An Introduction to Robust Optimization

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- Consider a simple production problem as follow

Parameter	DrugI	DrugII
Selling price, $ per 1000 packs	6,200	6,900
Content of agent A, g per 1000 packs	0.500	0.600
Manpower required, hours per 1000 packs	90.0	100.0
Equipment required, hours per 1000 packs	40.0	50.0
Operational costs, $ per 1000 packs	700	800

(a) Drug production data

Raw material	Purchasing price, $ per kg	Content of agent A, g per kg
RawI	100.00	0.01
RawII	199.90	0.02

(b) Contents of raw materials

Budget, $	Manpower, hours	Equipment, hours	Capacity of raw materials storage, kg
100,000	2,000	800	1,000

(c) Resources

Production problem

- The problem can be immediately posed as the following linear programming

$$
\begin{array}{lll}
\text{Opt} & = & \min\Big\{ \overbrace{[100 \cdot RawI + 199.90 \cdot RawII + 700 \cdot DrugI + 800 \cdot DrugII]}^{\text{purchasing and operational costs}} \\
& & \quad - \underbrace{[6200 \cdot DrugI + 6900 \cdot DrugII]}_{\text{income from selling the drugs}} \Big\} \qquad \text{[minus total profit]}
\end{array}
$$

$$
\begin{array}{lr}
\text{subject to} & \\
0.01 \cdot RawI + 0.02 \cdot RawII - 0.500 \cdot DrugI - 0.600 \cdot DrugII \geq 0 & \text{[balance of active agent]} \\
RawI + RawII \leq 1000 & \text{[storage constraint]} \\
90.0 \cdot DrugI + 100.0 \cdot DrugII \leq 2000 & \text{[manpower constraint]} \\
40.0 \cdot DrugI + 50.0 \cdot DrugII \leq 800 & \text{[equipment constraint]} \\
100.0 \cdot RawI + 199.90 \cdot RawII + 700 \cdot DrugI + 800 \cdot DrugII \leq 100000 & \text{[budget constraint]} \\
RawI, RawII, DrugI, DrugII \geq 0 &
\end{array}
$$

- The optimal solution of the LO problem is $Opt = -8819.658$, $RawI = 0$, $RawII = 438.789$ and $DrugI = 17.552$, $DrugII = 0$.

Question

- **Is the data accurate?**
- Of course not, Of course, no one can guarantee the accuracy of raw material data.
- **What will be the result of inaccurate data?**
- Loss of optimality or even feasibility
- **How to tackle the problem of data uncertainty?**
- Robust optimization

Notation

- An uncertain Linear Optimization problem is a collection as follow
$$LO_U = \{\min_x\{c^T x : Ax \leq b\}\}_{(A,b)\in U}$$
where U is a uncertainty set as follow
$$U = \left\{ \begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} A_0 & b_0 \end{bmatrix} + \sum_{\ell=1}^{L} \zeta_\ell \begin{bmatrix} A_\ell & b_\ell \end{bmatrix} : \zeta \in Z \subset \mathbf{R}^L \right\}$$
- ζ is a perturbation and Z is a perturbation set. Next, we will consider different perturbation set Z.
- PS:
Firstly, we always consider a affine uncertainty.
Secondly, if the objective is uncertain, then we can written the problem as a epigraph form. So we only consider the problem with certain objective and uncertain constraints.

