OSCILLATOR REPRESENTATIONS AND SEMISIMPLE PRE-TANNAKIAN CATEGORIES

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ABSTRACT. We investigate tensor products of oscillator representations of symplectic groups over finite fields, as well as of their semidirect products with Heisenberg groups. As an application, we construct non-trivial semisimple pre-Tannakian categories with an object of generic dimension whose second exterior and symmetric powers are simple, while determining the decomposition of its third tensor power into simple objects.

1. INTRODUCTION

In this paper, we study questions at the intersection of representation theory and category theory. We begin with the latter perspective, which is easier to describe in general terms.

Recall that by a *pre-Tannakian* category over $\mathbb{C}$, we mean a $\mathbb{C}$-linear abelian category with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product, which is locally finite in the sense that Hom-sets are finite dimensional $\mathbb{C}$-vector spaces, is rigid in the sense that objects have strong duals [3], and satisfies $End(1) = \mathbb{C}$. An abelian category is *semisimple* if every object is a direct sum of finitely many simple objects. Examples of semisimple pre-Tannakian categories with given properties are often difficult to construct.

Our main result on pre-Tannakian categories is the following theorem, answering a question of P. Deligne:

Theorem 1. *For every natural number q which is a power of a prime not equal to 2 or 3 and every $t \in \mathbb{C}$ such that $q^t \neq \pm 1, \pm q$, there exists a semisimple pre-Tannakian category $\mathscr{C}_{q,t}$ over $\mathbb{C}$ generated by an object X of dimension q^t such that X, $\Lambda^2(X)$, $Sym^2(X)$ are simple and*

$$dim(End(X^{\otimes 3})) = 2q + 2. \tag{1}$$

This theorem is proved by applying the technique of *interpolation* (see [1, 4, 10, 11]) to the category of direct summands of tensor powers of oscillator representations.

The *oscillator*, or *Weil-Shale*, representation ω_a of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$, where V_N is a $2N$-dimensional vector space over a finite field $\mathbb{F}_q$ with a symplectic form and $\mathbb{H}_N(\mathbb{F}_q)$ is the Heisenberg group determined by V_N, depends on a choice of $a \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$ (detemining, by Pontrjagin duality, a character $\psi_a : \mathbb{F}_q^\times \to \mathbb{C}^\times$). More detail on these objects is given in Secion 2 below. The proof of Theorem 1 uses certain facts about the oscillator representation, due to R. Howe [6, 7, 12]. The main point is a passage from representations of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ to representations of $Sp(V_N)$, which has the effect of reducing the exponent of tensor power of an oscillator representation by 1.

This allows us to prove the statements of Theorem 1 about semisimplicity of $\Lambda^2(X)$ and $Sym^2(X)$, as well as formula (1).

On the other hand, (1) raises the question [2] of decomposing $X^{\otimes 3}$ into simple summands which, in return, turns into a question of decompositionn of $\omega_a \otimes \omega_b$ as a representation of $Sp(V_N)$ into simple summands. Even before applying interpolation, endomorphism algebras can be studied using T-algebra data.

Using this approach, we prove

Theorem 2. *For q a power of a prime not equal to 2 or 3, as representations of $Sp(2N, \mathbb{F}_q) = Sp(V_N)$, for $a, b \in \mathbb{F}_q^\times$, the decomposition of $\omega_a \otimes \omega_b$ into irreducible representations can be calculated as follows:*

- **Case 1:** *If $a \equiv -b \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$, then, letting $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ denote a general multiplicative character of $\mathbb{F}_q^\times$ and letting ϵ denote the quadratic character, we have*

$$\omega_a \otimes \omega_b \cong$$

$$\left(\bigoplus_{\{\chi | \chi^2 \neq 1\}/\chi \sim 1/\chi} 2 \cdot Z_\chi \right) \oplus (Z_\epsilon^+ \oplus Z_\epsilon^-) \oplus (2 \cdot 1 \oplus \widetilde{Z}_1^+ \oplus \widetilde{Z}_1^-) \tag{2}$$

for simple non-isomorphic $Sp(2N, \mathbb{F}_q)$-representations $Z_\chi, Z_\epsilon^\pm, \widetilde{Z}_1^\pm$ of dimensions

$$dim(Z_\chi) = \frac{q^{2N} - 1}{q - 1}$$

$$dim(Z_\epsilon^+) = dim(Z_\epsilon^-) = \frac{q^{2N} - 1}{2(q - 1)}$$

$$dim(\widetilde{Z}_1^\pm) = \frac{1}{2} \cdot \left(\frac{q^{2N} - q}{q - 1} \pm q^N \right).$$

- **Case 2:** *If* $a \not\equiv -b \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$*, then, letting* $\theta : \mu_{q+1} \to \mathbb{C}^\times$ *denote a general character of the group* μ_{q+1} *of* $(q+1)$*th roots of unity in* $\mathbb{C}^\times$ *and letting* σ *denote the character of order 2, we have*

$$\omega_a \otimes \omega_b \cong \left(\bigoplus_{\{\theta|\theta^2\neq 1\}/\theta\sim 1/\theta} 2\cdot Y_\theta\right) \oplus (Y_\sigma^+ \oplus Y_\sigma^-) \oplus (Y_1^+ \oplus Y_1^-) \tag{3}$$

for simple, non-isomorphic $Sp(2N,\mathbb{F}_q)$*-representations* $Y_\theta, Y_\sigma^\pm, Y_1^\pm$ *of dimensions*

$$dim(Y_\theta) = \frac{q^{2N}-1}{q+1}$$

$$dim(Y_\sigma^+) = dim(Y_\sigma^-) = \frac{q^{2N}-1}{2(q+1)}$$

$$dim(Y_1^\pm) = \frac{1}{2}\cdot\left(\frac{q^{2N}+q}{q+1} \pm q^t\right).$$

Comments: 1. The statement of the theorem remains valid for $N=1$ (where $Sp(V_N) = SL_2(\mathbb{F}_q)$) with the modification that one of the $\widetilde{Z}_1^\pm$, $Y_1^\pm$ is 0.

2. Case 1 is more classical and uses a result of Howe that $\omega_a \otimes \omega_{-a}$ is isomorphic to the $\mathbb{C}$ linearization Ω of the basic (vector) representation V_N of $Sp(2N,\mathbb{F}_q)$. The characters χ involved in (2) are of "Harish-Chandra" type.

Case 2 involves constructing a μ_{q+1}-action on

$$End_{Sp(2N,\mathbb{F}_q)}(\omega_a \otimes \omega_b) \tag{4}$$

when $a \not\equiv -b \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$. Correspondingly, the characters θ in (3) are of "Deligne-Lusztig type," although we construct them directly by studying idempotents in (4).

Translating back into representations of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ and applying interpolation, we obtain the following result about the category $\mathscr{C}_{q,t}$ of Theorem 1:

Theorem 3. *In the category* $\mathscr{C}_{q,t}$*,* $q^t \neq \pm 1, \pm q$*, the decomposition of* $X^{\otimes 3}$ *into simple summands is as follows:*

- **Case 1:** *If* $q \equiv 1$ *mod* 3, *then*

(5)
$$X^{\otimes 3} \cong \left(\bigoplus_{\{\chi:\mathbb{F}_q^\times \to \mathbb{C}^\times | \chi^2 \neq 1\}/(\chi \sim 1/\chi)} 2 \cdot X_\chi \right) \oplus (X_\epsilon^+ \oplus X_\epsilon^-) \oplus (2 \cdot X_1^0 \oplus \widetilde{X}_1^+ \oplus \widetilde{X}_1^-)$$

for simple non-isomorphic objects X_χ, $X_\epsilon^\pm$, X_1^0, *and* $\widetilde{X}_1^\pm$ *whose dimensions are related to the dimensions of the objects of Theorem 2 by* $dim(X_\chi) = q^t \cdot dim(Z_\chi)$, $dim(X_\epsilon^\pm) = q^t \cdot dim(Z_\epsilon^\pm)$, $dim(X_1^0) = q^t$, *and* $dim(\widetilde{X}_1^\pm) = q^t \cdot dim(\widetilde{Z}_1^\pm)$.

- **Case 2:** *If* $q \equiv -1$ *mod* 3, *then*

$$X^{\otimes 3} \cong \left(\bigoplus_{\{\theta:\mu_{q+1} \to \mathbb{C}^\times | \theta \neq 1,\sigma\}/(\theta \sim 1/\theta)} 2 \cdot X_\theta \right) \oplus (X_\sigma^+ \oplus X_\sigma^-) \oplus (X_1^+ \oplus X_1^-) \tag{6}$$

for simple non-isomorphic objects X_θ *(for* $\theta \neq 1, \sigma$*),* $X_\sigma^\pm$, *and* $X_1^\pm$ *whose dimensions are related to the dimensions of the objects of Theorem 2 by* $dim(X_\theta) = q^t \cdot dim(Y_\theta)$, $dim(X_\sigma^\pm) = q^t \cdot dim(Y_\sigma^\pm)$, *and* $dim(X_1^\pm) = q^t \cdot dim(Y_1^\pm)$.

Comment: Again, the statement of Theorem 3 remains valid for $q^t = \pm q$ with the modification that one of the objects $\widetilde{X}_1^\pm$, resp. $X_1^\pm$, becomes 0. Therefore, in this case, $dim(End_{\mathscr{C}_{q,t}}(X^{\otimes 3})) = 2q + 1$.

The present paper is organized as follows: Section 2 contains preliminary computations with the oscillator representation. Section 3 discusses the proof of Case 1 of Theorem 2. The proof of Case 2 of Theorem 2 is completed in Section 4. Sections 5 and 6 treat interpolation: In Section 5, we discuss the T-algebra method and its applications; in Section 6, we discuss semisimplicity and prove Theorem 3.

2. The Oscillator Representation and its Powers

The purpose of this section is to introduce the Weil-Shale representation and its restriction to the symplectic group, and record some general computations, old and new, with the products of Weil-Shale representations. The key point is to establish the interplay between representations of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ and of $Sp(V_N)$ which is intrinsic throughout our discussion.

