

# QUANTUM DELANNOY CATEGORIES

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ABSTRACT. The main subject of this paper is the construction of quantum or “ $q > 1$ ” counterparts of the Delannoy category constructed by Harman, Snowden, and Snyder [7]. We investigate the remarkable properties of our new categories. As an application, we find a semisimple pre-Tannakian category of growth  $e^{e^{c \cdot n^2}}$ , which is the highest known.

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## 1. INTRODUCTION

The main result of this paper is the construction and investigation of certain surprising new categories. A new direction of category theory has recently been developing, which can be viewed as an extension of representation theory. It studies, for the most part, symmetric tensor (abelian) categories which are linear over some ring (often a field), whose *Hom*-sets are finitely generated modules, and which are rigid in the sense that all objects have strong duals [5]. Such categories are often referred to as *pre-Tannakian categories*, in reference to the paper by P. Deligne and J. S. Milne [3], which used the concept of *Tannakian categories*, meaning pre-Tannakian categories with a fiber functor gerbe, to construct candidates for categories of motives.

Pre-Tannakian categories turned out to be of independent interest. Specifically *semisimple* pre-Tannakian categories are mathematical entities which are both difficult to construct and have a rich internal geometry. P. Deligne called them “isolated diamonds.” New examples

of semisimple pre-Tannakian categories are the main subject of the present paper.

In particular, we construct *quantum* (or  $q > 1$ ) counterparts of the Delannoy category recently discovered by N. Harman, A. Snowden, and N. Snyder [7]. The quantum Delannoy categories have many striking properties. In particular, they imply the existence of a semisimple pre-Tannakian category of “growth”  $e^{e^{c \cdot n^2}}$ , which is currently the highest known.

To state our results more precisely and place them in context, we introduce some terminology. A natural generalization of a pre-Tannakian category is an additive category linear over a commutative ring with associative, commutative, unital (ACU) tensor product, finitely generated *Hom*-modules, and strong duality. We call them *quasi-pre-Tannakian (QPT) categories*. In a QPT category, one can define the *dimension* of an object  $X$  as the trace of  $Id_X$ , and the *growth* of  $X$  as the growth of the sequence

$$\text{rank}(\text{End}(X^{\otimes n})).$$

By the *growth of the category*, one means an (eventual) bound on the growth of all objects.

QPT categories are much easier to construct than pre-Tannakian ones. For example, the author in [11] constructed a QPT category of arbitrarily high growth. However, when a QPT category is semisimple, it is automatically abelian, and therefore, pre-Tannakian. P. Deligne and J. S. Milne [3] (Subsections 1.26, 1.27) constructed a QPT category  $\underline{\text{Rep}}(GL_t)$  in characteristic 0 which is free on an object in dimension  $t$ . After extending scalars to  $\mathbb{C}$ , it is semisimple when  $t$  is not a non-negative integer. It can be considered as an “interpolation”  $\underline{\text{Rep}}(GL_t)$  of algebraic representation categories of general linear groups.

The next important advancement in the study of semisimple pre-Tannakian categories was the paper of P. Deligne [2], which constructed, for  $k$  a field of characteristic 0, categories  $\underline{\text{Rep}}(S_t)$  for  $t \in k$ , “interpolating” the categories of finite dimensional  $k$ -representations of the symmetric groups  $S_n$ . The categories  $\underline{\text{Rep}}(S_t)$  thus constructed are semisimple for  $t$  not a non-negative integer.

Knop [9, 10], analogously to [2], defined categories  $\underline{\text{Rep}}(GL_t(\mathbb{F}_q))$  for a finite field  $\mathbb{F}_q$  with  $q = p^m$  for a prime  $p$ , “interpolating” the categories of finite-dimensional  $k$ -representations of  $GL_n(\mathbb{F}_q)$  for  $k$  a field of characteristic 0. Again, these categories are semisimple for generic values of  $t$ . In some sense, the category  $\underline{\text{Rep}}(GL_t(\mathbb{F}_q))$  can be considered a “ $q > 1$ ” analogue of the category  $\underline{\text{Rep}}(S_t)$ . However, while

the categories  $\underline{Rep}(S_t)$  and  $\underline{Rep}(GL_t)$  have growth about  $e^{c \cdot n \cdot \ln(n)}$ , the category  $\underline{Rep}(GL_t(\mathbb{F}_q))$  has growth about  $e^{c \cdot n^2}$ , which was the highest growth known at the time.

A major development in the subject was the work by N. Harman and A. Snowden [6], who generalized the interpolation method to a method of constructing locally finite additive  $k$ -linear categories with an ACU tensor product and strong duality using oligomorphic groups (see [1]) with “measures.” The above examples can be obtained from the oligomorphic groups  $S_\infty$ ,  $GL_\infty(\mathbb{F}_q)$ . However, a number of other examples exist, including the Delannoy category  $\mathcal{D}$  (N. Harman, A. Snowden, N. Snyder, [7]), which comes from a suitable measure on the oligomorphic group  $Aut(\mathbb{R}, <)$  of order-preserving bijections of  $\mathbb{R}$ . This category is semisimple (and hence, semisimple pre-Tannakian) over a field  $k$  of any characteristic and has similar growth as  $\underline{Rep}(S_t)$ . Another interesting class of examples constructed using oligomorphic groups with measures was recently described in [12].

A. Snowden [13] showed that for a semisimple pre-Tannakian category  $\mathcal{C}$  over a finite field  $k$  of growth  $f(n)$ , applying a non-zero idempotent of  $\mathbb{C}[k, \cdot]$  to the free  $\mathbb{C}$ -linear category  $\mathbb{C}[\mathcal{C}]$  gives a semisimple pre-Tannakian category of growth about  $e^{f(n)}$ . Using the Delannoy category, he obtained growth about  $e^{e^{c \cdot n \cdot \ln(n)}}$ .

The main result of this paper is a construction of “ $q > 1$ ” or *quantum* counterparts of the Delannoy category. This category combines the ideas of  $\underline{Rep}(GL_t(\mathbb{F}_q))$  and  $\mathcal{D}$ . I first discuss this construction using the formalism of *T-algebras* which I used in [11]. It is a general method for constructing QPT categories by studying the universal algebra structure on  $Hom(X^{\otimes m}, X^{\otimes n})$  with varying  $m, n$  for some generating object  $X$ . The data considered are (tensor) product, partial traces, and units in a particular sense. A T-algebra structure determines a QPT structure completely (assuming finitely generated *Hom*-modules).

When one studies the T-algebra of  $\underline{Rep}(S_t)$ , one finds that the *Hom*-modules  $Hom(X^{\otimes m}, X^{\otimes n})$  are the free modules on equivalence relations on

$$\{1, \dots, m\} \amalg \{1, \dots, n\}.$$

One has  $\dim(X) := tr(Id_X) = t$ . For the Delannoy category, the description of  $Hom(X^{\otimes m}, X^{\otimes n})$  is the same except that the equivalence classes are (totally) ordered. It turns out that this forces  $t = -1$ .

For  $\underline{Rep}(GL_t(\mathbb{F}_q))$ ,  $Hom(X^{\otimes m}, X^{\otimes n})$  can be described as the free module on quotient homomorphisms

$$(1.1) \quad \mathbb{F}_q^m \oplus \mathbb{F}_q^n \rightarrow V.$$

The basic object has dimension  $\dim(X) = q^t$ .

I defined the (Borel) quantum Delannoy category by adding the data of a choice of a maximal flag on  $V$  to (1.1). As it turns out, this, again, forces  $t = -1$ . The resulting category  $\mathcal{D}_{q,k}$  is semisimple for a target field  $k$  of characteristic not dividing  $q(q-1)$ .

This has an application to growth. Using the method of [13] on the Borel quantum Delannoy category  $\mathcal{B}_{q,k}$ , we obtain the following

**Theorem 1.1.** *There exists a semisimple pre-Tannakian category  $\mathcal{E}$  over  $\mathbb{C}$  with an object  $X$  such that  $\dim(\text{End}(X^{\otimes n}))$  grows at the rate  $e^{e^{c \cdot n^2}}$  for some fixed constant  $c \in \mathbb{R}_{>0}$ .*

These are the fastest growing examples of semisimple pre-Tannakian categories currently known.

It turns out that the category  $\mathcal{B}_{q,k}$  can also be constructed by the method of [6], using the oligomorphic group which is the semidirect product of  $\text{Aut}(\mathbb{R}, <)$  with an infinite Borel group.

Following a suggestion of P. Deligne [4], we also investigate a variant by using (in a suitably finitary version), the unipotent subgroup instead. The resulting category  $\mathcal{U}_{q,k}$  is semisimple for any field  $k$  of characteristic not equal to  $p$  and can, in some sense, be considered an even more “pure” definition of a quantum Delannoy category. We describe it again using both the T-algebra and oligomorphic group mechanisms. The construction of  $\mathcal{U}_{q,k}$  also has a variant where the “basic object” is the projective space  $\mathbb{P}(\mathbb{F}_q[\mathbb{R}])$ .

We study both categories  $\mathcal{B}_{q,k}$  and  $\mathcal{U}_{q,k}$  in some detail. In particular, we give the decomposition of the “basic object” into simple summands. In fact, over an algebraically closed field of characteristic not  $p$ , one can classify *all the simple objects* of  $\mathcal{U}_{q,k}$  (referring to the characters of the  $p$ -Sylow subgroup of  $GL_n(\mathbb{F}_q)$ ).

There is a natural tensor functor

$$\mathcal{B}_{q,k} \rightarrow \mathcal{U}_{q,k}.$$

Perhaps surprisingly, there also turns out to be a tensor functor

$$\mathcal{D} \rightarrow \mathcal{B}_{q,k}$$

(where  $\mathcal{D}$  is the Delannoy category), due to a certain striking universal property of the Delannoy category  $\mathcal{D}$ , which we describe.

The present paper is organized as follows: Section 2 is focused on the T-algebra method. There, we show how the T-algebra mechanism applies to the categories  $\underline{Rep}(S_t)$ ,  $\mathcal{D}$ , and  $\underline{Rep}(GL_t(\mathbb{F}_q))$ .

In Section 3, we then construct the Borel quantum Delannoy category  $\mathcal{B}_{q,k}$  using T-algebras and prove that it is semisimple when  $\text{char}(k)$  does not divide  $q(q-1)$ . We also describe the unipotent analogue for the construction, which gives  $\mathcal{U}_{q,k}$ .

Section 4 gives a construction of the Borel and unipotent quantum Delannoy categories  $\mathcal{B}_{q,k}$  and  $\mathcal{U}_{q,k}$  using the framework of oligomorphic groups. We review and verify the technical requirements to apply the results of A. Snowden and N. Harman to get an alternate proof of the semisimplicity of  $\mathcal{B}_{q,k}$  and  $\mathcal{U}_{q,k}$ , and give descriptions of orbits of the corresponding oligomorphic groups using matrices.

Section 5 discusses structural properties of the categories, including the simple decompositions of each category's basic objects, a classification of the simple objects of  $\mathcal{U}_{q,k}$ , and a universality property for the Delannoy category.

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## 2. T-ALGEBRA CONSTRUCTIONS

In this section, we discuss a general mechanism for describing a QPT category generated by a single object  $X$  using the modules of homomorphisms between its tensor powers. These modules form a universal algebra which we call a *T-algebra*. The initial T-algebra corresponds to the category  $\underline{Rep}(GL_t)$  constructed in [3], [2], Chapter 10. We also treat several other examples using the formalism of T-algebras, including  $\underline{Rep}(S_t)$  [2],  $\underline{Rep}(GL_t(\mathbb{F}_q))$  [9, 10], and the Delannoy category [7]. For general background on tensor categories, we refer the reader to [5].

**2.1. Definition of a T-algebra.** A  $k$ -linear abelian category  $\mathcal{C}$  (say for a field  $k$ ) with an ACU tensor product and strong duality (generated by an object  $X$ ) is, in fact, determined by the structure of the morphism  $k$ -vector spaces

$$(2.1) \quad \mathcal{C}_{S,T} := \text{Hom}_{\mathcal{C}}(X^{\otimes S}, X^{\otimes T})$$

for all finite sets  $S, T$ , which can be axiomatized by a type of a universal algebra structure which we call a *T-algebra*, which we define as follows:

**Definition 2.2.** *A T-algebra consists of the data of*

- (1) *A system of spaces  $\mathcal{T}_{S,T}$  for all finite sets  $S$  and  $T$ , which have functoriality (covariant in  $T$  and contravariant in  $S$ ) with respect to bijections in the  $S$  and  $T$  coordinates,*

- (2) *for all subsets  $S' \subseteq S, T' \subseteq T$  and any choice of a bijection*

$$\phi : S' \xrightarrow{\cong} T'$$

*a specified  $S_{S \setminus S'} \times S_{T \setminus T'}$ -equivariant trace map*

$$\tau_\phi : \mathcal{T}_{S,T} \rightarrow \mathcal{T}_{S \setminus S', T \setminus T'}$$

*(where  $S_U$  denotes the symmetric group on a set  $U$ ),*

- (3) *for finite sets  $S_1, S_2, T_1, T_2$ , product maps*

$$(2.2) \quad \pi : \mathcal{T}_{S_1, T_1} \otimes \mathcal{T}_{S_2, T_2} \rightarrow \mathcal{T}_{S_1 \amalg S_2, T_1 \amalg T_2},$$

- (4) *an element  $1 \in \mathcal{T}_{\emptyset, \emptyset}$  and an element  $\iota \in \mathcal{T}_{\{1\}, \{1\}}$ .*

*We require that this data satisfy the following axioms, which encode the functoriality of traces with respect to bijections in the remaining coordinates, ACU properties of the tensor product maps, compatibility of tensor product with traces, and “compositon unitality” for  $\iota$ :*

- (1) *For finite sets  $S, T$  with subsets  $S', S'' \subseteq S, T', T'' \subseteq T$  such that  $S' \cap S'' = T' \cap T'' = \emptyset$ , for choices of bijections*

$$\phi : S' \rightarrow T'$$

$$\phi' : S'' \rightarrow T'',$$

*we have*

$$\tau_{\phi \amalg \phi'} = \tau_\phi \circ \tau_{\phi'} = \tau_{\phi'} \circ \tau_\phi$$

*as maps*

$$\mathcal{T}_{S,T} \rightarrow \mathcal{T}_{S \setminus (S' \amalg S''), T \setminus (T' \amalg T'')}.$$

- (2) The product maps  $\pi$  satisfy the clear commutativity and associativity axioms, and are unital with respect to 1, which means, for finite sets  $S, T$ ,

$$\begin{array}{ccc} \mathcal{T}_{S,T} & \xrightarrow{Id_{\mathcal{T}_{S,T}} \otimes 1} & \mathcal{T}_{S,T} \otimes \mathcal{T}_{\emptyset,\emptyset} \\ & \searrow Id_{\mathcal{T}_{S,T}} & \downarrow \pi \\ & & \mathcal{T}_{S,T} \end{array}$$

- (3) For finite sets  $S_1, S_2, T_1$ , and  $T_2$ , and subsets  $S' \subseteq S_1 \amalg S_2$ ,  $T' \subseteq T_1 \amalg T_2$ , for a choices of bijection

$$\phi : S' \rightarrow T'$$

we have

$$\tau_\phi \circ \pi = \pi \circ (\tau_{\phi|_{S_1}} \otimes \tau_{\phi|_{S_2}})$$

(both mapping

$$\mathcal{T}_{S_1,T_1} \otimes \mathcal{T}_{S_2,T_2} \rightarrow \mathcal{T}_{(S_1 \amalg S_2) \setminus S', (T_1 \amalg T_2) \setminus T'}.$$

- (4) For any  $x \in \mathcal{T}_{\{1\},\{1\}}$ , “composing” with  $\iota$  gives  $x$ , meaning that if we take the product of  $\iota$  and  $x$ , consider it as an element of  $\mathcal{T}_{\{1,2\},\{1,2\}}$ , and take the partial trace  $\tau_{\{1\} \rightarrow \{2\}}$  with respect to the bijection sending 1 to 2, we recover  $x$  (and similarly if we take the product of  $x$  and  $\iota$ ).

