GROUP-COMPLETED FI-MODULES AND PRE-TANNAKIAN INTERPOLATIONS OF GA_n

SOPHIE KRIZ

1. Introduction

In this paper, FI-modules are functors from the category FI whose objects are finite sets (equivalently the sets $[n] = \{1, \dots, n\}$) and morphisms are injections to $\mathbb{C}$-vector spaces. The $\mathbb{C}$-linear abelian category FI-Mod of FI-modules (and natural transformations) has been studied extensively, see e.g. [1, 5, 6, 8, 11, 10, 12, 13, 14, 16, 17]. A particularly useful variant is the quotient FI-Mod$_{gen}$ of FI-Mod by the Serre subcategory of *torsion modules* consisting of *torsion* elements (i.e those which are sent to 0 by one of the morphisms of FI). The category FI-Mod$_{gen}$ is a useful tool for capturing phenomena of representation stability [1].

In this paper, we explore structures of a tensor category on FI-Mod and FI-Mod$_{gen}$ and the possibility of embedding those categories into *rigid* tensor categories, i.e. $\mathbb{C}$-linear abelian categories with strong duality. (Here and from now on, we restrict attention to finitely generated FI-modules, and will in fact change notation to use FI-Mod, resp. FI-Mod$_{gen}$ to denote their full subcategories on finitely generated FI-modules without further decorations.)

To begin with, there are *two* immediately visible tensor category structures on FI-Mod, FI-Mod$_{gen}$. One is the *level-wise structure*, i.e. for FI-modules M, N,

$$(M \otimes N)[n] = M[n] \otimes_{\mathbb{C}} N[n] \tag{1}$$

This structure was investigated for example in [1, 8, 11, 17]. The tensor structure $\otimes$ is most easily understood by the fact ([18], Exercise 5.29) that

$$M \mapsto \operatorname*{colim}_n M[n]$$

gives an equivalence between FI-Mod$_{gen}$ and smooth (finitely generated) representations of the countably infinite symmetric group Σ_∞, by which we mean the group of those permutations on $\mathbb{N} = \{1, 2, \dots\}$ that move only finitely many elements non-trivially, where *smooth* means that the stabilizer of any element contains a subgroup of the form $\Sigma_{\infty-n}$, which means the stabilizer of $[n]$. (In this note, we shall use Σ to denote a symmetric group, to avoid confusion with the notation for Specht modules.) Now smooth Σ_∞-representations have a tensor product (the tensor product of representations over $\mathbb{C}$) to which the tensor product in FI-Mod$_{gen}$ given by (1) corresponds.

While the resulting tensor category of smooth Σ_∞-representations is not rigid (i.e. does not have strong duality), it embeds naturally into any of the categories $\underline{Rep}(\Sigma_t)$ of P. Deligne [2], which are semisimple for $t \notin \mathbb{N}_0$. The embedding is defined as follows: Morphisms from $[m]$ to $[n]$ in $\underline{Rep}(\Sigma_t)$ can be identified with the free $\mathbb{C}$-vector spaces on *equivalence relations* on $[m] \amalg [n]$ (see [9] for more detail). Then morphisms of smooth Σ_∞-representations

$$V^{\otimes m} \to V^{\otimes n}$$

where $V = \mathbb{C}\mathbb{N}$ is the "standard" representations of Σ_∞ correspond to the free vector space on those equivalence relations whose every equivalence class has a non-empty intersection with $[m]$. Thus, for values $t \notin \mathbb{N}_0$, we obtain an embedding of FI-Mod$_{gen}$ with the tensor product $\otimes$ into the semisimple pre-Tannakian category $\underline{Rep}(\Sigma_t)$.

There is, however, another interesting tensor category structure on FI-Mod, FI-Mod$_{gen}$, which can be called the *Day product* and which we will denote by $\boxtimes$, where for FI-mdoules M, N, the FI-module $M \boxtimes N$ is defined by taking the $FI \times FI$-module $(M_m \otimes N_n)_{m,n \in \mathbb{N}_0^2}$ and applying the left Kan extension along the functor

$$FI \times FI \to FI$$

given by disjoint union. This is a tensor category structure which is, moreover, an exact functor (see Section 3 for details), so it defines a symmetric monoidal structure on FI-Mod$_{gen}$. (Note, while the isomorphism class of the corresponding smooth Σ_∞-representation is easy to define, it is not immediately obvious how to define this symmetric monoidal structure directly in that context: this is analogous to the difference between symmetric spectra and May spectra in stable homotopy theory.)

The question we investigate in the present paper is fully embedding FI-Mod$_{gen}$ into a pre-Tannakian category.

The basic idea is that we can easily embed the pseudo-abelian envelope of the category of (Day) tensor powers $X^{\boxtimes n}$ of the "basic" object into the category $(FI_c^{\pm})^{Op}$ whose objects are pairs $(m,n) \in \mathbb{N}_0^2$ and morphisms, for $(m,n),(p,q) \in \mathbb{N}_0^2$, are

$$Mor_{(FI_c^{\pm})^{Op}}((m,n),(p,q)) = \mathbb{C}Mor_{FI}([n] \amalg [p],[m] \amalg [q]).$$

When composing, one encounters "circles," which are replaced by multiplication by a given constant c. This is a variant (by replacing bijections with injections) of the category $\underline{Rep}(GL_c)$ considered by P. Deligne and J. S. Milne in [3], Subsections 1.26, 1.27, and later by P. Deligne in [2], Section 10. While this is a tensor embedding, the target category is not abelian. The main purpose of this paper is to show that one does in fact have a (non-semisimple) rigid pre-Tannakian category $FI_c^{\pm}$-Mod$_{gen}$ of *generic $FI_c^{\pm}$-modules* into which FI-Mod$_{gen}$ (with the Day product) embeds as a full tensor subcategory. In addition, we also identify the simple objects.