We also prove the claim of Theorem 1 for $t = N > 1 \in \mathbb{N}$, verifying (1) for basic object $X = \omega$ in the category of representations $Rep(Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q))$. To do this, we use a passage from the endomorphism algebras of tensor powers of ω in $Rep(Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q))$ to the endomorphism algebras of tensor powers of ω of one degree less in $Rep(Sp(V_N))$. This comes from the classical fact [5] that the tensor product of ω with its dual is the permutation representation $\mathbb{C}V_N$, which we include an explicit proof of. We also include a preliminary result of P. Deligne [2] (Theorem 6 below) about a tensor product of non-dual Weil-Shale representations, which will be needed later to explicitly decompose $\omega^{\otimes 3}$.

The construction of the classical Weil-Shale representation [12, 6, 7] goes as follows: For a q which is a power of a prime not equal to 2, suppose V_N is a $2N$-dimensional vector space over $\mathbb{F}_q$ endowed with a symplectic form. To avoid confusion, we specify V_N in the notation of the symplectic group $Sp(2N, \mathbb{F}_q) = Sp(V_N)$ from now on. Recall that the Heisenberg group is

$$\mathbb{H}_N(\mathbb{F}_q) = V_N \times \mathbb{F}_q$$

with the operation that for $v, w \in V_N$, $\lambda, \mu \in \mathbb{F}_q$,

$$(v, \lambda)(w, \mu) = (w + w, \lambda + \mu + \langle v, w \rangle)$$

(and therefore $0 \times \mathbb{F}_q$ is the center of $\mathbb{H}_N(\mathbb{F}_q)$). The symplectic group $Sp(V_N)$ acts on $\mathbb{H}_N(\mathbb{F}_q)$ by acting tautologically on the factor of V_N and trivially on the center. Consider the group

$$Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q).$$

For a non-trivial character

$$\psi : \mathbb{F}_q \ \to \mathbb{C}^\times,$$

there is a unique q^N-dimensional irreducible $\mathbb{H}_N(\mathbb{F}_q)$-representation ω_ψ (over $\mathbb{C}$). Then ω_ψ forms a representation of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ called the *Weil-Shale representation* (a priori projective but an actual representation for a finite field $\mathbb{F}_q$, see [6]). For the remainder of this note, when ψ is fixed, and we will omit it from the notation. When ψ is not fixed, we may also write ω_a where $a \in \mathbb{F}_q^\times$ is the element corresponding to ψ under a fixed identification of $\mathbb{F}_q^\times$ with its non-trivial multiplicative characters.

To avoid confusion, we write $\overline{\omega}$ to denote the restriction of ω to $Sp(V_N)$ and the inflation of this restriction to a representation of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ by letting $\mathbb{H}_N(\mathbb{F}_q)$ act trivially.

Considering the usual inclusion $GL_N(\mathbb{F}_q) \subset Sp(V_N)$, the restriction of ω to $GL_N(\mathbb{F}_q)$ gives the permutation representation $\mathbb{C}\mathbb{F}_q^N$ where $GL_N(\mathbb{F}_q)$ acts on $\mathbb{F}_q^N$ by matrix multiplication, tensored with

$$\epsilon(det) \tag{7}$$

where $det : GL_N(\mathbb{F}_q) \to \mathbb{F}_q^\times$ is the determinant map and $\epsilon : \mathbb{F}_q^\times \to \mathbb{C}^\times$ is the character of order 2.

We need to study the endomorphism algebras of tensor powers of ω. We begin by proving the claim of Theorem 1 for $t = N > 1 \in \mathbb{N}$, i.e. the following

Proposition 4. *For any $N > 1 \in \mathbb{N}$, for any Weil-Shale representation ω of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$, we have*

$$dim(End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes 2})) = 2 \tag{8}$$

and

$$dim(End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes 3})) = 2q + 2. \tag{9}$$

To calculate $dim(End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes n}))$, write

$$\Omega := \omega \otimes (\omega^\vee)$$

We first claim the following

Lemma 5. *For $N > 1$, as a representation of $Sp(V_N) \ltimes V_N$, Ω is isomorphic to the space of functions on V_N:*

$$Res^{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)\ltimes V_N}(\Omega) \cong \{f : V_N \to \mathbb{C}\}$$

where an element $(A, w) \in Sp(V_N) \ltimes V_N$ acts on a function $f : V_N \to \mathbb{C}$ by sending it to the function

$$(A, w)[f] : V_N \to \mathbb{C}$$

where for $v \in V_N$

$$((A, w)[f])(v) = \psi(\langle v, w\rangle) \cdot f(A(v)).$$

Proof. First note that we may write

$$\Omega = \bigoplus_{v \in V_N} \Omega_v$$

for lines Ω_v such that an element $w \in V_N = V_N \times \{0\} \subset \mathbb{H}_N(\mathbb{F}_q)$ preserves each Ω_v and acts by multiplication by the character

$$x \mapsto \psi(\langle v, w\rangle) \cdot x.$$

Ω can then be considered as the space of global sections of an $Sp(V_N)$-equivariant line bundle Ω_v over V_N (as a discrete set) where the action of $Sp(V_N)$ on Ω induces an action of $Sp(V_N)$ on the line bungle, i.e. for $\gamma \in Sp(V_N)$,

$$\gamma(\Omega_v) = \Omega_{\gamma(v)}.$$

However, Ω_v forms a trivial $Sp(V_N)$-equivariant line bundle, meaning that for every $v \in V_N$, the stabilizer subgroup $Sp(V_N)^v \subseteq Sp(V_N)$ fixing v acts trivially on Ω_v: At $v = 0$, for $N > 0$, $Sp(V_N)^0 = Sp(V_N)$, which is a perfect group (meaning that it has no non-trivial abelian quotients), and therefore acts trivially on Ω_0. For $v \neq 0$, taking W_v to be the quotient of the orthogonal space $V_N^{\perp v}$ of vectors perpendicular to v by $\mathbb{F}_q$-multiples of v,

$$Sp(V_N)^v = Sp(W_v),$$

which again is a perfect group (for $N > 0$).

Therefore, Ω is the space of global sections of the trivial $Sp(V_N)$-equivariant line bundle, i.e. a space of functions

$$\Omega = \{f : V_N \to \mathbb{C}\}, \tag{10}$$

and the action of $Sp(V_N) \ltimes V_N \subset Sp(V_N) \ltimes \mathbb{H}_n(\mathbb{F}_q)$ on a function f in (10) is given by $Sp(V_N)$ acting by composition, and $w \in V_N$ acting by sending f to the function

$$V_N \to \mathbb{C}$$
$$v \mapsto \psi(\langle v, w\rangle) \cdot f(v)$$

□

By Lemma 5, we therefore have

$$Res^{Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)\ltimes V_N}(\Omega^{\otimes k}) \cong \{f : V_N^k \to \mathbb{C}\},$$

which is generated by the character functions $\mathbb{1}_{(v_1,\dots,v_{k+1})}$ for $v_i \in V_N$ (which is 1 at $(v_1, \dots, v_{k+1})$ and 0 at all other elements of V_N^{k+1}).

The fixed points

$$(\Omega^{\otimes k+1})^{\mathbb{H}_N(\mathbb{F}_q)} \cong (Res^{Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)\ltimes V_N}(\Omega^{\otimes k+1}))^{V_N} \tag{11}$$

(the isomorphism follows since $\mathbb{F}_q \subset \mathbb{H}_N(\mathbb{F}_q)$ acts trivially on Ω) are then generated by $\mathbb{1}_{(v_1,\dots,v_{k+1})}$ for $v_i \in V_N$ such that for every $u \in V_N$,

$$\mathbb{1}_{(v_1,\dots,v_{k+1})} = u(\mathbb{1}_{(v_1,\dots,v_{k+1})}) =$$
$$\psi(\langle u, v_1\rangle) \cdot \dots \cdot \psi(\langle u, v_{k+1}\rangle) \cdot \mathbb{1}_{(v_1,\dots,v_{k+1})} =$$
$$\psi(\langle u, v_1 + \dots, v_{k+1}\rangle) \cdot \mathbb{1}_{(v_1,\dots,v_{k+1})}$$

meaning that for every $u \in V_N$

$$\langle u, v_1 + \dots, v_{k+1}\rangle = 0,$$

which is equivalent to $v_1 + \dots + v_{k+1} = 0$.

Therefore, (11) is identified with the space of functions on V_N^{k+1} with support on

$$\{(v_1,\dots,v_{k+1}) \in V_N^{k+1} \mid v_1 + \dots + v_{k+1} = 0 \in V_N\}. \tag{12}$$

Thus, for every $k \in \mathbb{N}$, there is an isomorphism

$$Res^{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)}(\Omega^{\otimes k}) \cong (\Omega^{\otimes k+1})^{\mathbb{H}_N(\mathbb{F}_q)} \tag{13}$$

as representations of $Sp(V_N)$ for $N >> 0$ following from Lemma 5 and the isomorphism between (12) and V_N^k.

Now, by duality,

$$End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes k+1}) \cong Hom_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(1, \Omega^{\otimes k+1}),$$

which is identified with the fixed points

$$(\Omega^{\otimes k+1})^{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)} \cong ((\Omega^{\otimes k+1})^{\mathbb{H}_N(\mathbb{F}_q)})^{Sp(V_N)}. \tag{14}$$

By (13), (14) is isomorphic to the $Sp(V_N)$ fixed points

$$(Res^{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}_{Sp(V_N)}(\Omega^{\otimes k}))^{Sp(V_N)},$$

which are isomorphic to

$$Hom_{Sp(V_N)}(1, \Omega^{\otimes k}).$$

Therefore, for $t \in \mathbb{C} \setminus \mathbb{Z}$, for $N >> 0$,

$$End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes k+1}) \cong$$
$$Hom_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(1, \Omega^{\otimes k+1}) \cong Hom_{Sp(V_N)}(1, \Omega^{\otimes k})$$

the dimension of which, by Lemma 5, can be calculated as the number of orbits of $Sp(V_N)$ on V_N^k.

For example, we can verify that

$$End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega) = 1$$

(the number of $Sp(V_N)$ orbits of $V_N^0 = 0$), and therefore that ω is simple.

The proof of Proposition 4 is also similarly elementary:

Proof of Proposition 4. We get (8) since the dimension of

$$End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes 2}) = 2$$

because V_N has two $Sp(V_N)$-orbits: 0 and $V_N \smallsetminus 0$.