It is clear that for a QPT category  $\mathcal{C}$ , the  $Hom$ -spaces  $\mathcal{C}_{S,T}$ , together with partial traces and (tensor) products on morphisms, form a T-algebra.

Conversely, the axioms of a T-algebra  $\mathcal{T}$  encode the morphisms of a category  $\mathcal{C}_{\mathcal{T}}$  by taking

$$\begin{aligned} Hom_{\mathcal{C}_{\mathcal{T}}}(X^{\otimes m_1} \otimes (X^\vee)^{\otimes n_1}, X^{\otimes m_2} \otimes (X^\vee)^{\otimes n_2}) = \\ = \mathcal{T}_{\{1, \dots, m_1+n_2\}, \{1, \dots, m_2+n_1\}} \end{aligned}$$

The axioms precisely encode the structure of an additive category with ACU tensor product and strong duality in this category.

In fact, one sees from axiom (1) that the traces are determined by “elementary traces”

$$(2.3) \quad \tau_{i,j} : \mathcal{T}_{S,T} \rightarrow \mathcal{T}_{S \setminus \{i\}, T \setminus \{j\}}$$

for choices of coordinates  $i \in S$ ,  $j \in T$ .

As in axiom (4), the composition in a category  $\mathcal{C}_{\mathcal{T}}$  for a T-algebra  $\mathcal{T}$  is described by taking

$$\mathcal{T}_{S,T} \otimes \mathcal{T}_{T,R} \rightarrow \mathcal{T}_{S,R}$$

by sending a choice of  $f \in \mathcal{T}_{S,T}$ ,  $g \in \mathcal{T}_{T,R}$  to the element obtained by taking the product  $\pi(f, g) \in \mathcal{T}_{S \amalg T, T \amalg R}$  and applying  $\tau_{Id_T}$ .

**Comments:** 1. T-algebras being a universal algebra means that they cannot encode local finiteness. However in explicit examples, this property is readily verified.

2. When discussing semisimplicity, one usually needs to create new objects which are images of idempotents. This step, known as the *pseudo-abelian* or *Karoubian* envelope described in [2], will be involved automatically without explicit mention.

We will begin by describing some examples of tensor categories which inform our constructions of the Borel and unipotent quantum Delannoy categories, from the point of view of T-algebras.

**2.3. The interpolated category of representations of the symmetric group.** Let us first discuss the example of the category  $\underline{Rep}(S_t)$  (see [2]) from the T-algebra point of view.

Letting  $X$  denote the basic generating object of  $\underline{Rep}(S_t)$ , we may choose bases of the morphism spaces between its tensor powers as partitions of the disjoint union of the indexing sets of the source and target:

$$\begin{aligned} \underline{Rep}(S_t)_{S,T} &= Hom_{\underline{Rep}(S_t)}(X^{\otimes S}, X^{\otimes T}) = \\ (2.4) \quad &= k\{\{U_1, \dots, U_\ell\} \mid \ell \leq |S| + |T|, \emptyset \neq U_i \subseteq S \amalg T, \\ &U_i \cap U_j = \emptyset \text{ for } i \neq j, \text{ and } \coprod_{i=1}^\ell U_i = S \amalg T\}. \end{aligned}$$

It is important in (2.4) to note that the set  $\{U_1, \dots, U_\ell\}$  is unordered, i.e. is equivalent to the data of an equivalence relation on  $S \amalg T$ .

For  $i \in S$ ,  $j \in T$ , the partial trace map  $\tau_{i,j}$  applied to a partition  $\{U_1, \dots, U_\ell\}$  of  $S \amalg T$  is defined by taking it to be 0 if  $i$  and  $j$  are not in the same equivalence class  $U_s$ , and, if  $i, j \in U_s$ , taking the partition

$$\{U_1, \dots, U_s \setminus \{i, j\}, \dots, U_\ell\}$$



when

$$(2.5) \quad U_s \setminus \{i, j\} \neq \emptyset$$

and

$$\{U_1, \dots, U_{s-1}, U_{s+1}, \dots, U_\ell\}$$

when

$$(2.6) \quad U_s = \{i, j\},$$

multiplied by a certain constant.

The constant is determined by the number of choices how the given configuration can arise when  $t = N$  is a large integer, where  $\{U_1, \dots, U_\ell\}$  corresponds to the orbit

$$(2.7) \quad S_{\text{SIT}}/S_{U_1} \times \dots \times S_{U_\ell}.$$

realized as the idempotent in  $\text{End}(k[1, \dots, N]^{\text{SIT}})$  which is identity on the orbit consisting of tuples whose coordinates belonging to the same set  $U_s$  are equal and 0 on other orbits. Therefore, the coefficient is 1 in the case of (2.5) and

$$(2.8) \quad t - \ell + 1$$

in the case of (2.6).

The products of  $\{U_1, \dots, U_\ell\}$  and  $\{V_1, \dots, V_{\ell'}\}$  are described by “gluings,” which are specified by surjections

$$(2.9) \quad \kappa : \{1, \dots, \ell\} \amalg \{1, \dots, \ell'\} \twoheadrightarrow \{1, \dots, \ell + \ell' - h\}$$

which are injective on each of the discrete summands in the source. The gluing is accomplished by forming a new equivalence relation with classes whose  $j$ th class is the union of  $U_s$ , resp.  $V_{s'}$  (whichever apply) where  $\kappa(s) = j$ , resp.  $\kappa(s') = j$ . Keep in mind, however, that the sets of equivalence classes are unordered, so gluing data are considered equal when they produce the same sets of equivalence classes. Compatibility of traces with product can be checked directly, but (since they correspond to numerical identities) also follow from considering  $t = N$  a large integer. Since one encounters factorials, the field  $k$  must be of characteristic 0.

Semisimplicity was proved in [2], Sections 3-5, assuming that

$$(2.10) \quad t(t-1) \dots (t-n+1)$$

are invertible for all  $n \in \mathbb{N}$ , i.e. that  $t$  is not a non-negative integer. The proof proceeds by considering the object  $U_n$  given by the idempotent in  $\text{End}(X^{\otimes n})$  corresponding to the partition

$$(2.11) \quad \{\{1_1, 1_2\}, \{2_1, 2_2\}, \dots, \{n_1, n_2\}\}$$

(where the subscript indicates which disjoint summand of

$$\{1, \dots, n\} \amalg \{1, \dots, n\}$$

we are in). One proves semisimplicity of the additive subcategory generated by  $U_m$ ,  $m \leq n$ . Using Proposition 3.8 of [2], one splits off  $U_n$  a direct sum of simple objects occurring for  $m < n$ , and one is left with the group algebra  $k[S_n]$ , which is semisimple since  $k$  has characteristic 0.

**2.4. The Delannoy category.** To describe the Delannoy category  $\mathcal{D}$ , for finite sets  $S, T$ , take the representations

$$\mathcal{D}_{S,T} = \text{Hom}_{\mathcal{D}}(X^{\otimes S}, X^{\otimes T})$$

to be the free  $k$ -modules on partitions of  $S \amalg T$ , similarly as in the case of  $\underline{\text{Rep}}(S_t)$ , but with a total ordering of the components of the partition.

More precisely, for finite sets  $S, T$ ,  $\mathcal{D}_{S,T}$  is

$$(2.12) \quad k\{(U_1, \dots, U_\ell) \mid \ell \leq |S| + |T|, \emptyset \neq U_i \subseteq S \amalg T, \\ U_i \cap U_j = \emptyset \text{ for } i \neq j, \text{ and } \coprod_{i=1}^\ell U_i = S \amalg T\}.$$

Thus, the only difference with (2.4) is that the tuple  $(U_1, \dots, U_\ell)$  is ordered.

Again, for  $i \in S$ ,  $j \in T$ , the partial trace map  $\tau_{i,j}$  applied to an ordered partition  $(U_1, \dots, U_\ell)$  of  $S \amalg T$  is defined by taking it to be 0 if  $i$  and  $j$  are not elements of the same  $U_s$ , and, if  $i, j \in U_s$ , taking

$$(2.13) \quad (U_1, \dots, U_s \setminus \{i, j\}, \dots, U_\ell)$$

when (2.5) occurs and

$$(2.14) \quad (U_1, \dots, U_{s-1}, U_{s+1}, \dots, U_\ell)$$

when (2.6) occurs, multiplied by suitable coefficients.

The coefficient is again 1 in the case of (2.5). In the case of (2.6), the number of equivalence classes becomes  $\ell - 1$ , but we should divide by the  $\ell$  choices of the number in  $\{1, \dots, \ell\}$  we omit. This suggests the coefficient

$$(2.15) \quad \frac{t - \ell + 1}{\ell}.$$

However, compatibility with product (which is the only non-trivial axiom to verify) turns out to force (2.15) to be independent of  $\ell$ , which occurs for  $t = -1$  (when the value of (2.15) is  $-1$ ).

The product is described by ordered gluings, which are surjections (2.9) that are strictly increasing on the discrete summands  $\{1, \dots, \ell\}$ ,  $\{1, \dots, \ell'\}$  in the source. To glue  $(U_1, \dots, U_\ell)$  to  $(V_1, \dots, V_{\ell'})$ , for  $1 \leq j \leq \ell + \ell' - h$ , take, again, the union of  $U_s$  and/or  $V_{s'}$  whenever  $\kappa(s) = j$  resp.  $\kappa(s') = j$  (whichever applies).

Proving the compatibility of trace with product is non-trivial only in the case of (2.6). Assume we take the trace  $\tau_{i,j}$  in the case (2.6), to produce the ordered set of equivalence classes (2.14). When taking product of (2.14) with

$$(2.16) \quad (V_1, \dots, V_\ell),$$

consider separately each case of  $\kappa|_{\{1, \dots, \ell-1\}}$ . Let  $r = \kappa(j-1)$ ,  $r' = \kappa(j)$ . In the case when we take the product with (2.16) first, there are  $2(r' - r) - 1$  choices of  $\kappa(j)$  (note that  $j$  becomes  $j+1$  and  $r$  may stay the same or increase by one depending on whether  $\kappa(j)$  is equal to  $\kappa(s)$  for  $s \in \{1, \dots, \ell'\}$  or not). There are  $r' - r - 1$  of the former cases (“clashes”) and  $r' - r$  of the latter cases (“non-clashes”). Thus, when taking the product first and then the trace, we obtain a coefficient of

$$(r' - r) \cdot (-1) + (r' - r + 1) = -1,$$

i.e. same as taking the trace first and then the product, as required.

Since the ordering eliminates products of the form (2.10), the Delaney category exists in any characteristic.

To prove semisimplicity, the analogue of (2.11) becomes the sum of its copies over all orderings i.e.

$$(2.17) \quad \sum_{\sigma \in S_n} (\{(\sigma(1))_1, (\sigma(1))_2\}, \dots, \{(\sigma(n))_1, (\sigma(n))_2\}).$$

Call the image of this idempotent, again,  $U_n$ . The analogue of Proposition 4.2 of [2] is describing the trace pairing matrix

$$(2.18) \quad \text{Hom}(U_0, U_n) \otimes \text{Hom}(U_n, U_0) \rightarrow k.$$

One has  $\dim(\text{Hom}(U_0, U_n)) = n!$  (corresponding to permutations on  $[n] = \{1, \dots, n\}$ ) and the matrix is diagonal with diagonal entries  $(-1)^n$ , so it is non-singular.

The analogue of Lemma 5.2 of [2] says that the “top part” of  $\text{End}_{\mathcal{D}}(U_n)$  is freely generated by elements of  $S_n$  composed with elements  $\phi_R$ , for all choices of subsets

$$R = \{r_1 < \dots < r_m\} \subseteq \{1, \dots, n\},$$

given by  
(2.19)

$$\begin{aligned} \phi_R = & (\{1_1, 1_2\}, \dots \\ & \dots \{(r_1 - 1)_1, (r_1 - 1)_2\}, \{(r_1)_1\}, \{(r_1)_2\}, \{(r_1 + 1)_1, (r_1 + 1)_2\}, \dots \\ & \dots \{(r_m - 1)_1, (r_m - 1)_2\}, \{(r_m)_1\}, \{(r_m)_2\}, \{(r_m + 1)_1, (r_m + 1)_2\}, \dots \\ & \dots, \{n_1, n_2\}). \end{aligned}$$

To see this, it suffices to note that morphisms from the image of (2.17) to itself are generated by equivalence relations with ordered equivalence classes on

$$\{1_1, \dots, n_1\} \amalg \{1_2, \dots, n_2\}$$

where the elements  $1_1, \dots, n_1$  are in distinct equivalence classes, and the elements  $1_2, \dots, n_2$  are in distinct equivalence classes.

One checks that such morphisms which factor through  $U_m$  for some  $m < n$  in fact must factor through the composition of the morphism

$$\begin{aligned} \sum_{\sigma \in S_n} & (\{(\sigma(1))_1, (\sigma(1))_2\}, \dots, \{(\sigma(i-1))_1, (\sigma(i-1))_2\}, \{(\sigma(i))_1\}, \\ & \{(\sigma(i+1))_1, (\sigma(i+1))_2\}, \dots, \{(\sigma(n))_1, (\sigma(n))_2\}) \end{aligned}$$

followed by

$$\begin{aligned} \sum_{\sigma \in S_n} & (\{(\sigma(1))_1, (\sigma(1))_2\}, \dots, \{(\sigma(i-1))_1, (\sigma(i-1))_2\}, \{(\sigma(i))_2\}, \\ & \{(\sigma(i+1))_1, (\sigma(i+1))_2\}, \dots, \{(\sigma(n))_1, (\sigma(n))_2\}), \end{aligned}$$

which is the sum over permutations  $\sigma \in S_n$  of the three terms

$$\begin{aligned} & (\{(\sigma(1))_1, (\sigma(1))_2\}, \dots, \{(\sigma(i))_1\}, \{(\sigma(i))_2\}, \dots, \{(\sigma(n))_1, (\sigma(n))_2\}) + \\ & (\{(\sigma(1))_1, (\sigma(1))_2\}, \dots, \{(\sigma(i))_1, (\sigma(i))_2\}, \dots, \{(\sigma(n))_1, (\sigma(n))_2\}) + \\ & (\{(\sigma(1))_1, (\sigma(1))_2\}, \dots, \{(\sigma(i))_2\}, \{(\sigma(i))_1\}, \dots, \{(\sigma(n))_1, (\sigma(n))_2\}). \end{aligned}$$

Imposing these relations, one gets free generators of the form of the morphisms (2.19), composed with permutations.