- **Robust feasible**: A vector $x \in \mathbf{R}^n$ is a robust feasible solution to LO_U, if it satisfies all realizations of the constraints from the uncertainty set.

$$Ax \leq b \quad \forall (c, d, A, b) \in U$$

- **Robust counterpart model**: Based on the definition of robust feasible and value, the robust counterpart model can be written as follow.

$$\min_x \left\{ c^T x : Ax \leq b \quad \forall (c, d, A, b) \in U \right\}$$

Box uncertainty

- The outline strategy allow us to focus on the problem with single uncertainty-affected constraint as $\min_x\{c^T x : a^T x \leq b, [a, b] \in U\}$
- Review the definition of uncertainty set as follow where ζ is perturbation vector and Z is perturbation set.

$$U = \left\{ \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a_0 & b_0 \end{bmatrix} + \sum_{\ell=1}^{L} \zeta_\ell \begin{bmatrix} a_\ell & a_\ell \end{bmatrix} : \zeta \in Z \subset \mathbf{R}^L \right\}$$

- Consider the case of interval uncertainty, where Z in a box. W.l.o.g. we can normalize the situation by follow

$$Z = \{\zeta \in \mathbf{R}^L : \|\zeta\|_\infty \leq 1\}$$

- Then the robust counterpart model can be written as follow

$$\min_x\{c^T x : a^T x \leq b \quad \forall (a, b) \in U\}$$

Box uncertainty

- Further, the uncertainty-affected constraint can be written as

$$
\begin{aligned}
&[a_0 + \sum_{\ell=1}^{L} \zeta_\ell a_\ell]^T x \leq b_0 + \sum_{\ell=1}^{l} \zeta_\ell b_\ell \\
\Leftrightarrow & a_0^T x + \sum_{\ell=1}^{L} \zeta_\ell a_\ell^T x \leq b_0 + \sum_{\ell=1}^{l} \zeta_\ell b_\ell \\
\Leftrightarrow & \sum_{\ell=1}^{L} \zeta_\ell [a_\ell^T x - b_\ell] \leq b_0 - a_0 x \\
\Leftrightarrow & \max_{\|\zeta\|_\infty \leq 1} \left\{ \sum_{\ell=1}^{L} \zeta_\ell [a_\ell^T x - b_\ell] \right\} \leq b_0 - a_0^T x \\
\Leftrightarrow & \sum_{\ell=1}^{L} |a_\ell^T x - b_\ell| \leq b_0 - a_0^T x
\end{aligned}
$$

$$\Leftrightarrow \begin{cases} -u_\ell \leq a^T x - b_\ell \leq u_\ell, \quad \forall \ell = 1, \cdots, L \\ a_0^T x + \sum_{\ell=1}^{L} u_\ell \leq b_0 \end{cases}$$

- In this way, the tractable representation of robust counterpart model can be rewritten as a tractable representation.

$$\begin{aligned} \min \quad & c^T x \\ s.t. - u_\ell \leq & a^T x - b_\ell \quad \forall \ell = 1, \cdots, L \\ & a^T x - b_\ell \leq u_\ell \quad \forall \ell = 1, \cdots, L \\ & a_0^T x + \sum_{\ell=1}^{L} u_\ell \leq b_0 \end{aligned}$$

Ball uncertainty

- Similar to the box uncertainty, but we consider a new perturbation set Z as follow where Ω is the radius.

$$Z = \{\zeta \in \mathbf{R}^L : \|\zeta\|_2 \leq \Omega\}$$

Further, the uncertainty-affected constraint can be written as

$$[a_0 + \sum_{\ell=1}^{L} \zeta_\ell a_\ell]^T x \leq b_0 + \sum_{\ell=1}^{l} \zeta_\ell b_\ell$$

$$\Leftrightarrow \max_{\|\zeta\|_\infty \leq 1} \left\{ \sum_{\ell=1}^{L} \zeta_\ell [a_\ell^T x - b_\ell] \right\} \leq b_0 - a_0^T x$$

$$\Leftrightarrow \Omega \sqrt{\sum_{\ell=1}^{L} (a_\ell^T x - b_\ell)^2} \leq b_0 - a_0^T x$$

Ball uncertainty

- In this way, the tractable representation of robust counterpart model can be rewritten as a tractable representation.

$$
\begin{aligned}
\min \quad & c^T x \\
s.t. \quad & a_0^T x + \Omega \sqrt{\sum_{\ell=1}^{L} (a_\ell^T x - b_\ell)^2} \leq b_0
\end{aligned}
$$