Finally, we get (9) since

$$End_{Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)}(\omega^{\otimes 3}) = 2q+2$$

because V_N^2 has $2q+2$ $Sp(V_N)$-orbits, which are as follows:

(A) 1 orbit of the form

$$\{(0,0)\},$$

(B) $q+1$ orbits of the form

$$\begin{array}{c}\{(v,0) \mid v \in V_N\},\ \{(0,v) \mid v \in V_N\},\\ \{(v,\lambda\cdot v) \mid v \in V_N\},\quad \text{for } \lambda \in \mathbb{F}_q^\times\end{array}$$

corresponding to "slopes" $\lambda \in \mathbb{P}^1(\mathbb{F}_q)$.

(C) q orbits of the form

$$\{(v,w) \mid \langle v,w\rangle = \mu,\ v,w \text{ linearly independent}\},\quad \text{for } \mu \in \mathbb{F}_q.$$

$\square$

To more explicitly calculate decomposition of tensor powers of the Weil-Shale representations, we will also need the following computation of P. Deligne, [2]:

Theorem 6. *For $a,b \in \mathbb{F}_q^\times$ such that $a+b \neq 0$, as representations of $Sp(V_N)\ltimes\mathbb{H}_N(\mathbb{F}_q)$ for any $N \in \mathbb{N}$ (as long as $V_N \neq \mathbb{F}_3^2$),*

$$\omega_a \otimes \omega_b \cong \overline{\omega}_{ab(a+b)} \otimes \omega_{a+b}. \tag{15}$$

Proof. N is fixed. To simplify notation, in this proof, we omit the subscript N of V. Let us denote by $S(v,w)$ for $v,w \in V$ the symplectic form of V. Write V_1, V_2 for two copies of V, with symplectic forms S_1, S_2 equivalent to S, and isomorphisms

$$\begin{array}{c} V \xrightarrow{\cong} V_i \\ v \mapsto v_i \end{array}$$

for $i = 1,2$.

It is enough to consider ω_a, ω_b as projective representation of the quotient
$$(Sp(V) \ltimes \mathbb{H}_N(\mathbb{F}_q))/Z(Sp(V) \ltimes \mathbb{H}_N(\mathbb{F}_q)),$$
which is the affine symplectic group
$$Sp(V) \ltimes V.$$

For every $a \in \mathbb{F}_q^\times$, the representation ω_a for (V, S) is isomorphic to ω_1 for $(V, a \cdot S)$ (replacing the symplectic form $S(v, w)$ on V by $a \cdot S(v, w)$). Thus, $\omega_a \otimes \omega_b$ for (V, S) can be considered the pullback of ω_1 for $(V_1 \oplus V_2, a \cdot S_1 + b \cdot S_2)$, using the diagonal embedding
$$\Delta : V \hookrightarrow V_1 \oplus V_2$$
$$v \mapsto (v_1, v_2)$$
Note that the pullback of the symplectic form $a \cdot S_1 + b \cdot S_2$ on $V_1 \oplus V_2$ along Δ is the form $(a + b) \cdot S$ on V.

Now we may also consider an antidiagonal embedding
$$\Delta' : V \hookrightarrow V_1 \oplus V_2$$
$$v \mapsto (bv_1, -av_2),$$
which has image $Im(\Delta')$ orthogonal to $Im(\Delta)$ using the form $a \cdot S_1 + b \cdot S_2$ on $V_1 \oplus V_2$. The pullback of the symplectic form $a \cdot S_1 + b \cdot S_2$ on $V_1 \oplus V_2$ along Δ' is the form $a \cdot b \cdot (a + b) \cdot S$ on V. Reparametrizing using the isomorphism
$$V_1 \oplus V_2 \cong Im(\Delta) \oplus Im(\Delta'),$$
we get an isomorphism between ω_1 for $(V_1 \oplus V_2, a \cdot S_1 + b \cdot S_2)$ and a tensor product of the pullbacks of ω_1 from $Im(\Delta)$ and $Im(\Delta')$. However, the embedding
$$\begin{tikzcd}
Sp(V) \ltimes V \arrow[r] & Sp(V_1 \oplus V_2) \ltimes (V_1 \oplus V_2) \arrow[d] \\
& (Sp(Im(\Delta)) \ltimes Im(\Delta)) \times (Sp(Im(\Delta')) \ltimes Im(\Delta'))
\end{tikzcd}$$
projects V bijectively into the first coordinate $Im(\Delta)$ (and to 0 on the second coordinate $Im(\Delta')$), though it continues to map $Sp(V) \hookrightarrow Sp(Im(\Delta')) \ltimes Im(\Delta')$ diagonally. Therefore, we obtain that as (projective) $Sp(V) \ltimes V$-representations,
$$\omega_a \otimes \omega_b \cong \overline{\omega}_{ab(a+b)} \otimes \omega_{a+b}$$
(where now $\overline{\omega}_{ab(a+b)}$ is considered as the $Sp(V) \ltimes V$-representation from letting V act trivially on the restriction of $\omega_{ab(a+b)}$ to $Sp(V)$).

Since there is only one way to extend back to representations of $Sp(V) \ltimes \mathbb{H}_N(\mathbb{F}_q)$, we can conclude (15). $\square$

By Theorem 6, in particular, $\omega^{\otimes 3}$ can be expressed as a tensor product of two inflations of restrictions $\overline{\omega}$ of some Weil-Shale representations, with one simple (unrestricted) Weil-Shale representation. Therefore, to decompose $\omega^{\otimes 3}$ in $Rep(Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$, we must decompose degree 2 tensor products of oscillator representations $\overline{\omega}$ in $Rep(Sp(V_N))$.

3. Proof of Theorem 2 - Case 1

In the current and next section, we will prove Theorem 2. We shall replace a, b in the statement of the theorem by A, B, to avoid conflict with some notation we will need later.

Theorem 2 concerns the tensor product

$$\overline{\omega}_A \otimes \overline{\omega}_B \tag{16}$$

for $A, B \in \mathbb{F}_q^\times$. We approach this calculation separately depending on whether $\overline{\omega}_A$ and $\overline{\omega}_B$ are dual $Sp(V_N)$-representations. Recall two classical facts about oscillator representations (see [6], for example): that

$$(\omega_A)^\vee = \omega_{-A}$$

(and therefore, $(\overline{\omega}_A)^\vee = \overline{\omega}_{-A}$, as well) and

$$\overline{\omega}_A \cong \overline{\omega}_B \text{ if and only if } A \equiv -B \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2. \tag{17}$$

Therefore, our cases of (16) separated based on whether or not

$$A \equiv -B \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2.$$

Proof of Case 1 of Theorem 2. Suppose $A \equiv -B \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$ (i.e. $\overline{\omega}_B \cong (\overline{\omega}_A)^\vee$). By Lemma 5, we then have

$$\overline{\omega}_A \otimes \overline{\omega}_B \cong \mathbb{C}V_N$$

where $Sp(V_N)$ acts on $\mathbb{C}V_N$ by matrix multiplication (identifying the character function).

To decompose $\mathbb{C}V_N$, first note that for every $\lambda \in \mathbb{F}_q^\times$, the multiplication map

$$\begin{gathered} \mathbb{C}V_N \to \mathbb{C}V_N \\ (v) \mapsto (\lambda v) \end{gathered}$$

commutes with the action of $Sp(V_N)$. Therefore, for a multiplicative character

$$\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times,$$

we may consider the $Sp(V_N)$-equivariant map

$$\begin{aligned} e_\chi : \mathbb{C}V_N &\to \mathbb{C}V_N \\ (v) &\mapsto \frac{1}{q-1} \sum_{\lambda \in \mathbb{F}_q^\times} \chi(\lambda) \cdot (\lambda v). \end{aligned}$$

For multiplicative characters $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$, the endomorphisms

$$e_\chi \in End_{Sp(V_N)}(\mathbb{C}V_N)$$

form a system of disjoint commuting idempotents, and therefore, denoting their images by

$$Z_\chi := e_\chi(\mathbb{C}V_N),$$

we obtain a decomposition

$$\mathbb{C}V_N \cong \bigoplus_{\chi:\mathbb{F}_q^\times \to \mathbb{C}^\times} Z_\chi. \tag{18}$$

Note that for every non-trivial $\chi \neq 1$,

$$dim(Z_\chi) = \frac{q^{2N}-1}{q-1}, \tag{19}$$

and

$$dim(Z_1) = \frac{q^{2N}-1}{q-1} + 1 \tag{20}$$

(the extra dimension arising from (0)).

Now consider the $Sp(V_N)$-equivariant map

$$\begin{aligned} f : \mathbb{C}V_N &\to \mathbb{C}V_N \\ (v) &\mapsto \sum_{S(v,w)=1} (w) \end{aligned}$$

where the sum for $f(v)$ runs over $w \in V_N$ for which $S(v,w) = 1$. We find that for $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$, $v \in V_N$,

$$f(e_\chi((v))) = \sum_{\lambda \in \mathbb{F}_q^\times} \sum_{S(\lambda v, w)=1} \chi(\lambda) \cdot (w) =$$
$$\sum_{\lambda \in \mathbb{F}_q^\times} \sum_{S(v,u)=1} \chi(\lambda) \cdot (\lambda^{-1} u) = \sum_{\mu \in \mathbb{F}_q^\times} \sum_{S(v,u)=1} \chi(\mu^{-1}) \cdot (\mu u) =$$
$$e_{1/\chi}(f((v)))$$

and therefore, f restricts to isomorphisms

$$Z_\chi \cong Z_{1/\chi}.$$

Therefore, we may rewrite (18) as

$$\mathbb{C}V_N = \left(\bigoplus_{\{\chi:\mathbb{F}_q^\times \to \mathbb{C}^\times | \chi \neq 1, \epsilon\}/(\chi \sim 1/\chi)} 2 \cdot Z_\chi \right) \oplus Z_\epsilon \oplus Z_1.$$

For $\chi = \epsilon$, resp. 1, the map f restricts to an endomorphism of Z_ϵ, Z_1. On Z_ϵ, we can use f to compute a further decomposition of Z_ϵ:

To find idempotents in $End(Z_\epsilon)$, we must first calculate $f|_{Z_\epsilon}^{\circ 2}$. For $v \in V_N$, we have

$$f(e_\epsilon(v)) = \sum_{\lambda \in \mathbb{F}_q^\times} \sum_{S(v,u)=\lambda^{-1}} \epsilon(\lambda)(u)$$

and therefore

$$f \circ f(e_\epsilon(v)) = \sum_{\lambda \in \mathbb{F}_q^\times} \sum_{S(v,u)=\lambda^{-1}} \sum_{S(u,w)=1} \epsilon(\lambda) \cdot (w).$$

