \leq n$ satisfying Condition 1.*

Proof of Claim 3. We begin by considering $x_i = x_1$, $x_j = x_3$. If there does not exist an edge $\{b \preceq a\}$ of $\mathcal{T}$ such that (19) holds, then we are done. Suppose there does exist an edge $\{b \preceq a\}$ such that (19) holds. Recall that we let $\ell(\mathcal{V}) = \ell(\mathcal{T}) \smallsetminus \{x_1, x_3\}$. We claim that then $a = x_2$. First note that then $a < x_3$, since $a \neq x_3$ (because by assumption, $a \in S_{\mathcal{V}}$) and if $a > x_3$, then we may consider the maximal node underneath the leaf x_2 in $\mathcal{T}$ and observe that every possible case of attaching this node (drawn below) is ruled out by (19) and the assumption that $\{b \preceq a\}$ is an edge of $\mathcal{V}$, since $x_2 \in \ell(\mathcal{V}) \in S_{\mathcal{V}}$:

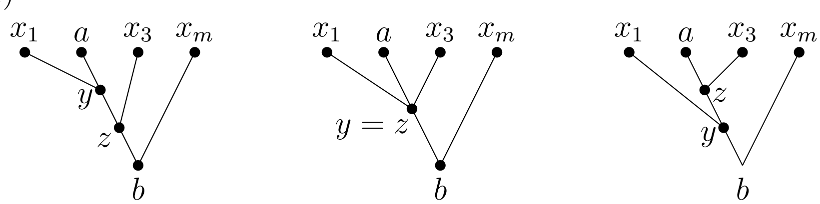


Therefore, we have $a < x_3$. Now we know that there exists an $i \neq 1, 3$ such that $a \preceq x_i$; suppose $i \neq 2$. Then we have $a < x_3 < x_i$ and $a \preceq x_i$. By (2), $a \preceq x_3$, contradicting (19). Hence, $a \preceq x_2$. Since there are no other leaves between x_1 and x_3, a cannot have valency ≥ 3, and is therefore a leaf, giving

$$a = x_2.$$

Note, also that there exists an $m > 3$ such that $b \preceq x_m$ (otherwise b is a vertex of valency 1). The only possible cases of y and z are then as follows:

(21)



Then $x_i = x_2 = a$, and $x_j = x_m$ satisfy Condition 1 in each case (21). Since $m > 3$, we have $j - i \geq 2$.

$\square$

Therefore, by Claims 2 and 3, the measure of any ordered planar tree $\mathcal{T}$ with ≥ 4 vertices can be computed from the measures of trees with fewer leaves using formula (20), thus concluding the proof of Theorem 3. □

5. Comparison with unordered arboreal measures

Consider groups $H \subseteq G$, acting oligomorphically on a set Ξ. If one has a regular measure μ on open H-orbits, one can ask if it automatically induces a measure $Ind_G^H(\mu)$ on open G-orbits, by taking μ_G of a G-orbit to be the sum of the measures of the H-orbits it decomposes into. However, it is possible for such a μ_G to fail to be a measure, which occurs in this case.

In this case, we consider the Fraïssé limit $\mathscr{T}^{\leq}$ of $\mathfrak{T}^{\leq}$. Its graphical representation, as a tree is isomorphic to the Fraïssé limit of the class of tree structures $\mathfrak{T}$ with unbounded valency considered in Remark 3.1 of [3]. Denote this unordered tree by $\mathscr{T}$ Therefore we have an inclusion of oligomorphic groups

$$Aut(\mathscr{T}^{\leq}) \subseteq Aut(\mathscr{T}),$$

acting on their sets of leaves.

In this case, for an unordered tree $\mathcal{T}$ with n nodes (i.e. non-leaf vertices) of valency $d_1, \dots, d_n \geq 2$, applying $Ind_{Aut(\mathscr{T})}^{Aut(\mathscr{T}^{\leq})}(\mu^{\geq})$ to $\mathcal{T}$ gives

$$(-1)^{n+1} \sum_{k=1}^{n} d_k! \cdot \prod_{i \neq k} (d_i - 1)! \; - \; (-1)^n 2e \prod_{i=1}^{n} (d_i - 1)!, \tag{22}$$