General Case

- Now consider a rather general case when the perturbation set Z is given by a conic representation
$$Z = \{\zeta \in \mathbf{R}^L : \exists u \in \mathbf{R}^K : P\zeta + Qu + p \in \mathbf{K}\}$$
- the uncertainty-affected constraint can be written as

$$[a_0 + \sum_{\ell=1}^{L} \zeta_\ell a_\ell]^T x \leq b_0 + \sum_{\ell=1}^{I} \zeta_\ell b_\ell$$
$$\Leftrightarrow \max_{\zeta \in Z} \left\{ \sum_{\ell=1}^{L} \zeta_\ell [a_\ell^T x - b_\ell] \right\} \leq b_0 - a_0^T x$$

- x is feasible if and only if the optimal value in the conic program is $\leq b_0 - a_0^T x$
$$\max_{\zeta, u} \left\{ \sum_{\ell=1}^{L} \zeta_\ell [a_\ell^T x - b_\ell] : P\zeta + Qu + p \in \mathbf{K} \right\}$$

General Case

- The optimal value of the conic program is $\leq b_0 - a_0^T x$ if and only if the conic dual problem as follow is attained and is $\leq b_0 - a_0^T x$

$$\begin{aligned} \min \quad & p^T y \\ s.t. \quad & Q^T y = 0 \\ & (P^T y)_\ell = -[a_\ell^T x - b_\ell] \ \ \forall \ell = 1, \cdots, L \\ & y \in K_* \end{aligned}$$

- Then the tractable representation of the robust counterpart model is as follow

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & p^T y + a_0^T x \leq b_0 \\ & Q^T y = 0 \\ & (P^T y)_\ell + a_\ell^T x = b_\ell \ \ \forall \ell = 1, \cdots, L \\ & y \in \mathbf{K}_* \end{aligned}$$

- Further, consider the perturbation set Z is a intersection of multiple cone conic as follow.

$$Z = \left\{\zeta : \exists u^1, \cdots, u^S : P_s\zeta + Q_s + p_s \in K^s, s = 1, \cdots, S\right\}$$

- The tractable representation of the robust counterpart model is as follow

$$\begin{aligned}
\min \quad & c^T x \\
s.t. \quad & \sum_{s=1}^{S} p_s^T y^s + a_0^T x \le b_0 \\
& Q_s^T y^s = 0 \quad \forall s = 1, \cdots, S \\
& \sum_{s=1}^{S} (P_s^T y^s)_\ell + a_\ell^T x = b_\ell \ \ \forall \ell = 1, \cdots, L \\
& y^s \in \mathbf{K}_*^s, \forall s = 1, \cdots, S
\end{aligned}$$

Example - Budgeted Uncertainty

- Consider the case where perturbation set Z is the intersection of $\|\cdot\|_\infty$ and $\|\cdot\|_1$, specifically,
$$Z = \{\zeta \in \mathbf{R}^L : \|\zeta\|_\infty \leq 1, \|\zeta\|_1 \leq \gamma\}$$
where γ is a given "uncertainty budget".
- The perturbation set Z can be rewirtten as
$$Z = \{\zeta \in \mathbf{R}^L : P_1\zeta + p_1 \in K^1, P_2\zeta + p_2 \in K^2\}$$
where
$(1) P_1\zeta = [\zeta; 0],\ p_1 = [0_{L\times 1}; 1],\ K^1 = \{[u;t] \in \mathbf{R}^L \times \mathbf{R} : \|u\|_\infty \leq t\}$
$(1) P_2\zeta = [\zeta; 0],\ p_1 = [0_{L\times 1}; \gamma],\ K^2 = \{[u;t] \in \mathbf{R}^L \times \mathbf{R} : \|u\|_1 \leq t\}$
- Further, $K_*^1 = \{[u;t] \in \mathbf{R}^L \times \mathbf{R} : \|u\|_1 \leq t\} = K^2$ and $K_*^2 = \{[u;t] \in \mathbf{R}^L \times \mathbf{R} : \|u\|_\infty \leq t\} = K^1$