Without loss of generality, (using the standard model for $V = (\mathbb{F}_q)^N \oplus (\mathbb{F}_q)^N$), take $v = (1, 0, \ldots, 0 \mid 0, \ldots, 0)$. Clearly, there are q^{2n-1} choices of u such that

$$S(v, u) = \lambda^{-1},$$

given by

$$u = (?, \ldots, ? \mid \lambda^{-1}, ?, \ldots, ?)$$

where the ? can be replaced by any element of $\mathbb{F}_q$. For each such choice of u, there is then one conributed term of the form

$$\epsilon(\lambda) \cdot (-\lambda, 0, \ldots 0 \mid 0, \ldots, 0),$$

since $S(u, (-\lambda, 0, \ldots 0 \mid 0, \ldots, 0)) = 1$. Terms involving vectors $w \in V_N$ which are not of the form

$$(-\lambda, 0, \ldots 0 \mid 0, \ldots, 0)$$

for $\lambda \in \mathbb{F}_q^\times$ will arise (with equal multiplicity) from different choices of u for each $\lambda \in \mathbb{F}_q^\times$, and therefore, will have a coefficient which is a multiple of $\displaystyle\sum_{\lambda \in \mathbb{F}_q^\times} \epsilon(\lambda) = 0$. Therefore, terms of this form cancel, and we have

$$f(f(e_\epsilon(v))) = \epsilon(-1) \cdot q^{2n-1} \cdot e_\epsilon(v)$$

(note that $\epsilon(-1) = \left(\frac{-1}{q}\right)$). In other words, we have

$$f|_{Z_\epsilon} \circ f|_{Z_\epsilon} = \epsilon(-1) \cdot q^{2n-1} \cdot e_\epsilon.$$

We then have idempotents

$$e_\epsilon^{\pm} := \frac{1}{2}\left(e_\epsilon \pm \frac{f|_{Z_\chi}}{\sqrt{\epsilon(-1)\cdot q^{2n-1}}}\right).$$

Since f has trace 0, both e_ϵ^+ and e_ϵ^- have trace

$$\frac{q^{2n}-1}{2(q-1)}.$$

Write

$$Z_\epsilon^{\pm} := Im(e_\epsilon^{\pm}).$$

Replacing Z_ϵ, we get

$$\mathbb{C}V_N = \left(\bigoplus_{\{\chi:\mathbb{F}_q^\times\to\mathbb{C}^\times|\chi\neq 1,\epsilon\}/(\chi\sim 1/\chi)} 2\cdot Z_\chi\right)\oplus(Z_\epsilon^+\oplus Z_\epsilon^-)\oplus Z_1.$$

Now, in Z_1, we immediately see two elements fixed under the action of $Sp(V_N)$: (0) and $\displaystyle\sum_{v\in V_N}(v)$. Therefore, there is a decomposition

$$Z_1 = 2\cdot 1\oplus \widetilde{Z}_1.$$

Finally, to decompose $\widetilde{Z}_1$, consider the morphism

$$\begin{aligned} g:\ & \mathbb{C}V_N\to\mathbb{C}V_N \\ & (v)\mapsto \sum_{S(u,v)=0,\ dim(\langle v,u\rangle)=2}(u) \end{aligned}$$

where for $v\in V_N$, the sum in $g(v)$ runs over all choices of $u\in V_N$ linearly independent to v with $S(u,v)=0$. Similarly as above, we will use g to calculate idempotents in $End(\widetilde{Z}_1)$. For $v\in V_N$, we have

$$g(e_1((v))) = \frac{1}{q-1}\sum_{\lambda\in\mathbb{F}_q^\times} g(\lambda v) = \frac{q-1}{q-1}g(v) = g((v)),$$

and therefore,

$$g\circ g(e_1((v))) = g\circ g((v)) = \sum_{S(u,v)=0,\ dim(\langle v,u\rangle)=2}\ \sum_{S(w,u)=0,\ dim(\langle w,u\rangle)=2}(w).$$

Again, without loss of generality, let us take $v=(1,0,\ldots,0\mid 0,\ldots,0)$ in the standard model of V_N. Then there are $q^{2N-1}-q$ choices of u such that $S(u,v)=0$ and $dim(\langle v,u\rangle)=2$. Each such u contributes one term of the form

$$(\lambda,0,\ldots,0\mid 0,\ldots,0)$$

for every $\lambda \in \mathbb{F}_q^\times$. Since we are working in $\widetilde{Z}_1$ (and are therefore identifying (0) and $\sum_{v\in V_N}(v)$ with 0), the remaining terms can be reduced to

$$-(q^{2N-2}-q)\cdot\sum_{\lambda\in\mathbb{F}_q^\times}(\lambda,0,\dots,0\mid 0,\dots 0).$$

Therefore, we see that in $\widetilde{Z}_1$,

$$g(g(e_1((v)))) = g(g(v)) = (q^{2N-1}-q^{2N-2})\cdot\textstyle\sum_{\lambda\in\mathbb{F}_q^\times}(\lambda v) = \\ (q-1)\cdot(q^{2N-1}-q^{2N-2})\cdot e_1((v)).$$

Hence,

$$g|_{\widetilde{Z}_1}\circ g|_{\widetilde{Z}_1} = (q^{N-1}\cdot(q-1))^2\cdot Id_{\widetilde{Z}_1}.$$

We may therefore construct idempotents

$$\widetilde{e}_1^{\pm} := \frac{1}{2}\cdot\left(\widetilde{e}_1 \pm \frac{g}{(q-1)\cdot q^{N-1}}\right).$$

The trace of $\widetilde{e}_1$ is

$$\frac{q^{2N}-q}{q-1}$$

and the trace of g can be calculated to be

$$q^{2N}-q^{2N-1}.$$

Therefore, writing

$$\widetilde{Z}_1^{\pm} := Im(\widetilde{e}_1^{\pm}),$$

we have

$$dim(\widetilde{Z}_1^{\pm}) = \frac{1}{2}\cdot\left(\frac{q^{2N}-q}{q-1}\pm q^N\right).$$

To summarize, so far, we have decomposed the $Sp(V_N)$-representation $\mathbb{C}V_N$ as

(21)

$$\mathbb{C}V_N = \\ \left(\bigoplus_{\{\chi:\mathbb{F}_q^\times\to\mathbb{C}^\times|\chi\neq 1,\epsilon\}/(\chi\sim 1/\chi)} 2\cdot Z_\chi\right)\oplus(Z_\epsilon^+\oplus Z_\epsilon^-)\oplus(2\cdot 1\oplus\widetilde{Z}_1^+\oplus\widetilde{Z}_1^-),$$

with

$$dim(Z_\chi) = \frac{q^{2N}-1}{q-1} \text{ for } \chi\in\{\chi:\mathbb{F}_q^\times\to\mathbb{C}^\times\mid\chi\neq 1,\epsilon\}/(\chi\sim 1/\chi)$$

$$dim(Z_\epsilon^+) = dim(X_\epsilon^-) = \frac{q^{2N}-1}{2(q-1)}$$

$$dim(\widetilde{Z}_1^{\pm}) = \frac{1}{2}\cdot\left(\frac{q^{2N}-q}{q-1}\pm q^N\right),$$

giving the claimed decomposition in Case 1 of Theorem 2.

If all these summands are non-isomorphic and are simple, we then can calculate that, as a $\mathbb{C}$-vector space,

$$End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_{-A}) =$$
$$\left(\frac{q-3}{2} \cdot M_2(\mathbb{C})\right) \oplus (\mathbb{C} \oplus \mathbb{C}) \oplus (M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C})$$

where $M_2(\mathbb{C})$ denotes the 2 by 2 matrix algebra on $\mathbb{C}$, and therefore we recover that

$$dim(End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_{-A})) = dim(End_{Sp(V_N)}(\mathbb{C}V_N)) =$$
$$dim(Hom_{Sp(V_N)}(1, \mathbb{C}(V_N \oplus V_N)) = 2q+2,$$

(since $\mathbb{C}V_n$ is self-dual), agreeing with the argument and count of $Sp(V_N)$-orbits in $\mathbb{C}V_N$ at the end of Proof of Proposition 4. If any of the summands of (21) are non-isomorphic or split further, the dimension of $End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_{-A})$ would larger, giving a contradiction. Thus, we have proved Case 1 of Theorem 2. $\square$

4. Proof of Theorem 2 - Case 2

We now consider the case $A \not\equiv -B \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$. Unlike in the case of $A \equiv -B \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$, now $\overline{\omega}_A \otimes \overline{\omega}_B$ is not isomorphic to Ω, so we need to develop a new formula for composition in

$$End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B).$$

This will be done in Lemma 7 below. We began with some preliminary dicussion.

By duality, we have an isommorphism of vector spaces

$$End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B) \cong Hom_{Sp(V_N)}(1, (\overline{\omega}_A \otimes \overline{\omega}_A^\vee) \otimes (\overline{\omega}_B \otimes \overline{\omega}_B^\vee)).$$

Again, we apply the classical fact that

$$\overline{\omega}_A \otimes \overline{\omega}_A^\vee \cong \omega_B \otimes \overline{\omega}_B^\vee \cong \mathbb{C}V_N,$$

to obtain an isomorphism

$$End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B) \cong Hom_{Sp(V_N)}(1, \mathbb{C}(V_N \oplus V_N)). \tag{22}$$

The space $Hom_{Sp(V_N)}(1, \mathbb{C}(V_N \oplus V_N))$ can be considered as a subspace of $\mathbb{C}(V_N \oplus V_N)$ generated by fixed points of $\mathbb{C}(V_N \oplus V_N)$ under the (diagonal) action of $Sp(V_N)$, i.e. sums

$$\sum_{(v,w)\in O} (v, w)$$

for $Sp(V_N)$-orbits $O \subset V_N \oplus V_N$. Equivalently, using the description of the orbits of $V_N \oplus V_N$ in the proof of Proposition 4, we may consider generators

$$\iota := (0,0),$$
$$f_\lambda := \sum_{v\in V_N} (v, \lambda\cdot v), \text{ for } \lambda \in \mathbb{P}^1(\mathbb{F}_q), \text{ (taking } f_\infty := \sum_{v\in V_N}(0,v))$$
$$g_\nu := \sum_{v,w\in V_N,\, S(v,w)=\nu} (v,w) \text{ for } \nu \in \mathbb{F}_q.$$

(note that here we consider "closed strata" elements, meaning that we allow v to be 0 in the definition of f_λ and allow v, w to be linearly dependent in the definition of g_ν).

