AN INTERPOLATION OF THE WEIL-SHALE REPRESENTATION

SOPHIE KRIZ

Abstract. In this note, we exhibit new examples of semisimple pre-Tannakian categories generated by a simple object whose second symmetric and exterior powers are also simple.

1. Introduction

By a *pre-Tannakian* category, we mean a $\mathbb{C}$-linear abelian category with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product, which is locally finite in the sense that Hom-sets are finite dimensional $\mathbb{C}$-vector spaces and is rigid in the sense that objects have strong duals [3]. An abelian category is *semisimple* if every object is a direct sum of finitely many simple objects. Examples of semisimple pre-Tannakian categories are, in general, difficult to construct, and their properties remain somewhat mysterious.

In this note, we prove the following result, answering a question of P. Deligne:

Theorem 1. *For every natural number q which is a power of a prime not equal to 2 and every $t \in \mathbb{C}$ such that $q^t \neq \pm 1, \pm q$, there exists a semisimple pre-Tannakian category over $\mathbb{C}$ generated by an object X of dimension q^t such that X, $\Lambda^2(X)$, $Sym^2(X)$ are simple and*

$$dim(End(X^{\otimes 3})) = 2q + 2. \tag{1}$$

To prove Theorem 1, we use the Weil-Shale representation, whose definition and key properties are taken from R. Howe [4], N. M. Katz [5], A. Weil [10], and P. Deligne [2]. The main point of this note is noticing that these examples for $n \in \mathbb{N}$ can be "interpolated" to $t \in \mathbb{C} \smallsetminus \mathbb{N}_0$ using the formalism of T-algebras [8, 9].

The construction of the classical Weil-Shale representation [10, 4, 5] goes as follows: For a q which is a power of a prime not equal to 2, suppose V_N is a $2N$-dimensional vector space over $\mathbb{F}_q$ endowed with a symplectic form. Recall that the Heisenberg group is

$$\mathbb{H}_N(\mathbb{F}_q) = V_N \times \mathbb{F}_q$$

with the operation that for $v, w \in V_N$, $\lambda, \mu \in \mathbb{F}_q$,

$$(v, \lambda)(w, \mu) = (w + w, \lambda + \mu + \langle v, w \rangle)$$

(and therefore $0 \times \mathbb{F}_q$ is the center of $\mathbb{H}_N(\mathbb{F}_q)$). The symplectic group $Sp(V_N)$ acts on $\mathbb{H}_N(\mathbb{F}_q)$ by acting tautologically on the factor of V_N and trivially on the center. Consider the group

$$Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q).$$

For a non-trivial character

$$\psi : \mathbb{F}_q \ \to \mathbb{C}^\times,$$

there is a unique q^N-dimensional irreducible $\mathbb{H}_N(\mathbb{F}_q)$-representation ω_ψ (over $\mathbb{C}$). Then ω_ψ forms a representation of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ called the *Weil-Shale representation* (a priori projective but an actual representation for a finite field $\mathbb{F}_q$, see [4]). For the remainder of this note, ψ will be fixed, and we will omit it from the notation.

Considering the usual inclusion $GL_N(\mathbb{F}_q) \subset Sp(V_N)$, the restriction of ω to $GL_N(\mathbb{F}_q)$ gives the permutation representation $\mathbb{C}\mathbb{F}_q^N$ where $GL_N(\mathbb{F}_q)$ acts on $\mathbb{F}_q^N$ by matrix multiplication, tensored with

$$\epsilon(det) \tag{2}$$

where $det : GL_N(\mathbb{F}_q) \to \mathbb{F}_q^\times$ is the determinant map and $\epsilon : \mathbb{F}_q^\times \to \mathbb{C}^\times$ is the character of order 2.

Our method of proving Theorem 1 is based on "interpolating" the Weil-Shale representation to $t \in \mathbb{C} \smallsetminus \mathbb{N}_0$.

The proof of Theorem 1 constitutes the remainder of this note. In Section 2, we describe the interpolation using the formalism of T-algebras [8, 9]. In Section 3, we prove our statements about the dimension of $End(X^{\otimes n})$, $n \leq 3$. In Section 4, we discuss the semisimplicity of the categories constructed.

