# ON THE STRUCTURE OF SIMPLE GENERIC $FI$-MODULES IN POSITIVE CHARACTERISTIC

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Abstract. We classify simple generic $FI$-modules in positive characteristic. We also give, in any characteristic $p > 0$, examples of simple generic $FI$-modules whose underlying representations are reducible in all sufficiently high degrees.

## 1. Introduction

In this paper, an *$FI$-module* (also a *$KFI$-module*) is a functor from the category $FI$ of finite sets and injections into $K$-modules for a field $K$. $FI$-modules were introduced by Church, Ellenberg, and Farb in [1], and have numerous applications in topology, algebra, and number theory. Stable phenomena of the representation theory of symmetric groups are encoded by the category of generic $FI$-modules, defined in a way to disregard elements which go to 0 in the representations of $\Sigma_n$ for $n >> 0$. This is analogous to the construction of the category of quasi-coherent sheaves from the category of graded modules over the projective coordinate ring of a projective scheme [14]. This analogy was in fact used by Sam and Snowden [13] to gain a good understanding of the category of generic $FI$-modules in characteristic 0. In particular, they identified all the simple objects of that category.

The case of characteristic $p > 0$ is more complicated and a generalization of the work of [13] was not known. The objective of this paper is to characterize all simple generic $FI$-modules in characteristic $p > 0$, and also to answer a question of R.Nagpal [11] asking if the $\Sigma_n$-representation terms of such a module are irreducible for infinitely many $n$. We find counterexamples for all primes $p$.

To discuss our results more precisely, we need some notation. Let $[n] = \{1, \dots, n\}$. For an $FI$-module $X$, we will sometimes write $X(N)$ instead of $X([N])$. For a given $N$, we identify a $K\Sigma_N$-module with the $KFI$-module equal to it in degree $N$ and 0 in other degrees. An $FI$-module $X$ is called *torsion* if each of the elements of every $X(n)$

goes to $0 \in X(m)$ for some $m >> 0$. Torsion $(K)FI$-modules form a Serre subcategory of the category of $(K)FI$-modules, and taking the Serre quotient by them gives the category of *generic* $(K)FI$*-modules* (see [4] for the details of this construction).

One might ask what the simple objects of the category of generic $(K)FI$-modules are. This is known when $K$ is a field of characteristic 0. In [13], Sam and Snowden proved, using the Schur-Weyl correspondence, that when $char(K) = 0$, simple generic $KFI$-modules are exactly the "Spechtral" $FI$-modules for Young diagrams $\lambda$, which consist of Specht modules (i.e., the irreducible $\Sigma_n$-representation in characteristic 0) corresponding to the Young diagrams given by adding a new first row to $\lambda$. However, in characteristic $p > 0$, Spechtral modules are not necessarily generically simple, and simple generic $KFI$-modules are not necessarily Spechtral.

In this paper, we classify simple generic $KFI$-modules in characteristic $p > 0$ into an explicitly constructed family, indexed by $p$-regular Young diagrams. We also use our construction to give examples of simple generic $KFI$-modules $X$ in all characteristics $p > 0$ such that $X(N)$ is a reducible $K\Sigma_N$-module for all but finitely many $N$.

We define two functors

$$\Psi' : FI\text{-Mod} \to FI\text{-Mod}$$

$$\Phi' : FI\text{-Mod} \to FI\text{-Mod}$$

by

$$\Psi'(M_\bullet) : [N] \mapsto \mathrm{Hom}_{FI\text{-Mod}}(K\mathrm{Map}_{FI}([\bullet],[N])^\vee, M_\bullet) \tag{1}$$

$$\Phi'(M_\bullet) : [N] \mapsto K\mathrm{Map}_{FI}([N],[\bullet])^\vee \otimes_{FI\text{-Mod}} M_\bullet \tag{2}$$

for an $FI$-module $M_\bullet$. By definition, $\Phi'$ is left adjoint to $\Psi'$. It is also easy to see that applying $\Phi'$ to a torsion $FI$-module gives 0 and that applying $\Phi'$ to any $FI$-module gives a torsion $FI$-module.

Let $KGFI$-Mod denote the category of generic finitely generated $KFI$-modules and let $KTFI$-Mod denote the full subcategory of $KFI$-modules on finitely generated torsion $KFI$-modules. Then $\Phi'$, $\Psi'$ induce a pair of functors

$$\Phi : KGFI\text{-Mod} \to KTFI\text{-Mod}$$

$$\Psi : KTFI\text{-Mod} \to KGFI\text{-Mod}$$

where $\Phi$ is left adjoint to $\Psi$.

In characteristic 0, by Schur-Weyl correspondence, the functors $\Psi$, $\Phi$ coincide with the functors of the same names in [13], where they are proved to be inverse equivalences of categories. This is false in characteristic $p > 0$. An easy argument is given in the beginning of Section 2.

To state our main results, we need some additional notation. A *Young diagram* is a $k$-tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ where $\lambda_1 \geq \dots \geq \lambda_k$ are positive integers (this can be visualized as a diagram of boxes with $k$ rows and $\lambda_i$ boxes in the $i$-th row). For a Young diagram $\lambda$, let $|\lambda|$ denote the number of its boxes (i.e. $|\lambda| = \lambda_1 + \dots + \lambda_k$). Let $S_\lambda$ denote the Specht module corresponding to a Young diagram $\lambda$. As a general reference for Specht modules, we recommend [6]. We denote by $M_\lambda$ the *Spechtral $FI$-module* consisting of the Specht modules of the Young diagrams obtained by adding a row to the top of $\lambda$ at each degree $\geq |\lambda| + \lambda_1$. A detailed construction of Spechtral modules independent of characteristic is given in [10].

A Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$ is called $p$-regular if at most $p-1$ of the numbers $\lambda_1, \dots, \lambda_k$ are equal to any given number $i$. Recall that the set of Young diagrams with $\ell$ boxes has a natural ordering called *dominance* given by saying, for two partitions $\mu = (\mu_1, \dots, \mu_n)$, $\nu = (\nu_1, \dots, \nu_m)$ of $\ell$, $\mu \trianglerighteq \nu$ when

$$\mu_1 + \dots + \mu_k \geq \nu_1 + \dots + \nu_k$$

for all $k \geq 1$. In this note, we will also call a Young diagram $\mu$ *strictly dominant* over $\nu$ (write $\mu \triangleright \nu$) if we have $\mu \trianglerighteq \nu$ and $\mu \neq \nu$.

For every $p$-regular Young diagram $\lambda$, $S_\lambda$ has a unique quotient $D_\lambda$ which is a simple $K\Sigma_{|\lambda|}$-module. These form a complete set of representatives of isomorphism classes of simple $K\Sigma_{|\lambda|}$-modules. Moreover, for a $p$-regular Young diagram $\lambda$, all the other constituent factors of $S_\lambda$ are $D_\mu$ with $\mu \triangleright \lambda$ ([6], Section 12).