Write $\alpha = B/A$. We claim the following

Lemma 7. *Using the identification (22), the endomorphism algebra* $(End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B), \circ)$*, is isomorphic to a sub-algebra of* $\mathbb{C}(V_N \oplus V_N)$*, generated by* ι*,* f_λ *for* $\lambda \in \mathbb{P}^1(\mathbb{F}_q)$*, and* g_ν *for* $\nu \in \mathbb{F}_q$*, with respect to the algebra operation*

$$\star : \mathbb{C}(V_N \oplus V_N) \otimes \mathbb{C}(V_N \oplus V_N) \to \mathbb{C}(V_N \oplus V_N)$$

given by

$$(v_1, w_1)\star(v_2, w_2) = \psi_A\Big(\frac{S(v_2, v_1) + \alpha S(w_2, w_1)}{2}\Big)\cdot(v_1+v_2, w_1+w_2), \tag{23}$$

where $\psi_A : \mathbb{F}_q \to \mathbb{C}^\times$ *denotes the non-trivial additive character corresponding to* $A \in \mathbb{F}_q^\times$*.*

Proof. First note that, identifying

$$End_{Sp(V_N)}(\overline{\omega}_A) \cong Hom_{Sp(V_N)}(1, \overline{\omega}_A \otimes \overline{\omega}_A^\vee) \subseteq \mathbb{C}V_N,$$

the composition operation

$$\circ : (End_{Sp(V_N)}(\overline{\omega}_A))^{\otimes 2} \to End_{Sp(V_N)}(\overline{\omega}_A)$$

can be described as the composition

$$\begin{array}{c}(Hom_{Sp(V_N)}(1, \overline{\omega}_A \otimes \overline{\omega}_A^\vee))^{\otimes 2} \\ \downarrow \\ Hom_{Sp(V_N)}(1, \overline{\omega}_A \otimes \underbrace{\overline{\omega}_A^\vee \otimes \overline{\omega}_A}\otimes \overline{\omega}_A^\vee) \\ \downarrow \\ Hom_{Sp(V_N)}(1, \overline{\omega}_A \otimes \overline{\omega}_A^\vee)\end{array}$$

with the top map being tensor product of morphisms (without reordering) and the bottom map being composition with

$$Id_{\overline{\omega}_A} \otimes \epsilon \otimes Id_{\overline{\omega}_A^\vee} : \overline{\omega}_A \otimes \underbrace{\overline{\omega}_A^\vee \otimes \overline{\omega}_A}\otimes \overline{\omega}_A^\vee \to \overline{\omega}_A \otimes \overline{\omega}_A^\vee$$

where

$$\epsilon : \overline{\omega}_A^\vee \otimes \overline{\omega}_A \to 1$$

denotes the unit map of the indicated tensor product of $\overline{\omega}_A^\vee$ and $\overline{\omega}_A$.

Now, let us consider Ω_A to be the quotient of the group algebra $\mathbb{C}\mathbb{H}_N(\mathbb{F}_q)$ such that Ω_A-modules are $\mathbb{H}_N(\mathbb{F}_q)$-representations with central character ψ_A. Then Ω_A has basis (v) for $v \in V_N$ with operation

$$(v) \star_A (w) = \psi_A(\frac{S(w,v)}{2}) \cdot (v+w).$$

Since ω_A is the only irreducible representation of $\mathbb{H}_N(\mathbb{F}_q)$ with central character ψ_A, its Ω_A-module structure gives an isomorphism of algebras from Ω_A to the vector space endomorphisms of ω_A

$$\Omega_A \cong End_{\mathbb{C}}(\omega_A). \tag{24}$$

By the unique definition of $Sp(V_N)$-action on ω_A, (24) is an isomorphism of $Sp(V_N)$-representations (with $Sp(V_N)$ acting Ω_A as the permutation representation and acting on $End_{\mathbb{C}}(\omega_A)$ as $\overline{\omega}_A \otimes \overline{\omega}_A^\vee$. Therefore,

$$End_{Sp(V_N)}(\overline{\omega}_A) = (End_{\mathbb{C}}(\omega_A))^{Sp(V_N)},$$

with composition, is a sub-algebra of Ω_A with $\star_A$.

Now in $End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B)$, the composition operation

$$\circ : (End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B))^{\otimes 2} \to End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B)$$
$$f \otimes g \to f \circ g$$

can be described as a composition of product, permutation, and trace:

$$
\begin{tikzcd}
(End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B))^{\otimes 2} \arrow[d, "\pi"'] & \\
End_{Sp(V_N)}((\overline{\omega}_A \otimes \overline{\omega}_B)^{\otimes 2}) \arrow[r, "? \circ \varsigma"] & End_{Sp(V_N)}((\overline{\omega}_A \otimes \overline{\omega}_B)^{\otimes 2}) \arrow[d, "\tau"] \\
& End_{Sp(V_N)}(\overline{\omega_A} \otimes \overline{\omega_B})
\end{tikzcd} \tag{25}
$$

where π denotes tensor product of morphisms (with no re-ordering of tensor factors), $? \circ \varsigma$ sends an $h \in End_{Sp(V_N)}((\overline{\omega}_A \otimes \overline{\omega}_B)^{\otimes 2})$ to $h \circ \varsigma$ where

$$\varsigma : (\overline{\omega}_A \otimes \overline{\omega}_B)^{\otimes 2} \to (\overline{\omega}_A \otimes \overline{\omega}_B)^{\otimes 2}$$

switches the two tensor factors $(\overline{\omega}_A \otimes \overline{\omega}_B)$, and τ maps an endomorphism $h \in End_{Sp(V_N)}((\overline{\omega}_A \otimes \overline{\omega}_B)^{\otimes 2})$ to its trace, matching the second factor of $(\overline{\omega}_A \otimes \overline{\omega}_B)$ in the source with the second factor of $(\overline{\omega}_A \otimes \overline{\omega}_B)$ in the target.

Again, this corresponds to the algebra operation on

$$Hom_{Sp(V_N)}(1, (\overline{\omega}_A \otimes \overline{\omega}_A^\vee) \otimes (\overline{\omega}_B \otimes \overline{\omega}_B^\vee)) \subseteq \mathbb{C}(V_N \oplus V_N). \tag{26}$$

Since in the composition (25) the factors $\overline{\omega}_A \otimes \overline{\omega_B}$ are preserved, the algebra (26) is a sub-algebra of $\Omega_A \otimes \Omega_B$ with respect to algebra operation $\star = \star_A \otimes \star_B$. Now for $(v_1, w_1), (v_2, w_2) \in \Omega_A \otimes \Omega_B$, we have

$$(v_1, w_1) \star (v_2, w_2) = \psi_A(\frac{S(v_2, v_1)}{2}) \cdot \psi_B(\frac{S(w_2, w_1)}{2}) \cdot (v_1 + v_2, w_1 + w_2). \tag{27}$$

By definition,

$$\psi_A(\frac{S(v_2, v_1)}{2}) \cdot \psi_B(\frac{S(w_2, w_1)}{2}) = \psi_A(\frac{S(v_2, v_1)}{2}) \cdot \psi_A(\frac{\alpha \cdot S(w_2, w_1)}{2}) = \psi_A(\frac{S(v_2, v_1) + \alpha \cdot S(w_2, w_1)}{2}).$$

Applying this to (27), we obtain the operation (23).

□

Now, since for every $(v, w) \in V_N \oplus V_N$,

$$(v, w) \star (0, 0) = \psi_A(0) \cdot (v, w) = (v, w),$$

the unit of $\star$ is $\iota = (0,0)$. For $\lambda \in \mathbb{P}^1(\mathbb{F}_q)$, we have

$$f_\lambda \star f_\lambda = \sum_{v,w\in V_N} (v, \lambda v) \star (w, \lambda w) =$$
$$\sum_{v,w\in V_N} \psi_A\Big(\frac{(1+\alpha\lambda^2)\cdot S(w,v)}{2}\Big)\cdot (v+w, \lambda(v+w)).$$

Reparametrizing this sum using $x = v + w$, we obtain that

$$f_\lambda \star f_\lambda = \sum_{v,x\in V_N} \psi_A\Big(\frac{(1+\alpha\lambda^2)\cdot S(x,v)}{2}\Big)\cdot (x, \lambda x). \tag{28}$$

Since, by assumption, $-\alpha \in \mathbb{F}_q^\times$ is not a square, there are no values of $\lambda \in \mathbb{P}^1(\mathbb{F}_q)$ with $1+\alpha\lambda^2 = 0$. Therefore, since in (28) the coefficient of each $(x, \lambda x)$ is a sum over all $v \in V_N$ of an additive character, applied to a non-zero multiple of $S(x, v)$, the only surviving term with non-zero coefficient is for $x = 0$. Thus,

$$f_\lambda \star f_\lambda = q^{2n}\cdot (0,0) = q^{2n}\cdot \iota.$$

Therefore, for $\lambda \in \mathbb{P}^1(\mathbb{F}_q)$, we have idempotents

$$e_\lambda^\pm := \frac{\iota \pm f_\lambda/q^n}{2}.$$

To calculate trace, note that the trace map

$$\mathbb{C}V_N \cong \omega_A \otimes \omega_A^\vee \to 1$$

is defined by sending (0) to q^n, and (v) to 0 for $v \neq 0 \in V_N$. Hence, $tr(\iota) = tr((0,0)) = q^{2n}$, and the only contributing term of f_λ is also $(0,0)$, so $tr(f_\lambda) = q^{2n}$, as well. Therefore,

$$tr(e_\lambda^\pm) = \frac{q^{2n} \pm q^n}{2}.$$

Now, for $\lambda \neq \mu$,

$$\begin{aligned} f_\lambda \star f_\mu = \sum_{v,w\in V_N} (v, \lambda v) \star (w, \mu w) = \\ \sum_{v,w\in V_N} \psi_A\Big(\frac{(1+\alpha\lambda\mu)\cdot S(w,v)}{2}\Big)\cdot (v+w, \lambda v + \mu w)). \end{aligned} \tag{29}$$

Writing $x = v + w$, $y = \lambda v + \mu w$, we have

$$v = \frac{\mu x - y}{\mu - \lambda},\ w = \frac{\lambda x - y}{\lambda - \mu},$$

and therefore

$$S(w,v) = \frac{S(y,x)}{\mu - \lambda}.$$

Reparametrizing (29), we have

$$f_\lambda \star f_\mu = \sum_{x,y \in V_N} \psi_A\left(\frac{(1+\alpha\lambda\mu)S(w,v)}{2(\mu-\lambda)}\right) \cdot (x,y) = \tag{30}$$
$$\sum_{\nu \in \mathbb{F}_q} \psi_A\left(\frac{\nu}{2} \cdot \frac{1+\alpha\lambda\mu}{\mu-\lambda}\right) \cdot g_\nu.$$

(Note that this implies that using $\star$, the f_λ generate all the g_ν and therefore all of $End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B)$.)

