TABLE E3.4

z_1	z_2	y
3	6	176
4	4	192
5	7	262
6	3	230
7	6	308
8	4	312

(e) If model (a) is fitted but model (b) is "true," what is $E(\hat{\theta}_1)$? What relationship does this have to your answer in (c)?
(f) Construct a 95% joint confidence region for θ_1 and θ_2.

3.5. (Source: Inactivation of adrenaline and nonadrenaline by human and other mammalian liver in vitro, by W. A. Bain and J. E. Batty, *Brit. J. Pharm. Chemotherapy*, **11**, 1956, 52–57.) The data in Table E3.5 are $n = 14$ epinephrine (adrenaline) concentrations Y (erg/mL) for five "times in the lower tissues," X (min), coded to x.
(a) What is the coding?
(b) Assume for the moment that the data consist of 14 independent observations. Fit $Y = \beta_0 + \beta_1 x + \varepsilon$, but show that lack of fit exists. What alternative model would you recommend?
(c) Fit a suitable alternative model.
(d) You now look up the original paper and find that each Y column in the table is a separate experiment, in each of which samples were taken successively in time from the same tube. Might this affect your analysis? If so, how and why?

TABLE E3.5

X	x	Y			ΣY
6	−2	30.0	28.6	28.5	87.1
18	−1	8.9	8.0	10.8	27.7
30	0	4.1	—	4.7	8.8
42	1	1.8	2.6	2.2	6.6
54	2	0.8	0.6	1.0	2.4
					132.6

CHAPTER 4

Factorial Designs at Two Levels

4.1. THE VALUE OF FACTORIAL DESIGNS

In the model-building process, the step of model specification or identification, as a result of which a model worthy of being entertained is put forward, is a somewhat tenuous one. It is a creative step in which the human mind, taking account of what is known about the system under study, must be allowed to interact freely with the data, making comparisons, seeking similarities, differences, trends, and so on. A class of experimental designs called "factorials" greatly facilitates this process.

We obtain a complete *factorial design* in k factors by choosing n_1 levels of factor 1, n_2 levels of factor 2, ..., n_k levels of factor k, and then selecting the $n = n_1 \times n_2 \times \cdots \times n_k$ runs obtained by taking all possible combinations of the levels selected. Figure 4.1*a* shows a $3 \times 2 \times 2$ factorial design in the factors:

Temperature	(three levels: 180°C, 190°C, 200°C)
Pressure	(two levels: 70 psi, 90 psi)
Catalyst	(two levels: type A, type B)

Suppose the numbers shown in Figure 4.1*b* are the measured amounts of an impurity which are produced at the various sets of conditions. By simply looking at the figure, an observer can make many different and informative comparisons. For example, suppose we knew that the experimental error was small (that the standard deviation σ was less than 1, say). Then the "horizontal" comparisons (7 → 8 → 10), (6 → 8 → 11), (3 → 5 → 5), (3 → 4 → 5) would suggest that, over the ranges studied, increasing temperature produced increasing impurity.

Also, the "vertical" comparisons (7 → 6), (8 → 8), (10 → 11), (3 → 3), (5 → 4), (5 → 5) would suggest that there is little to choose between the two

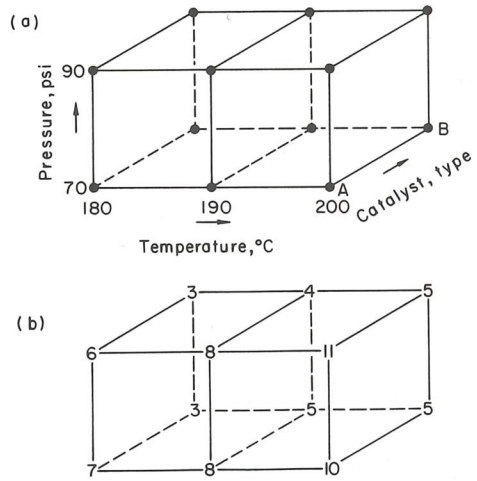

FIGURE 4.1. A $3 \times 2 \times 2$ factorial design. (*a*) The points of the design. (*b*) Impurity values observed at the 12 design point locations.

pressures examined. The six comparisons between the two types of catalysts would suggest that, over the ranges tested, B produces less impurity than A. Furthermore, the increase of impurity with increased temperature is less with B than with A (so that there is a temperature–catalyst interaction). The possibility is also seen that perhaps a fixed change in temperature produces a fixed *proportionate* response so that it is possible that the logarithm of the response is linearly and additively affected by temperature.

All of the above ideas can, of course, be subjected to more formal examination. It is the ability of the factorial design to suggest ideas to the investigator that we wish to emphasize.

In the above example, the factorial design is used to study two quantitative factors or variables (temperature and pressure) and one qualitative variable (type of catalyst). Most of this book will be concerned only with the study of quantitative variables. Factorial designs possess the following desirable properties:

1. They allow multitudes of comparisons to be made and so facilitate model creation and criticism.
2. They provide highly efficient estimates of constants (parameters), that is, estimates whose variances are as small, or nearly as small, as those that could be produced by any design occupying the same space.
3. They give rise to simple calculations.

On the mistaken supposition that simplicity of calculation is the *only* reason for their use, it is sometimes suggested that factorial designs are no longer needed because computers have made even complex calculations easy. Also it may be argued that "optimal" designs may now be computed (see Chapter 14 for details) which do not necessarily give nice patterns and yet, *if the model is assumed known*, will also give highly efficient estimates. However, this argument discounts the importance of allowing the mind to see patterns in the data at the model specification stage.

4.2. TWO-LEVEL FACTORIALS

Of particular importance for our purpose are the "two-level" factorial designs in which each variable occurs at just two levels. Such designs are especially useful at the exploratory stage of an investigation, when not very much is known about a system and the model is still to be identified. As we shall see in later examples, the initial two-level pattern can be a first building block in developing structures of many different sorts. A model-identification technique of great value that may be used in association with this type of design is the plotting of coefficients on normal probability paper (see Daniel, 1959, 1976). For additional details see Appendix 4B.

For a first illustration, we return to the 2^3 (i.e., $2 \times 2 \times 2$) factorial design introduced in Chapter 2 (see Table 2.2). This design was used to study Y = cycles to failure for combinations of three variables, each tested at two levels as follows.

Coded levels	x_i	-1	1
Length of test specimen (mm)	ξ_1	250	350
Amplitude of load cycle (mm)	ξ_2	8	10
Load (g)	ξ_3	40	50

As discussed in Chapter 2, it is convenient to code the lower and upper levels as x_1, x_2, x_3, taking the values -1 for the lower level and 1 for the upper level. Thus, for this example,

$$x_1 = \frac{(\xi_1 - 300)}{50}, \quad x_2 = \frac{(\xi_2 - 9)}{1}, \quad \text{and} \quad x_3 = \frac{(\xi_3 - 45)}{5}. \tag{4.2.1}$$

The 2^3 design is shown in Table 4.1 in both uncoded and coded form, together with the response values Y = cycles to failure and the values of $y = \log Y$. (The logarithms are taken to base 10.)

TABLE 4.1. Textile data. A 2^3 factorial design in uncoded and coded units with response Y, and $y = \log Y$ values

Uncoded			Coded			Cycles	
Specimen length (mm), ξ_1	Amplitude (mm), ξ_2	Load (g), ξ_3	Specimen length, x_1	Amplitude, x_2	Load, x_3	to failure, Y	$y =$ log Y
250	8	40	−1	−1	−1	674	2.83
350	8	40	1	−1	−1	3636	3.56
250	10	40	−1	1	−1	170	2.23
350	10	40	1	1	−1	1140	3.06
250	8	50	−1	−1	1	292	2.47
350	8	50	1	−1	1	2000	3.30
250	10	50	−1	1	1	90	1.95
350	10	50	1	1	1	360	2.56

Figure 4.2 shows the eight runs set out in the space of the three variables where the design points appear as the vertices of a cube. On the vertices of the left-hand cube are shown the values of the response Y, the number of cycles to failure, and on the right-hand cube are shown the corresponding transformed values $y = \log Y$. The geometrical display again illustrates the virtues of the factorial design. It is very easy to appreciate the many comparisons that can be made. In particular, one can see the consistent increases that occur in Y with increase in length (ξ_1) and the reductions that occur with increase in amplitude (ξ_2) and with increase in load (ξ_3). Moreover, the differences in magnitudes of these changes largely disappear after log transformation, indicating the regularity that occurs in the *proportional* increases in this example.