2. The T-Algebra Structure

Recall that a $\mathbb{C}$-linear additive category with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product and strong duality which is generated by a basic object X can be axiomatized by its *T-algebra*

$$\mathcal{T}(S,T) = Hom(X^{\otimes S}, X^{\otimes T})$$

(see [8, 9], following [1], Chapter 10):

A *T-algebra* $\mathcal{T}$ is a universal algebra structure which consists of the data of vector spaces $\mathcal{T}(S,T)$ corresponding (functorially) to pairs of finite sets S,T, along with the data of *partial trace* operations

$$\tau_\phi : \mathcal{T}(S,T) \to \mathcal{T}(S \smallsetminus S', T \smallsetminus T') \tag{3}$$

corresponding (functorially) to bijections $\phi : S' \to T'$ for subsets $S' \subseteq S$, $T' \subseteq T$, the data of *product* operations

$$\pi : \mathcal{T}(S_1,T_1) \otimes \mathcal{T}(S_2,T_2) \to \mathcal{T}(S_1 \amalg S_2, T_1 \amalg T_2)$$

for finite sets S_1, S_2, T_1, T_2, and the data of two types of "units" $1 \in \mathcal{T}(\emptyset,\emptyset)$ and $\iota \in \mathcal{T}(\{1\},\{1\})$ satisfying suitable axioms (see [8, 9] for details).

For a T-algebra $\mathcal{T}$, we may conversely construct an additive $\mathbb{C}$-linear category $\mathscr{C}(\mathcal{T})$ with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product and strong duality which is generated by a basic object X, by first constructing a category $\mathscr{C}(\mathcal{T})_0$ with objects

$$Obj(\mathscr{C}(\mathcal{T})_0) = \{X^{\otimes S} \otimes (X^\vee)^{\otimes T} \mid S,T \text{ finite sets}\}$$

and morphisms

$$Hom_{\mathscr{C}(\mathcal{T})_0}(X^{\otimes S_1} \otimes (X^\vee)^{\otimes T_1}, X^{\otimes S_2} \otimes (X^\vee)^{\otimes T_2}) = \mathcal{T}(S_1 \amalg T_2, S_2 \amalg T_1).$$

We construct $\mathscr{C}(\mathcal{T})$ by formally adding direct sums to $\mathscr{C}(\mathcal{T})_0$ and taking a pseudo-abelian envelope (for more details, see [8, 9]).

Recall from F. Knop [6, 7] that there is an interpolation $Rep(GL_t(\mathbb{F}_q))$ of the category of representations $Rep(GL_N(\mathbb{F}_q))$, which is generated by a basic object X of dimension q^t (interpolating $\mathbb{C}\mathbb{F}_q^N$), which is semisimple and pre-Tannakian for $t \in \mathbb{C} \smallsetminus \mathbb{N}_0$. One may consider the spaces of morphisms

$$Hom_{Rep(GL_t(\mathbb{F}_q))}(X^{\otimes S}, X^{\otimes T}), \tag{4}$$

for finite sets S,T. As a vector space, note that (4) is isomorphic to

$$Hom_{Rep(GL_N(\mathbb{F}_q))}((\mathbb{C}\mathbb{F}_q^N)^{\otimes S}, (\mathbb{C}\mathbb{F}_q^N)^{\otimes T}), \tag{5}$$

for $N >> 0$. Let us denote by

$$\mathcal{V}(S,T) \tag{6}$$

the subspace of (5) of morphisms that preserve the action of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ (taking V_N to be a symplectic space of dimension $2N$), for $N >> 0$.

We claim that $\mathcal{V}(S,T)$, considered as a subspace of (4), forms a sub-T-algebra of the T-algebra corresponding to $Rep(GL_t(\mathbb{F}_q))$ generated by X.

The T-algebra $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}$ corresponding to $Rep(GL_t(\mathbb{F}_q))$ generated by X is defined by taking spaces (4), describing partial trace by using the strong duality of X and composition with evaluation and coevaluation morphisms, and taking product to be the tensor product of morphisms (the two "units" then being $\iota = Id_X$ and $1 = Id_1$).