Theorem 1. *The simple objects $\mathscr{Y}_{\lambda,\mu}$ of the generic category of $FI_c^{\pm}$-modules are indexed by pairs of Young diagrams λ, μ. Further, the fusion rules of tensor products of these simple generic $FI_c^{\pm}$-modules exactly correspond to the fusion rules of tensor products of the simple objects $Y_{\lambda,\mu}$ in $\underline{Rep}(GL_{c-1})$.*

Comment: A. Snowden proposed a possible reinterpretation of the results of the present paper using Schur-Weyl duality. In [18], Theorem 6.3, it is proved that the category of generic FI-modules is equivalent to the category of polynomial representations of the infinite affine group $GA_\infty = \bigcup GA_n$. Here, *polynomial* means direct sums of summands of the basic representation. Moreover, the tensor product of representations corresponds to the Day product of FI-modules.

The suggestion is that one could embed FI-modules into interpolated categories of the form "$\underline{Rep}(GA_c)$." Furthermore, since GA_n is a semidirect product of GL_{n-1} with an additive group, the appearance of $\underline{Rep}(GL_{c-1})$ is explained.

On a technical level, a diagrammatic definition may amount to the same structures we describe here. It would be interesting to see if a general measure-theoretic approach can be used to construct non-semisimple interpolated pre-Tannakian categories.

2. The Category of $FI_c^{\pm}$-Modules

Fix a value $c \in \mathbb{C}$. Let us consider the category $FI_c^{\pm}$ defined by taking

$$Obj(FI_c^{\pm}) = \mathbb{N}_0^2$$

and for pairs $(m,n),(p,q) \in Obj(FI_c^{\pm})$, the morphisms

$$Mor_{FI_c^{\pm}}((m,n),(p,q)) = \mathbb{C}Mor_{FI}([m] \amalg [q],[p] \amalg [n]). \tag{2}$$

Composition will depend on the fixed value of $c \in \mathbb{C}$. To describe the composition in $FI_c^{\pm}$, we first give a graphical representation of the morphisms

$$(m,n) \to (p,q)$$

freely generating (2) in $FI_c^{\pm}$, following the diagramatic expressions of morphisms of the $\underline{Rep}(GL_c)$ described by P. Deligne in [2], Section 10.1:

On the left, we have m points labeled with the sign "+", and n points labeled with the sign "-," (representing the source of the morphism), and on the right, we have p and q points labeled "+" and "-," respectively (representing the target). The morphism then corresponds to a perfect pairing (or matching) of the disjoint union of the set of m points labeled "+" on the left, the set of q points labeled "-" on the right, a subset of the set of n points labeled "-" on the left, and a subset of the set of p points labeled "+" on the right, such that every point is paired with either a point of the same sign on the opposite side, or a point of opposite sign on the same side. For example, Figures 1 and 2 show an example of the graphical representation of a generating morphism $(2,5) \to (4,3)$. Figure 1 displays the injection, while Figure 2 shows the corresponding morphism in $FI_c^{\pm}$.

Composition in $FI_c^{\pm}$ is then described by defining it for free generating morphisms of (2) by placing two diagrams (as in Figure 2) next to each other, aligning the points corresponding to the intermediate pairs, and composing as in [2], Section 10.1: the lines and segments of the pairing are connected and a closed circuit is deleted and the obtained diagram is multiplied by c to form the final morphism of $FI_c^{\pm}$ forming the original morphisms' composition. We then may extend composition $\mathbb{C}$-linearly to define it for all composable morphisms $FI_c^{\pm}$.

It is formal to verify that $FI_c^{\pm}$ defined this way then indeed forms a category. We note that while $FI_c^{\pm}$, $(FI_c^{\pm})^{Op}$ pre-additive categories linear over $\mathbb{C}$, adding direct sums and taking a pseudoabelian envelope does not make an abelian category.

Definition 2. *We define $FI_c^{\pm}$-modules to be functors from $FI_c^{\pm}$ to vector spaces over $\mathbb{C}$*

$$FI_c^{\pm} \to \mathbb{C}\text{-}Vect. \tag{3}$$

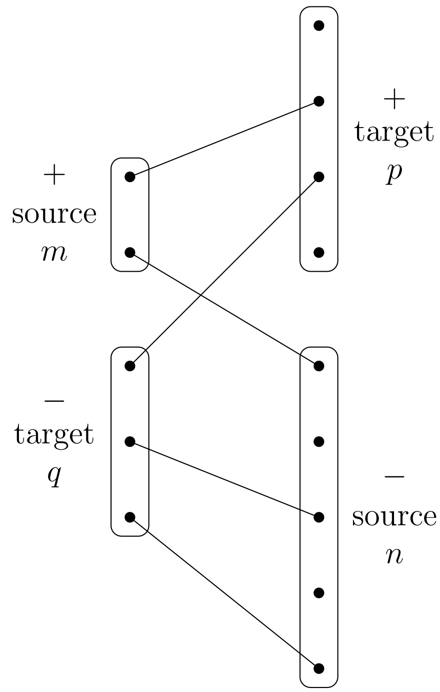

FIGURE 1. An injection $[m] \amalg [q] \to [p] \amalg [n]$, for $m = 2$, $n = 5$, $p = 4$, $q = 3$