- Define the dual variable as $y^1 = [u; \tau_1]$ and $y^2 = [v; \tau_2]$ where $u, v \in \mathbf{R}^L$ and $\tau_1, \tau_2 \in \mathbf{R}$.
- From the conclusion of general case, we can write the tractable representation of the robust counterpart model as follow

$$
\begin{aligned}
\min \quad & c^T x \\
s.t. \quad & \tau_1 + \gamma \tau_2 + a_0^T x \leq b_0 \\
& (u + v)_\ell = b_\ell - a_\ell^T x, \quad \forall \ell = 1, \cdots, L \\
& \|u\|_1 \leq \tau_1 \\
& \|v\|_\infty \leq \tau_2
\end{aligned}
$$

Example - Budgeted Uncertainty

- Further, the above model can be simplified by eliminating τ_1 and τ_2.

$$
\begin{aligned}
\min \quad & c^T x \\
s.t. \quad & \|u\|_1 + \gamma \|v\|_\infty + a_0^T x \le b_0 \\
& (u+v)_\ell = b_\ell - a_\ell^T x, \quad \forall \ell = 1, \cdots, L
\end{aligned}
$$

Scalar Chance Constraints

- We consider a scalar chance constraints as follow

$$Prob_{\zeta \sim P}\left\{\zeta : a_0^T x + \sum_{\ell=1}^{L} \zeta_\ell a_\ell^T x > b_0 + \sum_{\ell=1}^{L} \zeta_\ell b_\ell\right\} \leq \epsilon$$

- There is two assumption of the distribution of ζ as follow
  - 1. $\zeta_\ell, \ell = 1, \cdots, L$ are independent random variables;
  - 2. The distribution P_ℓ of the components ζ_ℓ are such that

  $$\int \exp\{t\zeta_\ell\} dP_\ell(\zeta_\ell) \leq exp\{|\mu_\ell t| + \tfrac{1}{2}\sigma_\ell^2 t^2\} \quad \forall t \in R$$

  with known constant $\mu_\ell^- \leq \mu_\ell^+$ and $\sigma_\ell \geq 0$

- For convenience, let z_0 denote $z_0 = a_0^T x - b_0$ and z_ℓ denote $a_\ell^T x - b_\ell$ for $\ell = 1, \cdots, L$. Then, the chance constraints can be written as

$$p(z) = Prob_{\zeta \sim P}\left\{\zeta : z_0 + \sum_{\ell=1} z_\ell \zeta_\ell > 0\right\} \leq \epsilon$$

Scalar Chance Constraints

$$
\begin{aligned}
& z_0 + \sum_{\ell=1}^{L} z_\ell \zeta_\ell > 0 \\
\Leftrightarrow & \exp\{\alpha[z_0 + \sum_{\ell=1}^{L} z_\ell \zeta_\ell]\} > 1 \quad \forall \alpha > 0 \\
\Rightarrow & \mathbf{E}\left\{\exp\{\alpha[z_0 + \sum_{\ell=1}^{L} z_\ell \zeta_\ell]\}\right\} \geq p(z) \quad \forall \alpha > 0 \\
\Rightarrow & \exp\{\alpha z_0 + \sum_{\ell=1}^{L} [\alpha |\mu_\ell z_\ell| + \frac{\alpha^2}{2} \sigma_\ell^2 z_\ell^2]\} \geq p(z) \quad \forall \alpha > 0 \\
\Leftrightarrow & \alpha z_0 + \sum_{\ell=1}^{L} [\alpha |\mu_\ell z_\ell| + \frac{\alpha^2}{2} \sigma_\ell^2 z_\ell^2] \geq \ln(p(z)) \quad \forall \alpha > 0
\end{aligned}
$$