where e denotes the number of edges in $\mathcal{T}$, with the first term coming from taking the kth node (of valency d_k) to be the root of possible ordered planar structure on $\mathcal{T}$, and the second term coming from taking the root to be on an edge of $\mathcal{T}$ (the number of choices of ordered planar structures on $\mathcal{T}$ is the factorial of the valency of the root, times the product of the valencies of all non-root nodes, minus 1). Denoting the number of leaves of $\mathcal{T}$ by ℓ, since $\sum_{k=1}^{n} d_k = 2e - \ell$, we may reduce (22) to give

$$(-1)^n \ell \prod_{i=1}^{n} (d_i - 1)! \tag{23}$$

It is easy to see that $Ind_{Aut(\mathscr{T})}^{Aut(\mathscr{T}^{\leq})}(\mu^{\geq})$ corresponds to no possible choice of u, v required in Theorem 3.5 of [3]. In particular, (23) fails to correspond to one of the "primary series" measures of [3], containing the regular measures (and hence giving semisimple pre-Tannakian categories). In fact, one easily sees from the measure of a tree with a single vertex that one would have to put $\tau = -1/2$. The formula (23) then agrees with Proposition 3.10 of [3] on trees with ≤ 3 leaves, but not beyond. This is, roughly speaking, due to the possibility of "gluing planar trees into a planar tree (up to isomorphism) in a non-planar way."

Proof of Proposition 2. By [5], all orbits of $Aut(\leq)$ are isomorphic to

$$C_n = \{x_1 < \cdots < x_n\}.$$

Recall that

$$\mu(C_n) = (-1)^n. \tag{24}$$

We proceed by induction on n. By definition, the statement is true for $n = 0, 1$. Suppose it is true for a given n. Then we need to sum over all the possibilities for attaching a leaf $x_{n+1} > x_n$ to an ordered planar tree $\mathcal{T}$ with leaves $x_1 < \cdots < x_n$. Let

$$w = sup\{y \in S_{\mathcal{T}} \mid y \preceq x_{n+1}\}.$$

Then, again, we have an alternating sum where in the top term, the only vertex z of T with $w \underset{\neq}{\preceq} z$ is a leaf, giving measure $-\mu^{\leq}(\mathcal{T})$. In the next term, w is a node of $\mathcal{T}$ directly below a leaf (if $n > 1$); the measure of this ordered planar tree is $\mu^{\leq}(\mathcal{T})$. The last term, where w does not exist (i.e. x_{n+1} is directly above the (new) root of the new tree with leaves $\ell(\mathcal{T}) \cup \{x_{n+1}\}$) has measure $-\mu^{\leq}(\mathcal{T})$. Therefore, the sum is

$$-\mu^{\leq}(\mathcal{T}),$$

as required for (24).

$\square$

6. Simple objects - the regular case

In this section, we discuss the simple objects of the tree Delannoy category.

Denote the object of $Rep(Aut(\mathscr{T}^{\leq}), \mu^{\leq})$ corresponding to an ordered planar tree $\mathcal{T}$ by $[\mathcal{T}]$. The basic object $X := [\{*\}]$ corresponding to the ordered planar tree on a single leaf tensor-generates $Rep(Aut(\mathscr{T}^{\leq}), \mu^{\leq})$, with $X^{\otimes n}$ equal to a sum of all possible iterated amalgamations of n single leaves $\{x_1\}, \dots, \{x_n\}$ (over empty trees). In particular, there is a summand of the form

$$\bigoplus_{|\ell(\mathcal{T})|=n} [\mathcal{T}] \subseteq X^{\otimes n}, \tag{25}$$

corresponding to each choice of orderings of the n leaves $\{x_1, \dots, x_n\}$. (All other terms in $X^{\otimes n}$ come from amalgamations where for some $i \neq j$, x_i and x_j are identified, leading to fewer total leaves). Let us write

$$X^{(n)} := \bigoplus_{|\ell(\mathcal{T})|=n} [\mathcal{T}],$$

corresponding to a choice of ordering $\{x_1 < \dots < x_n\}$. This plays the same role as $\mathscr{C}(\mathbb{R}^{(n)})$ in the case of the classical Delannoy category (see [5]), and, similarly, finding the simple summands of $X^{\otimes n}$ reduces to finding the simple summands of $X^{(n)}$.