Our main general result is the following

**Theorem 1.** *For every $p$-regular Young diagram $\lambda$, there exists a non-zero morphism of $KFI$-modules*

$$\iota_\lambda : M_\lambda \to \Psi(D_\lambda)$$

*such that $Im(\iota_\lambda)$ is a simple object in the category $KGFI$-Mod of generic finitely generated $KFI$-modules. Additionally, every simple generic finitely generated $KFI$-module is isomorphic to $Im(\iota_\lambda)$ for a unique $p$-regular Young diagram $\lambda$.*

Interesting examples can be constructed using this Theorem. R. Nagpal asked whether every simple generic $FI$-modules consists of simple $K\Sigma_n$-modules in infinitely many degrees $n$. This turns out to be false.

**Theorem 2.** *Suppose $K$ is a field of characteristic $p > 0$.*

*(1) If $p = 2$, then for every $N >> 0$, the $\Sigma_N$-representation*

$$(Im(\iota_{(3,1)}))(N)$$

*is reducible.*

*(2) If $p > 2$, then for every $N >> 0$, the $\Sigma_N$-representation*

$$(Im(\iota_{(p,2)}))(N)$$

*is reducible.*

We will prove Theorem 1 in Section 2. The proof of Theorem 2 requires different approaches depending on whether $p = 2$ or $p > 2$. The case of $p = 2$ is treated in Section 3, and the case of $p > 2$ is treated in Section 4. In both cases of the proof of Theorem 2, we will see several interesting examples of the behavior of the functor $\Psi$.

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## 2. Proof of Theorem 1

We begin with a closer discussion of the functors $\Phi$, $\Psi$. First, let us give an easy argument why in characteristic $p > 0$, they are not inverse equivalences of categories.

Consider the principal injective torsion $KFI$-module $K\Sigma_0$ (meaning $K = K\Sigma_0$ in degree $N = 0$ and 0 elsewhere). If $\Psi$ were an equivalence of categories, $\Psi(K\Sigma_0)$ (which is the $FI$-module $M_\emptyset$, i.e., equal to $K$ in every degree $N \geq 0$ where the structure maps are isomorphisms) would have to be an injective generic $KFI$-module. However, this is proved to be false in [5] (Section 3.2).

The authors of [5] consider the $FI$-module $A$ where

$$A_n = \mathbb{F}_p(\Sigma_n/(\Sigma_p \times \Sigma_{n-p})).$$

Then we have a diagram

$$
\begin{tikzcd}
A \arrow[r, "\subseteq"] \arrow[d] & \mathbb{F}_p\mathrm{Map}_{FI}([p],[\bullet]) \\
M_\emptyset &
\end{tikzcd}
\tag{3}
$$

where the top row inclusion is by corestriction and the vertical arrow is by augmentation (since $A_n$ is a permutation representation). They remark that the inclusion of $M_\emptyset$ into the pushout of (3) does not generically split, since any map of $\mathbb{F}_p\Sigma_N$-modules

$$\mathbb{F}_p\mathrm{Map}_{FI}([p],[N]) \to \mathbb{Z}/p$$

is 0 on $A$. In fact, it is proved in [12], Theorem 4.20 that the category $KGFI$-Mod does not have any non-trivial injectives.

Now let us describe $\Phi$, $\Psi$ more explicitly. (We shall drop the notation $\Phi'$, $\Psi'$, since from the point of view of an explicit description, they are the same.) By (2), we have, for an $FI$-module $X$,

$$\Phi(X)(m) = \mathrm{Hom}_{FI\text{-Mod}}(X(n), K\mathrm{Map}_{FI}([m],[n]))^\vee \tag{4}$$

(where $n$ ranges over all natural numbers $\geq m$). Now note that we also can consider $K\mathrm{Map}_{FI}([m],[n])$ (as an $K\Sigma_n$-module) as the induction from $K(\Sigma_{n-m}\times\Sigma_m)$ into $K\Sigma_n$, i.e.,

$$\begin{gathered}K\mathrm{Map}_{FI}([m],[n]) = \mathrm{Hom}_{\Sigma_m}(K\Sigma_n/\Sigma_{n-m}, K\Sigma_m) = (K\Sigma_n/\Sigma_{n-m})^\vee \\ \cong K\Sigma_n/\Sigma_{n-m}.\end{gathered}$$

Hence,

$$\begin{gathered}\mathrm{Hom}_{\Sigma_n}(X(n), K\mathrm{Map}_{FI}([m],[n])) = \\ \mathrm{Hom}_{\Sigma_n}(X(n), \mathrm{Hom}_{\Sigma_n}(K\Sigma_n/\Sigma_{n-m}, K)) = \\ (X(n)\otimes_{\Sigma_n} K\Sigma_n/\Sigma_{n-m})^\vee = ((X(n))_{\Sigma_{n-m}})^\vee,\end{gathered} \tag{5}$$

where the subscript in the last term denotes cofixed points. By (2), an element of $(\Phi(X)(m))^\vee$ corresponds to a system of elements

$$f_n \in \mathrm{Hom}_{\Sigma_n}(X(n), K\mathrm{Map}_{FI}([m],[n]))$$

such that for $n, n-k \geq m+\lambda_1$, the diagram

$$
\begin{tikzcd}
X(n) \arrow[r, "f_n"] & K\mathrm{Map}_{FI}([m],[n]) \\
X(n-k) \arrow[u] \arrow[r, "f_{n-k}"'] \arrow[ur, "h_{n,n-k}"] & K\mathrm{Map}_{FI}([m],[n-k]) \arrow[u]
\end{tikzcd}
$$

commutes. We denote the two equal compositions by

$$h_{n,n-k} \in \mathrm{Hom}_{\Sigma_{n-k}}(X(n-k), K\mathrm{Map}_{FI}([m],[n])).$$

Then, by (5), $(\Phi(X)(m))^\vee$ is equivalent to the limit of the diagram

$$\begin{tikzcd}
((X(n))_{\Sigma_{n-m}})^\vee \arrow[dr] & & ((X(n-k))_{\Sigma_{n-k-m}})^\vee \arrow[dl] \\
& ((Ind_{\Sigma_n}^{\Sigma_{n-k}\times\Sigma_k} X(n-k))_{\Sigma_{n-m}})^\vee &
\end{tikzcd}$$

for $n, n-k \geq m$. That is dual to the diagram

$$\begin{tikzcd}
(X(n))_{\Sigma_{n-m}} & & (X(n-k))_{\Sigma_{n-k-m}} \\
& (Ind_{\Sigma_n}^{\Sigma_{n-k}\times\Sigma_k} X(n-k))_{\Sigma_{n-m}} \arrow[ul, "\phi_+"] \arrow[ur, "\phi_-"'] &
\end{tikzcd} \tag{6}$$

where $\phi_+$ is given by taking $\Sigma_{n-m}$-cofixed points of the natural

$$Ind_{\Sigma_n}^{\Sigma_{n-k}\times\Sigma_k} X(n-k) \to X(n).$$

The map $\phi_-$ is defined to be the composition

$$\begin{tikzcd}
(Ind_{\Sigma_n}^{\Sigma_{n-k}\times\Sigma_k}(X(n-k)))_{\Sigma_{n-m}} \arrow[d] \\
(Ind_{\Sigma_n}^{\Sigma_{n-k}\times\Sigma_k}(X(n-k)))_{\Sigma_k\times\Sigma_{n-k-m}} \arrow[d] \\
(X(n-k))_{\Sigma_{n-k-m}}
\end{tikzcd}$$

where the top map is taking corestriction (i.e. summing over coset representatives of $\Sigma_{n-m}/\Sigma_k \times \Sigma_{n-m-k}$), and the lower map is the counit of adjunction of the induction as a right adjoint to cofixed points, followed by $\Sigma_{n-k-m}$-cofixed points.