Scalar Chance Constraints

- Therefore, we can say $p(z) \leq \epsilon$ if the following inequality holds
$$\exists \alpha : \quad \alpha z_0 + \sum_{\ell=1}^{L} [\alpha |\mu_\ell z_\ell| + \tfrac{\alpha^2}{2} \sigma_\ell^2 z_\ell^2] \leq \ln(\epsilon)$$
- The left side of above is a quadratic function of α. Then the above inequality holds if and only if
$$z_0 + \sum_{\ell=1}^{L} |\mu_\ell z_\ell| + \sqrt{2 \ln(1/\epsilon)} \sqrt{\sum_{\ell=1}^{L} \sigma_\ell^2 z_\ell^2} \leq 0$$
- In this way, we get a robust counterpart approximations of the chance constraints as above.

- Now, we consider a simple distribution as follow

$$\mathbf{E}\{\zeta_\ell\} = 0$$
$$|\zeta_\ell| \leq 1 \quad \forall \ell = 1, \cdots, L$$
$$\{\zeta_\ell\} \textit{ are independent}.$$

- Then, the robust counterpart approximations can be written as follow which is equivalent to the ball uncertainty.

$$z_0 + \sqrt{2\ln(1/\epsilon)}\sqrt{\sum_{\ell=1}^{L} \sigma_\ell^2 z_\ell^2} \leq 0$$

- Further, we can have the follow inequality

$$StD(\sum_{\ell=1}^{L} z_\ell \zeta_\ell) = \sqrt{\sum_{\ell=1}^{L} \sigma_\ell^2 z_\ell^2}$$

- The above approximation holds since

$$Prob\left\{\zeta : \sum_{\ell=1}^{L} z_\ell \zeta_\ell > \sqrt{2\ln(1/\epsilon)}\sqrt{\sum_{\ell=1}^{L} \sigma_\ell^2 z_\ell^2}\right\} \leq \epsilon$$

Another Approximation - Budgeted Uncertainty

- In the "General case", we get a robust corresponding constraints for budgetary uncertainty as follow.

$$\|u\|_1 + \gamma\|v\|_\infty + a_0^T x \leq b_0 \tag{1}$$

$$(u+v)_\ell = b_\ell - a_\ell^T x, \quad \forall \ell = 1, \cdots, L \tag{2}$$

- Then, we can prove the above satisfy $p(z) \leq \exp(\frac{-\gamma^2}{2L})$.
- Firstly, we should have $\|v\|_2 \leq \sqrt{L}\|v\|_\infty$ from the follow inequality

$$\|v\|_2^2 = \sum_{\ell=1}^{L} v_\ell^2 \leq \sum_{\ell=1}^{L} |v_\ell|\|v\|_\infty \leq \sqrt{L}\|v\|_2\|v\|_\infty$$

Another Approximation - Budgeted Uncertainty

Proof:

$$
\begin{aligned}
&\sum_{\ell=1}^{L} \zeta_\ell [a_\ell^T x - b_\ell] > b_0 - a_0^T x \\
\Rightarrow &-\sum_{\ell=1}^{L} u_\ell \zeta_\ell - \sum_{\ell=1}^{L} v_\ell \zeta_\ell > b_0 - a_0^T x \quad (\mathit{from}\ (2)) \\
\Rightarrow &\|u\|_1 - \sum_{\ell=1}^{L} v_\ell \zeta_\ell > b_0 - a_0^T x \\
\Rightarrow &-\sum_{\ell=1}^{L} v_\ell \zeta_\ell > b_0 - a_0^T x - \|u\|_1 \geq \gamma \|v\|_\infty \quad (\mathit{from}(1)) \\
\Rightarrow &-\sum_{\ell=1}^{L} v_\ell \zeta_\ell > \frac{\gamma}{\sqrt{L}} \|v\|_2 \Rightarrow Prob\{z_0 + \sum_{\ell=1}^{L} z_\ell \zeta_\ell > 0\} \leq \exp(\frac{-\gamma^2}{2L})
\end{aligned}
$$