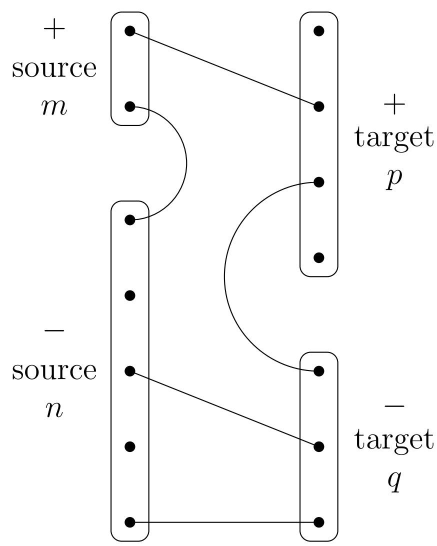

FIGURE 2. A generator of $Mor_{FI_c^{\pm}}((m, n), (p, q))$, for $m = 2$, $n = 5$, $p = 4$, $q = 3$

(Recall that, in this paper, we are restricting attention to *finitely generated* $FI_c^{\pm}$-modules.)

For a pair $(m,n) \in \mathbb{N}_0$, let us denote its endomorphism algebra in $FI_c^{\pm}$ by

$$\Sigma^c_{m,n} := End_{FI_c^{\pm}}((m,n)).$$

For $c \in \mathbb{C} \smallsetminus \mathbb{Z}$, the $\mathbb{C}$-algebra $\Sigma^c_{m,n}$ is semisimple (see [2], Section 10), and the irreducible $\Sigma^c_{m,n}$-representations are indexed by pairs of Young diagrams λ and μ such that $m-n = |\lambda| - |\mu|$. Write $\mathcal{Y}_{\lambda,\mu}(m,n)$ for the simple $\Sigma^c_{m,n}$-representation corresponding to λ, μ. This follows from the decomposition of the tensor of copies of the basic object and its dual

$$X^{\otimes m} \otimes (X^{\vee})^{\otimes n} \tag{4}$$

in $\underline{Rep}(GL_c)$ into simple objects $Y_{\lambda,\mu}$. The dimension of $\mathcal{Y}_{\lambda,\mu}(m,n)$ as a $\Sigma^c_{m,n}$-representation is the multiplicity of $Y_{\lambda,\mu}$ in (4):

$$dim(\mathcal{Y}_{\lambda,\mu}(m,n)) = dim(Hom_{\underline{Rep}(GL_c)}(Y_{\lambda,\mu}, X^{\otimes m} \otimes (X^{\vee})^{\otimes n}).$$

An $FI_c^{\pm}$-algebra F then determines, at each pair (m,n), a $\Sigma^c_{m,n}$-representation $F(m,n)$. Note that we have

$$dim(\Sigma^c_{m,n}) = |Mor_{FI}([m+n],[m+n])| = (m+n)!.$$

However, $\Sigma^c_{m,n}$ is not equal to the free representation of the symmetric group on $m+n$ elements, since its algebra structure is different and depends on c.

Similarly as for classical FI-modules, $FI_c^{\pm}$-modules form a category by taking morphisms between two functors of the form (3) to be natural transformations. This category, which we denote $FI_c^{\pm}$-Mod, is clearly a $\mathbb{C}$-additive category, by taking for $FI_c^{\pm}$-modules $F, G : \mathscr{C}^{Op} \to \mathbb{C}$-Vect, for $(m,n) \in \mathbb{N}_0^2$

$$(F \oplus G)(m,n) = F(m,n) \oplus G(m,n)$$

(the proof that this definition forms a $FI_c^{\pm}$-module is analogous to the proof for the similar statement for FI-modules).

For an element $(m,n) \in \mathbb{N}_0^2$, we may consider the corresponding *representable* $FI_c^{\pm}$*-module*

$$\underline{(m,n)} : FI_c^{\pm} \to \mathbb{C}\text{-Vect}$$

defined by sending a pair $(p,q) \in \mathbb{N}_0^2$ to the free $\mathbb{C}$-vector space

$$\underline{(m,n)}(p,q) = \mathbb{C}Mor_{FI_c^{\pm}}((m,n),(p,q)) =$$
$$\mathbb{C}Mor_{FI}([m] \amalg [q], [p] \amalg [n]).$$

Following the defintion of torsion FI-modules, we can make the following definition of *torsion $FI_c^{\pm}$-modules*:

Definition 3. *For an $FI_c^{\pm}$-module $F : FI_c^{\pm} \to \mathbb{C}$-Vect, for a pair $(m,n) \in \mathbb{N}_0^2$, an element $x \in F(m,n)$ is called* torsion *if there exists a pair $(p,q) \in \mathbb{N}_0^2$ and a morphism $f \in Mor_{FI_c^{\pm}}((m,n),(p,q))$ such that*

$$(F(f))(x) = 0.$$

At each pair $(m,n) \in \mathbb{N}_0^2$, the set of torsion elements of $F(m,n)$ forms a $\mathbb{C}$-subspace of $F(m,n)$, and sending (m,n) to this subspace defines a functor (3), and thus an $FI_c^{\pm}$-module, which we denote TF. We say that F is a torsion $FI_c^{\pm}$-module *if $TF = F$.*

We can take the category of torsion $FI_c^{\pm}$-modules, which we shall denote by $TFI_c^{\pm}$-Mod to be the full subcategory of $FI_c^{\pm}$-modules on these objects.