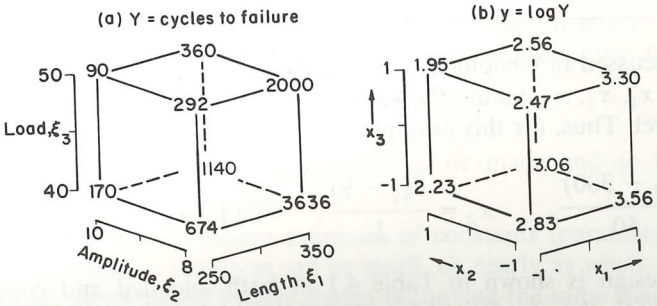

FIGURE 4.2. Textile data. A 2^3 design shown geometrically in (*a*) uncoded and (*b*) coded units, with responses (*a*) Y = cycles to failure and (*b*) $y = \log Y$.

In general, a 2^k factorial design consists of all the 2^k runs (points) with levels (coordinates)

$$(x_1, x_2, \ldots, x_k) = (\pm 1, \pm 1, \ldots, \pm 1), \tag{4.2.2}$$

where every possible combination of $\pm$ signs is selected in turn. Geometrically the design consists of the vertices of a hypercube in k dimensions. For purposes of analysis, it is convenient to list the runs in *standard order*, not in the order (usually randomized) in which they were made. This standard order is obtained by writing alternate − and + signs in the column headed x_1, alternate pairs − −, + +, in the x_2 column, alternate fours − − − −, + + + +, in the x_3 column, and so on. For illustration, the runs are set out in standard order in Table 4.1.

Analysis of the Factorial Design

The main effect of a given variable as defined by Yates (1937) is the average difference in the level of response as one moves from the low to the high level of that variable. For illustration, consider the calculation of the main effects for $y = \log Y$. A glance at Figure 4.2*b* shows that the main effect indicated by 1, of variable x_1, is*

$$1 \leftarrow \tfrac{1}{4}(3.56 + 3.06 + 3.30 + 2.56) - \tfrac{1}{4}(2.83 + 2.23 + 2.47 + 1.95)$$
$$= 0.75. \tag{4.2.3}$$

Similar calculations provide the results

$$2 \leftarrow -0.59, \tag{4.2.4}$$

$$3 \leftarrow -0.35. \tag{4.2.5}$$

A valuable property of the factorial design is that it makes possible not only the calculation of main effects (i.e., average effects) but also of interaction effects between variables as well.

Two variables, say 1 and 3, are said to interact (in their effect on the response) if the effect of 1 is different at the two different levels of 3. In the textile example, if we confine attention for the moment to the first four runs

*Here and in Chapter 5, we use the arrow pointing to the left (←) to mean "is estimated by," so that Eq. (4.2.3) is read as "the main effect of variable 1 is estimated by the number 0.75." Similarly, the reverse notation "0.75 → 1" is read as "0.75 is an estimate of the main effect of variable 1."

in which x_3 is at its lower level, the main (average) effect of 1 is

$$(1|x_3 = -1) \leftarrow \tfrac{1}{2}(3.56 + 3.06) - \tfrac{1}{2}(2.83 + 2.23) = 0.78. \quad (4.2.6)$$

However, for the last four runs, with x_3 at its upper level, the main effect of 1 is

$$(1|x_3 = 1) \leftarrow \tfrac{1}{2}(3.30 + 2.56) - \tfrac{1}{2}(2.47 + 1.95) = 0.72. \quad (4.2.7)$$

The interaction between variables 1 and 3 is defined as half the difference between the main effect of 1 at the upper level of x_3 and the main effect of 1 at the lower level of x_3. This interaction is denoted by the symbol 13, so that

$$13 = \tfrac{1}{2}\{(1|x_3 = 1) - (1|x_3 = -1)\} \leftarrow \tfrac{1}{2}\{0.72 - 0.78\} = -0.03. \quad (4.2.8)$$

(There is no ambiguity in this definition in the sense that interchanging the roles of variables 1 and 3 does not change the value of the interaction.)

Exercise. Evaluate $\tfrac{1}{2}\{(3|x_1 = 1) - (3|x_1 = -1)\}$ and show that it produces a result identical to the 13 value already obtained.

Via similar calculations we obtain the other two two-factor interactions as

$$12 \leftarrow -0.03, \quad (4.2.9)$$

$$23 \leftarrow -0.04. \quad (4.2.10)$$

Finally, the interaction 12 might be different at different levels of the variable 3. To check this we could evaluate and compare

$$(12|x_3 = -1) \leftarrow \tfrac{1}{2}\{(3.06 - 2.23) - (3.56 - 2.83)\} = 0.05,$$

$$(12|x_3 = 1) \leftarrow \tfrac{1}{2}\{(2.56 - 1.95) - (3.30 - 2.47)\} = -0.11. \quad (4.2.11)$$

Half the difference between these quantities is called the 123 interaction, that is,

$$123 = \tfrac{1}{2}\{(12|x_3 = 1) - (12|x_3 = -1)\} \leftarrow -0.08. \quad (4.2.12)$$

Exercise. Show that the same result occurs if we define the 123 interaction as half the difference between the interaction 23 at levels of $x_1 = 1$ and $x_1 = -1$, or as half the difference between the interaction 13 at levels of $x_2 = 1$ and $x_2 = -1$. Thus the three-way interaction 123 is unambiguous.

TABLE 4.2. Columns of signs and the divisors for systematically obtaining the factorial effects in a 2^3 factorial design

	I	1	2	3	12	13	23	123	y
	+	−	−	−	+	+	+	−	2.83
	+	+	−	−	−	−	+	+	3.56
	+	−	+	−	−	+	−	+	2.23
	+	+	+	−	+	−	−	−	3.06
	+	−	−	+	+	−	−	+	2.47
	+	+	−	+	−	+	−	−	3.30
	+	−	+	+	−	−	+	−	1.95
	+	+	+	+	+	+	+	+	2.56
Divisor	8	4	4	4	4	4	4	4	

There is an easier systematic way of making these calculations using the columns of signs of Table 4.2. Consider the calculation of the main effect 1 as in Eq. (4.2.3). This could be written

$$1 \leftarrow \tfrac{1}{4}(-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8). \quad (4.2.13)$$

In other words, the main effect 1 would be obtained by multiplying the column of data **y** by the column of − and + signs in the column labeled **1** in Table 4.2 and dividing by the divisor 4 indicated there. Note that the divisors are the numbers of + signs in the corresponding columns.

Similarly, it can readily be confirmed that

$$2 \leftarrow \tfrac{1}{4}(-y_1 - y_2 + y_3 + y_4 - y_5 - y_6 + y_7 + y_8), \quad (4.2.14)$$

where the signs and divisor are taken from the column of Table 4.2 labeled **2**. The main effect of 3 is similarly obtained.

If the operations leading to the calculation of the two-factor interaction 13 in Eq. (4.2.8) are similarly analyzed, it will be found that

$$13 \leftarrow \tfrac{1}{4}(y_1 - y_2 + y_3 - y_4 - y_5 + y_6 - y_7 + y_8). \quad (4.2.15)$$

This effect can be mechanically computed from the **13** column in the table; the signs in that column are simply the results of taking the products of the signs in the **1** and **3** columns, line by line. The interaction effects 12 and 23 are calculated in exactly similar fashion using their corresponding columns.

Finally, if the calculations leading to the 123 interaction are analyzed, we find that

$$123 \leftarrow \tfrac{1}{4}(-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8). \qquad (4.2.16)$$

The signs are those shown in the **123** column in Table 4.2 and this column is formed by taking the triple product of signs from the **1**, **2**, and **3** columns.