For a finite group  $G$ , consider the groupoid  $\Gamma$  of  $G$  acting on itself by translation. The “Drinfeld double” of  $G$  (note: terminology may vary) is  $k[Mor(\Gamma)]$  where composition of morphisms is defined as composition when morphisms are composable and as 0 otherwise. Then the Drinfeld double is isomorphic to the matrix algebra  $M_{|G|}(k)$  of  $|G| \times |G|$  matrices with entries in  $k$ . Using this fact, the “top part”

of  $End_{\mathcal{D}}(U_n)$  is isomorphic to the tensor product of  $M_{n!}(k)$  with the algebra of endomorphisms of  $U_n$  generated by elements (2.19), which is

$$\prod_n k[\mathbb{Z}/2, \cdot] = \prod_{2^n} k.$$

Thus, the top part of  $End_{\mathcal{D}}(U_n)$  is isomorphic to

$$\prod_{2^n} M_{n!}(k).$$

The formulas (2.19) are simpler than the formulas in Subsection 5.1 of [7]. This is due to the fact that we are suppressing terms going through  $U_m$  for  $m < n$ .

**2.5. The interpolated category of representations of the general linear group of a finite field.** We now describe the category  $\underline{Rep}(GL_t(\mathbb{F}_q))$  for  $q = p^m$  for a prime  $p$  and any  $t$  not a non-negative integer (not all non-negative integers need to be excluded, see [6, 9, 10]) using T-algebras.

For finite sets  $S, T$ , take the representation

$$\underline{Rep}(GL_t(\mathbb{F}_q))_{S,T} = Hom_{\underline{Rep}(GL_t(\mathbb{F}_q))}(X^{\otimes S}, X^{\otimes T})$$

to be the free  $k$ -module (where  $k$  is a field of characteristic 0) on the set of equivalence classes of vector space surjections

$$(2.20) \quad f : \mathbb{F}_q^{S \amalg T} = \mathbb{F}_q^S \times \mathbb{F}_q^T = \mathbb{F}_q\{x_i \mid i \in S\} \times \mathbb{F}_q\{y_j \mid j \in T\} \rightarrow V$$

where the equivalence relation is

$$f \sim g \circ f$$

for  $g \in GL(V)$ .

For  $i \in S, j \in T$ , the partial trace  $\tau_{i,j}(f)$  is defined to be 0 if

$$(2.21) \quad f(x_i) \neq f(y_j).$$

If

$$(2.22) \quad f(x_i) = f(y_j),$$

to define the partial trace, let  $f_{i,j}$  be the restriction of  $f$  to  $\mathbb{F}_q^{S \setminus \{i\}} \times \mathbb{F}_q^{T \setminus \{j\}}$ , considered as a map onto its image. Then in the case (2.22), the trace  $\tau_{i,j}(f)$  is a multiple of  $f_{i,j}$ . The coefficient is 1 if

$$(2.23) \quad V_{i,j} = V.$$

If

$$\dim(V_{i,j}) < \dim(V),$$

then necessarily

$$(2.24) \quad \dim(V_{i,j}) = \dim(V) - 1.$$

In the case (2.24), if  $t = N$  were a large integer (so we are working with  $\text{Rep}(GL_N(\mathbb{F}_q))$ ), the coefficient would be the number of choices how  $f_{i,j}$  can arise in this fashion. Thus, the coefficient is

$$(2.25) \quad q^t - q^{n-1}.$$

The product of (2.20) with

$$f' : \mathbb{F}_q^{S' \sqcup T'} \rightarrow V'$$

is again given by the sum of maps obtained by composing  $f \oplus f'$  with a surjection

$$(2.26) \quad \mu : V \oplus V' \rightarrow W$$

which is injective on each of the summands in the source. Again, two choices of  $\mu$  are considered equal if they are in the same orbit of the left action of  $GL(W)$ .

Verification of compatibility of partial traces with product in the case of (2.24) (which is the only non-trivial case of the axioms) again comes down to a polynomial identity which, in each degree, can be deduced from the case  $t = N$  for a large integer  $N$ , where it follows from the fact that we just described the structure arising in the case of  $\text{Rep}(GL_N(\mathbb{F}_q))$ .

To prove semisimplicity, we again use the method of [2], Sections 3-5. The object  $U_n$  is defined to be the image of the idempotent on  $\text{End}_{\text{Rep}(GL_t(\mathbb{F}_q))}(X^{\otimes n})$  given by the codiagonal

$$(2.27) \quad \begin{aligned} \mathbb{F}_q^n \times \mathbb{F}_q^n &\xrightarrow{\nabla} \mathbb{F}_q^n \\ (x, y) &\mapsto x + y \end{aligned}$$

The analogue of Lemma 5.2 says that the top part is  $k[GL_n(\mathbb{F}_q)]$ , so  $\text{End}_{\text{Rep}(GL_t(\mathbb{F}_q))}(U_n)$  is semisimple because  $k$  is of characteristic 0.

### 3. TWO QUANTUM DELANNOY CATEGORIES

This section contains the main construction of the present paper. The quantum (or  $q > 1$ ) Delannoy categories are defined by combining the ideas of  $\mathcal{D}$  (Subsection 2.4) and  $\text{Rep}(GL_t(\mathbb{F}_q))$  (Subsection 2.5). The first *Borel* version, denoted by  $\overline{\mathcal{B}}_{q,k}$ , is defined for a prime power  $q$  over a field  $k$  with

$$(3.1) \quad \text{char}(k) \nmid q(q-1)$$

is defined by specifying a maximal flag on the target of (2.20). The second *unipotent* version, denoted by  $\mathcal{U}_{q,k}$  is defined for a prime power  $q = p^m$  over a field  $k$  of characteristic not  $p$ , by specifying a basis of the target of (2.20) modulo the action of the unipotent subgroup of the Borel subgroup of  $GL(V)$  (where  $V$  denotes the target of (2.20)) of linear transformations preserving the flag. For the remainder of this paper,  $q = p^m$  will denote the power of a prime  $p$ .

**3.1. The T-algebra structure of  $\mathcal{B}_{q,k}$ .** We shall now define  $\mathcal{B}_{q,k}$  by taking the spaces  $\mathcal{B}_{S,T}$  to be the free  $k$ -modules on vector space surjections

$$(3.2) \quad f : \mathbb{F}_q^{S \amalg T} \rightarrow V$$

and choices of a maximal flag

$$(3.3) \quad 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{\dim(V)} = V$$

(i.e. where for each  $k = 1, \dots, \dim(V)$ ,

$$\dim(V_k) - \dim(V_{k-1}) = 1).$$

We identify the quotient maps (3.2) under the action of the Borel subgroup on the target.

We define  $\tau_{i,j}(f)$  again to be 0 in the case of (2.21). In the case of (2.22), we define  $\tau_{i,j}(f)$  to be a multiple of  $f_{i,j}$  (with the induced flag on the image in the case of (2.24)). The coefficient is 1 in the case of (2.23). In the case of (2.24), the coefficient should be (2.25) divided by the number of ways the restricted flag could arise, which is

$$(3.4) \quad \frac{q^t - q^{n-1}}{1 + q + \cdots + q^{n-1}}.$$

However, again, we will only get compatibility with products when (3.4) is independent of  $n$ , which occurs for

$$t = -1,$$

in which case the coefficient (3.4) becomes

$$(3.5) \quad q^{-1} - 1.$$

The product of (3.2) with

$$(3.6) \quad f' : \mathbb{F}_q^{S' \amalg T'} \rightarrow V'$$

with a maximal flag

$$(3.7) \quad 0 = V'_0 \subset V'_1 \subset \cdots \subset V'_{\dim(V')} = V'$$

is obtained by choosing an ordered surjection (2.9) (taking  $\ell = \dim(V)$ ,  $\ell' = \dim(V')$ ) and a surjection (2.26) which sends the  $V_i$  to  $W_{\kappa(i)}$  and  $V'_j$  to  $W_{\kappa(j)}$ . The product is then defined as a sum over all such choices together with a choice of a maximal flag on  $W$  compatible with the flags (3.3), (3.7).

To verify the compatibility of trace and product, the non-trivial case again is (2.24). Consider a case of (3.2) where (2.24) arises for a partial trace  $\tau_{i,j}$ , and consider a product with (3.6). Performing  $\tau_{i,j}$  on (3.2) first and then taking the product with (3.6), consider again separately each case of  $\kappa|_{\{1,\dots,\ell-1\}}$ . Let  $r = \kappa(s-1)$ ,  $r' = \kappa(s)$ . Then, again, in the case where we take the product first, there are  $2(r' - r) - 1$  choices of  $\kappa(s)$ . Note that  $s$  becomes  $s+1$  and  $r$  may stay the same or increase by one depending on whether  $\kappa(s)$  is equal to  $\kappa(u)$  for  $u \in \{1, \dots, \ell'\}$  or not. There are still  $r' - r - 1$  of the former cases (“clashes”) and  $r' - r$  of the latter cases (“non-clashes”).

In the clashing choice preceding a non-clashing choice, the number of choices of flags will be multiplied by

$$\frac{q-1}{q}.$$

Thus, the coefficient of the two choices (a clashing one preceding a non-clashing one) will be equal with opposite signs (see (3.5)). Hence, after tracing out the pair corresponding to the original  $(i, j)$ -pair after performing the product, all of the choices of  $\kappa(s)$  will again cancel out except the lowest choice, which is non-clashing and corresponds to what we get if we do  $\tau_{i,j}$  first, followed by the product.

**3.2. Semisimplicity of  $\mathcal{B}_{q,k}$ , Part I.** This construction makes sense in any characteristic not dividing

$$q(q-1).$$

One again proves that under this condition,  $\mathcal{B}_{q,k}$  is semisimple, following again the method of Sections 3-5 of [2]. The object  $U_n$  is defined as the image of the idempotent given by a sum of copies of (2.27) over all possible choices of maximal flags, which are indexed by  $GL_n(\mathbb{F}_q)/B_n$ , where  $B_n \subseteq GL_n(\mathbb{F}_q)$  is the Borel subgroup.

The analogue of Proposition 4.2 of [2] again requires proving that the matrix (2.18) is non-singular.  $\text{Hom}(U_0, U_n) = \text{Hom}(U_n, U_0)$  this time corresponds to choices of maximal flags on  $\mathbb{F}_q^n$  and the matrix (2.18) is diagonal with diagonal entries  $(q^{-1} - 1)^n$ . Thus, it is non-singular under our assumptions.



Similarly as in the Delannoy case, we now need to describe the “top part”  $\Xi_n$  of the algebra  $End_{\mathcal{D}}(U_n)$  and prove that it is semisimple. To this end, consider compositions of morphisms of the form

$$(3.8) \quad U_n \xrightarrow{F} U_{n-1} \xrightarrow{G} U_n.$$

Such morphism compositions can be described as follows:

Consider a finite dimensional  $\mathbb{F}_q$ -vector space  $V$  with a maximal flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V.$$

Let  $W \subseteq V$  be a subspace. Then we define the *symbol* of  $W$  to be the set

$$S_W = \{i \in [n] \mid V_i \cap W \supset V_{i-1} \cap W\}.$$

We shall describe the data of a composition (3.8) by linear maps

$$(3.9) \quad \begin{array}{ccc} \mathbb{F}_q^n \oplus \mathbb{F}_q^{n-1} & & \mathbb{F}_q^{n-1} \oplus \mathbb{F}_q^n \\ Id \oplus \phi \downarrow & & \downarrow \psi \oplus Id \\ \mathbb{F}_q^n & & \mathbb{F}_q^n \end{array}$$

with the standard maximal flags on the targets, with

$$\phi, \psi : \mathbb{F}_q^{n-1} \hookrightarrow \mathbb{F}_q^n$$

injective. The reason we chose the second summand of the second map in (3.9) to be  $Id$  is that we can always compose with an element of  $GL_n(\mathbb{F}_q)$ , so this is the case we need to consider without loss of generality. For the remainder of this subsection, let us denote the standard basis of  $\mathbb{F}_q^n$  by  $e_1, \dots, e_n$ . Note that the symbol of the image of  $\phi$  is of the form

$$(3.10) \quad S_{Im(\phi)} = \{1, \dots, i-1, i+1, \dots, n\}$$

for some specific element  $i$ .

In the composition, we will obtain data of the form

$$(3.11) \quad \begin{array}{c} \mathbb{F}_q^n \oplus \mathbb{F}_q^n \\ \downarrow \\ \mathbb{F}_q^n \end{array}$$

or

$$(3.12) \quad \begin{array}{c} \mathbb{F}_q^n \oplus \mathbb{F}_q^n \\ \downarrow \\ \mathbb{F}_q^{n+1}. \end{array}$$

The case of (3.11) is a “clash,” in which case we get nothing new (only an element of  $B_n$ ). In the case (3.12), we have  $q + 1$  choices of flags in the target: One choice arises from the case when the generator of the  $i$ th term of the flag comes from the first summand in the sense of (3.12) and  $q$  choices arise from when it comes from the second summand (for adding  $\mathbb{F}_q$ -multiples of the first summand in the flag).

In the source (3.12), decorate a vector  $v \in \mathbb{F}_q^n$  as  $(v)_1$ , resp.  $(v)_2$ , depending on whether it comes from the first or the second summand. Then the target of (3.12) can then be described as a quotient of  $\mathbb{F}_q^n \oplus \mathbb{F}_q^n$  by identifying

$$(\phi(e_j))_1 \sim (\psi(e_j))_2,$$

for  $j = 1, \dots, n - 1$ . The flag on the target of (3.12) corresponding to a choice of  $a \in \mathbb{F}_q$  can be described as

$$(3.13) \quad \begin{aligned} &(\phi(e_1))_1 = (\psi(e_1))_2, \\ &(\phi(e_2))_1 = (\psi(e_2))_2, \dots \\ &\dots, (\phi(e_{i-1}))_1 = (\psi(e_{i-1}))_2, \\ &(e_i)_1, (e_i)_2 + a \cdot (e_i)_1, \\ &(\phi(e_i))_1 = (\psi(e_i))_2, \dots \\ &\dots, (\phi(e_{n-1}))_1 = (\psi(e_{n-1}))_2 \end{aligned}$$

The most important point is that the source and target of the composition of (3.9) are considered different from the point of view of composing such morphisms, depending on the coefficients of  $e_i$  in

$$\phi(e_i), \dots, \phi(e_{n-1})$$

and

$$\psi(e_i), \dots, \psi(e_{n-1}).$$

**3.3. Semisimplicity of  $\mathcal{B}_{q,k}$ , Part II.** Generalizing to an arbitrary subset  $S \subseteq [n]$ , this structure can be described by a groupoid  $\mathbb{B}_S$  defined as follows:

Consider the set

$$\Phi_S = \{f : [n] \setminus S \rightarrow \mathbb{F}_q\}$$

and the subgroups of the Borel subgroup

$$B_S = \{b \in B_n \mid b_{i,i} = 1 \text{ for } i \in [n] \setminus S\}$$

$$B_{S,0} = \{b \in B_S \mid b_{i,j} = b_{j,i} = 0 \text{ for } i \in [n] \setminus S, i \neq j\}.$$

Then  $\mathbb{B}_S$  is the groupoid of the left action of  $B_S \times \Phi_S$  on the set of left cosets  $B_S/B_{S,0}$ , where  $\Phi_S$  acts trivially, and upon composition, elements of  $\Phi_S$  are multiplied coordinate-wise.