Dually, we have, by the definition (1),

$$\Psi(X)(N) = \mathrm{Hom}_{FI\text{-}\mathrm{Mod}}(K\mathrm{Map}_{FI}([n],[N])^\vee, X(n)).$$

Again, this corresponds to systems of maps $f^\ell$ such that the diagram

$$\begin{tikzcd}
(K\Sigma_N/\Sigma_{N-\ell+k})^\vee \arrow[r, "f^{\ell-k}"] \arrow[d] \arrow[dr, "h^{\ell-k,\ell}"] & X(\ell-k) \arrow[d] \\
(K\Sigma_N/\Sigma_{N-\ell})^\vee \arrow[r, "f^\ell"'] & X(\ell)
\end{tikzcd}$$

commutes where we denote the two equal compositions by $h^{\ell-k,\ell}$. Since each $K\Sigma_N/\Sigma_{N-\ell}$ is self-dual, we have

$$\mathrm{Hom}_{\Sigma_\ell}((K\Sigma_N/\Sigma_{N-\ell})^\vee, X(\ell)) = Ind_{\Sigma_N}^{\Sigma_\ell\times\Sigma_{N-\ell}}(X(\ell)).$$

Hence, $\Psi(X)(N)$ is the limit of the diagram

$$\begin{tikzcd}[column sep=small] Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_{N-\ell+k}}(X(\ell-k)) \arrow[dr, "\psi^+"] & & Ind_{\Sigma_N}^{\Sigma_\ell\times\Sigma_{N-\ell}}(X(\ell)) \arrow[dl, "\psi^-"'] \\ & Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_{N-\ell+k}}((X(\ell))^{\Sigma_k}) & \end{tikzcd} \tag{7}$$

where the map $\psi^+$ is given by applying $Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_{N-\ell+k}}$ to the natural map

$$X(\ell-k)\to(X(\ell))^{\Sigma_k}.$$

The map $\psi^-$ is defined as the composition

$$\begin{tikzcd} Ind_{\Sigma_N}^{\Sigma_\ell\times\Sigma_{N-\ell}}(X(\ell)) \arrow[d] \\ Ind_{\Sigma_N}^{\Sigma_\ell\times\Sigma_{N-\ell}}(Ind_{\Sigma_\ell}^{\Sigma_{\ell-k}\times\Sigma_k}(X(\ell)^{\Sigma_{\ell-k}})) \arrow[d] \\ Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_{N-\ell+k}}(X(\ell)^{\Sigma_{\ell-k}}) \end{tikzcd}$$

where the top map is given by induction applied to the unit of adjunction of fixed points as a right adjoint to induction, and the lower map, noting that

$$Ind_{\Sigma_N}^{\Sigma_\ell\times\Sigma_{N-\ell}}\circ Ind_{\Sigma_\ell}^{\Sigma_{\ell-k}\times\Sigma_k} = Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_k\times\Sigma_{N-\ell}},$$

is given by corestriction (i.e. summing over coset representatives of

$$(\Sigma_\ell\times\Sigma_{N-\ell})/(\Sigma_{\ell-k}\times\Sigma_k\times\Sigma_{N-\ell})).$$

Let $\lambda=(\lambda_1,\dots,\lambda_k)$ be a $p$-regular Young diagram and let $N\geq|\lambda|+\lambda_1$. Define

$$\lambda_N^+=(N-|\lambda|,\lambda_1,\dots,\lambda_k).$$

We will sometimes omit $N$ when it is implicit.

Our main technical tool is the following

**Lemma 3.** *Suppose $\lambda$ is a $p$-regular Young diagram, $N>|\lambda|+\lambda_1$.*
*(A) $\Psi(D_\lambda)(N)$ has a unique constituent factor $D_{\lambda_N^+}$.*

*(B) Let $X$ be a finitely generated $FI$-module. Suppose there exists a generic surjection $M_\lambda \twoheadrightarrow X$. Then there exists a surjection*

$$\Phi(X) \twoheadrightarrow D_\lambda. \tag{8}$$

*Additionally, the map*

$$X \to \Psi(D_\lambda) \tag{9}$$

*adjoint to (8) sends the constituent factor $D_{\lambda_N^+}$ to itself by an isomorphism.*

*Proof.* By [8], Theorem 1, the induction to $N >> 0$ of $D_\lambda$ contains $D_{\lambda_N^+}$ as a unique constituent factor, and all other constituent factors are of the form $D_\mu$ for $\mu \rhd \lambda_N^+$. Additionally, $D_{\lambda_N^+}$ is not a constituent factor in the induction of any $\Sigma_n$-module with $n < |\lambda|$. By the above description of the functor $\Psi$, this implies (A).

Also by [8], Theorem 1, for every $N$, the cofixed point $K\Sigma_{|\lambda|}$-module

$$(D_{\lambda_N^+})_{\Sigma_{N-|\lambda|}} \tag{10}$$

is $D_\lambda$ and the cofixed point module of $D_{\lambda_N^+}$ under $\Sigma_{N-i}$ with $i < |\lambda|$ is 0 (since $D_\lambda$ occurs at the "top branching level" of $L(\lambda_N^+)$). Thus, by the above description, $D_\lambda$ is by definition a quotient of the module of generators of $\Phi(X)$. Additionally, the assumption guarantees that these generators are not killed by the relations (again by [8], Theorem 1, since, if $\mu_N^+ \rhd \lambda_N^+$, then $\mu \rhd \lambda$ or $|\mu| < |\lambda|$). This implies the first statement of (B).

For the last statement, we also observe that by [8], Theorem 1, we cannot have $\lambda_N^+ = \mu_N^+$ for $|\mu| < |\lambda|$ and thus, by our above description of $\Psi$, $D_{\lambda_N^+}$ is a constituent factor of $\Psi(D_\lambda)(N)$. Additionally, all other constituent factors of $\Psi(D_\lambda)(N)$ are $D_\mu$ for $\mu \rhd \lambda_N^+$. Moreover, our construction of (8) from (10) implies that the adjoint (9) defines an isomorphism on the constituent factors $D_{\lambda_N^+}$.