The table of signs is completed by the addition of a column of plus signs, shown at the left with the heading **I**. This is needed for obtaining the average of the response values, and its divisor (the number of plus signs in the column) is 8 rather than 4 as in the other columns.

A table such as Table 4.2 is very easily constructed for any 2^k factorial design as follows. Begin with a column of 1's of length 2^k. The next k columns are ± 1 signs for the design, written down in standard order. Columns **12**, **13**, ..., **123**, ..., **123...k**, $(2^k - k - 1)$ in number, are then obtained by multiplying signs, row by row, in the way indicated by the headings. At the bottom are written the divisors, 2^k for the first column, and 2^{k-1} for all the others.

The table of signs is very useful for understanding the nature of the various factorial effects. However, a quicker method of obtaining the effects mechanically is available, due to Yates; this is described in Appendix 4A.

Variance and Standard Errors of Effects for 2^k Designs

For a complete 2^k design, if $V(y) = \sigma^2$,

$$V(\text{grand mean}) = \frac{\sigma^2}{2^k},$$

$$V(\text{effect}) = \frac{4\sigma^2}{2^k}. \qquad (4.2.17)$$

If the responses are not "y's" but are instead "$\bar{y}$'s", the averages of (say) r y-observations, then

$$V(\text{grand mean}) = \frac{\sigma^2}{n}, \qquad V(\text{effect}) = \frac{4\sigma^2}{n}, \qquad (4.2.17a)$$

where n is the total number of y-observations. Here $n = r2^k$. Formulas (4.2.17a) are also generally applicable to 2^{k-p} fractional factorial designs (see Chapter 5).

In practice, we shall need to obtain an estimate s^2 of the experimental error variance σ^2. We shall consider methods for doing this as we proceed. For the textile data, suppose that an estimate $s^2 = 0.0050$ were available. Then

$$\hat{V}(\bar{y}) = 0.000625, \qquad \hat{V}(\text{effect}) = 0.0025, \qquad (4.2.18)$$

and the corresponding standard errors are the square roots

$$s(\bar{y}) = 0.025, \qquad s(\text{effect}) = 0.05. \qquad (4.2.19)$$

The complete table of effects and their standard errors is as follows

$$\begin{aligned} I &\leftarrow \bar{y} = 2.745 \pm 0.025, \\ 1 &\leftarrow 0.75 \pm 0.05, \\ 2 &\leftarrow -0.59 \pm 0.05, \\ 3 &\leftarrow -0.35 \pm 0.05, \\ 12 &\leftarrow 0.03 \pm 0.05, \\ 13 &\leftarrow 0.03 \pm 0.05, \\ 23 &\leftarrow 0.04 \pm 0.05, \\ 123 &\leftarrow 0.08 \pm 0.05. \end{aligned} \qquad (4.2.20)$$

Effects and Regression Coefficients

If we fit a first degree polynomial to the textile data, as was done in Eq. (2.2.4), we obtain

$$\hat{y} = \underset{(0.025)}{2.745} + \underset{(0.025)}{0.375x_1} - \underset{(0.025)}{0.295x_2} - \underset{(0.025)}{0.175x_3}. \qquad (4.2.21)$$

where the numbers in parentheses are the standard errors of the coefficients. Notice that the estimated regression coefficients $b_1 = 0.375$, $b_2 = -0.295$, $b_3 = -0.175$ and their standard errors are exactly half the main effects and standard errors just calculated. The factor of one-half arises because an effect was defined as the difference in response on moving from the -1 level to the $+1$ level of a given variable x_i, which corresponds to the change in y when x_i is changed by *two* units. The regression coefficient b_i is, of course, the change in y when x_i is changed by *one* unit.

TABLE 4.3. Coding of variables in the 2^6 example

Variables	ξ_i	Coded levels, x_i -1	$+1$	x_i in terms of ξ_i
Polysulfide index	ξ_1	6	7	$x_1 = (\xi_1 - 6.5)/0.5$
Reflux rate	ξ_2	150	170	$x_2 = (\xi_2 - 160)/10$
Moles polysulfide	ξ_3	1.8	2.4	$x_3 = (\xi_3 - 2.1)/0.3$
Time (min)	ξ_4	24	36	$x_4 = (\xi_4 - 30)/6$
Solvent (cm^3)	ξ_5	30	42	$x_5 = (\xi_5 - 36)/6$
Temperature (°C)	ξ_6	120	130	$x_6 = (\xi_6 - 125)/5$

TABLE 4.4. A 2^6 factorial design and resulting observations of strength, hue, and brightness

Actual run order	Design levels: Polysulfide index, x_1	Reflux, rate x_2	Moles polysulfide, x_3	Time, (min), x_4	Solvent (cm^3), x_5	Temperature (°C), x_6	Responses: Strength, y_1	Hue, y_2	Brightness, y_3
26	−	−	−	−	−	−	3.4	15	36
3	+	−	−	−	−	−	9.7	5	35
11	−	+	−	−	−	−	7.4	23	37
5	+	+	−	−	−	−	10.6	8	34
42	−	−	+	−	−	−	6.5	20	30
18	+	−	+	−	−	−	7.9	9	32
41	−	+	+	−	−	−	10.3	13	28
14	+	+	+	−	−	−	9.5	5	38
17	−	−	−	+	−	−	14.3	23	40
27	+	−	−	+	−	−	10.5	1	32
19	−	+	−	+	−	−	7.8	11	32
56	+	+	−	+	−	−	17.2	5	28
23	−	−	+	+	−	−	9.4	15	34
8	+	−	+	+	−	−	12.1	8	26
32	−	+	+	+	−	−	9.5	15	30
7	+	+	+	+	−	−	15.8	1	28
46	−	−	−	−	+	−	8.3	22	40
13	+	−	−	−	+	−	8.0	8	30
58	−	+	−	−	+	−	7.9	16	35
38	+	+	−	−	+	−	10.7	7	35
43	−	−	+	−	+	−	7.2	25	32
55	+	−	+	−	+	−	7.2	5	35
6	−	+	+	−	+	−	7.9	17	36
64	+	+	+	−	+	−	10.2	8	32
22	−	−	−	+	+	−	10.3	10	20
4	+	−	−	+	+	−	9.9	3	35
16	−	+	−	+	+	−	7.4	22	35
47	+	+	−	+	+	−	10.5	6	28
63	−	−	+	+	+	−	9.6	24	27
51	+	−	+	+	+	−	15.1	4	36
20	−	+	+	+	+	−	8.7	10	36
29	+	+	+	+	+	−	12.1	5	35

4.3. A 2^6 DESIGN USED IN A STUDY OF DYESTUFFS MANUFACTURE

In the manufacture of a certain dyestuff, it is important to:

1. Obtain a product of a desired hue and brightness, and of maximum strength.
2. Know what changes in the process variables should be made if the requirements of the customer (as expressed by hue and brightness of the product) should change.
3. Know what sorts of changes in the process variables might compensate for given departures from specification.

TABLE 4.4. Continued

Actual run order	Design levels: Polysulfide index, x_1	Reflux, rate x_2	Moles polysulfide, x_3	Time (min), x_4	Solvent (cm^3), x_5	Temperature (°C), x_6	Responses: Strength, y_1	Hue, y_2	Brightness, y_3
62	−	−	−	−	−	+	12.6	32	32
1	+	−	−	−	−	+	10.5	10	34
37	−	+	−	−	−	+	11.3	28	30
61	+	+	−	−	−	+	10.6	18	24
44	−	−	+	−	−	+	8.1	22	30
24	+	−	+	−	−	+	12.5	31	20
59	−	+	+	−	−	+	11.1	17	32
60	+	+	+	−	−	+	12.9	16	25
35	−	−	−	+	−	+	14.6	38	20
50	+	−	−	+	−	+	12.7	12	20
48	−	+	−	+	−	+	10.8	34	22
36	+	+	−	+	−	+	17.1	19	35
21	−	−	+	+	−	+	13.6	12	26
9	+	−	+	+	−	+	14.6	14	15
33	−	+	+	+	−	+	13.3	25	19
57	+	+	+	+	−	+	14.4	16	24
10	−	−	−	−	+	+	11.0	31	22
39	+	−	−	−	+	+	12.5	14	23
25	−	+	−	−	+	+	8.9	23	22
40	+	+	−	−	+	+	13.1	23	18
30	−	−	+	−	+	+	7.6	28	20
31	+	−	+	−	+	+	8.6	20	20
28	−	+	+	−	+	+	11.8	18	20
49	+	+	+	−	+	+	12.4	11	36
52	−	−	−	+	+	+	13.4	39	20
15	+	−	−	+	+	+	14.6	30	11
34	−	+	−	+	+	+	14.9	31	20
53	+	+	−	+	+	+	11.8	6	35
2	−	−	+	+	+	+	15.6	33	16
12	+	−	+	+	+	+	12.8	23	32
45	−	+	+	+	+	+	13.5	31	20
54	+	+	+	+	+	+	15.8	11	20
			64 observations Σy				711.9	1085	1810
			64 observations Σy^2				8443.41	24,421	54,340

(These problems are essentially those we have listed in Section 1.7 as "Choice of operating conditions to achieve desired specifications" and "Approximate mapping of a surface within a limited region.") The particular process under study was one whose chemical mechanism was not well understood, so that an empirical approach was mandatory.