Proposition 4. *The category of torsion $FI_c^{\pm}$-modules $FI_c^{\pm}$-Mod$_{tor}$ forms a Serre subcategory of $FI_c^{\pm}$-Mod.*

□

Therefore, we can define the category $FI_c^{\pm}$-Mod$_{gen}$ of *generic $FI_c^{\pm}$-modules* as the Serre quotient of $FI_c^{\pm}$-Mod by $FI_c^{\pm}$-Mod$_{tor}$.

3. Symmetric Monoidal Structure - the "Day Product"

In this section, we will explicitly describe a tensor category structure on $FI_c^{\pm}$-modules which we call the *Day product*, defined as an analogue of the Day product on FI-Mod. For $FI_c^{\pm}$-modules

$$F, G : FI_c^{\pm} \to \mathbb{C}\text{-Vect},$$

we can first define the functor

$$F\overline{\boxtimes}G : FI_c^{\pm} \times FI_c^{\pm} \to \mathbb{C}\text{-Vect}$$

(defined by taking $F\overline{\boxtimes}G((m,n),(p,q)) = F(m,n) \otimes_{\mathbb{C}} G(p,q)$). We then define their Day product $F \boxtimes G$ as the left Kan extension

$$\begin{tikzcd} FI_c^{\pm} \times FI_c^{\pm} \arrow[r, "F\overline{\boxtimes}G"] \arrow[d, "+"'] & \mathbb{C}\text{-Vect} \\ FI_c^{\pm} \arrow[ur, dashed, "F\boxtimes G"'] & \end{tikzcd}$$

where the vertical map

$$+ : FI_c^{\pm} \times FI_c^{\pm} \to FI_c^{\pm}$$

is defined by taking coordinate-wise addition

$$(m, n) + (p, q) = (m + p, n + q).$$

Recall that, for $x \in \mathbb{N}_0^2$, $F \boxtimes G(x)$ is defined as the coequalizer of two functors

$$\phi, \psi : \bigoplus_{\alpha:a\to a', \beta:b\to b', \theta:a'+b'\to x} F(a) \otimes_{\mathbb{C}} G(b) \to \bigoplus_{\theta:a+b\to x} F(a) \otimes_{\mathbb{C}} G(b)$$

(all sums are over choices of morphisms in $FI_c^{\pm}$), where ϕ is defined by sending a direct summand $F(a) \otimes_{\mathbb{C}} G(b)$ of the source corresponding to morphisms $\alpha : a \to a', \beta : b \to b', \theta : a' + b' \to x$ in $FI_c^{\pm}$ to the summand $F(a') \otimes_{\mathbb{C}} G(b')$ of the target corresponding to θ by the linear map $F(a') \otimes G(b')$, and ψ is defined by sending such a summand of the source to the summand $F(a) \otimes G(b)$ corresponding to $\theta \circ (\alpha + \beta)$ by the identity.

This construction parallels the "Day product" of FI-modules already mentioned in the Introduction. In fact, the reader may use the above paragraphs as a review of the Day product of FI-modules by replacing $FI_c^{\pm}$ by $\mathbb{C}FI$. We also note that by using the characterization of FI-modules as modules over the twisted commutative algebra $\underline{\mathbb{C}}$ (see [15] and [18], Exercise 2.8), the Day product of FI-modules M, N can also be described as

$$M \circledast_{\underline{\mathbb{C}}} N.$$

We then have a functor

$$\Phi : FI\text{-Mod} \to FI_c^{\pm}\text{-Mod}$$

given by left Kan extension along the inclusion

$$\mathbb{C}FI \hookrightarrow FI_c^{\pm},$$

which is then automatically a tensor functor with respect to the Day product $\boxtimes$.

We now claim the following two standard facts about representable $FI_c^{\pm}$-modules:

Proposition 5. *For pairs $a_0, b_0 \in \mathbb{N}_0^2$, we have*

$$\underline{a_0} \boxtimes \underline{b_0} \cong \underline{(a_0 + b_0)}.$$

$\square$

Proposition 6. *Every (finitely generated) $FI_c^{\pm}$-module is, by definition, a quotient of a direct sum of (finitely many) objects of the form* $\underline{(m_i, n_i)}$.

$\square$

Note that, by Propositions 5 and 6, $\underline{(0,0)}$ is the unit of the symmetric monoidal structure on $FI_c^{\pm}$-modules.

Proposition 7. *The functors*

$$? \boxtimes M : FI\text{-}Mod \to FI\text{-}Mod \tag{5}$$

$$? \boxtimes M : FI_c^{\pm}\text{-}Mod \to FI_c^{\pm}\text{-}Mod \tag{6}$$

are exact for fixed objects M, N.

Proof. Since both functors are given by left Kan extensions, they are automatically right exact. To prove left exactness, we note the following general result:

Lemma 8. *Let $A : \mathscr{C} \to \mathscr{A}$ be a functor where $\mathscr{A}$ is an abelian category and let $F : \mathscr{C}^{Op} \to Set$ be a functor. Suppose further that for objects $a, b, c \in Obj(\mathscr{C})$ and morphisms*

$$f : a \to b$$

$$g : a \to c$$

in $\mathscr{C}$, and elements $y \in F(b)$, $z \in F(c)$ such that

$$(F(f))(y) = (F(g))(z),$$

there exists an object $d \in Obj(\mathscr{C})$ morphisms

$$h : b \to d,$$

$$k : c \to d$$

and $u \in F(d)$ such that

$$(F(h))(u) = y$$

$$(F(k))(u) = z$$

and $h \circ f = k \circ g$. Then

$$F \otimes_{\mathscr{C}} A \tag{7}$$

is an exact functor.