For $\lambda, \mu \in \mathbb{P}^1(\mathbb{F}_q)$, write

$$\lambda * \mu = \frac{\mu + \lambda}{1 - \alpha\lambda\mu}. \tag{31}$$

We need $-\alpha$ to not be a quadratic remainder to avoid $\frac{0}{0}$ when $\lambda = -\mu$. Otherwise, when the denominator is 0, we define the aswer to be ∞. Further, we define

$$\infty * \infty = 0,$$

$$\lambda * \infty = \frac{-1}{\alpha\lambda} \text{ for } \lambda \neq \infty.$$

Comment: Except for the factor α, this operation models the addition formula for $tan(x)$. One may, in fact, define the operation $*$ on $\mathbb{P}^1\mathbb{R}$ where $\tan(x)*\tan(y) = \tan(x+y)$, $x, y \in [-\pi/2, \pi/2]$ with $\pi/2 \sim -\pi/2$.

From the point of view of $\mathbb{F}_q$, the inclusion $\mathbb{F}_{q^2}^\times \subset GL_2(\mathbb{F}_q)$ induces $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times \subset PGL_2(\mathbb{F}_q)$, which gives a simple transitive action of $\mathbb{Z}/(q+1)$ on the rational points of $\mathbb{P}^1(\mathbb{F}_q)$, which is another way (up to reparametrization) of describing the operation $*$.

Then (30) can be expressed as

$$f_\lambda \star f_\mu = \sum_{\nu \in \mathbb{F}_q} \psi_A\left(\frac{\nu}{2 \cdot (\mu * (-\lambda))}\right) \cdot g_\nu.$$

We then make the following

Claim 1. *The operation* (31) *gives* $\mathbb{P}^1(\mathbb{F}_q)$ *the structure of an abelian group with unit* 0 *such that for every* $\lambda \in \mathbb{P}^1(\mathbb{F}_q)$,

$$\lambda * (-\lambda) = 0.$$

In fact,

$$(\mathbb{P}^1(\mathbb{F}_q), *) \cong (\mathbb{Z}/(q+1), +).$$

In particular, this claim implies that for every $x, \lambda, \mu \in \mathbb{P}^1(\mathbb{F}_q)$,

$$f_\lambda \star f_\mu = f_{x*\lambda} \star f_{x*\mu}.$$

Therefore, since f_λ algebra-generate all of $End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B)$ with respect to $\star$, for every $x \in \mathbb{P}^1(\mathbb{F}_q)$,

$$x : End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B) \to End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B)$$

$$f_\lambda \mapsto f_{x*\lambda}$$

is preserves the algebra structure. Thus $End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B)$ forms a $\mathbb{Z}/(q+1)$-algebra, and similarly as before, one can find idempotents $\iota_\theta \in End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B)$ corresponding to characters

$$\theta : \mathbb{Z}/(q+1) \to \mathbb{C}^\times,$$

with

$$tr(\iota_\theta) = \frac{q^{2N}-1}{q+1} \text{ for } \theta \neq 1,$$

$$tr(\iota_1) = \frac{q^{2N}+q}{q+1}$$

Writing

$$Y_\theta := Im(\iota_\theta),$$

the identification of $\theta = 1/\theta$ gives an isomorphism $Y_\theta \cong Y_{1/\theta}$ for $\theta \neq 1, \sigma$ and splits Y_σ into two non-isomorphic summands $Y_\sigma^\pm$ of dimension

$$\frac{q^{2N}+1}{2(q+1)}.$$

Finally, using $e_\lambda^\pm$, we find that Y_1 splits into two summands $Y_1^\pm$, of dimensions

$$\frac{1}{2}\left(\frac{q^{2N}+q}{q+1} \pm q^N\right).$$

We therefore obtain that $\overline{\omega}_A \otimes \overline{\omega}_B$ decomposes as

$$\left(\bigoplus_{\{\theta:\mathbb{Z}/(q+1)\to\mathbb{C}^\times | \theta\neq 1,\sigma\}/(\theta\sim 1/\theta)} Y_\theta\right) \oplus (Y_\sigma^+ \oplus Y_\sigma^-) \oplus (Y_1^+ \oplus Y_1^-),$$

As before, if these summands are non-isomorphic and simple, we have

$$End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B) = \left(\frac{q-1}{2} \cdot M_2(\mathbb{C})\right) \oplus (\mathbb{C} \oplus \mathbb{C}) \oplus (\mathbb{C} \oplus \mathbb{C}),$$

which is consistent with (22) and the count of $Sp(V_N)$-orbits of $\mathbb{C}(V_N \oplus V_N)$ in the end of the proof of Proposition 4. If any summands were isomorphic or split further, the dimension of $End_{Sp(V_N)}(\overline{\omega}_A \otimes \overline{\omega}_B)$ would become too large, giving the decomposition claimed in Case 2 of Theorem 2.

Thus, it remains to prove Claim 1:

Proof of Claim 1. It is clear the $*$ is commutative and associative. It is unital with respect to 0, since

$$\lambda * 0 = \frac{\lambda}{1} = \lambda,$$

and for every $\lambda \in \mathbb{P}^1(\mathbb{F}_q)$,

$$\lambda * (-\lambda) = \frac{0}{1 + \alpha\lambda^2} = 0$$

(again, using the fact that, by our assumptions, α is not a square in $\mathbb{F}_q^\times$, so the denomintaor is not 0. Therefore, $(\mathbb{P}^1(\mathbb{F}_q), *)$ forms an abelian group.

To prove that $(\mathbb{P}^1(\mathbb{F}_q), *)$ is cyclic, in then suffices to show that for every prime ℓ, there is at most one copy of $\mathbb{Z}/\ell$ forming a subgroup of it. For $\ell = 2$, the only λ which are their own inverses (i.e. satisfies $\lambda = -\lambda$, by above) are $\lambda = 0, \infty$. For $\ell > 2$,

$$\underbrace{\infty * \cdots * \infty}_{\ell} = \infty$$

since $\infty = -\infty$ and ℓ must be odd, and for $\lambda \in \mathbb{F}_q$,

$$\underbrace{\lambda * \cdots * \lambda}_{\ell} \tag{32}$$

can be expressed as

$$\lambda \cdot \frac{p_1(\lambda)}{p_2(\lambda)}$$

for polynomials p_1 and p_2 of degree $\ell - 1$. Therefore, there for every prime ℓ, there are at most ℓ elements of degree ℓ. Thus, $(\mathbb{P}^1(\mathbb{F}_q), *)$ is cyclic, and therefore isomorphic to $(\mathbb{Z}/(q+1), +)$.

□

Thus, we have proved both cases of Theorem 2.

5. Interpolation I: The T-Algebra Structure

In this section, we move away from classical rerpesentation theory, and describe the method of interpolation using the technique of *T-algebras.* The goal of this section is to begin constructing the categories claimed in Theorem 1 for values of $t \in \mathbb{C}$ not natural numbers.

Recall that a $\mathbb{C}$-linear additive category with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product and strong duality which is generated by a basic object X can be axiomatized by its *T-algebra*

$$\mathcal{T}(S,T) = Hom(X^{\otimes S}, X^{\otimes T})$$

(see [10, 11], following [1], Chapter 10):

A *T-algebra* $\mathcal{T}$ is a universal algebra structure which consists of the data of vector spaces $\mathcal{T}(S,T)$ corresponding (functorially) to pairs of finite sets S, T, along with the data of *partial trace* operations

$$\tau_\phi : \mathcal{T}(S,T) \to \mathcal{T}(S \smallsetminus S', T \smallsetminus T') \tag{33}$$

corresponding (functorially) to bijections $\phi : S' \to T'$ for subsets $S' \subseteq S$, $T' \subseteq T$, the data of *product* operations

$$\pi : \mathcal{T}(S_1, T_1) \otimes \mathcal{T}(S_2, T_2) \to \mathcal{T}(S_1 \amalg S_2, T_1 \amalg T_2)$$

for finite sets S_1, S_2, T_1, T_2, and the data of two types of "units" $1 \in \mathcal{T}(\emptyset, \emptyset)$ and $\iota \in \mathcal{T}(\{1\}, \{1\})$ satisfying suitable axioms (see [10, 11] for details).

For a T-algebra $\mathcal{T}$, we may conversely construct an additive $\mathbb{C}$-linear category $\mathscr{C}(\mathcal{T})$ with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product and strong duality which is generated by a basic object X, by first constructing a category $\mathscr{C}(\mathcal{T})_0$ with objects

$$Obj(\mathscr{C}(\mathcal{T})_0) = \{X^{\otimes S} \otimes (X^\vee)^{\otimes T} \mid S, T \text{ finite sets}\}$$

and morphisms

$$Hom_{\mathscr{C}(\mathcal{T})_0}(X^{\otimes S_1} \otimes (X^\vee)^{\otimes T_1}, X^{\otimes S_2} \otimes (X^\vee)^{\otimes T_2}) = \mathcal{T}(S_1 \amalg T_2, S_2 \amalg T_1).$$

We construct $\mathscr{C}(\mathcal{T})$ by formally adding direct sums to $\mathscr{C}(\mathcal{T})_0$ and taking a pseudo-abelian envelope (for more details, see [10, 11]).