Recall that all objects of $[\mathcal{T}]$ by definition are self-dual (see [4]). In particular, for ordered planar trees $\mathcal{S}$, $\mathcal{T}$, the space of morphisms

$$Hom([\mathcal{S}], [\mathcal{T}]) \tag{26}$$

is the free vector space on generators indexed by amalgamations $\mathcal{U}$ of $\mathcal{S}$ and $\mathcal{T}$ (over the empty tree) with leaves colored red or black (or both) if they are the images of leaves of $\mathcal{S}$ or $\mathcal{T}$ (or both) in the mutually surjective injections

$$\begin{aligned} \ell(\mathcal{S}) &\hookrightarrow \ell(\mathcal{U}) \\ \ell(\mathcal{T}) &\hookrightarrow \ell(\mathcal{U}). \end{aligned} \tag{27}$$

(In our figures, we render "black" leaves as solid and "red" leaves as empty circles.) In particular, in this convention, red leaves correspond to the "input" of $\mathcal{U}$ as a morphism, and black leaves correspond to its "output." The partial trace of such a morphism in (26) matching leaves $x \in \ell(\mathcal{S})$ and $y \in \ell(\mathcal{T})$ is 0 unless x and y are sent to the same element $z \in \ell(\mathcal{U})$, in which case the partial trace is the amalgamation $\mathcal{U}'$, which has leaves $\ell(\mathcal{U}') = \ell(\mathcal{U}) \smallsetminus \{z\}$ with the restricted ordered planar tree structure and colorings as in $\mathcal{U}$, multiplied by

$$(-1)^{|n(\mathcal{U})|-|n(\mathcal{U}')|}.$$

For example, the identity morphism $Id_{[\mathcal{T}]}$ is the colored tree $\mathcal{U} = \mathcal{T}$, with each leaf colored both black and red. (Note that by definition

$$tr(Id_{\mathcal{T}}) = \mu([\mathcal{T}]) = (-1)^{|n(\mathcal{T})|}.)$$

In particular, to compose colored amalgamations $\mathcal{U}, \mathcal{V}$, in $\mathcal{U} \circ \mathcal{V}$, the red leaves of $\mathcal{U}$ are matched with the black leaves of $\mathcal{V}$.

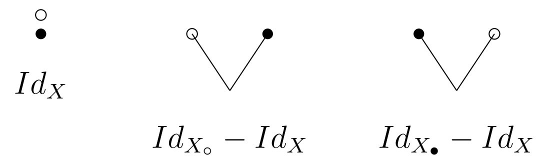


FIGURE 7. The three colored amalgamations of $\{*\}$ with itself (red and black leaves are drawn as empty and filled dots, respectively).

In particular, $End(X)$ has dimension 3, generated by the colored tree with one vertex colored red and black (i.e. Id_X), the colored tree with two leaves with the red leaf to the left of the black leaf, and the colored tree with two leaves with the black leaf to the right of the red leaf (see Figure 7). The sum of one of the colored trees with two leaves with Id_X, and -1 times the sum of all three colored trees gives three disjoint idempotents. We find that

$$X = X_\circ \oplus X_\bullet \oplus 1, \tag{28}$$

with $dim(X_\bullet) = dim(X_\circ) = -1$. The decomposition of X here is the same as the decomposition of the basic object of the Delannoy category (see [5, 4]).

To study the decomposition of $X^{(n)}$ into simple objects, we consider the endomorphism algebras

$$End([\mathcal{T}]) \tag{29}$$

for a fixed ordered planar tree $\mathcal{T}$ with n leaves. We define the *standard idempotents* of (29), as follows:

Consider an n-tuple

$$\varphi = (\varphi_1, \dots, \varphi_n)$$

of the formal symbols $\varphi_i \in \{\alpha, \beta, \epsilon\}$. Consider an amalgamation $\mathcal{U}$ of $\mathcal{T}$ with itself (over the empty tree), with mutually surjective injections

$$\iota_1 : \ell(\mathcal{T}) \hookrightarrow \ell(\mathcal{U})$$
$$\iota_2 : \ell(\mathcal{T}) \hookrightarrow \ell(\mathcal{U})$$

where leaves in the image of ι_1 are colored black, and leaves in the image of ι_2 are colored red. Let

$$\{b_1 < \cdots < b_n\}, \{r_1 < \cdots < r_n\} \subseteq \ell(\mathcal{U})$$

denote the (not necessarily disjoint) sets of black and red leaves, respectively. We say $\mathcal{U}$ is a *term of type* φ, if

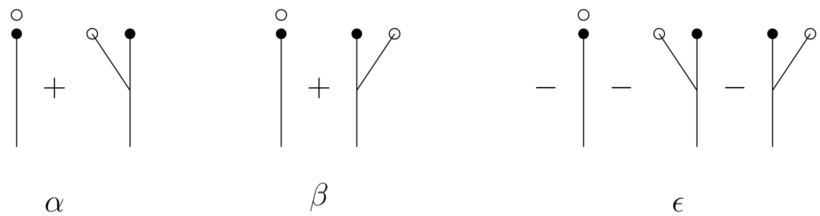


FIGURE 8. Endings of type α, β, and ϵ, respectively.