Proof. Under the assumptions, the left derived version of (7) is calculated by a homotopy colimit over a simplicial set which is homotopy discrete, which is why the higher left derived functors vanich. (Right exactness is, again, automatic.) □

Now to prove the left exactness of (5), apply the Lemma to $\mathscr{C} = FI \times FI$,

$$F(x,y) = Mor_{FI}(x \amalg y, z)$$

for z fixed.

To prove (6), apply similarly an $\mathbb{C}$-additive version of the Lemma with FI replaced by $FI_c^{\pm}$.

□

By Proposition 7, the Day tensor product $\boxtimes$ passes to a tensor structure on FI-Mod^{gen}, $FI_c^{\pm}$-Mod^{gen} and Φ induces a $\boxtimes$-tensor functor

$$\Phi^{gen} : FI\text{-Mod}^{gen} \to FI_c^{\pm}\text{-Mod}^{gen}.$$

We note that Φ^{gen} is also an exact functor by Lemma 8.

4. Simple Generic $FI_c^{\pm}$-Modules and Strong Duality

Proposition 9. *The objects $\underline{(m,n)}$ are strongly dualizable in $FI_c^{\pm}$-Mod^{gen}, and one has*

$$\underline{(m,n)}^{\vee} = \underline{(n,m)}.$$

Proof. The unit and counit

$$\eta : \underline{(0,0)} \to \underline{(m,n)} \boxtimes \underline{(n,m)} \cong \underline{(m+n,n+m)}$$

$$\epsilon : \underline{(m,n)} \boxtimes \underline{(n,m)} \cong \underline{(m+n,n+m)} \to \underline{(0,0)}$$

are represented by

$$\sigma : [m] \amalg [n] \to [n] \amalg [m]$$

where σ is the shuffle switching $[m]$ and $[n]$, see Figure 3.

One of the triangle identites is represented by Figure 4 below, the other is symmetrical.

□

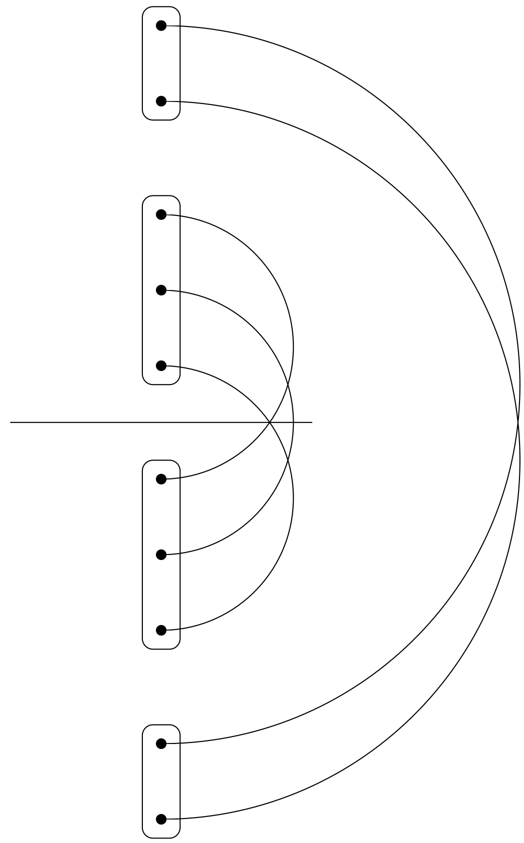

FIGURE 3. The (co)unit of duality of $\underline{(m,n)}^\vee = \underline{(n,m)}$ for $m = 2$, $n = 3$

For every pair of Young diagrams

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_i),$$

$$\mu = (\mu_1, \mu_2, \ldots, \mu_j),$$

define the *principal projective* generic $FI_c^{\pm}$-module $P_{\lambda,\mu}$ as the sub-$FI_c^{\pm}$-module of $\underline{(|\lambda|, |\mu|)}$ generated (in $FI_c^{\pm}$-Mod^{gen}) by $\mathcal{Y}_{\lambda,\mu}(|\lambda|, |\mu|)$.

Proposition 10. *For a pair* $(m,n) \in \mathbb{N}_0$

$$\underline{(m,n)} = \bigoplus_{|\lambda|=m, |\mu|=n} P_{\lambda,\mu}$$

where the direct sum runs over all Young diagrams λ, μ *with* $|\lambda| = m$, $|\mu| = n$.