Six variables were suggested as possibly of importance in affecting strength, hue, and brightness. They were ξ_1 = polysulfide index, ξ_2 = reflux rate, ξ_3 = moles of polysulfide, ξ_4 = reaction time, ξ_5 = amount of solvent, ξ_6 = reaction temperature. The ranges over which they were changed in the design, and the details of their coding to x's is shown in Table 4.3. The design employed was a 2^6 full factorial design.

Model Specification

Data from past manufacturing experience had indicated that effects of potential importance were not likely to be large compared with the (rather high) experimental error. Such evidence as existed suggested that, over the range of the region of interest, effects might be roughly linear and, in particular, no response was likely to be close to a maximum or other turning point. This was later checked by making additional runs both at the center of the 2^6 design and also at more extreme conditions along the six axial directions. No evidence of curved relationships was found, and we shall here use only the data from the principal part of the design, namely the complete 2^6 factorial. The results from these 64 runs (which were actually performed in random order) are shown in Table 4.4.

The preliminary statistical analysis for model specification consisted of calculating the factorial effects (see Table 4.5 and Appendix 4A) shown plotted on probability paper in Figure 4.3. For clarity, in the center of each plot only every fourth point is inserted. A brief discussion of the normal plotting of factorial effects is given in Appendix 4B. From this preliminary analysis, it appeared that the data were adequately explained in terms of linear effects in only three of the variables x_1, x_4, and x_6, that is, in polysulfide index, reaction time, and reaction temperature, respectively. The preliminary identification thus suggested that linear models might be tentatively entertained in the variables x_1, x_4, and x_6 with the residual variation ascribed principally to random error or "noise."

On this basis the data could be tentatively analyzed as if they came from a 2^3 factorial in x_1, x_4, and x_6 performed eight times over. The observations, appropriately rearranged to reflect this, are set out again in Table 4.6. The totals on the right of this table are used to facilitate the subsequent tentative analysis.

TABLE 4.5. Effects ranked in order for strength, hue, and brightness

Strength, y_1		Hue, y_2		Brightness, y_3	
Effect name	Effect value	Effect name	Effect value	Effect name	Effect value
Mean	11.12	Mean	16.95	Mean	28.28
4	2.98	1	−11.28	6	−8.87
6	2.69	6	10.84	4	−3.00
1	1.75	2456	−2.84	2346	−2.87
1245	−1.16	135	−2.78	12346	−2.81
12	0.89	2	−2.72	123456	−2.81
24	−0.86	13	2.66	234	−2.56
16	−0.82	2346	2.53	35	2.44
124	0.82	1235	2.47	1245	−2.37
1346	−0.77	46	2.34	1456	−2.37
12346	0.74	136	2.34	1356	2.19
345	0.72	256	−2.22	15	2.12
13456	0.71	12345	2.22	1246	1.94
2	0.70	1236	−2.16	56	−1.88
12345	0.70	234	2.09	126	1.75
123	−0.65	1245	−1.97	135	1.62
1235	0.65	3	−1.91	2456	−1.62
1356	−0.65	23	−1.91	125	−1.56
456	0.63	25	−1.91	1235	−1.56
23	0.60	36	−1.84	2	1.50
256	0.57	14	−1.78	1234	−1.50
146	−0.56	146	−1.72	5	−1.44
12356	0.50	23456	1.72	456	1.31
1456	−0.47	2345	−1.59	25	1.25
12456	0.47	1246	−1.47	146	1.25
5	−0.42	1356	−1.47	1256	1.25
15	−0.42	124	−1.41	134	−1.19
2346	−0.41	456	1.41	145	1.19
34	0.40	5	1.34	45	1.12
134	0.40	3456	1.34	14	1.06
234	−0.40	123	−1.22	26	1.06
2356	0.39	26	−1.16	246	1.00
236	0.35	345	1.16	3	−0.94
1345	0.34	45	1.09	12	0.94
123456	0.34	126	−1.09	16	0.94
25	−0.33	12346	1.09	136	−0.94
145	−0.33	236	1.03	156	0.94
245	−0.33	245	−1.03	346	−0.94
45	−0.32	15	−0.97	3456	−0.94
1246	−0.32	346	−0.97	235	−0.88
246	0.30	123456	−0.84	24	0.81
1236	0.29	16	0.78	2345	−0.81
3456	−0.28	34	−0.78	13	0.75
56	0.27	35	0.78	236	−0.69
125	−0.27	1256	−0.78	12345	−0.50
126	−0.25	12	0.72	12356	−0.50

TABLE 4.5. Continued.

Strength, y_1		Hue, y_2		Brightness, y_3	
Effect name	Effect value	Effect name	Effect value	Effect name	Effect value
2345	−0.24	12456	−0.66	245	0.44
356	−0.23	156	−0.53	256	−0.44
36	−0.22	1456	−0.53	1	0.38
1234	−0.22	2356	0.47	34	0.38
26	−0.18	56	0.41	345	0.38
346	0.17	4	−0.34	1236	−0.38
14	0.15	1345	0.34	23456	−0.38
13	0.14	12356	−0.34	46	−0.31
23456	0.14	1234	0.28	23	0.25
235	0.12	1346	−0.28	1346	0.25
136	0.11	246	0.22	13456	−0.25
3	0.10	356	0.22	36	0.13
156	0.10	24	0.16	123	0.06
35	0.08	125	0.16	1345	0.06
135	0.07	235	0.16	2356	−0.06
2456	0.03	13456	0.16	12456	0.06
46	0.02	134	0.03	124	0
1256	0.02	145	0.03	356	0

Model Fitting

The linear equations in x_1, x_4, and x_6 fitted by least squares to all 64 observations and rounded to two decimal places are*:

$$\text{Strength: } \hat{y}_1 = \underset{(0.24)}{11.12} + \underset{(0.24)}{0.87x_1} + \underset{(0.24)}{1.49x_4} + \underset{(0.24)}{1.35x_6} \tag{4.3.1}$$

$$\text{Hue: } \hat{y}_2 = \underset{(0.74)}{16.95} - \underset{(0.74)}{5.64x_1} - \underset{(0.74)}{0.17x_4} + \underset{(0.74)}{5.42x_6} \tag{4.3.2}$$

$$\text{Brightness: } \hat{y}_3 = \underset{(0.67)}{28.28} + \underset{(0.67)}{0.19x_1} - \underset{(0.67)}{1.50x_4} - \underset{(0.67)}{4.44x_6} \tag{4.3.3}$$

*Not all the estimated regression coefficients b_1, b_4, and b_6 in Eqs. (4.3.1) to (4.3.3) are significantly different from zero, but the values shown are the best estimates available in the subspace of greatest interest. We do not substitute values of zero for the nonsignificant coefficients since these would not be the best estimates. It might be thought that this judgment is somewhat contradictory since variables x_2, x_3, and x_5 have been dropped completely. In fact, however, we can simply regard the fitted equations as the best estimates in the three-dimensional subspace of the full six-dimensional space in which x_2, x_3, and x_5 are at their average values. Our treatment implies only that, had we prepared contour diagrams for the responses as functions of x_1, x_4, and x_6 for *other* levels of x_2, x_3, and x_5 over their relevant ranges, the appearances of the various diagrams would not be appreciably different.