In fact, the subalgebra  $\Xi_n^0 \subset \Xi_n$  generated by the morphisms not changing the flag on  $\mathbb{F}_q^n$  can be described as

$$k \left( \coprod_{S \subseteq [n]} \text{Mor}(\mathbb{B}_S) \right).$$

Even for  $\Xi_n^0$ , the composition formula can be complicated in general. However, we have a decreasing filtration  $F$  on  $\Xi_n^0$  where

$$F^k \Xi_n^0 = k \left( \coprod_{S \subseteq [n], |S| \leq n-k} \text{Mor}(\mathbb{B}_S) \right).$$

The composition then has the form

$$(3.14) \quad F^k \Xi_n^0 \otimes F^\ell \Xi_n^0 \rightarrow F^m \Xi_n^0$$

where  $m \geq \max(k, \ell)$ . If the associated graded algebra is semisimple, then so is  $\Xi_n^0$  (since it has 0 Jacobson radical). We can describe the part of (3.14) where  $k = \ell = m$ , which we will denote by  $\Xi_n^{0,k}$ .

We have

$$\Xi_n^{0,k} = k \left( \coprod_{S \subseteq [n], |S|=n-k} \text{Mor}(\mathbb{B}_S) \right),$$

where multiplication of two morphisms is given by composition when they are composable and is 0 otherwise. This algebra  $\Xi_n^{0,k}$  is thus semisimple, the endomorphism algebra being  $k[B_{S,0}]$ .

To describe the full algebra  $\Xi_n$ , note that  $B_n$  acts on  $\Xi_n^0$  by conjugation. The algebra  $\Xi_n$  then is the left Kan extension of  $\Xi_n^0$  from the group  $B_n$  to the groupoid  $GL_n(\mathbb{F}_q)$  acting on  $GL_n(\mathbb{F}_q)/B_n$  (where product of non-composable morphisms is again declared to be 0). The

algebra  $\Xi_n$  is semisimple because the endomorphism algebras  $\Xi_n^0$  are semisimple. We see that

$$\dim(\Xi_n^0) = q^{\binom{n}{2}} (2q-1)^n,$$

$$\dim(\Xi_n) = q^{\binom{n}{2}} ([n]_q!)^2 (2q-1)^n,$$

using the notation

$$[n]_q! = \frac{q^n - 1}{q - 1} \cdot \dots \cdot \frac{q - 1}{q - 1} = |GL_n(\mathbb{F}_q)/B_n|.$$

Thus, we have proved

**Theorem 3.4.** *Let  $q$  be a prime power and  $k$  be a field of characteristic not dividing  $q(q-1)$ . Then the category  $\mathcal{B}_{q,k}$  is semisimple.*

□

**3.5. The unipotent version  $\mathcal{U}_{q,k}$ .** We may also consider a unipotent variant  $\mathcal{U}_{q,k}$  (again, we omit the  $q, k$  subscript if  $q$  and  $k$  are fixed) of  $\mathcal{B}_{q,k}$ , with T-algebra structure defined as follows:

For finite sets  $S, T$ , we take the space  $\mathcal{U}_{S,T}$  to be the free  $k$ -vector space on equivalence classes of the data of a quotient map

$$f : \mathbb{F}_q^{S \amalg T} \rightarrow V$$

and a choice of ordered basis

$$(v_1, v_2, \dots, v_{\dim(V)})$$

generating  $V$ , over the equivalence relation that

$$(f, (v_1, \dots, v_{\dim(V)})) \sim (g \circ f, (g(v_1), \dots, g(v_{\dim(V)})))$$

for any isomorphism

$$g : V \rightarrow V$$

which is “unipotent” in  $GL(V)$ , i.e. such that we can express

$$g(v_i) = v_i + a_{i,i-1} \cdot v_{i-1} + \dots + a_{i,1} \cdot v_1$$

for some coefficients  $a_{i,j} \in \mathbb{F}_q$ .

The partial trace is defined by taking

$$\tau_{i,j}(f, (v_1, \dots, v_{\dim(V)}))$$

for  $i \in S, j \in T$  to be

- (1) 0 if  $f(x_i) \neq f(y_j)$  (denoting the free generators of  $\mathbb{F}_q^{S \amalg T}$  corresponding to the elements of  $S$  and  $T$  by  $x_s$  and  $y_t$ , respectively, for  $s \in S, t \in T$ ).

(2)

$$f_{i,j} := f|_{\mathbb{F}_q^{S \setminus \{i\}} \times \mathbb{F}_q^{T \setminus \{j\}}} : \mathbb{F}_q^{S \setminus \{i\}} \times \mathbb{F}_q^{T \setminus \{j\}} \twoheadrightarrow \text{Im}(f_{i,j})$$

if  $\dim(V_{i,j}) = \dim(V)$ , with the same basis as  $V$ .

(3) an  $-q^{-1} \cdot f_{i,j}$  is  $\dim(V_{i,j}) < \dim(V)$ , with the induced basis on  $V_{i,j}$ .

Product is defined analogously as for  $\mathcal{B}$ , and the compatibility of trace and product and semisimplicity in  $\mathcal{U}$  proceed similarly as above.

In the unipotent case, if we denote by  $\Theta_n^0$  the algebra of endomorphisms of the image of the idempotent

$$(\nabla : \mathbb{F}_q^n \times \mathbb{F}_q^n \twoheadrightarrow \mathbb{F}_q^n, (e_1, \dots, e_n))$$

(for  $(e_1, \dots, e_n)$  denoting the standard basis of  $\mathbb{F}_q^n$ ), we obtain

$$(3.15) \quad \dim(\Theta_n^0) = q^{\binom{n}{2}} (q+1)^n.$$

In particular, we have

**Theorem 3.6.** *Let  $q = p^m$  for a prime  $p$  and  $k$  be a field of characteristic not  $p$ . Then the category  $\mathcal{U}_{q,k}$  is semisimple.*

□

#### 4. QUANTUM DELANNOY CATEGORIES VIA OLIGOMORPHIC GROUPS

In this section, we give alternate descriptions of  $\mathcal{B}_{q,k}$  and  $\mathcal{U}_{q,k}$  using the theory of oligomorphic groups developed by N. Harman and A. Snowden in [6].

**4.1. The oligomorphic group for  $\mathcal{B}_{q,k}$ .** Let us consider the free  $\mathbb{F}_q$ -vector space  $V = \mathbb{F}_q[\mathbb{R}]$  and let us put

$$V^* = V \setminus \{0\}.$$

The vector space  $V$  has a natural  $\mathbb{R}$ -indexed filtration given by

$$(4.1) \quad F_r(V) = \mathbb{F}_q\{(s) \mid s \leq r \in \mathbb{R}\}.$$

We may consider the group

$$\Gamma = B \rtimes \text{Aut}(\mathbb{R}, <)$$

where  $Aut(\mathbb{R}, <)$  denotes the group of order-preserving bijections of  $\mathbb{R}$  and  $B \subset GL(V)$  denotes the subgroup of all linear isomorphisms from  $V$  to itself preserving the filtration.  $Aut(\mathbb{R}, <)$  acts on  $B$  by ordered permutation of the basis elements, which preserves the flag. One then sees that  $\Gamma$  acts on  $V^*$ , and in fact, forms an oligomorphic group. To obtain a semisimple pre-Tannakian category, we define the measure, in the sense of [6], of an orbit of  $\Gamma$  consisting of tuples of vectors generating an  $n$ -dimensional vector subspace of  $V$  to be

$$(4.2) \quad (q^{-1} - 1)^n.$$

Write  $\mu(\Gamma/H)$  for the measure of such an orbit  $\Gamma/H$ . An attractive feature of this approach is that N. Harman and A. Snowden [6], Theorem 13.2, prove that the condition that

$$\mu(\Gamma/H) \neq 0 \in k$$

for all open subgroups  $H \subseteq \Gamma$  implies semisimplicity of  $\mathcal{B}_{q,k}$  under the condition (3.1).

Now recall that an open subgroup (in the sense of [6]) of  $\Gamma$  is one which contains the stabilizer of some finite sequence of elements of  $V^*$ . However, we only specified measures of orbits  $\Gamma/H$  when there exists a finite-dimensional subspaces  $W \subset V$  such that

$$(4.3) \quad H = H_W = \{g \in \Gamma \mid \forall w \in W, g(w) = w\}.$$

Denote the class of such subgroup  $\mathcal{E}$ . Now the difficulty is that for an open subgroup  $K \subseteq \Gamma$ , there may not exist an  $H \in \mathcal{E}$  with

$$[K : H] < \infty.$$

For example, consider the subgroup

$$K = \{g \in \Gamma \mid g(F_0(V)) = F_0(V)\}.$$

This subgroup is open since  $K \supset H_{\langle(0)\rangle}$ , but has no subgroup  $H_W$  of finite index.

N. Harman and A. Snowden give a method for dealing with this situation by working *relative* to a class of open subgroups of  $\Gamma$ . The difficulty of working with the class  $\mathcal{E}$  directly is that in the relative case, the analogue of Theorem 13.2, guaranteeing semisimplicity, requires a technical condition on the class  $\mathcal{E}$  (Condition  $(*)$  of Remark 5.5 of [6]), which asserts that the stabilizers of finite subsets of an orbit  $\Gamma/H$  with  $H \in \mathcal{E}$  be in  $\mathcal{E}$ . This is false for the class  $\mathcal{E}$  defined by (4.3). We can solve this issue by passing to the class  $\bar{\mathcal{E}}$  consisting of subgroups of  $\Gamma$  of

the form  $H_{W,K}$  where for  $W \subset V$ ,  $K \subseteq B(W)$  (where  $B(W) \subseteq GL(W)$  is the set of elements preserving the induced flag on  $W$ ), we set

$$H_{W,K} = \{\gamma \in \Gamma \mid \gamma|_W \in K\}.$$

By assumption (3.1), for all choices of  $W, K$ , the index  $[H_{W,K} : H_W] \in k^\times$ , so we can put

$$\mu(\Gamma/H_{W,K}) = \frac{\mu(\Gamma/H_W)}{[H_{W,K} : H_W]}.$$

The axioms of [6] require that for every pullback diagram

$$(4.4) \quad \begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z, \end{array}$$

where  $X$  and  $Y$  are finite disjoint unions of  $\Gamma$ -orbits with stabilizers in  $\mathcal{E}$  and without loss of generality  $Z$  is a single  $\Gamma$ -orbit, that

$$(4.5) \quad \mu(X \times_Z Y) = \frac{\mu(X)\mu(Y)}{\mu(Z)}.$$

**4.2. A key Lemma.** To prove (4.5), one notes that there is an equivalence from the category whose objects are pairs of finite dimensional vector spaces  $A$  with a choice of a complete flag  $F_A$  and where a morphism

$$(A, F_A) \rightarrow (B, F_B)$$

is a pair of an inclusion  $f : A \hookrightarrow B$  such that intersecting the flag  $F_B$  with  $A$  gives  $F_A$  to the opposite category of  $\Gamma$ -orbits where all stabilizers are in  $\mathcal{E}$ . This equivalence of categories is the functor defined by taking an object  $(A, F_A)$  to the  $\Gamma$ -orbit  $O(A)$  of inclusions

$$\iota : A \rightarrow V$$

where the flag  $F_A$  is induced by the filtration (4.1), with  $\Gamma$  acting by composition (for any element  $\iota$ , the stabilizer is then the subgroup of  $\Gamma$  fixing  $\iota(A)$ , forming an element of the class  $\mathcal{E}$ ). This functor takes a morphism

$$f : (A, F_A) \rightarrow (B, F_B)$$

to a map of  $\Gamma$ -sets

$$O(B) \rightarrow O(A)$$

given by precomposing an element of  $O(B)$  with  $f$ .

Given this equivalence, a fiber product  $O(B) \times_{O(A)} O(B')$  (as in (4.4)) can be expressed as the  $\Gamma$ -set given as the disjoint union of  $O(C)$  for all choices of  $(C, F_C)$  which form a gluing, meaning a diagram

$$(4.6) \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ B' & \longrightarrow & C \end{array}$$

where each arrow is an injection and the flag on the target induces the flag on the source, and the maps from  $B, B'$  to  $C$  are jointly surjective.

Thus, to show (4.5), it suffices to show the following

**Lemma 4.3.** *For finite dimensional vector spaces  $A, B$ , and  $B'$  with flags*

$$F_A : 0 = A_0 \subset A_1 \subset \cdots \subset A_{\ell-1} \subset A_\ell = A$$

$$F_B : 0 = B_0 \subset B_1 \subset \cdots \subset B_{n-1} \subset B_n = B$$

$$F_{B'} : 0 = B'_0 \subset B'_1 \subset \cdots \subset B'_{m-1} \subset B'_m = B'$$

and flag-preserving inclusions

$$(4.7) \quad B \leftarrow A \hookrightarrow B',$$

letting  $G_{B \leftarrow A \hookrightarrow B'}$  denote the set of  $(C, F_C)$  which form a gluing of (4.7) as in (4.6), the following formula holds:

$$(4.8) \quad (q^{-1} - 1)^{\dim(B) + \dim(B') - \dim(A)} = \sum_{(C, F_C) \in G_{B \leftarrow A \hookrightarrow B'}} (q^{-1} - 1)^{\dim(C)}$$

([4]).