$\square$

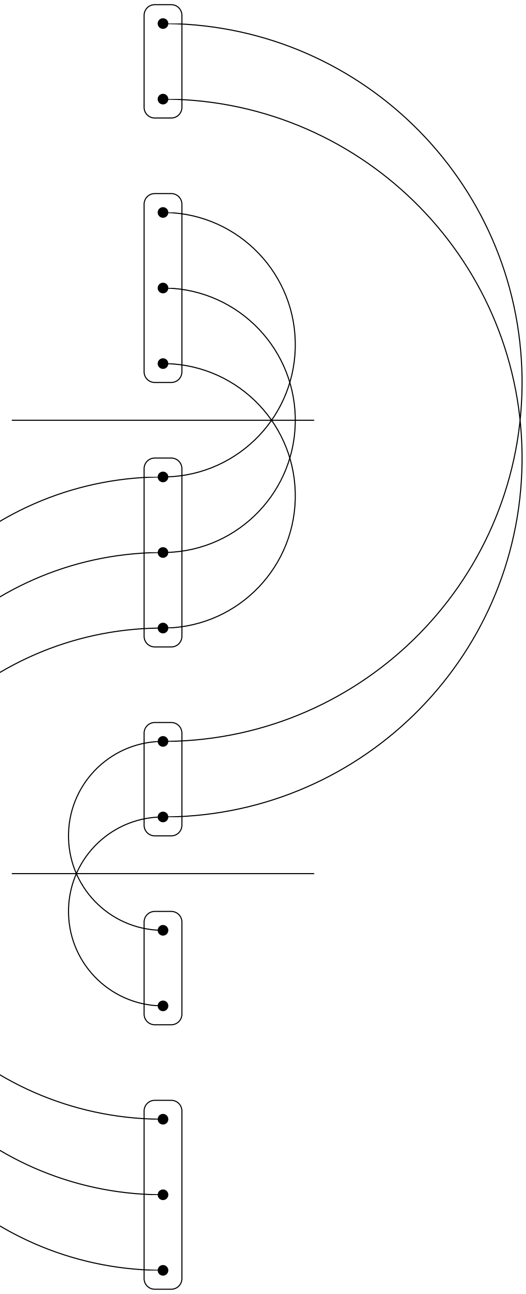

FIGURE 4. Triangle identity for $\underline{(m,n)}^\vee = \underline{(n,m)}$ for $m = 2$, $n = 3$

For $N \in \mathbb{N}_0$, denote

$$\lambda_N^+ = (N - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_i).$$

Let us now define the $FI_c^{\pm}$-module $\mathscr{Y}_{\lambda,\mu}$ as the quotient of the sub-$FI_c^{\pm}$-module of $P_{\lambda,\mu}$ generated (in $FI_c^{\pm}$-Mod^{gen}) by

$$\mathcal{Y}_{\lambda^+_{|\lambda|+\lambda_1},\mu}(|\lambda| + \lambda_1, |\mu|) \tag{8}$$

by the submodule generated by all $\mathcal{Y}_{?,\mu'}(|?|, |\mu'|)$ with $|\mu'| < |\mu|$.

This is analogous to the definition of the "Spechtral FI-module" $\mathscr{S}_\lambda$ corresponding to a Young diagram λ as the submodule generated by $S_{\lambda^+_{|\lambda|+\lambda_1}}$ of the principal projective FI-module P_λ in [18] before Proposition 2.7. The reason we need to consider subquotients is the duality, which we discuss in more detail below. In our present setting, Proposition 2.7 of [18] has the following analogue:

Proposition 11. *For a pair $(m, n) \in \mathbb{N}_0^2$ with $m \geq |\lambda| + \lambda_1$, $n \geq |\mu|$, we have*

$$\mathscr{Y}_{\lambda,\mu}(m, n) = \mathcal{Y}_{\lambda^+_m,\mu}(m, n). \tag{9}$$

Proof. Let $m \geq |\lambda| + \lambda_1$, $n \geq |\mu|$. By the Pieri rule, $P_{\lambda,\mu}(m, n)$ then contains a copy of

$$\mathcal{Y}_{\lambda^+_{m-n+|\mu|},\mu}(m, n) \tag{10}$$

along with $\Sigma^c_{m,n}$-representations of the form

$$\mathcal{Y}_{\lambda',\mu'}(m, n)$$

where $\lambda' = \overline{\lambda}^+_{m'}$, with $|\overline{\lambda}| < |\lambda|$, or $|\mu'| < |\mu|$. Thus, by the structure maps of $FI_c^{\pm}$, (8) can only map to (10), as desired.

$\square$

In particular, as generic $FI_c^{\pm}$-modules,

$$\mathscr{Y}_{(1),\emptyset} = \mathrm{Ker}(\underline{(1, 0)} \to \underline{(0, 0)})$$

and

$$\mathscr{Y}_{\emptyset,(1)} = \mathrm{Coker}(\underline{(0, 0)} \to \underline{(0, 1)}).$$

Theorem 12. *The generic $FI_c^{\pm}$-modules $\mathscr{Y}_{\lambda,\mu}$ are exactly the simple objects of $FI_c^{\pm}$-Mod$_{gen}$, and every finitely generated generic $FI_c^{\pm}$-module has a composition series with associated graded pieces isomorphic to a $\mathscr{Y}_{\lambda,\mu}$ for some λ, μ.*

Proof. As objects of $FI_c^{\pm}$-Mod$_{gen}$, the $\mathscr{Y}_{\lambda,\mu}$ must be simple, since at each pair (m,n), the corresponding $\Sigma^c_{m,n}$-representation is simple. We shall follow a proof analogous to the well-know analogue for classical generic FI-modules given, for example in [18].

To prove the claim that every finitely generated generic $FI_c^{\pm}$-module has a finite length filtration whose associated graded pieces are of the form $\mathscr{Y}_{\lambda,\mu}$, we will follow the terminology of [18].

For a Young diagram $\lambda = (\lambda_1, \dots, \lambda_n)$, recall that its *amplitude* is defined to be the sum $\lambda_2 + \lambda_3 + \dots + \lambda_n$. For pairs $(m,n) \in \mathbb{N}_0^2$, we consider the amplitude of a $\Sigma^c_{m,n}$-representation V to be the maximum amplitude of Young diagrams λ such that the simple $\mathcal{Y}_{\lambda,\mu}(m,n)$ $\Sigma^c_{m,n}$ is a summand of V. We additionally define its *coamplitude* to be the maximal total number of boxes $|\mu|$ of Young diagrams μ such that the simple $\mathcal{Y}_{\lambda,\mu}(m,n)$ $\Sigma^c_{m,n}$ is a summand of V. For an $FI_c^{\pm}$-module, we can then define its amplitude, resp. coamplitude, as the supremum of the amplitudes, resp. coamplitudes, of the $\Sigma^c_{m,n}$-representations they give at each pair $(m,n) \in \mathbb{N}_0^2$. Similarly as for FI-modules, every finitely generated $FI_c^{\pm}$-module has a finite amplitude. Since at each degree, the $\Sigma^c_{m,n}$-representations $\mathcal{Y}_{\lambda,\mu}(m,n)$ are defined only for Young diagrams λ, μ such that

$$|\lambda| - |\mu| = m - n,$$

we therefore know that the coamplitude of every finitely generated $FI_c^{\pm}$-module is also finite.