(1) when $\varphi_i = \alpha$, either $b_i = r_i$ or b_i is *directly to the right* of r_i, meaning that b_i is the minimal leaf with respect to the total ordering on $\ell(\mathcal{U})$ such that $r_i < b_i$.

(2) when $\varphi_i = \beta$, either $b_i = r_i$ or b_i is *directly to the left* of r_i, meaning that b_i is the maximal leaf with respect to the total ordering on $\ell(\mathcal{U})$ such that $b_i < r_i$.

(3) when $\varphi_i = \epsilon$, either $b_i = r_i$, b_i is directly to the left of r_i, or b_i is directly to the right or r_i.

We then define the *standard idempotent* ι_φ *of type* φ to be the sum of all colored amalgamation which are such terms of type φ, multiplied by the total coefficient

$$(-1)^{|\{i|\varphi_i=\epsilon\}|}. \tag{30}$$

Equivalently, ι_φ can be constructed by replacing the ith leaf with the "ending of type" φ (see Figure 8), and taking the linear combination of all possible terms.

For example, the idempotents of $End(X)$ giving the decomposition (28) are the standard idempotents ι_α, ι_β, ι_ϵ, with

$$Im(\iota_\alpha) = X_\circ,\ Im(\iota_\beta) = X_\bullet,\ Im(\iota_\epsilon) = 1.$$

Remarks: 1. The computation that the ι_φ are idempotents follows from the case of a single leaf, and since in all cases of φ, the corresponding pairs of red and black leaves are at worst directly next to each other. This implies that in any composition of two terms of type φ, the nodes of $\mathcal{T}$ will be preserved, and we may compute the composition "one leaf at a time" for each $i = 1, \dots, n$, i.e. we may compute the composition of the colored ordered planar trees restricted to the ith black and (not necessarily distinct) ith red leaves, and replace the ith leaf of the original tree $\mathcal{T}$ with the result.

2. Similarly, the above argument also implies that since, in the case of a single leaf, ι_α, ι_β, and ι_ϵ are disjoint, for two distinct choices of $\varphi, \psi \in \{\alpha, \beta, \epsilon\}^n$,

$$\iota_\varphi \circ \iota_\psi = \iota_\psi \circ \iota_\varphi = 0.$$

3. By examining one i at a time the possible cases (1)-(3) of the allowed leaves of terms of type φ corresponding to the possibilities of each φ_i (and noting that the sign (30) gives a minus sign for each possibility in case (3)), we see that

$$\sum_{\varphi \in \{\alpha,\beta,\epsilon\}^n} \iota_\varphi = Id_{[\mathcal{T}]}. \tag{31}$$

4. The trace of a standard idempotent ι_φ can be computed as

$$tr(\iota_\varphi) = (-1)^{|\{i|\varphi_i=\epsilon\}|} \cdot \mu(\mathcal{T})$$

since the only term of type φ that contributes to the trace is the one where the ith black leaf is equal to the ith red leaf for every $i = 1, \dots, n$. (In particular, we can verify that the trace of (31) is indeed $\mu(\mathcal{T})$.)

Definition 8. *We call the standard idempotents corresponding to n-tuples φ where no φ_i is ϵ the* regular idempotents *of $\mathcal{T}$.*

Theorem 9. *The image of a regular idempotent is simple and appears with multiplicity* 1 *in $[\mathcal{T}]$.*

Proof. Fix a $\varphi \in \{\alpha, \beta\}^n$. It suffices to prove that for any

$$f \in End_{Rep(Aut(\mathscr{T}^{\leq}),\mu^{\leq})}([\mathcal{T}]),$$

we have

$$f \circ \iota_\varphi = \iota_\varphi \circ f = c \cdot \iota_\varphi$$

for a constant c, or

$$f \circ \iota_\varphi = \iota_\varphi \circ f = 0.$$

Let f correspond to a colored amalgamation $\mathcal{U}$ of $\mathcal{T}$ with itself, with (not necessarily disjoint) sets of black and red leaves