We may rewrite the formula (4.8) as

$$(4.9) \quad \sum_{(C, F_C) \in G_{B \leftarrow 0 \hookrightarrow B'}} \left( \frac{q}{1-q} \right)^{\dim(B) + \dim(B') - \dim(C)} = 1.$$

*Proof Outline.* First, we can assume without loss of generality that  $A = 0$ , since

$$\begin{aligned} \dim(B/A) + \dim(B'/A) &= \dim(B) + \dim(B') - 2\dim(A), \\ \dim(C/A) &= \dim(C) - \dim(A), \end{aligned}$$

so quotienting out by  $A$  will simply divide both sides of (4.8) by

$$(q^{-1} - 1)^{\dim(A)}.$$



Since  $C$  is obtained by gluing (4.7) with  $A = 0$ , this determines a choice of a surjection

$$g : \{1, \dots, n\} \amalg \{1, \dots, m\} \rightarrow \{1, \dots, \dim(C)\},$$

injective and order-preserving on the discrete summands of the source, where there are exactly  $n + m - \dim(C)$  “clashes” (i.e. equalities of the images of elements of the first discrete summand with the second). Consider a choice of  $g$  which does involve a clash. Let the first clash of  $g$  send  $j \in \{1, \dots, n\}$ ,  $j' \in \{1, \dots, m\}$  to  $i \in \{1, \dots, \dim(C)\}$ . Now consider the associated summand of (4.9), (a multiple of

$$\left(\frac{q}{1-q}\right)^{n+m-\dim(C)}$$

by the number of gluings  $(C, F_C)$  corresponding to  $g$ ). This summand cancels with the associated summand to the gluing

$$g' : \{1, \dots, n\} \amalg \{1, \dots, m\} \rightarrow \{1, \dots, \dim(C) + 1\},$$

where

$$g'(s_1) = g(s_1), \text{ for } s \leq j$$

$$g'(s_1) = g(s_1) + 1, \text{ for } s > j$$

$$g'(s_2) = g(s_2), \text{ for } s < j'$$

$$g'(s_2) = g(s_2) + 1, \text{ for } s \geq j'$$

(where, again, the subscript indicates which discrete summand of the source  $s$  is considered as), by the same argument as the compatibility of partial traces and products for  $\mathcal{B}_{q,k}$  in Subsection 3.1.

Then the only term that remains is the one associated to the gluing

$$\{1, \dots, n\} \amalg \{1, \dots, m\} \rightarrow \{1, \dots, n + m\}$$

sending the second discrete summand of the source to the first  $m$  elements of the target and the first discrete summand to the last  $n$  elements. This also corresponds to a single choice of  $C$ , giving (4.9).  $\square$

**4.4. A Detailed Proof of the Consistency of Measure for  $\Gamma$ .** A detailed proof of Lemma 4.3, following P. Deligne [4], goes as follows:

By induction on the numbers  $n, m$ , it suffices to prove Lemma 4.3 when

$$\dim(B) = \dim(B') = \dim(A) + 1.$$

It is beneficial to rewrite the statement in basis notation. Note that a finite totally ordered set  $(S, \leq)$  corresponds to the finite dimensional vector space  $\mathbb{F}_q^S$ , with the  $S$ -indexed filtration given by, for  $s \in S$ ,

$$F_s(\mathbb{F}_q^S) = \mathbb{F}_q^{\{t \in S \mid t \leq s\}}.$$

An order-preserving injection

$$(S, \leq) \rightarrow (T, \leq)$$

induces a morphism

$$(4.10) \quad \mathbb{F}_q^S \hookrightarrow \mathbb{F}_q^T$$

preserving these filtrations. In fact, for every finite dimensional vector spaces  $A$  and  $B$  with complete flags  $F_A$  and  $F_B$  (resp.), every flag-preserving inclusion

$$(4.11) \quad A \hookrightarrow B$$

will be the same, up to isomorphism, as an order-preserving map (4.10). Since any flag-preserving automorphism on  $A$  in (4.11) will extend to a flag-preserving morphism on  $B$ , any diagram (4.7) is isomorphic to one coming from a diagram of order-preserving injections

$$(4.12) \quad \begin{array}{ccc} & (T, \leq) & \\ \uparrow & & \\ (S, \leq) & \longrightarrow & (T', \leq). \end{array}$$

It suffices, then, to prove (4.8) for a diagram (4.7) arising from a diagram (4.12) for sets  $S, T, T'$  with

$$|T| = |T'| = |S| + 1.$$

Without loss of generality,  $S = \{1, \dots, n\}$  for some  $n \in \mathbb{N}$ , with the standard ordering. Note that we may then write

$$T = S \amalg \{t\}$$

$$T' = S \amalg \{t'\}$$

where  $t$  is inserted in the  $i$ th place, i.e.  $i - 1 < t < i$  in the total ordering of  $T$ , and  $t'$  is inserted in the  $j$ th place, i.e.  $j - 1 < t' < j$  in the total ordering of  $T'$ .

**Case 1:**  $i \neq j$ . It suffices to prove that there is only a single gluing which is

$$\begin{array}{ccc} (T, \leq) & \longrightarrow & (R, \leq) \\ \uparrow & & \uparrow \\ (S, \leq) & \longrightarrow & (T', \leq) \end{array}$$

of (4.12), with

$$(4.13) \quad R = S \amalg \{t\} \amalg \{t'\}$$

inserting  $t$  and  $t'$  so that  $i - 1 < t < i$ ,  $j - 1 < t < j$ , since then

$$|R| = |T| + |T'| - |S|.$$

For a gluing diagram

$$(4.14) \quad \begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ A & \longrightarrow & B' \end{array}$$

of flag-preserving inclusions of (4.7), we must have  $C = B \oplus_A B'$  (since, otherwise,  $C \cong B \cong B'$ , and the flag-preserving inclusions

$$A \hookrightarrow B$$

$$A \hookrightarrow B'$$

would be isomorphic, which contradicts the assumption of this case).

Without loss of generality, let us assume that  $i < j$ . We can choose an ordered basis

$$(4.15) \quad e_1, \dots, e_i, \dots, e_{j+1}, \dots, e_{n+2}$$

of  $B \oplus_A B'$  such that for  $0 \leq k \leq n + 2$ , the first  $k$  elements

$$A = \langle e_1, \dots, \widehat{e_i}, \dots, \widehat{e_{j+1}}, \dots, e_{n+2} \rangle$$

$$B = \langle e_1, \dots, \widehat{e_{j+1}}, \dots, e_{n+2} \rangle$$

$$B' = \langle e_1, \dots, \widehat{e_i}, \dots, e_{n+2} \rangle$$

To prove that the only case we need to consider is (4.13), it suffices to show that the only complete flag

$$(4.16) \quad 0 = F_0(B \oplus_A B') \subset \dots \subset F_{n+2}(B \oplus_A B') = B \oplus_A B'$$

inducing the given flags on  $B$  and  $B'$  is the one given by the ordered basis (4.15). For a flag (4.16), we must have

$$(4.17) \quad F_1(B \oplus_A B') \not\subseteq B \text{ or } F_1(B \oplus_A B') \not\subseteq B'$$

(both are not possible, by dimension). Without loss of generality,  $i = 1$ . The possibilities (4.17) then imply that

$$(4.18) \quad F_k(B \oplus_A B') = F_1(B \oplus_A B') \oplus F_{k-1}(B)$$

or

$$(4.19) \quad F_k(B \oplus_A B') = F_1(B \oplus_A B') \oplus F_{k-1}(B'),$$

respectively. We have

$$B = \langle e_1, \dots, \widehat{e_j}, \dots, e_{n+2} \rangle$$

$$B' = \langle e_2, \dots, e_{n+2} \rangle.$$

If  $F_1(B \oplus_A B') \subseteq B$ , then  $F_1 = \langle e_1 \rangle$  and  $F_1(B \oplus_A B') \not\subseteq B'$ , so

$$F_k(B \oplus_A B') = F_1(B \oplus_A B') \oplus F_{k-1}(B') = \langle e_1 \rangle \oplus \langle e_2, \dots, e_k \rangle,$$

giving the claim. If  $F_1(B \oplus_A B') \not\subseteq B$ , then  $F_1(B \oplus_A B') \subseteq B'$ , so

$$F_1(B \oplus_A B') = \langle e_2 \rangle,$$

which is impossible since then it can not induce the given filtration on  $B$ .

**Case 2:**  $i = j$ . Let us choose an ordered basis for  $B$

$$e_1, \dots, e_{n+1}$$

such that

$$F_k(B) = \langle e_1, \dots, e_k \rangle$$

and

$$A = \langle e_1, \dots, \widehat{e_i}, \dots, e_{n+1} \rangle$$

Gluing of  $B$  and  $B'$  along  $A$  will be of dimension  $n + 1$  or  $n + 2$ . For a gluing of dimension  $n + 1$ , the exponent of  $\frac{q}{1-q}$  in the corresponding summand of (4.9) is 1. Each such gluing corresponds to a flag-preserving automorphism

$$B \rightarrow B$$

which is identity when restricted to  $A$ , and must therefore send  $e_k$  to itself for all  $k \neq i$  and send  $e_i$  to a linear combination

$$a_1 \cdot e_1 + \dots + a_i \cdot e_i$$

for coefficients  $a_1, \dots, a_{i-1} \in \mathbb{F}_q$ ,  $a_i \in \mathbb{F}_q^\times$ . Thus, there are  $q^{i-1}(q-1)$  choices of gluings of dimension  $n + 1$ , contributing

$$(4.20) \quad \frac{q}{1-q} \cdot (q^{i-1}(q-1)) = -q^i.$$

For the gluings of dimension  $n + 2$ , again a gluing diagram (4.14) will have  $C = B \oplus_A B'$ , and we can choose ordered bases

$$\begin{aligned} B &= \langle e_1, \dots, e_{i-1}, e_i, e_{i+1}, \dots, e_{n+1} \rangle \\ B' &= \langle e_1, \dots, e_{i-1}, e'_i, e_{i+1}, \dots, e_{n+1} \rangle \\ A &= \langle e_1, \dots, \widehat{e}_i, \dots, e_{n+1} \rangle, \end{aligned}$$

(inducing the given flags on  $A$ ,  $B$ ,  $B'$ ). Gluings  $C$  then are given by a choice of complete flag inducing the given flags on  $B$  and  $B'$ , each of which will contribute a term of  $(\frac{q}{1-q})^0 = 1$  in (4.9).

First note that the number of choices of such flags

$$(4.21) \quad 0 = F_0(B \oplus_A B') \subset \dots \subset F_{n+2}(B \oplus_A B') = B \oplus_A B'$$

depends only on  $i$ . There must be exactly one  $0 \leq k_0 \leq n + 1$  such that

$$(4.22) \quad F_{k_0}(B \oplus_A B') \cap B = F_{k_0+1}(B \oplus_A B') \cap B$$

(and similarly, there exists exactly one  $0 \leq k'_0 \leq n + 1$  for which this holds with  $B$  replaced by  $B'$ ). Thus, for all  $k > i$ , by (4.22),

$$e_1, \dots, e_{i-1}, e_i, e_{i+1}, \dots, e_{k-1} \in F_k(B \oplus_A B')$$

and, by the analogue for  $B'$ ,

$$e_1, \dots, e_{i-1}, e'_i, e_{i+1}, \dots, e_{k-1} \in F_k(B \oplus_A B').$$

Thus, by dimension,

$$\langle e_1, \dots, e_{i-1}, e_i, e'_i, e_{i+1}, \dots, e_{j-1} \rangle = F_j(B \oplus_A B').$$

For  $k \leq i$ ,

$$F_k(B \oplus_A B') \subseteq F_i(B \oplus_A B'),$$

and hence the data of the flag (4.21) is in bijective correspondence with a flag on

$$(4.23) \quad \langle e_1, \dots, e_{i-1}, e_i, e'_i \rangle$$

giving the induced flags on

$$(4.24) \quad \langle e_1, \dots, e_{i-1}, e_i \rangle$$

$$(4.25) \quad \langle e_1, \dots, e_{i-1}, e'_i \rangle$$

from the flags on  $B$  and  $B'$ .

In fact, the number of such flags on (4.23) is

$$(4.26) \quad q^{i-1}(q-1) + q^{i-2}(q-1) + \dots + q(q-1) + (q+1)$$

(interpreted to be  $q + 1$  if  $i = 1$ ), which we prove by induction on  $i$ :

Without loss of generality, by the above argument, let us suppose

$$A = \langle e_1, \dots, e_{i-1} \rangle$$

and  $B$  and  $B'$  are (4.24) and (4.25), respectively. For  $i = 1$ ,  $\dim(A) = 0$  and  $\dim(B) = \dim(B') = 1$ , so any complete flag on  $B \oplus B'$ , of which there are  $q + 1$ , will work.

For  $i > 1$ , if  $F_1(B \oplus_A B') \not\subseteq B, B'$ , then there exists a vector  $v$  of the form

$$(4.27) \quad a_1 \cdot e_1 + \dots + a_i \cdot e_i + e'_i$$

with  $a_1, \dots, a_{i-1} \in \mathbb{F}_q$ ,  $a_i \in \mathbb{F}_q^\times$ , with

$$F_1(B \oplus_A B') = \langle v \rangle$$

and for  $k \geq 1$

$$F_k(B \oplus_A B') = \langle v, e_1, \dots, e_{k-1} \rangle.$$

There are  $q^{i-1}(q - 1)$  choices of (4.27).

If  $F_1(B \oplus_A B')$  is in  $B$  or  $B'$ , then

$$F_1(B \oplus_A B') = \langle e_1 \rangle \subseteq B, B'.$$

Quotienting out  $F_1(B \oplus_A B')$  then gives a flag on

$$\langle e_2, \dots, e_{i-1}, e_i, e'_i \rangle$$

of which the number of choices is

$$q^{i-2}(q - 1) + q^{i-3}(q - 1) + \dots + q(q - 1) + (q + 1)$$

by the induction hypothesis. Summing these two cases together, we get (4.26).

Summing (4.20) with (4.26) gives that the left hand side of (4.9) is

$$-q^i + q^{i-1}(q - 1) + q^{i-2}(q - 1) + \dots + q(q - 1) + (q + 1) = 1.$$

□

**4.5. A Matrix Description of Orbits of  $\Gamma$  and the Unipotent Analogue.** The orbits of  $(V^*)^n$  with respect to the action of  $\Gamma$  can be described as equivalence classes of matrices with  $n$  non-zero columns with entries in  $\mathbb{F}_q$  under the equivalence relation  $\sim_b$  generated by operations

- (1) Addition of an  $\mathbb{F}_q$ -multiple of the  $i$ th row to the  $j$ th row for  $i < j$

- (2) Omission of zero rows
- (3) Multiplication of a row by an element of  $\mathbb{F}_q^\times$

Equivalence classes have unique representatives in *semi-echelon form* which means that the matrix contains no zero rows and if we call the left-most non-zero entry of each row a *pivot*, then we require that

- (1) All entries below a pivot are 0
- (2) Every pivot is equal to 1

The measure of an orbit corresponding to a matrix in semi-echelon form is  $(q^{-1} - 1)^\ell$  where  $\ell$  is the number of pivots.

A variant of the construction where we use  $V$  instead of  $V^*$  for the oligomorphic group action gives the same characterization of orbits by matrices except that we allow matrices with 0 columns.