To describe this approach in more detail, we can identify the vector space $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S,T)$, for finite sets S,T, with the free $\mathbb{C}$-vector space generated by equivalence classes of quotients

$$f : \mathbb{F}_q^{S \amalg T} = \mathbb{F}_q\{e_i \mid i \in S \amalg T\} \twoheadrightarrow V \tag{7}$$

with the equivalence relation that f is equivalent to any composition of f with an automorphism of the target.

The product π of two quotient maps is a sum of all possible "amalgamations" of the target vector spaces. For more detail, see [9].

Partial trace can be defined by, for subsets $S' \subseteq S$, $T' \subseteq T$, and a bijection $\phi : S' \to T'$, describing $\tau_\phi(f)$ for f as in (7), and extending linearly.

If there exists an $i \in S'$ such that $f(e_i) \neq f(e_{\phi(i)})$, then take $\tau_\phi(f) = 0$.

If for every $i \in S'$ we have $f(e_i) = f(e_{\phi(i)})$, then we take $\tau_\phi(f)$ to be a multiple of the restriction of $f|_{\mathbb{F}_q^{(S \amalg T) \smallsetminus (S' \amalg T')}}$, where the coefficient is determined by the difference of dimensions

$$\ell = dim(V) - dim(Im(f|_{\mathbb{F}_q^{(S \amalg T) \smallsetminus (S' \amalg T')}})),$$

by being 1 if $\ell = 0$, and

$$(q^t - q^{dim(V)-1}) \cdot \dots \cdot (q^t - q^{dim(V)-\ell})$$

if $\ell \neq 0$. This formula for general t is obtained by polynomially (in q^t) interpolating the respective formulas for $N >> 0$.

For the purposes of this paper, we may restrict attention to the Hom-spaces $Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T})$ where $|S| = |T|$ (the *graded* context – we can set the other $Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T})$ to 0.) We shall assume this convention throughout the rest of this paper.

Now recall the space $\mathcal{V}(S,T)$ of (6).

Lemma 2. *For $t \in \mathbb{C} \smallsetminus \mathbb{N}_0$, the restriction of the partial trace maps*

$$\tau_\phi : \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S,T) \to \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S \smallsetminus S', T \smallsetminus T')$$

for finite sets $S' \subseteq S$, $T' \subseteq T$, and bijections $\phi : S' \to T'$ ($|S| = |T|$, $|S'| = |T'|$), and the product maps

$$\pi : \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S_1,T_1) \otimes \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S_2,T_2) \to$$
$$\to \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S_1 \amalg S_2, T_1 \amalg T_2)$$

for finite sets S_1, S_2, T_1, T_2, to $\mathcal{V}(S,T)$ and $\mathcal{V}(S_1,T_1) \otimes \mathcal{V}(S_2,T_2)$, respectively, have images contained in $\mathcal{V}(S \smallsetminus S', T \smallsetminus T')$ and $\mathcal{V}(S_1 \amalg S_2, T_1 \amalg T_2)$, respectively.

Proof. The statement holds for $N >> 0$ (note that tensoring with (2) can be neglected for our purposes since we are only considering the graded part of the T-algebra for $Rep(GL_t(\mathbb{F}_q))$). Therefore, it holds for a general t since the constants involved are polynomial in q^t for $t = N$. $\square$

Write $\mathcal{V}_t$ for the T-algebra formed by the vector spaces $\mathcal{V}(S,T)$ and the partial trace and product maps of $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}$.

We may therefore consider the category $\mathscr{C}(\mathcal{V})$, which can be considered as generated by an interpolation ω of the Weil-Shale representation. We write

$$\mathscr{C}_{Sp(V_t) \ltimes \mathbb{H}_t(\mathbb{F}_q)} := \mathscr{C}(\mathcal{V}_t).$$

Since $\mathcal{V}_t$ forms a sub-T-algebra of $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}$, the category $\mathscr{C}_{Sp(V_t) \ltimes \mathbb{H}_t(\mathbb{F}_q)}$ forms a subcategory of $Rep(GL_t(\mathbb{F}_q))$.