□

Now, by this Lemma, for a $p$-regular Young diagram $\lambda$, we have a natural (non-zero) surjection

$$\beta_\lambda : \Phi(M_\lambda) \to D_\lambda.$$

Then since $\Phi$ and $\Psi$ are adjoint, we obtain a non-zero map

$$\iota_\lambda : M_\lambda \to \Psi(D_\lambda).$$

*Proof of Theorem 1.* To prove the first statement of Theorem 1, we need to show that the image $Im(\iota_\lambda)$ is a irreducible generic $KFI$-module. Suppose there exists some non-zero $FI$-module $E$ with a surjection

$$Im(\iota_\lambda) \twoheadrightarrow E. \tag{11}$$

First, composing with the natural surjection

$$M_\lambda \twoheadrightarrow Im(\iota_\lambda)$$

gives a surjection

$$M_\lambda \twoheadrightarrow E.$$

Hence, at each $FI$-degree $N$, we have a surjection

$$E(N) \twoheadrightarrow D_{\lambda_N^+}.$$

Then, by Lemma 3, we obtain a non-zero surjection

$$\Phi(E) \twoheadrightarrow D_\lambda.$$

Thus, the adjunction between $\Phi$ and $\Psi$ gives a non-zero map

$$E \to \Psi(D_\lambda). \tag{12}$$

On the other hand, we also have a natural injection

$$Im(\iota_\lambda) \hookrightarrow \Psi(D_\lambda),$$

which factors through (12), giving a commuting diagram

$$\begin{tikzcd} Im(\iota_\lambda) \arrow[r] \arrow[dr] & E \arrow[d] \\ & \Psi(D_\lambda). \end{tikzcd}$$

(since the diagram commutes on the adjoint level). Thus, (11) must also be injective and hence, an isomorphism. Thus, $Im(\iota_\lambda)$ is irreducible.

We shall now prove that every simple generic finitely generated $FI$-modules $E$ is isomorphic to $Im(\iota_\lambda)$ for some $\lambda$.

Let $D_\mu$ be the bottom factor of some $E(n)$. By the Noetherian property, we may assume that $D_\mu$ is non-torsion and hence without loss of generality, it generates $E(n)$. Thus, by the Pieri rule applied to $S_\mu$, which surjects onto $D_\mu$, we have $M_\nu \twoheadrightarrow E$ for some Young diagram $\nu$ (not necessarily $p$-regular). However, by James [7], Theorem A, all constituent factors $D_\rho$ of $M_\mu(m)$ (and hence $E(m)$) have $\rho \trianglerighteq (\nu_m^+)^r$ (where the superscript $r$ denotes the Young diagram obtained by shifting the boxes as far as possible along each ladder). Also, for

$m >> 0$, $(\nu_m^+)^r = (\nu^r)_m^+$. Therefore, if we denote $\rho = (\rho_1, \rho_2, \dots, \rho_k)$, $\overline{\rho} = (\rho_2, \dots, \rho_k)$, the number $|\overline{\rho}| = \rho_2 + \dots + \rho_k$ is bounded above by a number $B$ (independent of $m$).

Now without loss of generality, $E(m)$ generates $E$ for every $m$. Then $\Phi(E)(N)$ is the colimit of the sequence

$$\Phi(E)(N) = \mathrm{colim}(\dots \twoheadrightarrow E(m)_{\Sigma_{m-N}} \twoheadrightarrow E(m+1)_{\Sigma_{m+1-N}} \twoheadrightarrow \dots). \tag{13}$$

Let $D_{\mu_m}$ be a top factor of $E(m)$. Then by [8], Theorem 1, $D_{\overline{\mu_n}}$ is a top factor of $E(m)_{\Sigma_{m-N}}$. However, by the upper bound on the total number of boxes $|\overline{\mu_m}|$, the Young diagrams $\overline{\mu_m} = \lambda$ coincide for infinitely many values of $m$. We see from (13) that for $N = |\overline{\mu_m}|$, $\Phi(E)(N)$ has top factor $D_\lambda$.

Thus, we have a morphism of $FI$-modules

$$\Phi(E) \twoheadrightarrow D_\lambda.$$

Now consider the non-zero adjoint morphism

$$E \to \Psi(D_\lambda), \tag{14}$$

which is therefore (generically) injective, since $E$ was assumed to be simple. By Lemma 3, for infinitely many $n$, the top constituent factor

$$D_{\mu_n} = D_{\lambda_n^+}$$

survives in the image of (14) (meaning that it will be present in the image of $E_n$ divided by the image of $Ker(E_n \to D_{\lambda_n^+})$). Therefore, the image of (14) cannot have 0 intersection with the image of

$$M_\lambda \to \Psi(D_\lambda)$$

(whose constituent factors are strictly dominant to $\lambda_n^+$ for all $n$). Thus, by the Schur lemma,

$$E \cong Im(\iota_\lambda).$$

To show that $Im(\iota_\lambda) \ncong Im(\iota_\mu)$ for two different $p$-regular Young diagrams $\lambda$, $\mu$, we note that $\lambda_N^+$, $\mu_N^+$ are both $p$-regular for $N >> 0$ and the surjection

$$M_\lambda \twoheadrightarrow Im(\iota_\lambda)$$

gives a surjection

$$S_{\lambda_N^+} \twoheadrightarrow Im(\iota_\lambda)(N)$$

thus giving a surjection

$$Im(\iota_\lambda)_N \twoheadrightarrow D_{\lambda_N^+}.$$

However, there is no surjection $S_{\lambda_N^+} \twoheadrightarrow D_{\mu_N^+}$ for $\mu \neq \lambda$ $p$-regular, $N > |\lambda| + \lambda_1, |\mu| + \mu_1$.

$\square$

## 3. Proof of Theorem 2 at $p = 2$

First, note that we have a short exact sequence

$$0 \to S_{(4)} \to S_{(3,1)} \to D_{(3,1)} \to 0. \tag{15}$$

Thus,

$$dim(D_{(3,1)}) = dim(S_{(3,1)}) - dim(S_{(4)}) = 3 - 1 = 2,$$

which is also the dimension of $S_{(2,2)}$. Since, at $p = 2$, we have

$$(3,1) = (2,2)^r$$

(where $\lambda^r$ denotes the Young diagram obtained from shifting the boxes of $\lambda$ as high as possible along each ladder (see [7]), $D_{(3,1)}$ is a constituent factor of $S_{(2,2)}$ (by [7], Theorem A). Thus,

$$D_{(3,1)} = S_{(2,2)}.$$

By Lemma 3, we have a natural surjection

$$\Phi(M_{(3,1)}) \twoheadrightarrow D_{(3,1)} = S_{(2,2)}.$$

Now we claim the following

**Proposition 4.** *There is a short exact sequence*

$$0 \to M_{(2,2)} \to \Psi(D_{(3,1)}) \to M_{(2)} \to 0.$$

First, note that by the Pieri rule, the restriction of the $K\Sigma_4$-module $D_{(3,1)} = S_{(2,2)}$ to $\Sigma_3$ is the Specht module $S_{(2,1)}$ (since the only removable box in $(2,2)$ is the bottom right corner). We thus obtain that the induction of $S_{(2,1)}$ has constituent factors

$$D_{(3,1)},\ D_{(4)},\ D_{(3,1)},\ D_{(4)},\ D_{(3,1)}, \tag{16}$$

listed from top to bottom (i.e., with the piece that can be considered as a quotient listed first, and the piece that can be considered a submodule listed last).