Symmetric group action on orbits is given by permuting the columns of a matrix and putting the resulting matrix in semi-echelon form. Product of a matrix with rows  $\{1, \dots, i\}$  and a matrix with rows  $\{1, \dots, j\}$  is a sum indexed by “ordered gluings of rows,” i.e. surjections

$$\phi : \{1, \dots, i\} \amalg \{1, \dots, j\} \rightarrow \{1, \dots, i + j - h\}$$

which are order-preserving injections on each disjoint summand. We arrange the matrices side by side while moving their rows according to  $\phi$ . It follows that the resulting matrix is in semi-echelon form.

Partial trace of a matrix  $M$  with respect to the  $i$ th and  $j$ th columns (without loss of generality, assume  $i < j$ ) is defined by summing over all matrices  $M'$  that are equivalent to  $M$  where the  $i$ th and  $j$ th columns coincide (this does not depend on the representative) such that after deleting the  $j$ th column,  $M'$  is in semi-echelon form. The corresponding summand to  $M'$  is the matrix in semi-echelon form which is equivalent to  $M'$  after deleting the  $i$ th and  $j$ th columns, with coefficient equal to the sum of measures of the orbits of the omitted columns.

To construct the category  $\mathcal{U}_{q,k}$ , let us first note that  $B \subset GL(V)$  also can be replaced by a “finitary” variant

$$B_{fin} = \bigcup_{S \subset \mathbb{R}, |S| < \infty} B_S$$

where  $B_S \subseteq GL(\mathbb{F}_q[S])$  is the Borel subgroup of lower triangular matrices. The resulting group

$$\Gamma_{fin} = B_{fin} \rtimes Aut(\mathbb{R}, <)$$

is also oligomorphic and in fact the  $\Gamma_{fin}$ -orbits of  $(V^*)^n$  (or  $V^n$ ) are the same as the orbits of  $\Gamma$ .

The advantage of the “finitary” approach is that it also has a *unipotent version*

$$U_{fin} = \bigcup_{S \subset \mathbb{R}, |S| < \infty} U_S$$

where  $U_S \subseteq GL(\mathbb{F}_q[S])$  is the maximal unipotent subgroup of the Borel subgroup  $B_S$ . One can then form the group

$$\Gamma_u = U_{fin} \rtimes Aut(\mathbb{R}, <),$$

which also acts oligomorphically on  $V$  and  $V^*$ .

Let us now describe the orbits of  $(V^*)^n$  with respect to  $\Gamma_u$  and a non-zero measure, which gives the semisimple pre-Tannakian category  $\mathcal{U}_{q,k}$ .

The description of the orbits is by an equivalence of matrices, which we denote by  $\sim_u$ , analogous to that for  $\Gamma$  except that we drop the operation (3). Accordingly, the representatives of equivalence classes are described by matrices in *weak semi-echelon form*, which satisfy the conditions (1), (2) (dropping condition (3)).

The symmetric group action on orbits and product can be described directly analogously as for  $\mathcal{B}_{q,k}$ . Again, we take the partial trace of a matrix  $M$  with respect to the  $i$ th and  $j$ th columns (without loss of generality, assume  $i < j$ ) to be the sum over all matrices  $M'$  equivalent (now only using operations (1) and (2)) to  $M$  where the  $i$ th and  $j$ th columns coincide (this does not depend on the representative) such that after deleting the  $j$ th column, the matrix is in weak semi-echelon form. The corresponding summand in the partial trace of  $M$  is the matrix in weak semi-echelon form which is equivalent to  $M'$  with the  $i$ th and  $j$ th columns deleted, with coefficient equal to the sum of measures of the orbits of the omitted columns.

The measure of an orbit corresponding to a matrix in weak semi-echelon form with  $\ell$  pivots is defined to be  $(-q)^{-\ell}$ . This is a measure in the sense of [6] relative to the class of subgroups  $\mathcal{E}_u$  consisting, for finite subsets  $S \subset \mathbb{R}$ , of subgroups  $G \subset \Gamma_n$  which send

$$\mathbb{F}_q[S] \rightarrow \mathbb{F}_q[S]$$



by a transformation in  $U_S$ .

It then suffices to prove multiplicativity of the measure on orbits given by stabilizers of finite sequences. The proof is analogous as for  $\Gamma$ , with the product summand given by a term containing a clash canceling with the summand obtained by shifting the lowest clashing term of the first factor to a neighboring non-clashing term to the right. This again leaves only one summand whose measure is the required product by definition.

Since the measures of  $\Gamma_u/H$  with  $H \in \mathcal{E}_u$  have denominators given by powers of  $q$ , the category  $\mathcal{U}_{q,k}$  is defined and semisimple over any field  $k$  of characteristic  $\neq p$ .

## 5. STRUCTURAL RESULTS

In this section, we will investigate the structure of the categories  $\mathcal{B}_{q,k}$ ,  $\mathcal{U}_{q,k}$ . First, we study the decompositions of the “basic objects” into simple summands, which is surprisingly subtle. Also, it is helpful to consider further variants of the basic object. Eventually, it is possible to use the same method to actually characterize all the isomorphism classes of the simple objects of  $\mathcal{U}_{q,k}$ .

In Subsection 5.8, we discuss comparison functors between the various categories considered. In particular, we construct a functor

$$\mathcal{D} \rightarrow \mathcal{B}_{q,k}$$

from a remarkable universal property of the Delannoy category, which we prove.

**5.1. The Decomposition of the Basic Object of  $\mathcal{B}_{q,k}$  into Simple Objects.** Let  $k$  be a field satisfying (3.1) and containing  $(q-1)$ th roots of unity. In this section, we shall give the semisimple decompositions of the basic objects of  $\mathcal{B}_{q,k}$  (Proposition 5.2 below).

We have  $q-1$  multiplicative characters

$$\psi : \mathbb{F}_q^\times \rightarrow k^\times.$$

Let  $\Omega = [V^*] \in \text{Obj}(\mathcal{B}_{q,k})$  in the notation of [6]. Then the action of  $\mathbb{F}_q^\times$  on  $V^*$  by multiplication induces a decomposition

$$\Omega = \bigoplus_{\psi} \Omega_{\psi}$$

into pieces on which  $\mathbb{F}_q^\times$  acts by  $\psi$ . The corresponding idempotent on  $\Omega$  is

$$(5.1) \quad \iota_\psi = \frac{1}{q-1} \sum_{\psi \in \mathbb{F}_q^\times} \psi(a) \cdot (a),$$

where  $(a)$  denotes the action of  $a \in \mathbb{F}_q^\times$  by multiplication. The action of  $a \neq 1 \in \mathbb{F}_q^\times$  has trace 0, and the trace of 1 is

$$\dim(\Omega) = (q-1) \left( -\frac{1}{q} \right).$$

Hence,

$$\text{tr}(\iota_\psi) = -\frac{1}{q},$$

The orbits of  $(V^*)^2$  have the following representatives in semi-echelon form:

$$(5.2) \quad \begin{aligned} (>) &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (<) &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ (a) &:= \begin{pmatrix} 1 & a \\ & \end{pmatrix} \\ (a)^\sim &:= \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

where  $a \in \mathbb{F}_q^\times$ . One has

$$\begin{aligned} (<)(a) &= (a)(<) = (<) \\ (>)(a) &= (a)(>) = (>). \end{aligned}$$

It follows that  $(>)$ ,  $(<)$  map  $\Omega_1$  to itself and acts by 0 on  $\Omega_\psi$ , for all non-trivial characters  $\psi \neq 1$ . We have

$$(5.3) \quad (a)^\sim(b) = (b)(a)^\sim = (a \cdot b)^\sim.$$

It follows that  $(a)^\sim$  maps  $\Omega_\psi$  to itself and in  $\Omega_\psi$ ,

$$(a)^\sim = \psi(a) \cdot (1)^\sim.$$

One concludes that

$$(5.4) \quad \begin{aligned} \text{Hom}(\Omega_\psi, \Omega_\phi) &= 0 \quad \text{for } \psi \neq \phi \\ \text{Hom}(\Omega_\psi, \Omega_\psi) &= \begin{cases} \langle \text{Id}_{\Omega_\psi}, (1)^\sim|_{\Omega_\psi} \rangle & \text{if } \psi \neq 1 \\ \langle \text{Id}_{\Omega_\psi}, (<), (>), (1)^\sim|_{\Omega_1} \rangle & \text{if } \psi = 1. \end{cases} \end{aligned}$$

We next calculate the coefficients  $s, t \in k$  such that

$$((1)^\sim)^2 = s \cdot (1)^\sim + t \cdot \text{Id}.$$

To calculate  $t$ , we note that

$$(5.5) \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so the fiber above  $(v, v) \in V^* \times V^*$  in  $(V^*)^3$  is isomorphic to  $V^*$  and hence has measure  $q^{-1} - 1$ , so

$$t = q^{-1} - 1.$$

To calculate  $s$ , one notes that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix}, \text{ for } x \in \mathbb{F}_q$$

which gives  $q$  orbits of measure  $q^{-1} - 1$  in  $(V^*)^3$  above a representative with semi-echelon form  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and also

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which gives 1 additional orbit of measure  $q^{-1} - 1$  in  $(V^*)^3$  over a representative with semi-echelon form  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Also, using

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix} \text{ for } x \neq 1, 0$$

(we exclude 0 to avoid the case (5.5)) gives additional  $q - 2$  orbits of measure 1 in  $(V^*)^3$  over a representative with semi-echelon form  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Thus,

$$s = (q + 1)(q^{-1} - 1) + (q - 2) = q^{-1} - 2,$$

and hence,

$$(5.6) \quad ((1)^\sim)^2 = (q^{-1} - 2) \cdot (1)^\sim + (q^{-1} - 1) \cdot Id.$$

Now solving the equation (5.6), we get roots  $(1)^\sim = -1, q^{-1} - 1$ . Thus, 1 decomposes into idempotent multiples of

$$(1)^\sim + 1, (1)^\sim + 1 - q^{-1}.$$

Noting that

$$1 = q \cdot ((1)^\sim + 1) - q \cdot ((1)^\sim + 1 - q^{-1}),$$

we get that the idempotents are

$$(5.7) \quad \iota^+ := q((1)^\sim + 1), \quad \iota^- := -q((1)^\sim + 1 - q^{-1}),$$

giving a decomposition

$$\Omega_\psi^+ \oplus \Omega_\psi^- = \Omega_\psi$$

for objects  $\Omega_\psi^+, \Omega_\psi^-$ , of dimensions  $-1, 1 - q^{-1}$ , respectively, corresponding to the idempotents

$$\iota_\psi^+ := \iota_\psi \circ \iota^+ = \iota^+ \circ \iota_\psi$$

$$\iota_\psi^- := \iota_\psi \circ \iota^- = \iota^- \circ \iota_\psi$$

(recalling that  $\iota^+$  and  $\iota^-$  commute with  $\iota_\psi$  since  $(1)^\sim$  does, by (5.3)).

It follows from (5.4) that  $\Omega_\psi^+, \Omega_\psi^-$  are simple for  $\psi \neq 1$ . For  $\psi = 1$ , we additionally have the idempotents

$$\bar{e}_0 = \frac{(<)}{q^{-1} - 1}, \quad \bar{e}_\infty = \frac{(>)}{q^{-1} - 1}$$

of trace 0. One notes that  $\bar{e}_0, \bar{e}_\infty$  commute, have trace 0, and

$$tr(\bar{e}_0 \bar{e}_\infty) = 1.$$

Thus, the idempotents

$$e_0 := \bar{e}_0 - \bar{e}_0 \bar{e}_\infty, \quad e_\infty := \bar{e}_\infty - \bar{e}_0 \bar{e}_\infty, \quad \text{and} \quad \bar{e}_0 \bar{e}_\infty$$

are disjoint and give a decomposition

$$\Omega_1^+ \cong \Omega_{1,0}^+ \oplus \Omega_{1,\infty}^+ \oplus 1$$

where the summands correspond to the idempotents respectively and are of dimensions  $-1, -1$ , and  $1$ .

By the description (5.2) of the orbits of  $(V^*)^2$ , we have

$$\dim(\text{End}_{\mathcal{B}_{q,k}}(\Omega) = 2q,$$

and thus  $\Omega_\psi^+, \Omega_\psi^-, \Omega_1^-, \Omega_{1,0}^+, \Omega_{1,\infty}^+$  and  $1$  are all simple and non-isomorphic objects.

In fact, we have

$$[V] = \Omega \oplus (0)$$

and we know that

$$\dim(\text{End}_{\mathcal{B}_{q,k}}([V]) = 2q + 3.$$

By counting dimensions of the simple summands, one can therefore deduce

$$1 \cong (0)$$

(which can also be checked directly by noting that

$$(0, 0), (1, 0), (0, 1), (1, 0)(0, 1)$$

form a two-by-two matrix algebra).

Thus, we have proved the following

**Proposition 5.2.** *The decomposition of  $\Omega$  into simple objects is*

$$\Omega = \left( \bigoplus_{\psi \neq 1} (\Omega_{\psi}^{-} \oplus \Omega_{\psi}^{+}) \right) \oplus \Omega_1^{-} \oplus \Omega_{1,0}^{+} \oplus \Omega_{1,\infty}^{+} \oplus 1$$

$$[V] = \left( \bigoplus_{\psi \neq 1} (\Omega_{\psi}^{-} \oplus \Omega_{\psi}^{+}) \right) \oplus \Omega_1^{-} \oplus \Omega_{1,0}^{+} \oplus \Omega_{1,\infty}^{+} \oplus 2 \cdot 1$$

where all the summands are non-isomorphic and

$$\dim(\Omega_{\psi}^{-}) = 1 - q^{-1}$$

$$\dim(\Omega_{\psi}^{+}) = \dim(\Omega_{1,0}^{+}) = \dim(\Omega_{1,\infty}^{+}) = -1$$

$$\dim(1) = 1.$$

□

**5.3. The Simple Objects of  $\mathcal{U}_{q,k}$ .** To investigate the case of  $\mathcal{U}_{q,k}$ , we begin by noting that we have a  $\Gamma_u$ -invariant map

$$\theta : V^* \rightarrow \mathbb{F}_q^{\times},$$

given by sending a vector to the coefficient of its coordinate which has the highest index in  $\mathbb{R}$ , i.e. sending

$$a_1 \cdot (r_1) + \cdots + a_n \cdot (r_n) \in \mathbb{F}_q[\mathbb{R}] \setminus \{0\}$$

for  $a_1, \dots, a_n \in \mathbb{F}_q^{\times}$ ,  $r_1 > \cdots > r_n \in \mathbb{R}$ , to  $a_1 \in \mathbb{F}_q^{\times}$ . Each fiber  $\theta^{-1}(i)$  gives a suborbit of  $\Omega$ , which is isomorphic to the projective space

$$P := \mathbb{P}(\mathbb{F}_q[\mathbb{R}] \setminus \{0\}) = (\mathbb{F}_q[\mathbb{R}] \setminus \{0\})/\mathbb{F}_q^{\times}.$$

The basic objects  $\Omega$ ,  $[V]$  then splits as

$$\Omega \cong (q-1)[P],$$

$$[V] \cong (q-1)[P] \oplus (0).$$

Similarly to  $\Omega$ , we can use  $[P]$  as the basic object for  $\mathcal{U}_{q,k}$ , since it generates  $[V]$  (and  $\Omega$ ) and therefore generates an equivalent category.