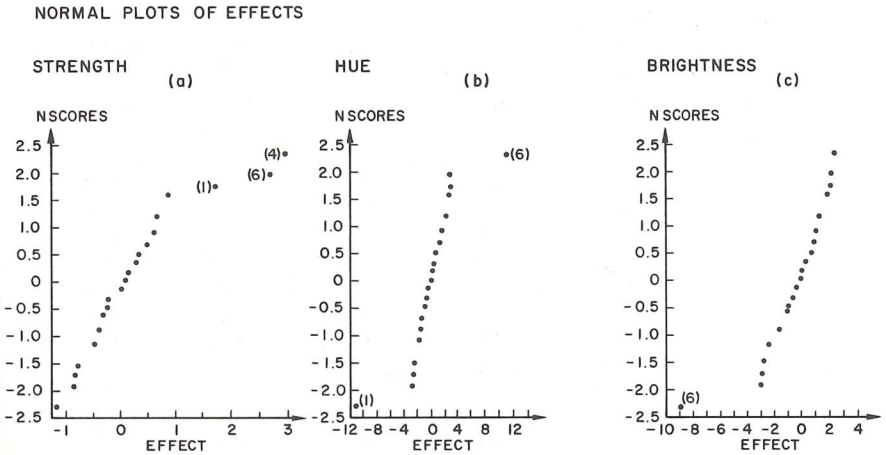

FIGURE 4.3. Normal plots for (*a*) strength, (*b*) hue, (*c*) brightness. (The effects plotted are those of orders 1, 2, 3, 4, 8, 12, . . . , 60, 61, 62, and 63.)

The appropriate analyses of variance are set out in Table 4.7. The estimates of the error standard deviations obtained from the residual sums of squares after fitting these equations are, respectively,

$$s_1 = 1.90, \qquad s_2 = 5.93, \qquad s_3 = 5.39. \tag{4.3.4}$$

These estimates of the standard deviations might be biased somewhat upward because a number of small main effects and interactions have been ignored, and perhaps biased slightly downward because of the effect of selection, that is, only the large estimates were taken to be of real effects. However, these are the figures we have used in estimating the standard errors of the coefficients shown in parentheses beneath the coefficients.

4.4. DIAGNOSTIC CHECKING OF THE FITTED MODELS, 2^6 DYESTUFFS EXAMPLE

Table 4.8 shows the residuals from the models (4.3.1) to (4.3.3). The time order of the observations is also shown. Figures 4.4*a–c* show residuals for the fitted models plotted in various ways. There is some evidence that the variance of the response hue (y_2) is not homogeneous (see Figure 4.4*b*, the residuals versus $\hat{y}_2$ plot). However, in general, it appears that the tentative assumption that the changes in the responses are associated mainly with the linear effects of the three variables polysulfide index (x_1), reaction time (x_4), and reaction temperature (x_6) is borne out.

TABLE 4.6. The 64 observations on each of three responses grouped in sets of eight to reflect their dependence on the variables x_1 = polysulfide index, x_4 = time, and x_6 = temperature

x_1	x_4	x_6	x_2, x_3, x_5												Row averages	Row totals
			− − −	+ − −	− + −	+ + −	− − +	+ − +	− + +	+ + +						
Strength																
−	−	−	3.4	7.4	6.5	10.3	8.3	7.9	7.2	7.9					7.3625	58.9
+	−	−	9.7	10.6	7.9	9.5	8.0	10.7	7.2	10.2					9.2250	73.8
−	+	−	14.3	7.8	9.4	9.5	10.3	7.4	9.6	8.7					9.6250	77.0
+	+	−	10.5	17.2	12.1	15.8	9.9	10.5	15.1	12.1					12.9000	103.2
−	−	+	12.6	11.3	8.1	11.1	11.0	8.9	7.6	11.8					10.3000	82.4
+	−	+	10.5	10.6	12.5	12.9	12.5	13.1	8.6	12.4					11.6375	93.1
−	+	+	14.6	10.8	13.6	13.3	13.4	14.9	15.6	13.5					13.7125	109.7
+	+	+	12.7	17.1	14.6	14.4	14.6	11.8	12.8	15.8					14.2250	113.8
Hue																
−	−	−	15	23	20	13	22	16	25	17					18.875	151
+	−	−	5	8	9	5	8	7	5	8					6.875	55
−	+	−	23	11	15	15	10	22	24	10					16.250	130
+	+	−	1	5	8	1	3	6	4	5					4.125	33
−	−	+	32	28	22	17	31	23	28	18					24.875	199
+	−	+	10	18	31	16	14	23	20	11					17.875	143
−	+	+	38	34	12	25	39	31	33	31					30.375	243
+	+	+	12	19	14	16	30	6	23	11					16.375	131
Brightness																
−	−	−	36	37	30	28	40	35	32	36					34.250	274
+	−	−	35	34	32	38	30	35	35	32					33.875	271
−	+	−	40	32	34	30	20	35	27	36					31.750	254
+	+	−	32	28	26	28	35	28	36	35					31.000	248
−	−	+	32	30	30	32	22	22	20	20					26.000	208
+	−	+	34	24	20	25	23	18	20	36					25.000	200
−	+	+	20	22	26	19	20	20	16	20					20.375	163
+	+	+	20	35	15	24	11	35	32	20					24.000	192

TABLE 4.7. Analysis of variance tables, for strength, hue, and brightness

Strength

Source of variation	SS	df	MS	F ratio
Total SS $= \Sigma y^2$	8,443.4100	64		
Correction factor, SS due to $b_0 = (\Sigma y)^2/64$	7,918.7752	1		
Corrected total SS	524.6348	63		
Due to $b_1 = b_1 \Sigma x_1 y = (\Sigma x_1 y)^2/\Sigma x_1^2$	48.8252	1	48.8252	13.47*
Due to $b_4 = b_4 \Sigma x_4 y = (\Sigma x_4 y)^2/\Sigma x_4^2$	142.5039	1	142.5039	39.32*
Due to $b_6 = b_6 \Sigma x_6 y = (\Sigma x_6 y)^2/\Sigma x_6^2$	115.8314	1	115.8314	31.96*
Residual	217.4743	60	3.6246	
			$s_1 = 1.9038$	

$\Sigma y = 711.9$; $\Sigma x_1 y = 55.9$; $\Sigma x_4 y = 95.5$; $\Sigma x_6 y = 86.1$

Hue

Source of variation	SS	df	MS	F ratio
Total SS $= \Sigma y^2$	24,421.0000	64		
SS due to $b_0 = (\Sigma y)^2/64$	18,394.1406	1		
Corrected total SS	6,026.8594	63	6,026.8594	
Due to $b_1 = b_1 \Sigma x_1 y = (\Sigma x_1 y)^2/\Sigma x_1^2$	2,036.2656	1	2,036.2656	57.98*
Due to $b_4 = b_4 \Sigma x_4 y = (\Sigma x_4 y)^2/\Sigma x_4^2$	1.8906	1	1.8906	0.05
Due to $b_6 = b_6 \Sigma x_6 y = (\Sigma x_6 y)^2/\Sigma x_6^2$	1,881.3906	1	1,881.3906	53.57*
Residual	2,107.3126	60	35.1219	
			$s_2 = 5.9264$	

$\Sigma y = 1085$; $\Sigma x_1 y = -361$; $\Sigma x_4 y = -11$; $\Sigma x_6 y = 347$

Brightness

Source of variation	SS	df	MS	F ratio
Total SS $= \Sigma y^2$	54,340.0000	64		
SS due to b_0	51,189.0625	1		
Corrected total SS	3,150.9375	63		
Due to b_1	2.2500	1	2.2500	0.08
Due to b_4	144.0000	1	144.0000	4.95*
Due to b_6	1260.2500	1	1260.2500	43.35*
Residual	1744.4375	60	29.0740	
			$s_3 = 5.3920$	

$\Sigma y = 1810$; $\Sigma x_1 y = 12$; $\Sigma x_4 y = -96$; $\Sigma x_6 y = -284$

*Significant at $\alpha = 0.05$ or smaller α-level; $P[F(1, 60) \geq 4.00] = 0.05$.