3. The Counting Argument

In this section, we prove that, for $N > 1$

$$dim(End_{Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes 2})) = 2 \tag{8}$$

and

$$dim(End_{Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes 3})) = 2q + 2. \tag{9}$$

To calculate $dim(End_{Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes n}))$, consider the object

$$\Omega := \omega \otimes (\omega^\vee) \in \mathscr{C}_{Sp(V_t)\ltimes \mathbb{H}_t(\mathbb{F}_q)}$$

For every $k \in \mathbb{N}$, there is an isomorphism

$$Res^{Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)}(\Omega^{\otimes k}) \cong (\Omega^{\otimes k+1})^{\mathbb{H}_N(\mathbb{F}_q)} \tag{10}$$

as representations of $Sp(V_N)$ for $N >> 0$ (taking Ω to denote $\omega_\psi \otimes (\omega_\psi)^\vee$), since by [4] Proposition 2 (i-ii), their characters are equal. In fact, we have the following

Lemma 3. *For $N > 1$, as a representation of $Sp(V_N) \ltimes V_N$, Ω is isomorphic to the space of functions on V_N:*

$$Res^{Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)\ltimes V_N}(\Omega) \cong \{f : V_N \to \mathbb{C}\}$$

where an element $(A, w) \in Sp(V_N) \ltimes V_N$ acts on a function $f : V_N \to \mathbb{C}$ by sending it to the function

$$(A, w)[f] : V_N \to \mathbb{C}$$

where for $v \in V_N$

$$((A, w)[f])(v) = \psi(\langle v, w\rangle) \cdot f(A(v)).$$

Proof. First note that we may write

$$\Omega = \bigoplus_{v\in V_N} \Omega_v$$

for lines Ω_v such that an element $w \in V_N = V_N \times \{0\} \subset \mathbb{H}_N(\mathbb{F}_q)$ preserves each Ω_v and acts by multiplication by the character

$$x \mapsto \psi(\langle v, w\rangle) \cdot x.$$

Ω can then be considered as the space of global sections of an $Sp(V_N)$-equivariant line bundle Ω_v over V_N (as a discrete set) where the action of $Sp(V_N)$ on Ω induces an action of $Sp(V_N)$ on the line bungle, i.e. for $\gamma \in Sp(V_N)$,

$$\gamma(\Omega_v) = \Omega_{\gamma(v)}.$$

However, Ω_v forms a trivial $Sp(V_N)$-equivariant line bundle, meaning that for every $v \in V_N$, the stabilizer subgroup $Sp(V_N)^v \subseteq Sp(V_N)$ fixing v acts trivially on Ω_v: At $v = 0$, for $N > 0$, $Sp(V_N)^0 = Sp(V_N)$, which is a perfect group (meaning that it has no non-trivial abelian quotients), and therefore acts trivially on Ω_0. For $v \neq 0$, taking W_v to be the quotient of the orthogonal space $V_N^{\perp v}$ of vectors perpendicular to v by $\mathbb{F}_q$-multiples of v,

$$Sp(V_N)^v = Sp(W_v),$$

which again is a perfect group (for $N > 0$).

Therefore, Ω is the space of global sections of the trivial $Sp(V_N)$-equivariant line bundle, i.e. a space of functions

$$\Omega = \{f : V_N \to \mathbb{C}\}, \tag{11}$$

and the action of $Sp(V_N) \ltimes V_N \subset Sp(V_N) \ltimes \mathbb{H}_n(\mathbb{F}_q)$ on a function f in (11) is given by $Sp(V_N)$ acting by composition, and $w \in V_N$ acting by sending f to the function

$$\begin{gathered} V_N \to \mathbb{C} \\ v \mapsto \psi(\langle v, w\rangle) \cdot f(v) \end{gathered}$$

□

By Lemma 3, we have

$$Res^{Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)\ltimes V_N}(\Omega^{\otimes k}) \cong \{f : V_N^k \to \mathbb{C}\},$$

which is generated by the character functions $\mathbb{1}_{(v_1,\dots,v_{k+1})}$ for $v_i \in V_N$ (which is 1 at $(v_1, \dots, v_{k+1})$ and 0 at all other elements of V_N^{k+1}).