**Lemma 5.** *The unit of adjunction*

$$S_{(2,2)} \to Ind_{\Sigma_4}^{\Sigma_3}(S_{(2,2)}|_{\Sigma_3})$$

*maps $S_{(2,2)}$ isomorphically to the bottom $D_{(3,1)}$ piece (16) (coming from $S_{(2,1,1)}$).*

*Proof.* We can identify the non-zero elements of $S_{(2,2)}$ with 4-cycle subgraphs of the complete graph on vertices $[4] = \{1,2,3,4\}$. On the other hand, $S_{(2,1)}$ can be identified with the submodule of $K^{[3]}$ consisting of vectors whose coordinates have sum 0. Thus, $Ind_{\Sigma_4}^{\Sigma_3}(S_{(2,1)})$ is a submodule of $Ind_{\Sigma_4}^{\Sigma^3}(K^{[3]})$, which is identified with $\mathrm{Map}_{FI}([3],[4])$ (where by our convention, the image of 1 is the new coordinate and the image of 2 comes from the coordinate in [3]). We encode an injective map $[2] \to [4]$ by a 4-tuple where we write $i$ for the image of $i = 1, 2$, and 0's in the remaining places. Under these conventions, our unit of adjunction maps

$$S_{(2,1)} \ni \{1,2\} + \{2,3\} + \{3,4\} + \{4,1\} \longmapsto$$

$$\begin{gathered}(2,0,0,1) + (0,0,1,2) + (1,0,0,2)+\\ +(0,1,2,0) + (0,0,2,1) + (0,2,1,0)+\\ +(1,2,0,0) + (2,1,0,0).\end{gathered} \tag{17}$$

On the other hand, in this notation, the generators of the Specht module $S_{(2,1,1)} \subseteq \mathrm{Map}_{FI}([2],[4])$ can be identified with, choosing $i \in [4]$, the sum $q_i$ of the six 4-tuples which are non-zero on $i$. We then see that (17) lies in this submodule, and namely, is equal to $q_1 + q_3$.

The images under the unit of adjunction of other elements of $S_{(2,2)}$ then also lie in the submodule

$$S_{(2,1,1)} \subseteq Ind_{\Sigma_4}^{\Sigma_3}(S_{(2,1)}).$$

□

*Proof of Proposition 4.* Now for induction from $S_{(2,2)}$ to a degree $N >> 0$, the Pieri rule gives pieces (from top to bottom)

$$S_{(N-2,2)},\ S_{(N-3,2,1)},\ S_{(N-4,2,2)}.$$

The middle summand is eliminated by the above observation using the description of the functor $\Psi$ in the beginning of Section 2 as the limit of the Diagram (7). Thus, we get generically

$$0 \to M_{(2,2)} \to \Psi(D_{(3,1)}) \to M_{(2)} \to 0.$$

□

Now any map of $FI$-modules

$$M_{(3,1)} \to M_{(2)}$$

is 0, since the map is necessarily 0 in degree 7 (since the composition factors of $S_{(3,3,1)}$ are $D_{(7)}$ and $D_{(4,2,1)}$, while $S_{(5,2)}$ is irreducible. Hence, the map $\iota_{(3,1)}$ facotrs through

$$\begin{tikzcd} & & M_{(3,1)} \arrow[dl, dotted, "\kappa"'] \arrow[d, "\iota_{(3,1)}"] \\ 0 \arrow[r] & M_{(2,2)} \arrow[r] & \Psi(S_{(2,2)}) \end{tikzcd}$$

for some map

$$\kappa : M_{(3,1)} \to M_{(2,2)}.$$

At an $FI$-degree $N$, denote the cokernel

$$C = Coker(\kappa).$$

We claim the following

**Lemma 6.** *In degrees $>> 0$, generically,*

$$C = M_{\emptyset}.$$

To prove this Lemma, we will need calculations of $\Psi(S_{(4)})$ and $\Psi(S_{(3,1)})$, which we make in the following propositions:

**Proposition 7.** *Generically, there is a short exact sequence*

$$0 \to M_4 \to \Psi(S_{(4)}) \to M_{\emptyset} \to 0.$$

*Proof.* First, the restriction of the Specht module $S_{(4)}$ to $\Sigma_3$ is exactly the Specht module $S_{(3)}$, whose induction to $\Sigma_4$ has pieces (listed from top to bottom) $S_{(4)}$, $S_{(3,1)}$. The unit of adjunction (between restriction and induction) sends $S_{(4)}$ monomorphically to the lowest piece.

Now the induction of $S_{(4)}$ to $N \geq 8$ has pieces (listed from top to bottom)

$$S_{(N)},\ S_{(N-1,1)},\ S_{(N-2,2)},\ S_{(N-3,3)},\ S_{(N-4,4)}.$$

The above observation, along with our description of the functor $\Psi$, eliminates all but the first and last piece. Thus, we get generically

$$0 \to M_4 \to \Psi(S_{(4)}) \to M_{\emptyset} \to 0.$$

$\square$

**Proposition 8.** *We have*

$$\Psi(S_{(3,1)}) = M_{(3,1)}.$$

*Proof.* First, note that the restriction of the Specht module $S_{(3,1)}$ to $\Sigma_3$ has pieces $S_{(3)},\ S_{(2,1)}$. The induction back to $\Sigma_4$ of the first piece is $S_{(3,1)}$, to which the bottom piece $D_{(4)}$ of $S_{(3,1)}$ injects by the unit of adjunction. The piece $S_{(2,1)}$ inducts to $S_{(3,1)}$ and $S_{(2,1,1)}$, to which the top piece $S_{(2,2)}$ of $S_{(3,1)}$ injects.