TABLE 4.8. **Residuals from individual and from average observations, fitted values $\hat{y}$ obtained by using Eqs. (4.3.1) to (4.3.3), and time order of observations. (The layout is similar to that of Table 4.6 and all figures are rounded to the length of the original observations)**

			x_2, x_3, x_5								From	
x_1	x_4	x_6	− − −	+ − −	− + −	+ + −	− − +	+ − +	− + +	+ + +	average	$\hat{y}$
Strength												
−	−	−	−4.0	0	−0.9	2.9	0.9	0.5	−0.2	0.5	0	7.4
+	−	−	0.5	1.4	−1.3	0.3	−1.2	1.5	−2.0	1.0	0	9.2
−	+	−	3.9	−2.6	−1.0	−0.9	−0.1	−3.0	−0.8	−1.7	−0.8	10.4
+	+	−	−1.6	5.1	0	3.7	−2.2	−1.6	3.0	0	0.8	12.1
−	−	+	2.5	1.2	−2.0	1.0	0.9	−1.2	−2.5	1.7	0.2	10.1
+	−	+	−1.4	−1.3	0.6	1.0	0.6	1.2	−3.3	0.5	−0.3	11.9
−	+	+	1.5	−2.3	0.5	0.2	0.3	1.8	2.5	0.4	0.6	13.1
+	+	+	−2.1	2.3	−0.2	−0.4	−0.2	−3.0	−2.0	1.0	−0.6	14.8
Hue												
−	−	−	−2	6	3	−4	5	−1	8	0	2	17
+	−	−	−1	2	3	−1	2	1	−1	2	1	6
−	+	−	6	−6	−2	−2	−7	5	7	−7	−1	17
+	+	−	−5	−1	2	−5	−3	0	−2	−1	−2	6
−	−	+	4	0	−6	−11	3	−5	0	−10	−3	28
+	−	+	−7	1	14	−1	−3	6	3	−6	1	17
−	+	+	10	6	−16	−3	11	3	5	3	2	28
+	+	+	−5	2	−3	−1	13	−11	6	−6	−1	17
Brightness												
−	−	−	2	−3	−4	−6	6	1	−2	2	0	34
+	−	−	1	0	−2	4	−4	1	1	−2	0	34
−	+	−	9	1	3	−1	−11	4	−4	5	1	31
+	+	−	1	−3	−5	−3	4	−3	5	4	0	31
−	−	+	7	5	5	7	−3	−3	−5	−5	1	25
+	−	+	8	−2	−6	−1	−3	−8	−6	10	−1	26
−	+	+	−2	0	4	−3	−2	−2	−6	−2	−2	22
+	+	+	−3	12	−8	1	−12	12	9	−3	1	23
Time order												
−	−	−	26	11	42	41	46	58	43	6		
+	−	−	3	5	18	14	13	38	55	64		
−	+	−	17	19	23	32	22	16	63	20		
+	+	−	27	56	8	7	4	47	51	29		
−	−	+	62	37	44	59	10	25	30	28		
+	−	+	1	61	24	60	39	40	31	49		
−	+	+	35	48	21	33	52	34	2	45		
+	+	+	50	36	9	57	15	53	12	54		

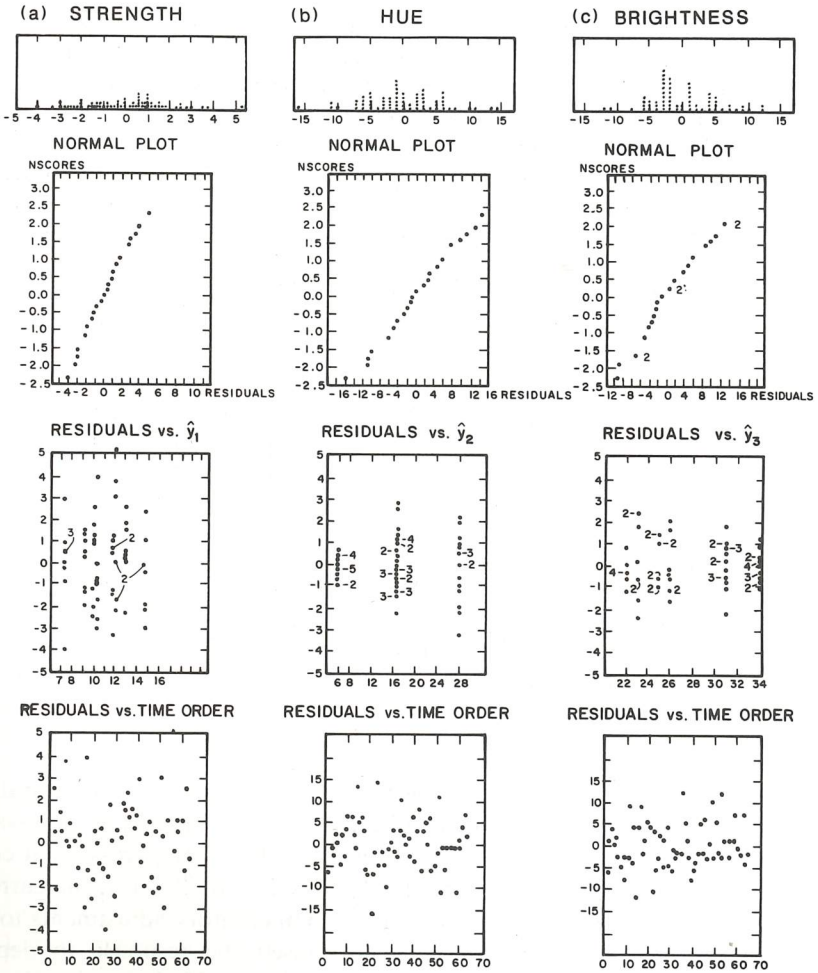

FIGURE 4.4. Plots of residuals for (*a*) strength, (*b*) hue, and (*c*) brightness.

4.5. RESPONSE SURFACE ANALYSIS OF THE 2^6 DESIGN DATA

We now consider the model's implications for the manufacturing problem, assuming that, to a sufficient approximation, the fitted models are adequate to represent the data. The immediate objective of these experiments was to determine operating conditions, if any such existed, at which a standard hue of 20 units and a standard brightness of 26 units could be obtained with maximum strength. A second objective was to understand the way in which

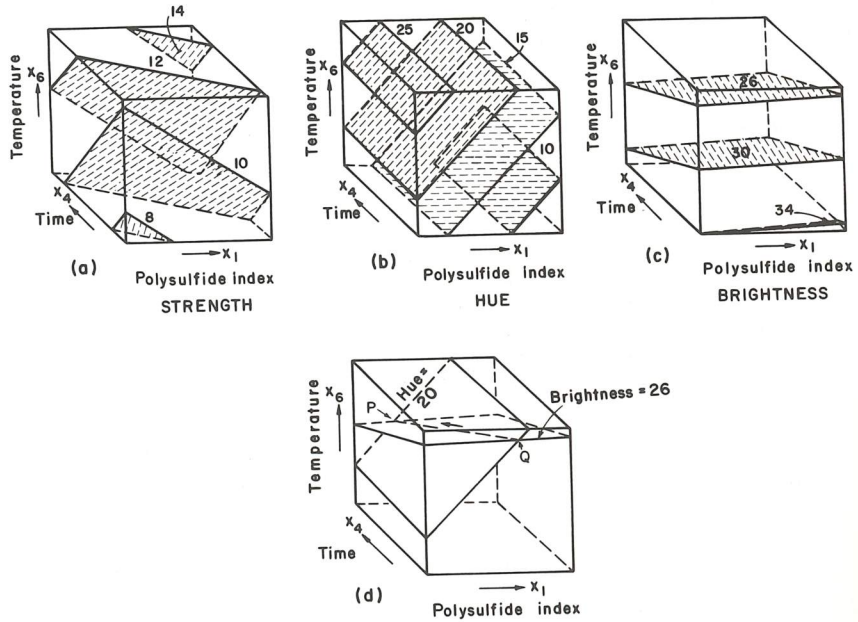

FIGURE 4.5. Contours of (*a*) strength, (*b*) hue, (*c*) brightness with (*d*) intersection of the hue = 20 and brightness = 26 planes.

the three responses were affected by the variables. Such an understanding would allow future modifications to be made in the process as necessary. Such modifications might be needed, for example, to prepare special consignments of dye whose hue and brightness had to differ from current specifications, and also to allow intelligent compensatory adjustments to be made when unknown disturbances had caused the response to depart temporarily from the standard. What was needed in this latter situation in essence was some kind of "navigation chart."