- There are 200 assets. The 200th Asset has yearly return $r_{200} = 1.05$ and zero variability.
- The yearly returns r_ℓ, $i = 1, \cdots, 199$ of the remaining assets are independent random variables taking values in $[\mu_\ell - \sigma_\ell, \mu_\ell + \sigma_\ell]$ where

$$\mu_\ell = 1.05 + 0.3\frac{(200-i)}{199} \qquad and \qquad \sigma_\ell = 0.05 + 0.06\frac{(200-i)}{199}$$

- We want to solve the uncertain LO problem as follow.

$$\begin{aligned} \max_{x,R} \quad & R \\ s.t. \quad & \sum_{\ell=1}^{199} r_\ell x_\ell + r_{200}x_{200} \geq R \\ & \sum_{\ell=1}^{200} x_\ell = 1 \\ & \mathbf{x} \succeq 0 \end{aligned}$$

- The uncertain data are $r_\ell = \mu_\ell + \sigma_\ell \zeta_\ell$, $\forall \ell = 1, \cdots, 199$

Robust Counterpart Model

- The box RC is as follow:

$$\max_{x,R} \left\{ R : \begin{array}{c} \sum_{\ell=1}^{199} (\mu_\ell - \sigma_\ell) x_\ell + r_{200} x_{200} \geq R \\ \sum_{\ell=1}^{200} x_\ell = 1 \\ \sum_{\ell=1}^{200} x_\ell = 1 \mathbf{x} \succeq 0 \end{array} \right\}$$

- The ball RC is as follow:

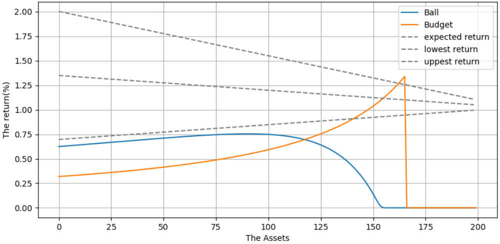
$$\max_{x,R,z,w} \left\{ R : \begin{array}{c} \sum_{\ell=1}^{199} \mu_\ell x_\ell + r_{200} x_{200} - \|z\|_1 - \Omega \sqrt{\sum_{\ell=1}^{L} w_\ell^2} \geq R \\ z_\ell + w_\ell = \sigma_\ell x_\ell \quad \forall \ell = 1, \cdots, 199 \\ \sum_{\ell=1}^{200} x_\ell = 1 \quad \mathbf{x} \succeq 0 \end{array} \right\}$$

- The budgeted RC is as follow:

$$\max_{x,R,z,w} \left\{ R : \begin{array}{c} \sum_{\ell=1}^{199} \mu_\ell x_\ell + r_{200} x_{200} - \|z\|_1 - \gamma \|w\|_\infty \geq R \\ z_\ell + w_\ell = \sigma_\ell x_\ell \quad \forall \ell = 1, \cdots, 199 \\ \sum_{\ell=1}^{200} x_\ell = 1 \quad \mathbf{x} \succeq 0 \end{array} \right\}$$

Performance

- Take $\epsilon = 0.005$, then the correspending $\Omega = 3.255$ and $\gamma = 45.921$.
- The solution of box is $x_{200} = 1$ and $x_\ell = 0$ for all $\ell = 1, \cdots, 199$. The solution of "ball RC" and "budget RC" is shown as follow. And $R_{box} = 1.05$, $R_{ball} = 1.12$ and $R_{budget} = 1.10$
- The solutions are robust since the failure times of three model in 10^6 simulations are 0.



Conservatism	*Box* > *Budget* > *Ball*
Risk	*Ball* = *Budget* > *Box*
Return	*Ball* > *Budget* > *Box*

Thanks!