The fixed points

$$(\Omega^{\otimes k+1})^{\mathbb{H}_N(\mathbb{F}_q)} \cong (Res^{Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)\ltimes V_N}(\Omega^{\otimes k+1}))^{V_N} \tag{12}$$

(the isomorphism follows since $\mathbb{F}_q \subset \mathbb{H}_N(\mathbb{F}_q)$ acts trivially on Ω) are then generated by $\mathbb{1}_{(v_1,\dots,v_{k+1})}$ for $v_i \in V_N$ such that for every $u \in V_N$,

$$\begin{gathered} \mathbb{1}_{(v_1,\dots,v_{k+1})} = u(\mathbb{1}_{(v_1,\dots,v_{k+1})}) = \\ \psi(\langle u, v_1\rangle) \cdot \dots \cdot \psi(\langle u, v_{k+1}\rangle) \cdot \mathbb{1}_{(v_1,\dots,v_{k+1})} = \\ \psi(\langle u, v_1 + \dots, v_{k+1}\rangle) \cdot \mathbb{1}_{(v_1,\dots,v_{k+1})} \end{gathered}$$

meaning that for every $u \in V_N$

$$\langle u, v_1 + \dots, v_{k+1}\rangle = 0,$$

which is equivalent to $v_1 + \dots + v_{k+1} = 0$.

Therefore, (12) is identified with the space of functions on V_N^{k+1} with support on

$$\{(v_1, \dots, v_{k+1}) \in V_N^{k+1} \mid v_1 + \dots + v_{k+1} = 0 \in V_N\}. \tag{13}$$

Therefore, the isomorphism (10) follows from Lemma 3 the isomorphism between (13) and V_N^k.

Now, by duality,

$$End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes k+1}) \cong Hom_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(1,\Omega^{\otimes k+1}),$$

which is identified with the fixed points

$$(\Omega^{\otimes k+1})^{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)} \cong ((\Omega^{\otimes k+1})^{\mathbb{H}_N(\mathbb{F}_q)})^{Sp(V_N)}. \tag{14}$$

By (10), (14) is isomorphic to the $Sp(V_N)$ fixed points

$$(Res^{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)}(\Omega^{\otimes k}))^{Sp(V_N)},$$

which are isomorphic to

$$Hom_{Sp(V_N)}(1,\Omega^{\otimes k}).$$

Therefore, for $t\in\mathbb{C}\smallsetminus\mathbb{Z}$, for $N>>0$,

$$End_{\mathscr{C}_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}}(\omega^{\otimes k+1}) \cong$$
$$Hom_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(1,\Omega^{\otimes k+1}) \cong Hom_{Sp(V_N)}(1,\Omega^{\otimes k})$$

the dimension of which, by Lemma 3, can be calculated as the number of orbits of $Sp(V_N)$ on V_N^k.

For example, we can verify that

$$End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega) = 1$$

(the number of $Sp(V_N)$ orbits of $V_N^0 = 0$), and therefore that $\omega \in \mathscr{C}_{Sp(V_t)\ltimes\mathbb{H}_t(\mathbb{F}_q)}$ is simple.

We get (8) since the dimension of

$$End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes 2}) = 2$$

because V_N has two $Sp(V_N)$-orbits: 0 and $V_N\smallsetminus 0$.

Finally, we get (9) since

$$End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes 3}) = 2q+2$$

because V_N^2 has $2q+2$ $Sp(V_N)$-orbits, which are as follows:

(A) 1 orbit of the form
$$\{(0,0)\},$$

(B) $q+1$ orbits of the form
$$\{(v,0)\mid v\in V_N\},\ \{(0,v)\mid v\in V_N\},$$
$$\{(v,\lambda\cdot v)\mid v\in V_N\},\quad \text{for } \lambda\in\mathbb{F}_q^\times$$
corresponding to "slopes" $\lambda\in\mathbb{P}^1(\mathbb{F}_q)$.

(C) q orbits of the form
$$\{(v,w)\mid \langle v,w\rangle=\mu,\ v,w \text{ linearly independent}\},\quad \text{for } \mu\in\mathbb{F}_q.$$

4. Semisimplicity

In this section, we address the semisimplicity statement of Theorem 1. We will begin with some general observations.

In an algebra $\mathcal{A}$ of the form

$$\mathcal{A} = \prod_{k=1}^{n} M_k(\mathbb{C}), \tag{15}$$

the general trace is of the form

$$\widetilde{tr}(A) = \sum_{k=1}^{n} b_k \cdot tr(A_k),$$

for $A = (A_1, \ldots, A_n) \in \mathcal{A}$ with $A_k \in M_k(\mathbb{C})$, for some $b_k \neq 0$.