**Proposition 5.4.** *For a field  $k$  of characteristic not equal to  $p$  containing  $p$ th roots of unity, we have decompositions*

$$\begin{aligned} [P] &\cong 1 \oplus \Omega_{1,0}^+ \oplus \Omega_{1,\infty} \oplus \bigoplus_{\alpha \neq 1: \mathbb{F}_q \rightarrow k^\times} \tilde{\Omega}_\alpha \\ \Omega &\cong (q-1) \cdot \left( 1 \oplus \Omega_{1,0}^+ \oplus \Omega_{1,\infty}^+ \oplus \bigoplus_{\alpha \neq 1: \mathbb{F}_q \rightarrow k^\times} \tilde{\Omega}_\alpha \right), \\ [V] &\cong q \cdot 1 \oplus (q-1) \cdot \left( \Omega_{1,0}^+ \oplus \Omega_{1,\infty}^+ \oplus \bigoplus_{\alpha \neq 1: \mathbb{F}_q \rightarrow k^\times} \tilde{\Omega}_\alpha \right) \end{aligned}$$

where the direct sum runs over non-trivial additive characters

$$\alpha : \mathbb{F}_q \rightarrow k^\times$$

for simple non-isomorphic objects  $\tilde{\Omega}_\alpha$ , with

$$\begin{aligned} \dim(\tilde{\Omega}_\alpha) &= q^{-1} \\ \dim(\Omega_{1,0}^+) &= \dim(\Omega_{1,\infty}^+) = -1 \\ \dim(1) &= 1. \end{aligned}$$

*Proof.* First note that the orbits of  $P^2$  have the following representatives in (weak) semi-echelon form:

$$\begin{aligned} (>) &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (<) &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ (1) &:= \begin{pmatrix} 1 & 1 \end{pmatrix} \\ [a] &:= \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \end{aligned} \tag{5.8}$$

for  $a \in \mathbb{F}_q^\times$  (note that, for every  $a \in \mathbb{F}_q^\times$ , we have  $[a] \sim_b (1)^\sim$ ).

First, let us compute  $[a][b]$  for  $a, b \in \mathbb{F}_q^\times$ . Note that for any choice of  $a, b \in \mathbb{F}_q^\times$ , we have

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ 0 & a \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ a & a \\ 0 & b \end{pmatrix},$$

giving  $-q^{-1}[a]$  as a summand of  $[a][b]$  (since the partial trace of the product with respect to the middle columns is

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ 0 & a \\ 0 & b \end{pmatrix}$$

with coefficient  $-q^{-1}$  because the dimension of the corresponding fiber is 1, i.e. we delete one row to put it into weak semi-echelon form). Similarly, the fact that

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & a \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ 0 & b \\ a & 0 \end{pmatrix}$$

gives a multiple of  $[b]$ , since

$$\begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ 0 & b \\ 0 & 0 \end{pmatrix},$$

as a summand of  $[a][b]$ . Again, the coefficient of  $[b]$  will be  $-q^{-1}$  since the dimension of the corresponding fiber is 1 (again, we delete one row to put the product matrix into weak semi-echelon form after deleting the traced columns). So, for any choice of  $a, b \in \mathbb{F}_q^\times$ , the composition  $[a][b]$  will have summands  $-q^{-1}[a]$  and  $-q^{-1}[b]$ .

If  $b \neq -a$ , then the only other possible summand is a multiple of  $[a+b]$ . The term  $[a+b]$  can arise from considering

$$\begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ a & a+b \end{pmatrix},$$

giving a copy of  $[a+b]$  with coefficient 1 since the dimension of a corresponding fiber will be 0 (the product matrix is already in weak semi-echelon form after deleting the traced columns). We also get an additional  $q-1$  copies of  $[a+b]$  with coefficient  $-q^{-1}$  arising from

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ 0 & a \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ a & a+b \\ 0 & d \end{pmatrix}$$

for any choice of  $d \in \mathbb{F}_q^\times$ . Thus, if  $a \neq -b$ ,

$$(5.9) \quad [a][b] = q^{-1}[a+b] - q^{-1}[a] - q^{-1}[b].$$

If  $b = -a$ , then the only possible summands of  $[a][b]$  other than the terms  $-q^{-1}[a]$  and  $-q^{-1}[-a]$  which we described above, are (1) and  $[d]$  for  $d \in \mathbb{F}_q^\times$ . By considering

$$\begin{pmatrix} 1 & 1 \\ 0 & -a \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ a & 0 \end{pmatrix},$$

we obtain the term (1) with coefficient  $-q^{-1}$  since the dimension of the corresponding is 1 (we delete one row when putting the product matrix into weak semi-echelon form after deleting the traced columns). For  $d \in \mathbb{F}_q^\times$ , we also obtain a copy of  $[d]$  with coefficient  $-q^{-1}$  by considering

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ 0 & a \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ a & 0 \\ 0 & d \end{pmatrix}$$

(the coefficient again arises since we lose one row when putting the combined matrix into weak semi-echelon form). Thus,

$$(5.10) \quad [a][-a] = -q^{-1}(1) - q^{-1}[a] - q^{-1}[-a] - q^{-1} \sum_{d \in \mathbb{F}_q^\times} [d].$$

These multiplication formulas give that, if we write, for  $a \in \mathbb{F}_q^\times$ ,

$$[a]' := q \cdot [a] + (1),$$

then we have, for  $a, b \in \mathbb{F}_q^\times$  with  $b \neq -a$ ,

$$(5.11) \quad [a]'[b]' = [a+b]'$$

and

$$(5.12) \quad [a]'[-a]' = - \left( \sum_{d \in \mathbb{F}_q^\times} [d]' \right).$$



Let us write

$$[0]' := - \left( \sum_{d \in \mathbb{F}_q^\times} [d]' \right).$$

Note that for every  $a \in \mathbb{F}_q^\times$ ,

$$\text{tr}([a]') = -q^{-1}$$

and

$$\text{tr}([0]') = -(q-1) \cdot (-q^{-1}) = 1 - q^{-1}.$$

In fact, the formulas (5.11) and (5.12) imply that  $[0]'$  is an idempotent, and moreover, for every  $a \in \mathbb{F}_q^\times$ ,

$$[a]'[0]' = [0]'[a]' = [a]'$$

For a non-trivial additive character

$$\alpha : \mathbb{F}_q \rightarrow k^\times$$

(of which there are  $q-1$ ), we may then write

$$z_\alpha := \sum_{a \in \mathbb{F}_q^\times} \alpha(a) \cdot [a]',$$

and get that composing  $z_\alpha$  with itself gives

$$z_\alpha^2 = (q-2) \cdot z_\alpha + (q-1) \cdot [0]'$$

Solving this equation for idempotents gives an idempotent

$$e_\alpha := q^{-1} \cdot z_\alpha + q^{-1} \cdot [0]' = q^{-1} \cdot \sum_{a \in \mathbb{F}_q} \alpha(a) \cdot [a]',$$

and its complement  $[0]' - e_\alpha$ , for every choice of  $\alpha$ . It follows elementarily from the independence of characters that for two distinct choices  $\alpha \neq \beta$ ,

$$e_\alpha e_\beta = 0.$$

Take

$$\tilde{\Omega}_\alpha = \text{Im}(e_\alpha),$$

which then has dimension equal to the trace of  $e_\alpha$ , which is  $q^{-1}$ .

Similarly as in the proof of Proposition 5.2, we may also consider idempotents

$$\bar{e}_0 = -\frac{(<)}{q^{-1}}, \quad \bar{e}_\infty = -\frac{(>)}{q^{-1}}$$

of trace 0, which, again, commute and satisfy

$$\text{tr}(\bar{e}_0 \bar{e}_\infty) = 1.$$

Thus we have disjoint idempotents

$$e_0 := \bar{e}_0 - \bar{e}_0 \bar{e}_\infty, \quad e_\infty := \bar{e}_\infty - \bar{e}_0 \bar{e}_\infty, \quad \text{and } \bar{e}_0 \bar{e}_\infty$$

which therefore give summands

$$\Omega_{1,0}^+ \oplus \Omega_{1,\infty}^+ \oplus 1 \subseteq [P],$$

of dimensions  $-1$ ,  $-1$ , and  $1$ , respectively. Since these summands and the  $\tilde{\Omega}_\alpha$  are all disjoint, and (5.8) implies that

$$\dim(\text{End}_{\mathcal{U}_{q,k}}([P])) = q + 2,$$

they must be simple, giving the stated simple decomposition of  $[P]$ , and thus, also, the simple decomposition of  $\Omega$ .

The number of orbits of  $V \times V$  with respect to the  $\Gamma_u$ -action is

$$q^3 - q + 1$$

(the orbits being

$$(0,0), (a,0), (0,b), (a,b), \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}, \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$$

with  $a, b \in \mathbb{F}_q^\times$ ,  $x \in \mathbb{F}_q$ , giving

$$1 + 2(q-1) + 2(q-1)^2 + (q-1)^2 q = q^3 - q + 1$$

orbits). Hence,

$$\begin{aligned} \dim(\text{End}_{\mathcal{U}_{q,k}}([V])) &= q^3 - q + 1 = \\ &= (q-1)^2(q+2) + 2q - 1 = \\ &= \dim(\text{End}_{\mathcal{U}_{q,k}}(\Omega)) + q^2 - (q-1)^2, \end{aligned}$$

and thus, by counting dimensions, we must have

$$1 \cong (0).$$

(Again, this can also be reasoned directly since the elements

$$(0,0), (a,0), (0,b), (a,0)(0,b), \quad \text{for } a, b \in \mathbb{F}_q^\times$$

form a  $q$ -by- $q$  matrix algebra).

□

**5.5. The Simple Objects of  $\mathcal{U}_{q,k}$ .** Let us now assume that  $k$  is an algebraically closed field of characteristic not  $p$ . Let us also write  $Q_n$  for the object of  $\mathcal{U}_{q,k}$  corresponding to the  $\Gamma_u$ -orbit in  $P^n$  generated by elements

$$(1 \cdot (r_1), \dots, 1 \cdot (r_n)) \in (\mathbb{F}_q[\mathbb{R}])^n$$

for  $r_1 > \dots > r_n$ .

There are  $(q+1)^n$  disjoint idempotents in  $\text{End}_{\mathcal{U}_{q,k}}(Q_n)$  corresponding to a linear combination of orbits of

$$(1 \cdot (r_1), \dots, 1 \cdot (r_n), 1 \cdot (s_1), \dots, 1 \cdot (s_n))$$

such that

$$r_1, s_1 > r_2, s_2 > \dots > r_n, s_n,$$

which are described as follows:

The idempotents  $e_{(\alpha_1, \dots, \alpha_n)}$  are indexed by choosing

$$(5.13) \quad \alpha_i \in \{0, \infty\} \cup (\text{Hom}(\mathbb{F}_q, k^\times) \setminus \{1\})$$

(where, as above, 1 denotes the trivial additive character). By extending linearly direct sums of the matrices (5.8), we may write direct sums of idempotents. We then put

$$(5.14) \quad e_{(\alpha_1, \dots, \alpha_n)} = e_{\alpha_1} \oplus \dots \oplus e_{\alpha_n}$$

where the columns of the matrices correspond to

$$r_1, s_1, \dots, r_n, s_n.$$

The simple central idempotents of the group algebra  $k[U_n]$  (where  $U_n$  denotes the unipotent subgroup of the Borel subgroup  $B_n \subseteq GL_n(\mathbb{F}_q)$ ) commute with (5.14). Therefore, by (3.15), these idempotents correspond to all the non-isomorphic simple summands of  $\Theta_n^0$ .

They are non-trivial because they have non-zero trace. The trace of an element  $\sum a_g \cdot g$  of the group algebra is

$$a_1 \cdot \text{tr}(1).$$

In the present setting,

$$\text{tr}(1) = \text{tr}(e_{(\alpha_1, \dots, \alpha_n)}) = (-1)^{|\{i | \alpha_i \in \{0, \infty\}\}|} \cdot q^{-|\{j | \alpha_j \notin \{0, \infty\}\}|}.$$

Using the results of I. M. Isaacs [8], we know that the dimensions of the simple representation of  $U_n$  are powers of  $q$ . Thus, we have proved the following

**Theorem 5.6.** *Let  $q = p^m$  and let  $k$  be an algebraically closed field of characteristic not equal to  $p$ . The non-isomorphic simple objects of  $\mathcal{U}_{q,k}$  are obtained by taking a simple  $U_n$ -summand of  $e_{(\alpha_1, \dots, \alpha_n)}$  with  $\alpha_i$  as in (5.13). The dimensions of these objects are all of the form  $\pm q^{-\ell}$  for  $\ell \in \mathbb{N}_0$ .*

□

**5.7. Comparison of the simple objects of  $\mathcal{B}_{q,k}$  and  $\mathcal{U}_{q,k}$ .** Note that, despite the similarity of the statements, Proposition 5.2 assumes the target field  $k$  has  $(q-1)$ th roots of unity, while Proposition 5.4 assumes it has  $p$ th roots of unity. If we assume that the field  $k$  has characteristic not dividing  $q(q-1)$ , then there is a canonical tensor functor

$$\mathcal{B}_{q,k} \rightarrow \mathcal{U}_{q,k}$$

(given by morphisms of T-algebras). If, in addition, both  $(q-1)$ th and  $p$ th roots of unity are present in  $k$ , we can compare the summands in Proposition 5.2 and Proposition 5.4.