$$\{b_1 < \dots < b_n\}, \{r_1 < \dots < r_n\} \subseteq \ell(\mathcal{U}).$$

First, note that, unless at each $i = 1, \dots, n$, we have that $b_i = r_i$, b_i is directly to the left of r_i, or b_i is directly to the right of r_i, we will have

$$f \circ \iota_\varphi = \iota_\varphi \circ f = 0,$$

since to compose we must glue $\mathcal{U}$ to each term of type φ, over identifying the red (resp. black) leaves of $\mathcal{U}$ with the black (resp. red) leaves of the terms of type φ, and taking the partial trace over the pairs of identified red and black leaves. For a given term of type φ, this reproduces $\mathcal{U}$, multiplied by -1 to the power of the number of $i = 1, \ldots, n$ for which, in the term of type φ, the ith black vertex is not equal to the ith red vertex (since this will lead to one node being deleted in the partial trace). Since $\varphi \in \{\alpha, \beta\}^n$, the sum of these terms is 0.

Suppose that for every $i = 1, \ldots, n$, we have that $b_i = r_i$, or b_i is directly to the left of r_i, or b_i is directly to the right of r_i. Then, again by the reasoning in Remark 1, it suffices to compose f and ι_φ one leaf at a time, in which case, by considering the possible non-zero compositions in the case of one black and one red leaf, we find that $\mathcal{U}$ must be a term of type φ for either composition with ι_φ to be non-zero.

In that case, the claim follows directly from Remark 1 and the proof that ι_φ are idempotents.

$\square$

Taking the images of the regular idempotents gives 2^n simple summands of multiplicity one in $[\mathcal{T}]$. For every $\varphi \in \{\alpha, \beta\}^n$, let us denote the image of ι_φ in $[\mathcal{T}]$ by $[\mathcal{T}]$ with the subscript corresponding to φ by replacing every α with $\circ$ and every β with $\bullet$, e.g. for $\mathcal{T}$ the unique ordered planar tree with two leaves, the regular simple summands are

$$[\mathcal{T}]_{\circ\circ} = Im(\iota_{\alpha,\alpha}),\ [\mathcal{T}]_{\circ\bullet} = Im(\iota_{\alpha,\beta})$$
$$[\mathcal{T}]_{\bullet\circ} = Im(\iota_{\beta,\alpha}),\ [\mathcal{T}]_{\bullet\bullet} = Im(\iota_{\beta,\beta})$$

However, $[\mathcal{T}]$ contains other simple summands which lie in the images of standard idempotents ι_φ when φ contains an ϵ coordinate. We make the following

Definition 10. *Call an idempotent $\iota \in End([\mathcal{T}])$ a* singular simple idempotent *if $Im(\iota)$ is simple, and ι is a composition of morphisms*

$$[\mathcal{T}] \to [\mathcal{S}] \to [\mathcal{T}]$$

for some ordered planar tree $\mathcal{S}$ with fewer leaves than $\mathcal{T}$. If an idempotent $\iota \in End([\mathcal{T}])$ a residual simple idempotent *if $Im(\iota)$ is simple, but ι is not a regular idempotent or a singular idempotent.*

We will now show that all simple objects (even the images of the residual simple idempotents) have dimension ± 1.

Proof of Theorem 4. In general, in a semisimple pre-Tannakian category, for an object Z and a basis $A_1, \dots, A_N$ of its endomorphism algebra $End(Z)$, the dimensions of the simple subobjects of Z are, up to sign, the eigenvalues of the symmetric matrix

$$(tr(A_iA_j)) = \begin{pmatrix} tr(A_1A_1) & \dots & tr(A_1A_N) \\ \vdots & & \vdots \\ tr(A_NA_1) & \dots & tr(A_NA_N) \end{pmatrix}. \tag{32}$$

We consider $Z = [\mathcal{T}]$ for any ordered planar tree $\mathcal{T}$. We take the basis $\{A_1, \dots, A_N\}$ to be the set of all colored amalgamations of $\mathcal{T}$ with itself.

For such a colored amalgamation $\mathcal{U}$, let $\overline{\mathcal{U}}$ denote the colored amalgamation whose underlying ordered planar tree is the same as that of $\mathcal{U}$, whose black leaves are exactly the red leaves of $\mathcal{U}$, and whose red leaves are exactly the black leaves of $\mathcal{U}$.