Lemma 4. *For $A \in \mathcal{A}$, if $\widetilde{tr}(A) \neq 0$, then for every $N \in \mathbb{N}$, there exists an $M > N$ such that $\widetilde{tr}(A^M) \neq 0$.*

Proof. Write $A = (A_1, \ldots, A_n) \in \mathcal{A}$, for $A_k \in M_k(\mathbb{C})$. Without loss of generality, the matrices A_k are in Jordan form. Thus, our statement reduces to the following:

Claim: Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ be different numbers. If, for some numbers $\alpha_1, \ldots \alpha_m \in \mathbb{C}$

$$\alpha_1\lambda_1 + \cdots + \alpha_m\lambda_m \neq 0,$$

then for all N, there exists an $M > N$ such that

$$\alpha_1\lambda_1^M + \cdots + \alpha_m\lambda_m^M \neq 0.$$

Without loss of generality $\lambda_1, \ldots, \lambda_m \neq 0$. Then the matrix

$$\Lambda = \begin{pmatrix} \lambda_1^N & \lambda_1^{N-1} & \ldots & \lambda_1^{N+m-1} \\ \vdots & \vdots & & \vdots \\ \lambda_m^N & \lambda_m^{N-1} & \ldots & \lambda_m^{N+m-1} \end{pmatrix}$$

is non-singular by the Vandermonde determinant.

Thus, there exists a vector $v = (v_1, \ldots v_m)^T$ such that $(\lambda_1, \ldots, \lambda_m)^T = \Lambda \cdot v$. Thus, if

$$(\alpha_1, \ldots, \alpha_m) \cdot \Lambda = 0,$$

then

$$\alpha_1\lambda_1 + \cdots + \alpha_m\lambda_m = (\alpha_1, \ldots, \alpha_m)\Lambda v =$$
$$(\alpha_1, \ldots, \alpha_m)0 = 0.$$

Contradiction.

□

Recall that, for a locally finite, $\mathbb{C}$-linear additive category with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product and strong duality, one can form the *semisimplification* (see [1], Section 6.1) by quotienting out *negligible* morphisms (i.e. morphisms $f : X \to Y$ such that for every morphism $g : Y \to X$, the trace $tr(g \circ f)$ is 0).

Lemma 4 then gives the following

Proposition 5. *The (pseudo-abelian envelope of the) semisimplification of a locally finite, $\mathbb{C}$-linear additive category $\mathscr{C}$ with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product and strong duality which is generated by an object X, is semisimple if and only if for every $a \in End(X^{\otimes n})$,*

$$tr(a) \neq 0 \;\Rightarrow\; \forall N \in \mathbb{N}\, \exists M > N \;\; tr(a^M) \neq 0. \tag{16}$$

Proof. Necessity follows from Lemma 4 (since this is a general form of a trace in a semisimple algebra).

To prove sufficiency, given the assumption, if $a \in End(X^{\otimes n})$ is non-negligible, say, $tr(ab) \neq 0$ for a $b \in End(X^{\otimes n})$, then for every $N \in \mathbb{N}$ there exists an $M > N$ such that

$$tr((ab)^M) \neq 0,$$

and hence $a \notin Jac(End(X^{\otimes n}))$. Thus, the semisimplification of $\mathscr{C}$ is semisimple.

□

This implies the following

Proposition 6. *If the category $\mathscr{C}(\mathcal{T})$ for a T-algebra $\mathcal{T}$ is semisimple (pre-Tannakian), then for every sub-T-algebra $\mathcal{V} \subseteq \mathcal{T}$, the (pseudo-abelian envelope of the) semisimplification of $\mathscr{C}(\mathcal{V})$ is semisimple pre-Tannakian.*

Proof. The condition (16) remains true in $\mathcal{V}$. □

Applying this to our sub-T-algebra $\mathcal{V}_t$ (corresponding to the Weil-Shale representation) of the T-algebra $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}$ corresponding to $Rep(GL_t(\mathbb{F}_q))$, we obtain that the semisimplification

$$\widetilde{\mathscr{C}}_{Sp(V_t)\ltimes\mathbb{H}_t(\mathbb{F}_q)}$$

of $\mathscr{C}_{Sp(V_t)\ltimes\mathbb{H}_t(\mathbb{F}_q)}$ is a semisimple category, giving the existence of a semisimple pre-Tannakian category as claimed in Theorem 1.