Now the induction of $S_{(3,1)}$ to $N \geq 8$ has pieces

$$S_{(N-1,1)},\ S_{(N-2,2)},\ S_{(N-2,1,1)},\ S_{(N-3,3)},\ S_{(N-3,2,1)},\ S_{(N-4,3,1)}.$$

The first, second, and fourth price are eliminated by the first part of the unit of adjunction (to the induction of $S_{(3)}$) and the third and fourth pieces are eliminated by the second part of the unit of adjunction (to the induction of $S_{(2,1,1)}$), similarly as in the proofs of Proposition 4 and Proposition 7. Thus,

$$\Psi(S_{(3,1)}) = M_{(3,1)}.$$

□

*Proof of Lemma 6.* Recall again the exact sequence

$$0 \to S_{(4)} \to S_{(3,1)} \to S_{(2,2)} \to 0.$$

Since $\Psi$ is right adjoing to $\Phi$, it is left exact, so we obtain

$$0 \longrightarrow \Psi(S_{(4)}) \longrightarrow \Psi(S_{(3,1)}) \xrightarrow{\rho} \Psi(S_{(2,2)}).$$

Then $\rho$ factors through $\kappa$ (since by above, $\Psi(S_{(3,1)}) = M_{(3,1)}$).

Thus, at every $FI$-degree $N >> 0$, the dimension of $C(N)$ equals

$$dim(M_{(2,2)}(N)) - dim(M_{(3,1)}(N)) + dim(\Psi(S_{(4)})(N)) =$$

$$= dim(M_{(2,2)}(N)) - dim(M_{(3,1)}(N)) + dim(M_{\emptyset}(N)) + dim(M_{(4)}(N)) =$$

$$= dim(M_{\emptyset}(N)) = dim(S_{(N)}) = 1$$

(since, by the hook length formula,

$$dim(S_{(k,3,1)}) = \frac{(k+4)(k+3)(k+1)(k-2)}{8}$$

$$dim(S_{(k,4)}) = \frac{(k+4)(k+3)(k+2)(k-3)}{24}$$

and

$$dim(S_{(k,3,1)}) - dim(S_{(k,4)}) = \frac{(k+4)(k+3)k(k-1)}{12}) = dim(S_{(k,2,2)}).$$

Hence, $C(N)$ is a $K\Sigma_N$-module with dimension 1. Thus, for every $N$, $C(N) = S_{(N)}$, proving that, as $FI$-modules,

$$C = M_{\emptyset}.$$

$\square$

Finally, to prove Theorem 2, we let $R_\lambda = K\Sigma^\lambda_{\text{row}}$ where $\Sigma^\lambda_{\text{row}}$ is the subgroup of $\Sigma_{|\lambda|}$ of permutations preserving the rows of a Young diagram $\lambda$.

*Proof of Theorem 2.* Suppose $N \geq 8$ is odd. We consider the morphism

$$\theta_{T_1} : R_{(N-3,2,1)} \to R_{(N-4,2,2)} \tag{18}$$

of [6] given by the tableau $T_1$ with rows

| | | | | | |
|---|---|---|---|---|---|
| 3 | 3 | 2 | 1 | ... | 1 |
| 2 | 1 | | | | |
| 1 | | | | | |

We calculate that, using the notation of [6],

$$N_{1,1}(T_1) = N-6,\ N_{2,1}(T_1) = 1,\ N_{3,1}(T_1) = 2,$$
$$N_{1,2}(T_1) = 1,\ N_{2,2}(T_1) = 1,\ N_{3,2}(T_1) = 0,$$
$$N_{1,3}(T_1) = 1,\ N_{2,3}(T_1) = 0,\ N_{3,3}(T_1) = 0,$$

and thus $T_1$ satisfies the condition of Theorem 24.6, (ii), [6] (since $N$ is assumed to be odd). Hence, by Theorem 24.6, (ii), [6], the restriction of $\theta_{T_1}$ is a non-zero homomorphism

$$\theta_{T_1}|_{S_{(N-3,2,1)}} : S_{(N-3,2,1)} \to S_{(N-4,2,2)}.$$

Since $T_1$ is reverse semistandard, by the proof of Theorem 24.6,

$$Im(\theta_{T_1}|_{S_{(N-3,2,1)}}) \subseteq S_{(N-4,2,2)}$$

contains the constituent factor $D_{(N-3,2,1)}$. Therefore, this constituent factors must be present in $Im(\iota_{(3,1)})(N) \cong Im(\kappa)(N)$, which is therefore not simple, since it also contains the constituent factor $D_{(N-4,3,1)}$.

Suppose $N \geq 8$ is even. We consider the morphism

$$\theta_{T_2} : R_{(N-2,1,1)} \to R_{(N-4,2,2)} \tag{19}$$

given by the tableau $T_2$ with rows

$$\begin{array}{|c|c|c|c|c|c|}\hline 3 & 3 & 2 & 1 & \dots & 1 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

We calculate, using the notation of [6],

$$N_{1,1}(T_1) = N - 5,\ N_{2,1}(T_1) = 1,\ N_{3,1}(T_1) = 2,$$
$$N_{1,2}(T_1) = 0,\ N_{2,2}(T_1) = 2,\ N_{3,2}(T_1) = 0,$$
$$N_{1,3}(T_1) = 1,\ N_{2,3}(T_1) = 0,\ N_{3,3}(T_1) = 0,$$

and thus, again, $T_2$ satisfies the considiton of Theorem 24.6, (ii), [6] (since $N$ is assumed to be even). Hence, the restriction of $\theta_{T_2}$ is a non-zero homomorphism

$$\theta_{T_2}|_{S_{(N-2,1,1)}} : S_{(N-2,1,1)} \to S_{(N-4,2,2)}.$$

Now all constituent factors of $S_{(N-2,1,1)}$ are of the form $D_\lambda$ where $\lambda \rhd (N-2,1,1)$ (by Theorem 12.1 of [6]). Then $\theta_{T_2}|_{S_{(N-2,1,1)}}$ must be non-zero on at least one such $D_\lambda$, and therefore $D_\lambda$ must be a constituent factor of $Im(\theta_{T_2}|_{S_{(N-2,1,1)}}) \subseteq S_{(N-4,2,2)}$. Hence, this $D_\lambda$ is also a constituent factor of $Im(\iota_{(3,1)}) \cong Im(\kappa)$. By Theorem 24.4 of [6], $\lambda \neq (N)$. In addition, since $\lambda \rhd (N-2,1,1)$, we also have $\lambda \neq (N-4,3,1)$. Therefore, since $Im(\iota_{(3,1)})(N) \cong Im(\kappa)(N)$ also contains the constituent factor $D_{(N-4,3,1)}$, it can not be simple.

□

## 4. Proof of Theorem 2 at $p > 2$

Suppose $p > 2$. First, we have the following

**Proposition 9.** *There is a short exact sequence*

$$0 \to S_{(p+1,1)} \to S_{(p,2)} \to D_{(p,2)} \to 0.$$

*Proof.* First note that since $(p,2)$ contains a bad box (see [2, 3]), $S_{(p,2)}$ must be reducible. It therefore contains a submodule of the form $D_\lambda$ where $\lambda \rhd (p,2)$. The only options for $\lambda$ are $(p+1,1)$ and $(p+2)$. By [6], Theorem 24.4, $D_{(p+2)} = S_{(p+2)}$ is not a submodule of $S_{(p,2)}$ since $p$ is not $-1$ mod $p$. Thus, $D_{(p+1,1)} = S_{(p+1,1)}$ (the equality holds since $(p+1,1)$ has no bad boxes) is a submodule of $S_{(p,2)}$.