Now, by the definition of trace, for

$$tr(\mathcal{U} \circ \mathcal{V}) = tr(\mathcal{V} \circ \mathcal{U}) \tag{33}$$

to be non-zero, the composition of $\mathcal{V}$ and $\mathcal{U}$ must have a term which, as a colored tree, is $\mathcal{T}$ with every leaf colored both black and red, which we recall is $Id_{[\mathcal{T}]}$ as an endomorphism of $[\mathcal{T}]$. The only choice of $\mathcal{V}$ for which this is possible is for $\mathcal{V} = \overline{\mathcal{U}}$. Therefore, the matrix (32) decomposes as a direct sum of 2-by-2 blocks for every pair of colored amalgamations $\{\mathcal{U}, \overline{\mathcal{U}}\}$:

$$\bigoplus_{\{\mathcal{U}, \overline{\mathcal{U}}\}} \begin{pmatrix} 0 & tr(\mathcal{U} \circ \overline{\mathcal{U}}) \\ tr(\overline{\mathcal{U}} \circ \mathcal{U}) & 0 \end{pmatrix}. \tag{34}$$

The coefficient of $Id_{[\mathcal{T}]}$ in the composition of $\mathcal{U}$ and $\overline{\mathcal{U}}$ is ± 1, so

$$tr(\mathcal{U} \circ \overline{\mathcal{U}}) = tr(\overline{\mathcal{U}} \circ \mathcal{U}) = \pm \mu^{\leq}(\mathcal{T}) \in \{\pm 1\} \tag{35}$$

Thus, each block in (34) is of the form

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

and therefore has eigenvalues ± 1. Hence, so does (34). $\square$

7. The decomposition of the tree with two leaves into simple summands

The occurrence of the residual simple idempotents (not appearing in the classical Delannoy category) is substantial in trees with more than one leaf, but we do not understand them well. We will see this effect by working out the smallest non-trivial example.

Let $\mathcal{T}$ be the unique ordered planar tree with two leaves. In this case,

$$X^{(2)} = [\mathcal{T}],$$
$$X^{\otimes 2} = X \oplus 2 \cdot [\mathcal{T}].$$

Recall that $\mu(\mathcal{T}) = 1$.

The endomorphism algebra $End([\mathcal{T}])$ has total dimension 85: For a colored amalgamation $\mathcal{U}$ of $\mathcal{T}$ with itself, writing $\{r_1 < r_2\}$, $\{b_1 < b_2\}$ for the red and black sets of leaves of $\mathcal{U}$, there are 11 possibilities of an ordered planar tree with four leaves for each of the six choices of orderings of r_i, b_i with no equalities, there are 3 possibilities of an ordered planar tree with three leaves for each of the six choices of orderings of r_i, b_i where for exactly one pair $(i,j) \in \{1,2\}^2$, $r_i = b_j$, and finally there is one ordered planar tree with two leaves for when $r_1 = b_1$, $r_2 = b_2$.

We have four non-isomorphic regular simple idempotents

$$\iota_{\alpha,\alpha}, \iota_{\alpha,\beta}, \iota_{\beta,\alpha}, \iota_{\beta,\beta} \in End([\mathcal{T}]).$$

To count the singular simple idempotents, we note that

$$Hom(X, [\mathcal{T}]) \tag{36}$$

generated freely by all possible colored amalgamations $\mathcal{U}$ of the ordered planar tree with a single leaf and $\mathcal{T}$ (write $\{r_1\}, \{b_1 < b_2\} \subseteq \ell(\mathcal{U})$ for the single red and two black leaves of $\mathcal{U}$). There is one possibility of $\mathcal{U}$ for the each case

$$\{r_1 = b_1 < b_2\} \text{ or } \{b_1 < r_1 = b_2\}, \tag{37}$$

(corresponding to the unique ordered planar tree with two leaves), and there are three possibilities of $\mathcal{U}$ for each case

$$\{r_1 < b_1 < b_2\} \text{ or } \{b_1 < r_1 < b_2\} \text{ or } \{b_1 < b_2 < r_1\} \tag{38}$$

(corresponding to the three possible ordered planar trees with three leaves). Therefore, the dimension of (36) is 11. On the other hand,

$$dim(Hom(1, [\mathcal{T}])) = 1,$$

since the unit corresponds to the empty tree, of which there is only a single colored amalgamation with $\mathcal{T}$ ($\mathcal{U} = \mathcal{T}$, with all leaves colored black). Hence,

$$dim(Hom(X_\circ \oplus X_\bullet, [\mathcal{T}])) = 10,$$

and therefore, by symmetry,

$$dim(Hom(X_\circ, [\mathcal{T}])) = dim(Hom(X_\bullet, [\mathcal{T}])) = 5.$$

Thus, we have

$$1 \oplus 5 \cdot (X_\circ) \oplus 5 \cdot (X_\bullet) \subseteq [\mathcal{T}]. \tag{39}$$

The endomorphism algebra of (39) is the subalgebra of the endomorphism algebra of $[\mathcal{T}]$ generated by the singular simple idempotents, and it has dimension 51.