This implies that for every finitely generated generic $FI_c^{\pm}$-module M there exists a finite length filtration

$$0 = F_0M \subset F_1M \subset F_2M \subset \dots \subset F_{n-1}M \subset F_NM = M \tag{11}$$

such that for every $1 \leq n \leq N$, there exist some pair of Young diagrams λ, μ such that the associated graded piece is a subobject (in the category $GFI_c^{\pm}$-Mod) of its corresponding simple object

$$F_iM/F_{i-1}M \subseteq \mathscr{Y}_{\lambda,\mu}. \tag{12}$$

Since we have already proved that the $\mathscr{Y}_{\lambda,\mu}$ are simple as generic $FI_c^{\pm}$-modules, and thus, (12) must be equality, proving the claim.

It follows formally from the fact that every generic $FI_c^{\pm}$-module has a fitlration of the form (11) with associated graded pieces of the form $\mathscr{Y}_{\lambda,\mu}$, that every simple object in the category in $FI_c^{\pm}$-Mod$_{gen}$ is of the form $\mathscr{Y}_{\lambda,\mu}$.

□

Our next goal is to describe the "fusion rules" for the generic $FI_c^\pm$-modules $\mathscr{Y}_{\lambda,\mu}$, and prove their strong dualizability. As a warm-up, let us first describe the fusion rules of Spechtral FI-modules with respect to the Day poduct $\boxtimes$.

Proposition 13. *For Young diagrams*

$$\mathscr{S}_{\lambda_1}\boxtimes\mathscr{S}_{\lambda_2}\cong\bigoplus_{\lambda}\kappa_\lambda^{\lambda_1,\lambda_2}\mathscr{S}_\lambda$$

where $\kappa_\lambda^{\lambda_1,\lambda_2}$ denote the Littlewood-Richardson numbers.

Proof. By the Pieri rule, this statement is equvalent to the statement that

$$(S_\lambda\otimes_{LR}S_\mu)^{\Sigma_n}=\bigoplus_{m+\ell=n}S_\lambda^{\Sigma_m}\otimes_{LR}S_\mu^{\Sigma_\ell}$$

(as $\Sigma_{|\lambda|+|\mu|-n}$-representation, the superscripts denoting fixed points). Since we have assumed that the ground field is $\mathbb{C}$, we can dualize to obtain an equivalent statement involving cofixed points, which holds for all representations V, W (instead of simple S_λ, S_μ), since

$$\begin{gathered}Ind_{\Sigma_N}^{\Sigma_M\times\Sigma_L}((V\otimes W)\otimes_{\mathbb{C}\Sigma_n}\mathbb{C})=\\ \bigoplus\nolimits_{m+\ell=n}Ind_{\Sigma_{N-n}}^{\Sigma_{M-m}\times\Sigma_{L-\ell}}((V\otimes_{\mathbb{C}\Sigma_m}\mathbb{C})\otimes(W\otimes_{\mathbb{C}\Sigma_\ell}\mathbb{C}))\end{gathered}\tag{13}$$

for all L,M,N with $N=L+M$. This statement follows from the fact that, for a bijection

$$f:[M]\amalg[L]\to N,$$

the bijection obtained by restricting away from elements that f sends to a certain choice of n elements, is exactly equivalent to restricting f away from some m elements of $[M]$ and some ℓ elements of $[L]$ for some choice of m,ℓ with $m+\ell=n$.

□

The analogous statement for $FI_c^\pm$-modules is the following

Theorem 14. *The generic simple $FI_c^\pm$-modules $\mathscr{Y}_{\lambda,\mu}$ are strongly dualizable. In fact, there is a tensor functor*

$$\Xi:\underline{Rep}(GL_{c-1})\to FI_c^\pm\text{-}Mod_{gen}$$

(with respect to the Day product on the right hand side) such that

$$\Xi(Y_{\lambda,\mu})=\mathscr{Y}_{\lambda,\mu}.$$

Consequently, the category $FI_c^\pm$-Mod^{gen} of finitely generated generic $FI_c^\pm$-modules has strong duality.

Proof. We have a short exact sequence

$$0 \to \mathscr{S}_{(1)} \to \underline{1} \to \mathscr{S}_\emptyset \to 0, \tag{14}$$

which implies

$$0 \to \mathscr{Y}_{(1),\emptyset} \to \underline{(1,0)} \to \mathscr{Y}_{\emptyset,\emptyset} \to 0, \tag{15}$$

where we note that $\mathscr{Y}_{\emptyset,\emptyset} = \underline{(0,0)}$.

We have "two out of three" for strong duality in tensor categories where the tensor product is exact. Thus, $\mathscr{Y}_{(1),\emptyset}$ is strongly dualizable and has dimesnion $c-1$ (by additivity of dimensions in short exact sequences of strongly dualizable objects), and its strong dual is $\mathscr{Y}_{\emptyset,(1)}$. This already defines a tensor functor

$$\Xi : \underline{Rep}(GL_{c-1}) \to FI_c^{\pm}\text{-Mod}^{gen}, \tag{16}$$

which maps

$$\Xi(X) = \mathscr{Y}_{(1),\emptyset}$$
$$\Xi(X^\vee) = \mathscr{Y}_{\emptyset,(1)}.$$

Moreover, from the case of FI-Mod^{gen} (Proposition 13), we already know that $\mathscr{Y}_{\lambda_1,\emptyset}$, $\mathscr{Y}_{\lambda_2,\emptyset}$ fuse according to the Littlewood-Richardson rule (and in particular are strongly dualizable). So what remains to prove is that the image of $Y_{\lambda,\mu}$ under (16) is $\mathscr{Y}_{\lambda,\mu}$.