The contours of the fitted equations are shown in Figures 4.5*a–c* in cubes whose corners are the design points.

In Figure 4.5*d* are shown, together, the predicted contour planes for brightness and hue equal to their desired levels of 26 units and 20 units, respectively; the line PQ along which these two planes intersect has predicted strengths ranging from about 11.08 at Q to about 12.46 at P, as Table 4.9 indicates. Approximately, then, we expect that, anywhere along the line of intersection, satisfactory brightness and hue would be obtained and that higher strengths would result by moving in the direction indicated by the arrow in the figure. The estimated difference in strengths at the points P

TABLE 4.9. Predicted strengths at points along the line PQ in Figure 4.5*d*

	Coordinates on line PQ			
	x_1	x_4	x_6	Predicted strength, $\hat{y}_1$
Q	0.32	−1.00	0.87	11.08
	0.09	−0.39	0.65	11.50
	−0.05	0.00	0.51	11.77
	−0.18	0.34	0.39	12.00
P	−0.42	1.00	0.16	12.46

and Q in Figure 4.5*d* is given by

$$\begin{aligned}\hat{y}_P - \hat{y}_Q &= b_1(x_{1P} - x_{1Q}) + b_4(x_{4P} - x_{4Q}) + b_6(x_{6P} - x_{6Q}) \\ &= 12.46 - 11.08 \\ &= 1.38 \qquad (4.5.1)\end{aligned}$$

with a variance of

$$\begin{aligned}V(\hat{y}_P - \hat{y}_Q) &= \left\{(x_{1P} - x_{1Q})^2 + (x_{4P} - x_{4Q})^2 + (x_{6P} - x_{6Q})^2\right\}V(b_i) \\ &= \overline{PQ}^2 V(b_i), \qquad (4.5.2)\end{aligned}$$

where $V(b_i) = \sigma^2/n$, and $\overline{PQ}^2$ is the squared distance between the points P and Q in the scale of the x's. Therefore the standard deviation of $(\hat{y}_P - \hat{y}_Q) = 1.38$ is $\overline{PQ}\sigma/n^{1/2}$, which we can evaluate as

$$\left\{(-0.42 - 0.32)^2 + (1 + 1)^2 + (0.16 - 0.87)^2\right\}^{1/2}\sigma/64^{1/2} = 0.2809\sigma. \qquad (4.5.3)$$

From Table 4.7 we substitute $s_1 = 1.9038$ for σ, giving a standard error of about 0.53. The difference $\hat{y}_P - \hat{y}_Q = 1.38$ is thus about 2.6 times its standard error and we conclude that the strength is significantly higher at the position P than at the position Q.

[We note, in passing, that this design happens to be what is called a rotatable first-order design (see Chapter 14) and therefore has the interesting property that the variance of differences in estimated yields between two

points is simply a function of the distance between the two points concerned. See Box and Draper, 1980.]

It is obvious that our analysis can be only an approximate one, since the data on which it is based are only approximate. However, on the assumption that the form of the prediction equation is not seriously at fault, we can readily calculate confidence limits at the points P and Q, or at any other set of values, for $E(y_1)$, $E(y_2)$, and $E(y_3)$. For any of the three responses, at a general point (x_1, x_4, x_6), we can write

$$V(\hat{y}_i) = V(b_0) + x_1^2 V(b_1) + x_4^2 V(b_4) + x_6^2 V(b_6)$$

$$= \sigma_i^2 [1 + x_1^2 + x_4^2 + x_6^2]/64, \qquad (4.5.4)$$

where the σ_i^2 is the appropriate one for the response desired, and is estimated by s_i^2 from the appropriate part of Table 4.7. Thus, for example, the estimated variance of $\hat{y}_1$ at the point P with coordinates $(-0.42, 1.00, 0.16)$ is

$$V(\hat{y}_1) = \frac{3.6246(2.202)}{64} = 0.1247 \qquad (4.5.5)$$

where we have substituted for σ_1^2, $s_1^2 = 3.6246$ based on 60 df from Table 4.7. Taking the square root, we then have to two decimal places that

$$\text{standard error } (\hat{y}_1) = 0.35. \qquad (4.5.6)$$

Proceeding in exactly similar fashion and using $t_{0.025}(60) = 2.00$, we obtain at P and Q 95% confidence limits for $E(y_1)$, $E(y_2)$ and $E(y_3)$ as follows:

	At point P	At point Q
Yield	12.46 ± 0.70	11.08 ± 0.80
Hue	20 ± 2.20	20 ± 2.51
Brightness	26 ± 2.00	26 ± 2.28

(In Appendix 4C, methods are given for constructing confidence regions both for contour planes and for their lines of intersection.)

Confirmatory runs made in the regions indicated by this analysis verified the general conclusions given above. A three-dimensional model with contours of strength, hue, and brightness shown by different colors was constructed and proved to be of great value in improving the product and in day-to-day operation of the process.

In this investigation we were somewhat fortunate in finding that first-order expressions were sufficient to represent the three response functions. The area of application of such methods is greatly widened by allowing the possibility of nonlinear transformation in either the response or predictor variables or, sometimes, in both. Quite frequently, although the relationships may not be adequately represented by first-order expressions in the original variables, approximate first-order representation can be obtained by employing, for example, the logarithmic or inverse transformation. Methods whereby appropriate transformations may be determined are discussed later (see Chapters 8 and 13).

APPENDIX 4A. YATES' METHOD FOR OBTAINING THE FACTORIAL EFFECTS FOR A TWO-LEVEL DESIGN

Yates' method is best understood by considering an example. Table 4A.1 shows the calculations for the textile data previously discussed in Chapter 4. Column C_1 is developed from the y_u column by these operations:

In row 1 of C_1 place the entry $y_2 + y_1$,
2 $y_4 + y_3$,
3 $y_6 + y_5$,
4 $y_8 + y_7$,
5 $y_2 - y_1$,
6 $y_4 - y_3$,
7 $y_6 - y_5$,
In row 8 of C_1 place the entry $y_8 - y_7$.

The identical operations performed on the C_1 column now give rise to the C_2 column, and the latter is used to calculate the C_3 column in the same manner. Each entry in the C_3 column is then divided by a divisor. The first divisor is the number of runs in the design, here 8. All the other divisors are half of this. The results of this mechanical procedure are the effects of the variables in the standard order corresponding to the plus signs in the variable settings; here, for example, the order is $I, 1, 2, 12, 3, 13, 23, 123$. Rearranging this order, we obtain the estimates already given.

Grand mean	$I \leftarrow 2.745 \quad (= \bar{y})$
Main effects	$1 \leftarrow 0.75$
	$2 \leftarrow -0.59$
	$3 \leftarrow -0.35$

TABLE 4A.1. Yates' method for obtaining factorial effects

Row	Setting of variable x_1	x_2	x_3	Observations, y_u	C_1	C_2	C_3	Divisor	Effect Value	Effect Name
1	−1	−1	−1	2.83	6.39	11.68	21.96	8	2.745	I
2	1	−1	−1	3.56	5.29	10.28	3.00	4	0.75	1
3	−1	1	−1	2.23	5.77	1.56	−2.36	4	−0.59	2
4	1	1	−1	3.06	4.51	1.44	−0.12	4	−0.03	12
5	−1	−1	1	2.47	0.73	−1.10	−1.40	4	−0.35	3
6	1	−1	1	3.30	0.83	−1.26	−0.12	4	−0.03	13
7	−1	1	1	1.95	0.83	0.10	−0.16	4	−0.04	23
8	1	1	1	2.56	0.61	−0.22	−0.32	4	−0.08	123

Two-factor interactions	12 ← −0.03
	13 ← −0.03
	23 ← −0.04
Three-factor interaction	123 ← −0.08

Yates' method may be applied to any two-level 2^k factorial design by carrying out the sum and difference operations on k columns. (For its implementation on 2^{k-p} fractional factorial designs, see Section 5.5.)