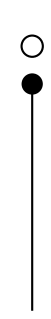
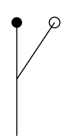
To see where the summands (39) lie with respect to the standard (non-regular) idempotents of $End([\mathcal{T}])$, we may consider compositions of singular idempotents with the standard idempotents

$$\iota_{\alpha,\epsilon}, \iota_{\beta,\epsilon}, \iota_{\epsilon,\alpha}, \iota_{\epsilon,\beta}. \tag{40}$$

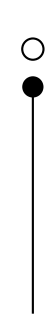
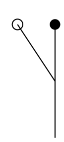
However, again, by the argument in Remark 1, all compositions can be performed one leaf at a time. To count multiplicities of $X_\circ$, $X_\bullet$ in the images of (40), it suffices to compose each of the idempotents (40) with the basis elements of

$$Hom(X, [\mathcal{T}]). \tag{41}$$

We observe that each basis element of (41) (i.e. ordered planar tree with three leaves, two of which are labeled black and one red) has at least one leaf which connects directly with its root. This leaf must be black and match the ϵ of (40), otherwise the composition is 0. In fact, the composition is also 0 unless the remaining leaves are

 or  to match α

or

 or  to match β

In any case, the composition can only be equal to a tree with one black leaf connecting to the root and α or β on the other leaf. This implies that $X_\circ = Im(\iota_\alpha)$ or $X_\bullet = Im(\iota_\beta)$ lies in the image of each idempotent (40)

$$X_\circ \subseteq Im(\iota_{\alpha,\epsilon}), \quad X_\circ \subseteq Im(\iota_{\epsilon,\alpha})$$
$$X_\bullet \subseteq Im(\iota_{\beta,\epsilon}), \quad X_\bullet \subseteq Im(\iota_{\epsilon,\beta})$$

with multiplicity one. Therefore, there are three remaining copies of $X_\bullet$, $X_\circ$ in $Im(\iota_{\epsilon,\epsilon})$ each. A similar argument from examining compositions one leaf at a time (or, alternatively, arguing by symmetry) gives that the single copy of 1 also must be in the image of $\iota_{\epsilon,\epsilon}$:

$$1 \oplus 3 \cdot (X_\bullet) \oplus 3 \cdot (X_\circ) \subseteq Im(\iota_{\epsilon,\epsilon}) \tag{42}$$

There are still $30 = 85-4-51$ remaining unaccounted for dimensions of the endomorphism algebra of $[\mathcal{T}]$, which must come from residual simple idempotents.

Lemma 11. *When $\varphi \in \{(\alpha,\epsilon),(\beta,\epsilon),(\epsilon,\alpha),(\epsilon,\beta)\}$, then the left ideal in $End([\mathcal{T}])$ generated by ι_φ has dimension* 11.

Assume for the moment that Lemma 11 is true. Then five of the eleven dimensions correspond to the five copies of $X_\circ$ (resp. $X_\bullet$). The remaining six must correspond to simple summands whose dimensions add up to 0 (since $tr(\iota_\varphi) = dim(X_\circ) = dim(X_\bullet) = -1$). We therefore see that no multiplicity greater than 1 is possible, and the six dimensions must correspond to six non-isomorphic simple summands, three of which have dimension 1 and three of which have dimension -1.

This leaves dimension $85-4-51-24 = 6$ of the left ideal of $End([\mathcal{T}])$ generated by $\iota_{\epsilon,\epsilon}$. The sum of the dimensions of the corresponding simple summands is $1-(1-6) = 6$, and thus, again, no multiplicity greater than 1 is possible and we have six new non-isomorphic simple summands of dimension 1.