To show that, note that $Y_{\lambda,\emptyset} \otimes Y_{\emptyset,\mu}$ contains $Y_{\lambda,\mu}$ plus summands of the form $Y_{\lambda',\mu'}$, $|\lambda'| < |\lambda|$, $|\mu'| < |\mu|$. Thus, if we can show that $\mathscr{Y}_{\lambda,\emptyset} \boxtimes \mathscr{Y}_{\emptyset,\mu}$ has, in each degree, the exact same summands as

$$\bigoplus_{i=1}^{N} \mathscr{Y}_{\lambda_i,\mu_i} \tag{17}$$

where

$$Y_{\lambda,\emptyset} \otimes Y_{\emptyset,\mu} = \bigoplus_{i=1}^{N} Y_{\lambda_i,\mu_i},$$

then we can argue by induction. Now (17) can be seen by induction from studying

$$\mathscr{Y}_{\lambda,\mu} \boxtimes P_{(1),\emptyset}. \tag{18}$$

Indeed, one sees, by definition, that in degree $(m+1,n)$, (18) has the $\Sigma^c_{m+1,n}$-representation

$$\bigoplus_{i=1}^{N} \mathcal{Y}_{\overline{\lambda^+},\overline{\mu}}(m+1,n), \tag{19}$$

where $\overline{\lambda^+}$ and $\overline{\mu}$ are obtained from λ^+ and μ (where $\mathcal{Y}_{\lambda^+,\mu}(m,n)$ is the summand of $\mathscr{Y}_{\lambda,\mu}$ in degree (m,n)) by the Pieri rule, adding one square to some row to λ^+ or subtracting a square from some row of μ, while still obtaining a Young diagram.

The summands where we add a square to the first row of λ^+ match a copy of $\mathscr{Y}_{\lambda,\mu}$ again, while the others match the precise copies of

$$\bigoplus_{i=1}^{N'} \mathscr{Y}_{\lambda_i',\mu_i'}$$

where

$$Y_{\lambda,\mu} \otimes Y_{(1),\emptyset} = \bigoplus_{i=1}^{N'} Y_{\lambda_i',\mu_i'}$$

(using, again, the Pieri rule). This is, of course, the right number, by exactness and the short exact sequence (15).

□

References

[1] T. Church, J. S. Ellenberg, B. Farb: FI-modules and stability for representations of symmetric groups, *Duke Math. J.* 164 (2015), no. 9, 1833-1910.

[2] P. Deligne. La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel. *Algebraic groups and homogeneous spaces*, 209-273, Tata Inst. Fund. Res. Stud. Math., 19, Tata Inst. Fund. Res., Mumbai, 2007.

[3] P. Deligne, J. S. Milne. Tannakian categories, in *Hodge Cycles, Motives and Shimura Varieties*, Lecture Notes in Math. 900, Springer Verlag, 101-228, 1982.

[4] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik: *Tensor categories.* Math. Surveys Monogr., 205 American Mathematical Society, Providence, RI, 2015, xvi+343 pp.

[5] N. Gadish. Categories of FI type: a unified approach to generalizing representation stability and character polynomials. *J. Algebra* 480 (2017), 450-486.

[6] W. L. Gan, L. Li: Coinduction functor in representation stability theory, *J. London Math. Soc.*, 92 (2015), no. 3, 689-711.

[7] N. Harman: Virtual specht stability for FI-modules in positive characteristic, *J. Algebra* 488 (2017), 29-41.

[8] S. Kriz: On the local cohomology of L-shaped integral FI-modules. *J. Algebra* 611 (2022), 149-174.

[9] S. Kriz: Quantum Delannoy Categories, 2023, preprint.

[10] L. Li: Upper bounds of homological invariants of FI_G-modules, *Arch. Math. (Basel)* 107 (2016), no. 3, 201-211

[11] L.Li, E.Ramos: Depth and the local cohomology of FI_G-modules. *Adv. Math.* 329 (2018), 704-741.

[12] J. Miller, J.C.H. Wilson. Quantitative representation stability over linear groups. *Int. Math. Res. Not. IMRN* 2020, no. 22, 8624-8672.

[13] R. Nagpal: VI-modules in non-describing characteristic, part I. *Algebra Number Theory* 13 (2019), no. 9, 2151-2189.

[14] R. Nagpal: VI-modules in non-describing characteristic, part II. *J. Reine Angew. Math.* 781 (2021), 187-205.

[15] R. Nagpal, S. Sam, A. Snowden: Noetherianity of some degree two twisted commutative algebras. *Selecta Mathematica* 22 (2016), 913-937.

[16] E. Ramos: Homological invariants of FI-modules and FI_G-modules, *Journal of Algebra* 502 (2018), 163- 195.

[17] S. V. Sam, A. Snowden: GL-Equivariant Modules Over Polynomial Rings in Infinitely Many Variables, *Trans. Am. Math. Soc.* 368 (2016), no. 2, 1097-1158.

[18] A. Snowden. Algebraic Structures in Representation Stability, lecture notes, 2010.