What remains to show is that (1) holds for values of q^t not equal to $\pm 1, \pm q$:

Lemma 7. *For all t with $q^t \neq \pm 1, \pm q$,*

$$dim(End_{\widetilde{\mathscr{C}}_{Sp(V_t)\ltimes\mathbb{H}_t(\mathbb{F}_q)}}(\omega^{\otimes 3})) = 2q+2. \tag{17}$$

Proof. Since we have proved that the semisimplification of $\mathscr{C}_{Sp(V_t)\ltimes\mathbb{H}_t(\mathbb{F}_q)}$ is semisimple, it suffices to prove that

$$det(tr(a_i \circ a_j)) \tag{18}$$

is non-zero, where $a_1, \dots, a_{2q+2}$ is a basis of $End_{\mathscr{C}_{Sp(V_t)\ltimes\mathbb{H}_t(\mathbb{F}_q)}}(\omega^{\otimes 3})$.

The generators $a_1, \dots, a_{2q+2}$ can be identified with the orbits of types (A), (B), (C) at the end of Section 3. The number

$$tr(a_i \circ a_j) \tag{19}$$

can be non-zero when the orbit a_i contains a vector

$$(u,v) = \left(\begin{array}{c|c} u_1 & v_1 \\ u_2 & v_2 \end{array}\right)$$

(where $u_1, u_2, v_1, v_2 \in \mathbb{F}_q^N$) and the orbit a_j contains the vector

$$\left(\begin{array}{c|c} u_2 & v_2 \\ u_1 & v_1 \end{array}\right).$$

In that case, the number (19) is equal to the number of vectors in the orbit a_i (equivalently a_j). We shall refer to the orbits a_i, a_j as *contragradient*. We find that each orbit of type (A), (B) is contragradient to iteself, while the orbit of type (C) corresponding to λ is contragradient to the orbit of type (C) corresponding to $-\lambda$.

The number of elements of the orbit of type (A) is 1. The number of elements of an orbit of type (B) is equal to

$$q^{2n} - 1.$$

The number of elements of the orbit of type (C) corresponding to a $\lambda \neq 0$ is equal to

$$(q^{2n}-1)q^{2n-1}.$$

The number of elements of the orbit of type (C) corresponding to $\lambda = 0$ is

$$(q^{2n}-1)(q^{2n-1}-q).$$

Thus, the number (18) is a polynomial in q^{2n} with factors q^{2n}, $q^{2n}-1$, $q^{2n}-q^2$. Therefore, its zeros in $x=q^n$ are $x=0,\pm 1,\pm q$. The statement follows. $\square$

References

[1] P. Deligne. La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel. *Algebraic groups and homogeneous spaces*, 209-273, Tata Inst. Fund. Res. Stud. Math., 19, Tata Inst. Fund. Res., Mumbai, 2007.

[2] P. Deligne: Letter to S. Kriz, November 22, 2023.

[3] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik: *Tensor categories.* Math. Surveys Monogr., 205 American Mathematical Society, Providence, RI, 2015, xvi+343 pp.

[4] R. Howe. On the character of Weil's representation, *Trans. A. M. S.* 177 (1973), 287-298.

[5] N. M. Katz. Larsen's alternative, moments, and the monodromy of Lefschetz pencils. *Contributions to automorphic forms, geometry, and number theory*, 521-560. Johns Hopkins University Press, Baltimore, MD, 2004.

[6] F. Knop. A construction of semisimple tensor categories. *C. R. Math. Acad. Sci. Paris C.* 343, 2006.

[7] F. Knop. Tensor Envelopes of Regular Categories. *Adv. Math.* 214, 2007.

[8] S. Kriz. Arbitrarily High Growth in Quasi-Pre-Tannakian Categories, 2023. Available at https://krizsophie.github.io/ACUCategoryFinal23123.pdf

[9] S. Kriz. Quantum Delannoy Categories, 2023. Available at https://krizsophie.github.io/SquaredGrowth23088.pdf

[10] A. Weil. Sur certains groupes d'operateurs unitaires, *Acta Math.* 111 (1964), 143-211.