Thus, we have proved

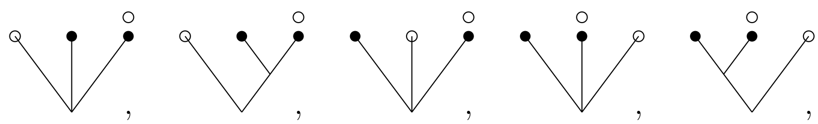
Proposition 12. *The object $[\mathcal{T}]$ corresponding to the ordered planar tree $\mathcal{T}$ with two leaves has* 4 *regular non-isomorphic simple summands of dimension* 1 *and multiplicity one, two singular non-isomorphic simple summands $X_\circ, X_\bullet$ of dimension* -1 *and multiplicity* 5 *each, one further singular simple summand $X_\emptyset = 1$ of dimension* 1 *and multiplicity one, and* 30 *residual non-isomorphic simple summands,* 18 *of which have dimension* 1 *and* 12 *of which have dimension* -1.

□

Proof of Lemma 11. We will treat the idempotent $\iota_{\epsilon,\alpha}$ (the other cases are symmetrical). When we compose $\iota_{\epsilon,\alpha}$ with a tree $\mathcal{U}$ where each of the two black leaves, $\mathcal{U} \circ \iota_{\epsilon,\alpha}$ is either also colored red or attached to a node of multiplicity 3 to which a red leaf is also attached. It is easy to see that the composition is a (possibly 0) multiple of $\iota_{\epsilon,\alpha}$.

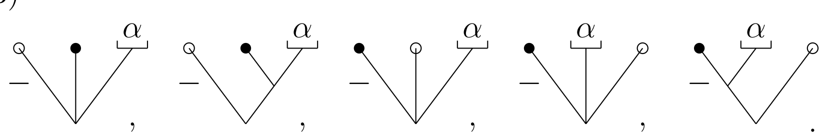
Now we observe that every other tree generator of $End([\mathcal{T}])$ has a black leaf x which is not colored red and is attached to a node to which another black leaf is attached. It follows then that the leaf x must match the ϵ ending of $\iota_{\epsilon,\alpha}$, otherwise the composition is 0.

One possibility is that the other black leaf is attached to the same node as x, and is also colored red. There are five such trees

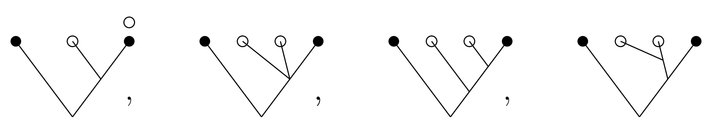


which, when applying $\circ\iota_{\epsilon,\alpha}$, produce

(43)

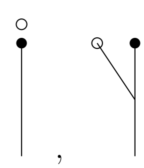


The element $\iota_{\epsilon,\alpha}$ together with the generators (43) span a vector space of dimension 6. Among the remaining trees,



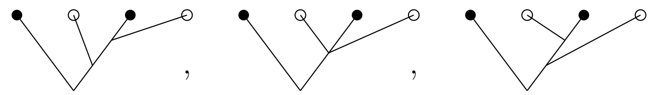
give 0. Briefly, composing with

(44)

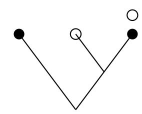


creates equal terms with opposite signs; in the case of composing with the second term of (44), this is due to an alternating sum starting and ending with -1.

The trees where at least one of the red leaves attaching to the right branch of the black $\mathcal{T}$ tree is on the right, give non-zero outcomes when composed $\circ\iota_{\epsilon,\alpha}$ due to exceptional terms arising when a red leaf matches with the black leaf in the second term of (44). In case of the trees

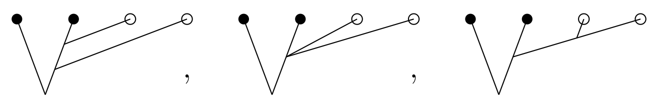
(45) 

the exceptional summand of the composition with the second term (44) is



canceling the remaining summand of that composition. Thus, the original term (45) survives, and we set three further independent generators (for a total of 9).

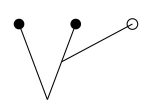
In case of the trees

(46) 

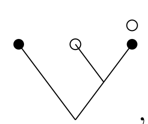
composing with the second term of (44) creates two exceptional summands depending on which of the red leaves merges with the black one of the second term of (44). This however means that the construction of composition with the first term of (44) still cancels (due to the second term of (44) producing the same term with an opposite sign).

We conclude that while the three trees (46) do not compose to 0 when applying $\circ\iota_{\epsilon,\alpha}$, they produce only one additional dimension in the outcome, for a total dimension of 10.

The last tree to consider is



which, when composed $\circ\iota_{\epsilon,\alpha}$, produces an additional new element



for a total dimension 11. This completes the proof of the Lemma.

$\square$

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