Recall from F. Knop [8, 9] that there is an interpolation $Rep(GL_t(\mathbb{F}_q))$ of the category of representations $Rep(GL_N(\mathbb{F}_q))$, which is generated by a basic object X of dimension q^t (interpolating $\mathbb{C}\mathbb{F}_q^N$), which is semisimple and pre-Tannakian for $t \in \mathbb{C} \smallsetminus \mathbb{N}_0$. One may consider the spaces of morphisms

$$Hom_{Rep(GL_t(\mathbb{F}_q))}(X^{\otimes S}, X^{\otimes T}), \tag{34}$$

for finite sets S, T. As a vector space, note that (34) is isomorphic to

$$Hom_{Rep(GL_N(\mathbb{F}_q))}((\mathbb{C}\mathbb{F}_q^N)^{\otimes S}, (\mathbb{C}\mathbb{F}_q^N)^{\otimes T}), \tag{35}$$

for $N >> 0$. Let us denote by

$$\mathcal{V}(S, T) \tag{36}$$

the subspace of (35) of morphisms that preserve the action of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ (taking V_N to be a symplectic space of dimension $2N$), for $N >> 0$.

We claim that $\mathcal{V}(S, T)$, considered as a subspace of (34), forms a sub-T-algebra of the T-algebra corresponding to $Rep(GL_t(\mathbb{F}_q))$ generated by X.

The T-algebra $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}$ corresponding to $Rep(GL_t(\mathbb{F}_q))$ generated by X is defined by taking spaces (34), describing partial trace by using the strong duality of X and composition with evaluation and coevaluation morphisms, and taking product to be the tensor product of morphisms (the two "units" then being $\iota = Id_X$ and $1 = Id_1$).

To describe this approach in more detail, we can identify the vector space $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S, T)$, for finite sets S, T, with the free $\mathbb{C}$-vector space generated by equivalence classes of quotients

$$f : \mathbb{F}_q^{S \amalg T} = \mathbb{F}_q\{e_i \mid i \in S \amalg T\} \twoheadrightarrow V \tag{37}$$

with the equivalence relation that f is equivalent to any composition of f with an automorphism of the target.

The product π of two quotient maps is a sum of all possible "amalgamations" of the target vector spaces. For more detail, see [11].

Partial trace can be defined by, for subsets $S' \subseteq S$, $T' \subseteq T$, and a bijection $\phi : S' \to T'$, describing $\tau_\phi(f)$ for f as in (37), and extending linearly.

If there exists an $i \in S'$ such that $f(e_i) \neq f(e_{\phi(i)})$, then take $\tau_\phi(f) = 0$.

If for every $i \in S'$ we have $f(e_i) = f(e_{\phi(i)})$, then we take $\tau_\phi(f)$ to be a multiple of the restriction of $f|_{\mathbb{F}_q^{(S \amalg T) \smallsetminus (S' \amalg T')}}$, where the coefficient is determined by the difference of dimensions

$$\ell = dim(V) - dim(Im(f|_{\mathbb{F}_q^{(S \amalg T) \smallsetminus (S' \amalg T')}})),$$

by being 1 if $\ell = 0$, and

$$(q^t - q^{dim(V)-1}) \cdot \dots \cdot (q^t - q^{dim(V)-\ell})$$

if $\ell \neq 0$. This formula for general t is obtained by polynomially (in q^t) interpolating the respective formulas for $N >> 0$.

For the purposes of this paper, we may restrict attention to the Hom-spaces $Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T})$ where $|S| = |T|$ (the *graded* context – we can set the other $Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T})$ to 0.) We shall assume this convention throughout the rest of this paper.

Now recall the space $\mathcal{V}(S,T)$ of (36).

Lemma 8. *For $t \in \mathbb{C} \smallsetminus \mathbb{N}_0$, the restriction of the partial trace maps*

$$\tau_\phi : \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S,T) \to \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S \smallsetminus S', T \smallsetminus T')$$

for finite sets $S' \subseteq S$, $T' \subseteq T$, and bijections $\phi : S' \to T'$ ($|S| = |T|$, $|S'| = |T'|$), and the product maps

$$\pi : \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S_1,T_1) \otimes \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S_2,T_2) \to$$
$$\to \mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}(S_1 \amalg S_2, T_1 \amalg T_2)$$

for finite sets S_1, S_2, T_1, T_2, to $\mathcal{V}(S,T)$ and $\mathcal{V}(S_1,T_1) \otimes \mathcal{V}(S_2,T_2)$, respectively, have images contained in $\mathcal{V}(S \smallsetminus S', T \smallsetminus T')$ and $\mathcal{V}(S_1 \amalg S_2, T_1 \amalg T_2)$, respectively.

Proof. The statement holds for $N >> 0$ (note that tensoring with (7) can be neglected for our purposes since we are only considering the graded part of the T-algebra for $Rep(GL_t(\mathbb{F}_q))$). Therefore, it holds for a general t since the constants involved are polynomial in q^t for $t = N$. □

Write $\mathcal{V}_t$ for the T-algebra formed by the vector spaces $\mathcal{V}(S,T)$ and the partial trace and product maps of $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}$.

We may therefore consider the category $\mathscr{C}(\mathcal{V})$, which can be considered as generated by an interpolation ω of the Weil-Shale representation. We write

$$\overline{\mathscr{C}}_{q,t} := \mathscr{C}(\mathcal{V}_t).$$

Since $\mathcal{V}_t$ forms a sub-T-algebra of $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}$, the category $\overline{\mathscr{C}}_{q,t}$ forms a subcategory of $Rep(GL_t(\mathbb{F}_q))$.

6. Interpolation II: Semisimplicity

In this section, we complete the construction of the categories in Theorem 1 and complete the proof of Theorem 1. We also prove the decomposition of $X^{\otimes 3}$ claimed in Theorem 3 using Theorem 2. We begin with some general observations.

In an algebra $\mathcal{A}$ of the form

$$\mathcal{A} = \prod_{k=1}^{n} M_k(\mathbb{C}), \tag{38}$$

the general trace is of the form

$$\widetilde{tr}(A) = \sum_{k=1}^{n} b_k \cdot tr(A_k),$$

for $A = (A_1, \ldots, A_n) \in \mathcal{A}$ with $A_k \in M_k(\mathbb{C})$, for some $b_k \neq 0$.

Lemma 9. *For $A \in \mathcal{A}$, if $\widetilde{tr}(A) \neq 0$, then for every $N \in \mathbb{N}$, there exists an $M > N$ such that $\widetilde{tr}(A^M) \neq 0$.*

Proof. Write $A = (A_1, \ldots, A_n) \in \mathcal{A}$, for $A_k \in M_k(\mathbb{C})$. Without loss of generality, the matrices A_k are in Jordan form. Thus, our statement reduces to the following:

Claim: Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ be different numbers. If, for some numbers $\alpha_1, \ldots \alpha_m \in \mathbb{C}$

$$\alpha_1 \lambda_1 + \cdots + \alpha_m \lambda_m \neq 0,$$

then for all N, there exists an $M > N$ such that

$$\alpha_1 \lambda_1^M + \cdots + \alpha_m \lambda_m^M \neq 0.$$

Without loss of generality $\lambda_1, \ldots, \lambda_m \neq 0$. Then the matrix

$$\Lambda = \begin{pmatrix} \lambda_1^N & \lambda_1^{N-1} & \ldots & \lambda_1^{N+m-1} \\ \vdots & \vdots & & \vdots \\ \lambda_m^N & \lambda_m^{N-1} & \ldots & \lambda_m^{N+m-1} \end{pmatrix}$$

is non-singular by the Vandermonde determinant.

Thus, there exists a vector $v = (v_1, \dots v_m)^T$ such that $(\lambda_1, \dots, \lambda_m)^T = \Lambda \cdot v$. Thus, if

$$(\alpha_1, \dots, \alpha_m) \cdot \Lambda = 0,$$

then

$$\alpha_1\lambda_1 + \cdots + \alpha_m\lambda_m = (\alpha_1, \dots, \alpha_m)\Lambda v = (\alpha_1, \dots, \alpha_m)0 = 0.$$

Contradiction.

□

Recall that, for a locally finite, $\mathbb{C}$-linear additive category with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product and strong duality, one can form the *semisimplification* (see [1], Section 6.1) by quotienting out *negligible* morphisms (i.e. morphisms $f : X \to Y$ such that for every morphism $g : Y \to X$, the trace $tr(g \circ f)$ is 0).

Lemma 9 then gives the following

Proposition 10. *The (pseudo-abelian envelope of the) semisimplification of a locally finite, $\mathbb{C}$-linear additive category $\mathscr{C}$ with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product and strong duality which is generated by an object X, is semisimple if and only if for every $a \in End(X^{\otimes n})$,*

$$tr(a) \neq 0 \;\Rightarrow\; \forall N \in \mathbb{N}\; \exists M > N \;\; tr(a^M) \neq 0. \tag{39}$$

Proof. Necessity follows from Lemma 9 (since this is a general form of a trace in a semisimple algebra).

To prove sufficiency, given the assumption, if $a \in End(X^{\otimes n})$ is non-negligible, say, $tr(ab) \neq 0$ for a $b \in End(X^{\otimes n})$, then for every $N \in \mathbb{N}$ there exists an $M > N$ such that

$$tr((ab)^M) \neq 0,$$

and hence $a \notin Jac(End(X^{\otimes n}))$. Thus, the semisimplification of $\mathscr{C}$ is semisimple.

□

This implies the following

Proposition 11. *If the category $\mathscr{C}(\mathcal{T})$ for a T-algebra $\mathcal{T}$ is semisimple (pre-Tannakian), then for every sub-T-algebra $\mathcal{V} \subseteq \mathcal{T}$, the (pseudo-abelian envelope of the) semisimplification of $\mathscr{C}(\mathcal{V})$ is semisimple pre-Tannakian.*

Proof. The condition (39) remains true in $\mathcal{V}$. □

Applying this to our sub-T-algebra $\mathcal{V}_t$ (corresponding to the Weil-Shale representation) of the T-algebra $\mathcal{T}_{Rep(GL_t(\mathbb{F}_q))}$ corresponding to $Rep(GL_t(\mathbb{F}_q))$, we obtain that the semisimplification

$$\mathscr{C}_{q,t}$$

of $\overline{\mathscr{C}}_{q,t}$ is a semisimple category, giving the existence of a semisimple pre-Tannakian category as claimed in Theorem 1.