Since the elements

$$(0,0), (a,0), (0,b), (a,0)(0,b), \text{ for } a, b \in \mathbb{F}_q^\times$$

form a matrix algebra  $M_q(k)$ , corresponding to the  $q$  copies of 1 in  $[V]$ , some of the summands in Proposition 5.2 must have a summand of 1 in  $\mathcal{U}_{q,k}$ . We have that

$$(0,1)\iota^+ \neq 0$$

$$(0,1)\iota^- = 0,$$

so we can conclude that each of the summands  $\Omega_\psi^+$ , for a multiplicative character

$$\psi : \mathbb{F}_q^\times \rightarrow k^\times,$$

has a summand of 1 in  $\mathcal{U}_{q,k}$ . In fact, since  $[V]$  has a commutative algebra structure as in [2], Section 8, we have

$$\Omega_\psi^+ \cong 1 \oplus \Omega_{1,0}^+ \oplus \Omega_{1,\infty}^+$$

for every multiplicative character

$$\psi : \mathbb{F}_q^\times \rightarrow k^\times.$$

The remaining summands described in Proposition 5.4 give a decomposition

$$\Omega_\psi^- \cong \bigoplus_{\alpha \neq 1: \mathbb{F}_q \rightarrow k^\times} \tilde{\Omega}_\alpha,$$

where the direct sum is over non-trivial additive characters

$$\alpha : \mathbb{F}_q \rightarrow k^\times.$$

In particular, for all multiplicative characters  $\psi, \phi : \mathbb{F}_q^\times \rightarrow k^\times$ , as objects of  $\mathcal{U}_{q,k}$ ,

$$\Omega_\psi^- \cong \Omega_\phi^-.$$

**5.8. Comparison Functors and the Universality of the Delannoy Category.** One has a commutative diagram of tensor functors of the following form, where all categories are over a field  $k$  of characteristic 0:

$$\begin{array}{ccccccc} \underline{Rep}(S_{-1}) & \xrightarrow{\alpha} & \mathcal{D} & & & & \\ & & \downarrow & & & & \\ \underline{Rep}(S_{q-1}) & \xrightarrow{\epsilon} & \underline{Rep}(GL_{-1}(\mathbb{F}_q)) & \xrightarrow{\gamma} & \mathcal{B}_{q,k} & \xrightarrow{\delta} & \mathcal{U}_{q,k}. \end{array}$$

The functors  $\alpha$  and  $\epsilon$  send the basic objects of their sources to the basic objects of their targets, and their existence follows from Proposition 8.3 of [2]. The functors  $\gamma, \delta$  follow directly from the construction of the T-algebra, and also from the oligomorphic group method. The existence of the functor  $\beta$  follows from the universality of the Delannoy category of [7], which we state and prove in this section.

Suppose  $\mathcal{C}$  is a pseudo-abelian category with ACU tensor product and strong duality with an object  $X$  such that there are multiplication and unit morphisms

$$(5.15) \quad \mu : X \otimes X \rightarrow X, \quad \eta : 1 \rightarrow X$$

in  $\mathcal{C}$  giving  $X$  the structure of an associative, commutative, unital (ACU) algebra. Recall that, following Section 8.1 of [2], we may construct a morphism  $Tr : X \rightarrow 1$  as the composition

$$X \xrightarrow{Id_X \otimes coev_X} X \otimes X \otimes X^* \xrightarrow{\mu \otimes Id_{X^*}} X \otimes X^* \cong X^* \otimes X \xrightarrow{ev_X} 1$$

**Theorem 5.9.** *For a pseudo-abelian category  $\mathcal{C}$  over a commutative ring  $R$  with ACU tensor product and strong duality, tensor functors*

$$(5.16) \quad \mathcal{D} \rightarrow \mathcal{C}$$

*are in bijective correspondence with the following data:*

- (1) An object  $X \in \text{Obj}(\mathcal{C})$  which is an ACU algebra via morphisms (5.15) such that

$$(5.17) \quad X \otimes X \xrightarrow{\mu} X \xrightarrow{Tr} 1$$

makes  $X$  its own dual.

- (2) A splitting

$$(5.18) \quad X = X_+ \oplus 1 \oplus X_-$$

where

- (a)  $\dim(X_+) = \dim(X_-) = -1$
- (b)  $X_+ \oplus 1, X_- \oplus 1$  are subalgebras of  $X$  with ideals  $X_+, X_-$ , respectively.
- (c) The self-duality of  $X$  switches  $X_+$  and  $X_-$  in the composition (5.18)

If 2 is not invertible in  $R$ , we also assume

- (d) Let  $\pi_+ : X \rightarrow X_+ \oplus 1$  be the morphism of  $\mathcal{C}$  which is identity on  $X_+ \oplus 1$  and 0 on  $X_-$ . Then the composition

$$(5.19) \quad \begin{array}{c} X \longrightarrow X \otimes X \otimes X^\vee \xrightarrow{\mu} X \otimes X^\vee \\ \downarrow \pi_+ \otimes Id_{X^\vee} \\ (X_+ \oplus 1) \otimes X^\vee \\ \downarrow \subseteq \\ X \otimes X^\vee \\ \downarrow \\ 1 \end{array}$$

is 0 (see Figure 1 below).

**Comment:** Let also  $\pi_- : X \rightarrow X_- \oplus 1$  denote the morphism which is identity on  $X_- \oplus 1$  and 0 on  $X_+$ , and let  $\pi_0 : X \rightarrow X$  denote the morphism which is identity on 1 and 0 on  $X_+ \oplus X_-$ . One can check that the assumption about  $X_+$  and  $X_-$  being ideals implies that  $\pi_+$  is dual to the inclusion  $X_- \oplus 1 \subseteq X$  and similarly with  $+$  and  $-$  reversed.

This actually implies that the composition (5.19) is equal to its analogue where we replace  $\pi_+$  by  $\pi_-$ . We also know that if we replace  $\pi_+$

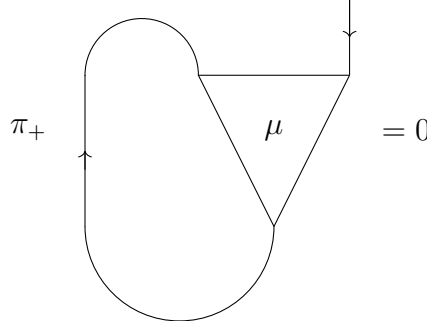


FIGURE 1. Condition (d)

(or  $\pi_-$ ) by  $\pi_1$ , we get the same as when we replace it by  $Id_X$ , i.e. the composition (5.19) is  $Tr$ . This implies that (5.19) multiplied by 2 is 0.

P. Deligne [4] found a counterexample to Theorem 5.9 if we do not make the additional assumption (d) in characteristic 2. Let  $\mathcal{C}$  be the category of finite dimensional vector spaces over a field  $k$  of characteristic 2 with the usual tensor product. Take  $X = k^3$  with the product ring structure. Then  $X_+ \oplus 1$ , resp.  $X_- \oplus 1$ , is the subring given by the equation  $x_1 = x_2$ , resp.  $x_2 = x_3$ . As ideals,  $X_+$ , resp.  $X_-$ , is given by the equation  $x_3 = 0$ , resp.  $x_1 = 0$ . Then the composition (5.19) is not 0, so there exists no tensor functor

$$\mathcal{D} \rightarrow \mathcal{C}$$

sending the basic object of  $\mathcal{D}$  to  $X$ , while  $\mathcal{C}$  satisfies every assumption of Theorem 5.9 but (d).

*Proof of Theorem 5.9.* Recalling the description of the T-algebra corresponding to the Delannoy category described in Subsection 2.4, we must first construct elements of  $Hom_{\mathcal{C}}(X^{\otimes S}, X^{\otimes T})$  that correspond to the ordered partitions

$$(5.20) \quad (U_1, \dots, U_\ell)$$

of SIIT generating (2.12). We will begin by constructing an idempotent of  $End_{\mathcal{C}}(X^{\otimes n})$  corresponding to the ordered partition

$$(5.21) \quad (\{1_1, 1_2\}, \{2_1, 2_2\}, \dots, \{n_1, n_2\})$$

of  $\{1, \dots, n\} \amalg \{1, \dots, n\}$  (where, again, the subscripts indicate which disjoint summand we are considering the element to be in).

First, we identify the ordered partition

$$(5.22) \quad (\{1_1\}, \{1_2\}) \in \mathcal{D}_{\{1\}, \{1\}}$$

of  $\{1\} \amalg \{1\}$  with  $-\pi_+ \in \text{End}_{\mathcal{C}}(X)$ , and, similarly, we identify

$$(5.23) \quad (\{1_2\}, \{1_1\})$$

with  $-\pi_-$ .

Now the assumptions on  $X$  in particular imply those required in the universality property of  $\underline{\text{Rep}}(S_{-1})$  (see Section 8.2, [2]), so we have a tensor functor

$$\underline{\text{Rep}}(S_{-1}) \rightarrow \mathcal{C}$$

(sending the basic object of  $\text{Rep}(S_{-1})$  to  $X$ ). In particular, we can construct elements

$$\mu \in \text{Hom}_{\mathcal{C}}(X^{\otimes 2}, X)$$

$$\nu \in \text{Hom}_{\mathcal{C}}(X, X^{\otimes 2})$$

$$\kappa \in \text{Hom}_{\mathcal{C}}(X^{\otimes 2}, X^{\otimes 2})$$

corresponding to (unordered) partitions

$$\{\{1_1, 2_1, 1_2\}\} \in \underline{\text{Rep}}(S_{-1})_{\{1,2\}, \{1\}}$$

$$\{\{1_1, 1_2, 2_2\}\} \in \underline{\text{Rep}}(S_{-1})_{\{1\}, \{1,2\}}$$

$$\{\{1_1, 2_1, 1_2, 2_2\}\} \in \underline{\text{Rep}}(S_{-1})_{\{1,2\}, \{1,2\}},$$

respectively (recalling the description (2.4)). (Note that  $\mu, \nu$  correspond to the multiplication map and its dual, respectively.) We may consider the product

$$(5.24) \quad \underbrace{\nu \otimes \kappa \otimes \cdots \otimes \kappa \otimes \mu}_n \in \text{End}_{\mathcal{C}}(X^{\otimes (2n-1)}),$$

and trace it with

$$(5.25) \quad (-1)^{n-1} \cdot \underbrace{(\pi_+ \otimes \cdots \otimes \pi_+)}_{n-1} \in \text{End}_{\mathcal{C}}(X^{\otimes (n-1)})$$

where the source of each  $\pi_+$  is plugged in to the target of a tensor factor of (5.24) and its target is plugged into the source of the next term in (5.25). This trace we take to be the idempotent of  $\text{End}_{\mathcal{C}}(X^{\otimes n})$  corresponding to (5.21). Given (5.21), we may compose with multiplication, the dual of multiplication, the unit, and the augmentation to get morphisms in  $\mathcal{C}$  corresponding to all ordered partitions (5.20).

To verify the arising functor definition, write

$$A = Id_X - \pi_+ + \pi_- = -\pi_-$$



$$B = Id_X - \pi_- + \pi_1 = -\pi_+$$

(the elements corresponding to (5.23) and (5.22), respectively). We then have

$$A^2 = -A, B^2 = -B, A \circ B = -A - B - Id_X = \pi_1.$$

Note that the categorical traces of  $A$  and  $B$  are 0.

By taking partial traces of morphisms in  $\underline{Rep}(S_{-1})$  (see Subsection 2.3) with product of  $A$  or  $B$  repeatedly, we can order the equivalence classes defining a generating morphism of the T-algebra of  $\underline{Rep}(S_{-1})$ . Multiple sequences of partial traces with  $A$  resp.  $B$  can imply the same ordering, and we need to prove that the answers are indeed equal using our axioms. Similarly, some ordering definitions can be inconsistent, and we need to prove that the respective partial traces are 0.

This can be reduced to two specific statements which we will now describe. Denote for  $f, g : X \rightarrow X$  the composition

$$X \xrightarrow{\nu} X \otimes X \xrightarrow{f \otimes g} X \otimes X \xrightarrow{\mu} X$$

by  $f \odot g$ . The we need to show

$$(5.26) \quad A \odot A = A$$

and

$$(5.27) \quad A \odot B = 0 \in \text{End}_{\mathcal{C}}(X).$$

In fact, it turns out that (5.27) implies (5.26).

To this end, note that we may express  $\pi_1$  as the composition

$$X \xrightarrow{Tr} 1 \xrightarrow{\eta} X$$

(where, again,  $\eta : 1 \rightarrow X$  denotes the unit of the algebra structure of  $X$ , which is dual to  $Tr$ ). This, since  $X$  has dimension  $-1$ ,

$$A \odot \pi_1 = -A.$$

On the other hand, since  $A$  has trace 0,

$$A \odot Id_X = 0.$$

Now, to prove (5.27), it suffices to show

$$(A \odot B) \circ A = 0,$$

since, by symmetry (and the commutativity of the algebra structure on  $X$ ), then  $(A \odot B) \circ B = 0$ , and  $(A \odot B) \circ \pi_1$  automatically. We may express  $(A \odot B) \circ A$  as a partial trace of the composition

$$X \otimes X \xrightarrow{A \otimes A} X \otimes X \xrightarrow{\mu} X \xrightarrow{A} X \xrightarrow{\nu} X \otimes X$$

which is the same as the partial trace of

$$(5.28) \quad X \otimes X \xrightarrow{A \otimes A} X \otimes X \xrightarrow{\mu} X \xrightarrow{\nu} X \otimes X.$$

We may replace  $\nu \circ \mu$  by the composition

$$X \otimes X \xrightarrow{Id_X \otimes \nu} X \otimes X \otimes X \xrightarrow{\mu \otimes Id_X} X \otimes X,$$

the partial trace of which, after composing with  $A \otimes A$  as in (5.28), is

$$X = 1 \otimes X \xrightarrow{\Psi \otimes Id_X} X \otimes X \xrightarrow{\mu} X,$$

where  $\Psi \in Hom_{\mathcal{C}}(1, X)$  is the partial trace of

$$X \xrightarrow{\nu} X \otimes X \xrightarrow{Id_X \otimes A} X \otimes X,$$

(in the second coordinate of the target) which is 0.

□

## REFERENCES

- [1] P. Cameron. Oligomorphic permutation groups. *London Mathematical Society Lecture Note Series*, vol. 152, Cambridge University Press, Cambridge, 1990.
- [2] P. Deligne. La catégorie des représentations du groupe symétrique  $S_t$ , lorsque  $t$  n'est pas un entier naturel. *Algebraic groups and homogeneous spaces*, 209-273, Tata Inst. Fund. Res. Stud. Math., 19, Tata Inst. Fund. Res., Mumbai, 2007.
- [3] P. Deligne, J. S. Milne. Tannakian categories, in *Hodge Cycles, Motives and Shimura Varieties*, Lecture Notes in Math. 900, Springer Verlag, 101-228, 1982.
- [4] P. Deligne: Letters to S. Kriz, 2022-2023.
- [5] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik: *Tensor categories*. Math. Surveys Monogr., 205 American Mathematical Society, Providence, RI, 2015, xvi+343 pp.

- [6] N. Harman, A. Snowden. Oligomorphic groups and tensor categories. arXiv:2204.04526, 2022.
- [7] N. Harman, A. Snowden, N. Snyder. The Delannoy Category. arXiv:2211.15392, 2023.
- [8] I. M. Isaacs. Characters of Groups Associated with Finite Algebras. *J. Algebra*. 177, 708-730, 1995.
- [9] F. Knop. A construction of semisimple tensor categories. *C. R. Math. Acad. Sci. Paris C*. 343, 2006.
- [10] F. Knop. Tensor Envelopes of Regular Categories. *Adv. Math.* 214, 2007.
- [11] S. Kriz. Arbitrarily High Growth in Additive Rigid Categories with ACU Tensor Product, 2023. Available at <https://krizsophie.github.io/ACUCategory23085.pdf>
- [12] A. Snowden. On the Representation Theory of the Symmetry Group of the Cantor Set, 2023, preprint.
- [13] A. Snowden. Some Fast-Growing Tensor Categories. arXiv:2305.18230, 2023.