To prove the Proposition, by [6], Section 11, it suffices to show

$$S^{\perp}_{(p,2)} \cap S_{(p,2)} = S_{(p+1,1)}, \tag{20}$$

where $S^{\perp}_{(p,2)}$ is the orthogonal complement of $S_{(p,2)}$ in $R_{(p,2)}$ (the standard permutation module basis of $R_{(p,2)}$ is orthonormal). By the above discussion, we already know $S^{\perp}_{(p,2)} \cap S_{(p,2)} \supseteq S_{(p+1,1)}$ in (20).

To prove the other inclusion in (20), first, by the hook formula, we have

$$dim(S_{(p,2)}) = \frac{(p+2)!}{(p+1)p(p-2)!2} = \frac{(p+2)(p-1)}{2},$$

and we also have

$$dim(R_{(p,2)}) = \frac{(p+2)!}{p!2} = \frac{(p+2)(p+1)}{2}.$$

So

$$dim(R_{(p,2)}) - dim(S_{(p,2)}) = \frac{2(p+2)}{2} = p+2. \tag{21}$$

Let

$$V_n = K\Sigma_n/\Sigma_{n-1} = R_{(n-1,1)}.$$

Then we have a homomorphism

$$\psi_{1,1} : R_{(p,2)} \to V_{p+2}$$

and $S_{(p,2)} \subseteq ker(\psi_{1,1})$ (by [6], Corollary 17.18), where $\psi_{1,1}$ is defined as a sum of standard basis elements obtained by moving one box from the second row to the first row. In fact, in this case $\psi_{1,1}$ is surjective since its image contains sums of every pair of standard basis elements in $V_{p+2}$ and $p > 2$.

Thus, since $dim(V_{p+2}) = p+2$, by (21), we have a short exact sequence

$$0 \longrightarrow S_{(p,2)} \longrightarrow R_{(p,2)} \xrightarrow{\psi_{1,1}} V_{p+2} \longrightarrow 0.$$

Hence, $S^{\perp}_{(p,2)} \cong V_{p+2}$, and in particular,

$$S^{\perp}_{(p,2)} \cap S_{(p,2)} \leq p+2.$$

To prove (20), since we already know the $\supseteq$-inclusion, it suffices to show

$$S_{(p,2)}^{\perp} \cap S_{(p,2)} \leq p+1 = dim(S_{(p+1,1)}).$$

To this end, it suffices to find an element in $S_{(p,2)}^{\perp} \smallsetminus S_{(p,2)}$. Consider the map

$$R_{(p,2)} \to K,$$

given by sending a basis element to $1 \in K$ if it has a 2 in a given position and to $0 \in K$ else. This is equivalent to taking the dot product with the sum $v$ of such basis elements, of which there are $p+1$. Thus, the dot product of the element $v$ with itself is $p+1$ which is non-zero, and thus, $v$ is not in $S_{(p,2)} = ker(\psi_{1,1})$. Thus, (20) is proven, concluding the proof of the Proposition.

$\square$

Again, since $\Psi$ is a right adjoint, it is left exact, giving

$$0 \to \Psi(S_{(p+1,1)}) \to \Psi(S_{(p,2)}) \to \Psi(D_{(p,2)}). \tag{22}$$

We then claim the following

**Proposition 10.** *We have*

$$\Psi(S_{(p+1,1)}) = M_{(p+1,1)}.$$

*Proof.* Letting

$$V_n = K(\Sigma_n/\Sigma_{n-1}) \cong K^n,$$

we have

$$S_{(p+1,1)} = K\{(v_1, \dots, v_{p+2}) \in V_{p+2} | \sum_{i=1}^{p+2} v_i = 0\}.$$

Consider the unit of adjunction between induction and restriction

$$S_{(p+1,1)} \to Ind_{\Sigma_{p+2}}^{\Sigma_{p+1}} Res_{\Sigma_{p+1}}^{\Sigma_{p+2}} S_{(p+1,1)}. \tag{23}$$

Using the isomorphism

$$Ind_{\Sigma_{p+2}}^{\Sigma_{p+1}} Res_{\Sigma_{p+1}}^{\Sigma_{p+2}} S_{(p+1,1)} \cong K(\Sigma_{p+2}/\Sigma_{p+1}) \otimes_K S_{(p+1,1)}$$

the map (23) can be described as sending $(v_1, \dots, v_{p+2}) \in S_{(p+1,1)}$ to $(1, 1, \dots, 1) \otimes (v_1, \dots, v_{p+2})$.

Now the restriction of $S_{(p+1,1)}$ to $\Sigma_{p+1}$ has pieces $S_{(p+1)}$, $S_{(p,1)}$, with $S_{(p+1)}$ above $S_{(p,1)}$. The image of (23) must be contained in the induction of $S_{(p,1)}$ since any $(1,\dots,1)\otimes(v_1,\dots,v_{p+2})$ in the image of (23) can be expressed as the sum

$$\sum_{i=1}^{p+2}(0,\dots,0,1,0,\dots,0)\otimes(v_1,\dots,v_{i-1},0,v_{i+1},\dots,v_{p+2})$$

(where in the $i$th summand, the 1 is in the $i$th place).

The only piece of the induction of $S_{(p+1,1)}$ to $N >> 0$ that is not a piece in the induction of $S_{(p,1)}$ is $S_{(N-p-2,p+1,1)}$. Thus, by the description (7) of $\Psi$,

$$\Psi(S_{(p+1,1)}) = M_{(p+1,1)}.$$

□

*Proof of Theorem 2:* Fix some $N >> 0$. Denote by $\varphi$ the first map of (22). By Proposition 10, the injection is of the form

$$\varphi : S_{(N-p-2,p+1,1)} \to \Psi(S_{(p,2)})(N).$$

We therefore obtain the short exact sequence

$$0 \to \varphi^{-1}(S_{(N-p-2,p,2)}) \to S_{(N-p-2,p,2)} \to (Im(\iota_{(p,2)}))(N) \to 0. \tag{24}$$

(For the sake of brevity, let us write $k = N-p-2$.)