What remains to show is that (1) holds for values of q^t not equal to $\pm 1, \pm q$:

Lemma 12. *For all t with $q^t \neq \pm 1, \pm q$,*

$$dim(End_{\mathscr{C}_{q,t}}(\omega^{\otimes 3})) = 2q + 2. \tag{40}$$

Proof. Since we have proved that the semisimplification $\mathscr{C}_{q,t}$ of $\overline{\mathscr{C}}_{q,t}$ is semisimple, it suffices to prove that

$$det(tr(a_i \circ a_j)) \tag{41}$$

is non-zero, where $a_1, \ldots, a_{2q+2}$ is a basis of $End_{\overline{\mathscr{C}}_{q,t}}(\omega^{\otimes 3})$.

The generators $a_1, \ldots, a_{2q+2}$ can be identified with the orbits of types (A), (B), (C) at the end of the proof of Proposition 4. The number

$$tr(a_i \circ a_j) \tag{42}$$

can be non-zero when the orbit a_i contains a vector

$$(u, v) = \left(\begin{array}{c|c} u_1 & v_1 \\ u_2 & v_2 \end{array} \right)$$

(where $u_1, u_2, v_1, v_2 \in \mathbb{F}_q^N$) and the orbit a_j contains the vector

$$\left(\begin{array}{c|c} u_2 & v_2 \\ u_1 & v_1 \end{array} \right).$$

In that case, the number (42) is equal to the number of vectors in the orbit a_i (equivalently a_j). We shall refer to the orbits a_i, a_j as *contragradient*. We find that each orbit of type (A), (B) is contragradient to iteself, while the orbit of type (C) corresponding to λ is contragradient to the orbit of type (C) corresponding to $-\lambda$.

The number of elements of the orbit of type (A) is 1. The number of elements of an orbit of type (B) is equal to

$$q^{2n}-1.$$

The number of elements of the orbit of type (C) corresponding to a $\lambda \neq 0$ is equal to

$$(q^{2n}-1)q^{2n-1}.$$

The number of elements of the orbit of type (C) corresponding to $\lambda = 0$ is

$$(q^{2n}-1)(q^{2n-1}-q).$$

Thus, the number (41) is a polynomial in q^{2n} with factors q^{2n}, $q^{2n}-1$, $q^{2n}-q^2$. Therefore, its zeros in $x=q^n$ are $x=0,\pm 1,\pm q$. The statement follows. $\square$

We have therefore completed the proof of Theorem 1

Finally, we prove Theorem 3:

Proof of Theorem 3. We begin by proving Theorem 3 for $t=N$. Fix an $a \in \mathbb{F}_q^\times$, and consider the representation ω_a of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$.

In particular, we obtain that

$$\omega_a \otimes \omega_a = \overline{\omega}_{2a^3} \otimes \omega_{2a}.$$

Therefore, by (17), we have

$$\omega_a \otimes \omega_a = \overline{\omega}_{2a} \otimes \omega_{2a}.$$

Applying (15) again, we obtain that

$$\omega_a^{\otimes 3} = \overline{\omega}_{2a} \otimes (\omega_{2a} \otimes \omega_a) = \overline{\omega}_{2a} \otimes \overline{\omega}_{6a^3} \otimes \omega_{3a} = (\overline{\omega}_{2a} \otimes \overline{\omega}_{6a}) \otimes \omega_{3a}. \tag{43}$$

Therefore, we need to calculate

$$\overline{\omega}_{2a} \otimes \overline{\omega}_{6a}.$$

To apply Theorem 2, we must consider whether or not

$$2a \equiv -6a \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2,$$

or, equivalently, whether or not -3 is a square in $\mathbb{F}_q^\times$, which is determined by q mod 3.

Case 1: $q \equiv 1$ mod 3.

Then since, by assumption, -3 is a square in $\mathbb{F}_q$, we in fact have

$$\overline{\omega}_{6a} \cong \overline{\omega}_{2a}^\vee,$$

and therefore we get that a decomposition of $\omega_a^{\otimes 3}$ as ω_{3a} tensored with (2) (where we consider $Sp(V_N)$ representations as representations of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ by letting $\mathbb{H}_N(\mathbb{F}_q)$ act trivially).

Therefore, by taking

$$X_\chi = Z_\chi \otimes \omega_{3a} \text{ for } \chi \in \{\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times \mid \chi \neq 1, \epsilon\}/(\chi \sim 1/\chi)$$
$$X_\epsilon^\pm = Z_\epsilon^\pm \otimes \omega_{3a}$$
$$X_1^0 = 1 \otimes \omega_{3a} = \omega_{3a}$$
$$\widetilde{X}_1^\pm = \widetilde{Z}_1^\pm \otimes \omega_{3a},$$

we obtain the decomposition and dimensions stated in Case 1 of Theorem 3 for $t = N$, $X = \omega$.

If all the summands of (5) are non-isomorphic and are simple, we then can calculate that, as a $\mathbb{C}$-vector space,

$$End_{Rep(Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q))}(\omega^{\otimes 3}) = \left(\frac{q-3}{2} \cdot M_2(\mathbb{C})\right) \oplus (\mathbb{C}\oplus\mathbb{C}) \oplus (M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C})$$

(where $M_2(\mathbb{C})$ denotes the 2 by 2 matrix algebra on $\mathbb{C}$), and therefore

$$dim(End_{Rep(Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q))}(\omega^{\otimes 3}) = 2q + 2,$$

agreeing with Proposition 9. If there were further decompositions or additional isomorphisms, we would obtain more terms or larger matrix algebras, contradicting Proposition 4. Hence, we may conclude that all X_χ for $\chi \in \{\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times \mid \chi \neq 1, \epsilon\}/(\chi \sim 1/\chi)$, $X_\epsilon^\pm$, X_1^0, and $\widetilde{X}_1^\pm$ are all simple and non-isomorphic, completing the proof of Case 1 of Theorem 3 at $t = N$.

Case 2: $q \equiv -1 \bmod 3$.

Then -3 is not a square in $\mathbb{F}_q$ and therefore we get a decomposition of $\omega_a^{\otimes 3}$ as ω_{3a} tensored with (3) (where again we consider $Sp(V_N)$ representations as representations of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ by letting $\mathbb{H}_N(\mathbb{F}_q)$ act trivially).

Therefore, by taking

$$X_\theta = Y_\theta \otimes \omega_{3a} \text{ for } \theta \in \{\theta : \mu_{q+1}^\times \to \mathbb{C}^\times \mid \theta \neq 1, \sigma\}/(\theta \sim 1/\theta)$$
$$X_\sigma^\pm = Y_\sigma^\pm \otimes \omega_{3a}$$
$$X_1^\pm = Y_1^\pm \otimes \omega_{3a}$$

we obtain the decomposition and dimensions stated in Case 2 of Theorem 3 for $t = N$, $X = \omega$.

If all the summands of (6) are non-isomorphic and simple, then, as a $\mathbb{C}$-vector space,

$$End_{Rep(Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q))}(\omega^{\otimes 3}) = \left(\frac{q-1}{2}\cdot M_2(\mathbb{C})\right) \oplus (\mathbb{C}\oplus\mathbb{C}) \oplus (\mathbb{C}\oplus\mathbb{C})$$

(where, again, $M_2(\mathbb{C})$ denotes the 2 by 2 matrix algebra on $\mathbb{C}$). Therefore we again recover that

$$dim(End_{Rep(Sp(V_N)\ltimes \mathbb{H}_N(\mathbb{F}_q))}(\omega^{\otimes 3}) = 2q+2,$$

agreeing with Proposition 9. As in the previous case, if there were further decompositions or additional isomorphisms, we would obtain more terms or larger matrix algebras, contradicting Proposition 4. Hence, we may conclude that all X_θ for $\chi \in \{\theta : \mu_{q+1} \to \mathbb{C}^\times \mid \chi \neq 1, \sigma\}/(\theta \sim 1/\theta)$, $X_\sigma^\pm$, and $X_1^\pm$ are all simple and non-isomorphic, completing the proof of Case 2 of Theorem 3 at $t = N$.

Therefore we have proved Theorem 3 at $t = N$. Note the by definition, Theorem 3 also holds in

$$\overline{\mathscr{C}}_{q,t}$$

at all values of $t \in \mathbb{C}$. In particular, we recover the bad values $q^t = \pm 1$ and $\pm q$ for Lemma 12 and Theorem 1 since the dimensions of X_χ, X_θ, and $\widetilde{X}_1^\pm$, $X_q^\pm$ are 0 precisely in these cases. Away from these bad values, since the dimensions of all simple summands of $X^{\otimes 3}$ we can conclude that the decomposition of Theorem 3 holds in the semisimplification

$$\mathscr{C}_{q,t},$$

as well.

$\square$

References

[1] P. Deligne. La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel. *Algebraic groups and homogeneous spaces*, 209-273, Tata Inst. Fund. Res. Stud. Math., 19, Tata Inst. Fund. Res., Mumbai, 2007.

[2] P. Deligne: Letters to S. Kriz, 2023-2024.

[3] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik: *Tensor categories.* Math. Surveys Monogr., 205 American Mathematical Society, Providence, RI, 2015, xvi+343 pp.

[4] N. Harman, A. Snowden. Oligomorphic groups and tensor categories. arXiv:2204.04526, 2022.

[5] R. Howe. Invariant Theory and Duality for Classical Groups over Finite Fields, with Applications to their Singular Representation Theory, preprint, Yale University.

[6] R. Howe. On the character of Weil's representation, *Trans. A. M. S.* 177 (1973), 287-298.

[7] N. M. Katz. Larsen's alternative, moments, and the monodromy of Lefschetz pencils. *Contributions to automorphic forms, geometry, and number theory*, 521-560. Johns Hopkins University Press, Baltimore, MD, 2004.

[8] F. Knop. A construction of semisimple tensor categories. *C. R. Math. Acad. Sci. Paris C.* 343, 2006.

[9] F. Knop. Tensor Envelopes of Regular Categories. *Adv. Math.* 214, 2007.

[10] S. Kriz. Arbitrarily High Growth in Quasi-Pre-Tannakian Categories, 2023. Available at https://krizsophie.github.io/ACUCategoryFinal23123.pdf

[11] S. Kriz. Quantum Delannoy Categories, 2023. Available at https://krizsophie.github.io/SquaredGrowth23088.pdf

[12] A. Weil. Sur certains groupes d'operateurs unitaires, *Acta Math.* 111 (1964), 143-211.