Now consider the map

$$\theta_T : R_{(\lfloor \frac{k}{p}\rfloor p+p,k-\lfloor \frac{k}{p}\rfloor p+1,1)} \to R_{(k,p,2)} \tag{25}$$

(again using the notation and definitions given in [6]) where $T$ is the reverse semistandard tableau

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 2 | ... | 2 | 2 | ... | 2 | 1 | ... | 1 |
| 2 | 1 | 1 | ... | 1 | | | | | | |
| 1 | | | | | | | | | | |

which has

$$N_{1,1}(T) = \left\lfloor \frac{k}{p} \right\rfloor p-1,\ N_{2,1}(T) = p-1,\ N_{3,1}(T) = 2$$

$$N_{1,2}(T) = k - \left\lfloor \frac{k}{p} \right\rfloor p,\ N_{2,2}(T) = 1,\ N_{3,2}(T) = 0$$

$$N_{1,3}(T) = 1,\ N_{2,3}(T) = 0,\ N_{3,3}(T) = 0.$$

This satisfies the conditions of Theorem 24.6, (ii), [6] and therefore (25) restricts to a non-zero map

$$\widehat{\theta_T} : S_{(\lfloor \frac{k}{p} \rfloor p+p, k-\lfloor \frac{k}{p} \rfloor p+1,1)} \to S_{(k,p,2)}.$$

It therefore suffices to show $\widehat{\theta_T}$ does not lift to a map

$$S_{(\lfloor \frac{k}{p} \rfloor p+p, k-\lfloor \frac{k}{p} \rfloor p+1,1)} \to \varphi^{-1}(S_{(k,p,2)}) \subseteq S_{(k,p+1,1)}, \tag{26}$$

for (24) (since then $(Im(\iota_{(p,2)}))(N)$ will have constituent factors $D_{(k,p,2)}$ and $D_\lambda$ for some $\lambda$ dominant or equal to $(\left\lfloor \frac{k}{p} \right\rfloor p+p, k-\left\lfloor \frac{k}{p} \right\rfloor p+1,1)$ and therefore be reducible).

Suppose a lifting (26) exists. If $p$ divides $k$, then $(k,p+1,1)$ contains no bad boxes, so $S_{(k,p+1,1)}$ is irreducible, thus already forming a contradiction since then (26) is 0. So, suppose $p$ does not divide $k$. By [6], Theorem 13.13, it suffices to show all linear combinations of $\widehat{\theta}_T$ for semistandard $(\left\lfloor \frac{k}{p} \right\rfloor p+p, k-\left\lfloor \frac{k}{p} \right\rfloor p+1,1)$-tableaux $T$ of type $(k,p+1,1)$ which have image contained in the Specht module $S_{(k,p+1,1)}$ are 0. The only semistandard $(\left\lfloor \frac{k}{p} \right\rfloor p+p, k-\left\lfloor \frac{k}{p} \right\rfloor p+1,1)$-tableau $T$ of type $(k,p+1,1)$ is

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & \dots & 1 & 1 & \dots & 1 & 2 & \dots & 2 \\
\hline
2 & 2 & \dots & 2 \\
\cline{1-4}
3 \\
\cline{1-1}
\end{array} . \tag{27}$$

We will prove that $Im(\widehat{\theta_T}) \nsubseteq S_{(k,p+1,1)}$ using [6], Corollary 17.18 by finding $i,v$ with $\psi_{i-1,v}(Im(\widehat{\theta_T})) \neq 0$, where

$$\psi_{i-1,v} : R_\lambda \to R_{(\lambda_1,\dots,\lambda_{i-2},\lambda_{i-1}+\lambda_i-v,v,\lambda_{i+1},\dots)}$$

is obtained by moving $\lambda_i - v$ boxes from the $i$th row to the $(i-1)$th row.

Let us choose $i=2$, $v=p$. Applying $\psi_{i-1,v}$ then involves summing over the different tableaux $T'$ arising from taking un-signed row permutations and then taking the sum of signed column permutations of tableaux $T''$ arising from $T'$ by replacing one 2 in (27) by a 1.

It then suffices to show that there exists a $T''$ with no two numbers the same in any column and this $T''$ arises a number of times that is not divisible by $p$. Consider the $T''$ given as the $(\left\lfloor \frac{k}{p} \right\rfloor p+p, k-\left\lfloor \frac{k}{p} \right\rfloor p+1,1)$-tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 2 & 1 & \dots & 1 & 1 & \dots & 1 & 2 & \dots & 2 \\ \hline 1 & 2 & \dots & \multicolumn{1}{c|}{2} & \multicolumn{6}{c}{} \\ \cline{1-4} \multicolumn{1}{|c|}{3} & \multicolumn{9}{c}{} \\ \cline{1-1} \end{array} \; . \tag{28}$$

This can arise in two fashions:

1. $T'$ arises by moving the first 2 in the first row to the first column and $T''$ then arises by replacing the first 2 in the second row with a 1. This yields one positive summand.

2. $T'$ arises by moving the first 2 in the first row to any of the first $k+1$ spots of the first row (including the possibility of letting it stay in the same spot), and $T''$ then arises by replacing this same 2 by a 1, and switching the 1 and 2 in the first column. This gives $k+1$ negative summands.

Thus, the summand $T''$ arises exactly $-k$ times. By our assumption, $p$ does not divide $k$ (and thus also does not divide $-k$), hence concluding the proof of the Theorem. $\square$

## References


[1] T.Church, J.S.Ellenberg, B.Farb: $FI$-modules and stability for representations of symmetric groups, *Duke Math. J.* 164 (2015), no. 9, 1833-1910.

[2] M.Fayers: Reducible Specht modules, *J. Algebra*, 280 (2004), no. 2, 500-504.

[3] M.Fayers: Irreducible Specht modules for Hecke algebras of type A, *Adv. Math.*, 193 (2005), no. 2, 438-452.

[4] P.Gabriel. Des catégories abéliennes, *Bull. Soc. Math. France*, 90 (1962), 323-448.

[5] W.L.Gan, L.Li: Coinduction functor in representation stability theory, *J. London Math. Soc.*, 92 (2015), no. 3, 689-711.

[6] G.D.James: *The Representation Theory of the Symmetric Groups,* Springer Lecture Notes in Mathematics 692, Springer (1980)

[7] G.D.James: On the Decomposition Matrices of the Symmetric Groups, II, *J. Algebra* 43 (1976), 45-54.

[8] G.D.James: On the Decomposition Matrices of the Symmetric Groups, III, *J. Algebra* 71 (1981), 115-122.

[9] G.D.James, A.Mathas: The Irreducible Specht Modules in Characteristic 2, *Bull. London Math. Soc.* 31 (1999), no. 4, 457-462

[10] S.Kriz: On the local cohomology of L-shaped integral FI-modules, preprint, 2021

[11] R.Nagpal: private communication, 2021.

[12] L.Li, E.Ramos: Depth and the local cohomology of $FI_G$-modules, *Adv. Math.* 329 (2018), 704-741.

[13] S.V.Sam, A.Snowden: $GL$-Equivariant Modules Over Polynomial Rings in Infinitely Many Variables, *Trans. Am. Math. Soc.* 368 (2016), no. 2, 1097-1158.

[14] J-P.Serre: Faisceaux Algébriques Cohérents, *Ann. Math.* 2nd Ser., 61, No. 2. (1955), 197-278.

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