APPENDIX 4B. NORMAL PLOTS ON PROBABILITY PAPER

When two-level factorial and fractional factorial designs are run, it is frequently the case that no independent estimate of error is available. A useful technique that is often effective in distinguishing real effects from noise employs *normal probability paper*, available in technical bookstores. (Ask, for example, for Keuffel and Esser 46-8003, or Codex 3227, or Team 3211 or Dietzgen 340-PS-90.)

What Is Normal Probability Paper?

Observations y having a normal distribution with mean η and variance σ^2 have the probability density function

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\eta)^2/(2\sigma^2)}. \tag{4B.1}$$

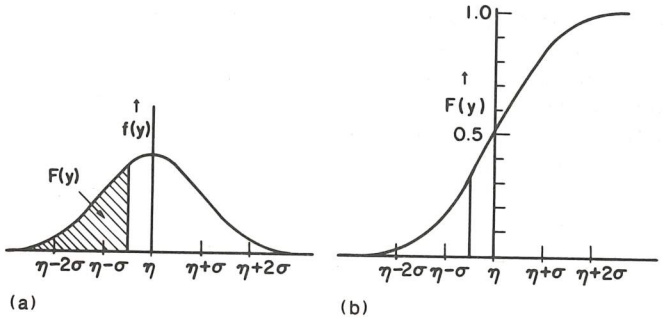

FIGURE 4B.1. Plots of (a) the normal density function $f(y)$ and (b) the cumulative normal density function $F(y)$. Note, for example, that prob$(y \le \eta - 0.5\sigma) = 0.3085$ is the area shaded in (a) and is the vertical height marked in (b).

The corresponding cumulative density function is

$$F(y) = \int_{-\infty}^{y} \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\eta)^2/(2\sigma^2)}\, dt. \tag{4B.2}$$

Both the normal density function and its cumulative are plotted in Figure 4B.1. We see that the cumulative density function $F(y)$ is the area under the probability density function $f(y)$ from $-\infty$ to y, and that $0 \le F(y) \le 1$. The normal cumulative distribution function is an "ogive," that is, an "S" shaped curve that begins at $(-\infty, 0)$ and passes through points such as $(\eta - \sigma, 0.1587)$, $(\eta, 0.5)$, $(\eta + \sigma, 0.8413)$, and $(\infty, 1.0)$, and in general through the points $\{\eta + u\sigma, F(\eta + u\sigma)\}$ where u is called the *normal score*. Normal probability paper shown in Figure 4B.2 is arranged so that the ogive curve $F(y)$ is made into a straight line by stretching outward from the center symmetrically the vertical dimension of the paper. Shown on the vertical axis are both the probability $F(y) = F(\eta + u\sigma)$ and the normal score u.

We illustrate use of the probability paper by drawing a random sample of 20 observations from a normal distribution with mean $\eta = 7$ and variance $\sigma^2 = 0.5$. The sample is rearranged in ascending order in Table 4B.1 and plotted as a dot diagram at the bottom of Figure 4B.2. With the ith in order, $y_{(i)}$, is associated

$$F(i) = F(y_{(i)}) = \frac{(i - \frac{1}{2})}{n}, \tag{4B.3}$$

evaluated in Table 4B.1 for $n = 20$. The points $\{y_{(i)}, F(i)\}$ are then plotted on the probability paper in Figure 4B.2.

On the assumption that the data are a sample from a normal distribution, an "eye-fitted" straight line can be used to provide rough estimates of the mean and standard deviation. The estimate of the mean η is simply the abscissa value associated with $F(y) = 0.50$. To obtain an estimate of σ, recall that prob$(y \le \eta +$

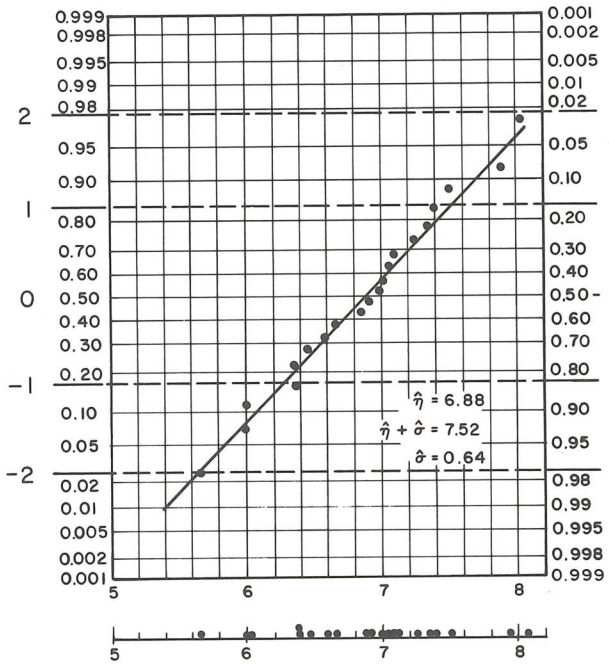

FIGURE 4B.2. Plots of 20 randomly drawn observations from a normal distribution with mean $\eta = 7.0$ and standard deviation $\sigma = 0.707$.

TABLE 4B.1. The 20 ordered observations $y_{(i)}$ and the corresponding values of $F(i) = (i - \frac{1}{2}) / 20$ for $i = 1, 2, \ldots, 20$

$y_{(i)}$	$F(i)$	$y_{(i)}$	$F(i)$
5.66	0.025	6.97	0.525
6.03	0.075	7.04	0.575
6.04	0.125	7.08	0.625
6.38	0.175	7.10	0.675
6.38	0.225	7.24	0.725
6.46	0.275	7.33	0.775
6.59	0.325	7.39	0.825
6.64	0.375	7.51	0.875
6.88	0.425	7.94	0.925
6.91	0.475	8.08	0.975

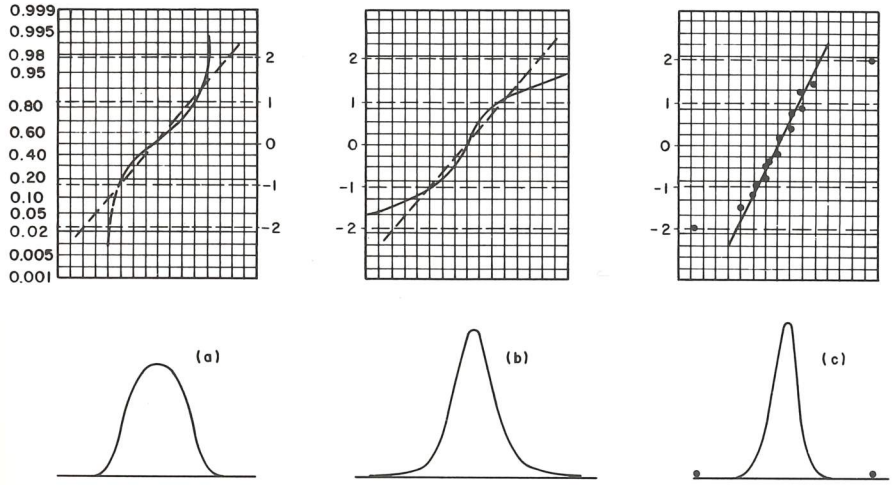

FIGURE 4B.3. (*a*) Light-tailed distribution. (*b*) Heavy-tailed distribution. (*c*) Normal distribution with two outliers.

$\sigma) = 0.8413$. The abscissa values for $F(y) = 0.50$ and $F(y) = 0.8413$ are, respectively, 6.88 and 7.52; thus for this set of data, $\hat{\eta} = 6.88$ and $\hat{\sigma} = 0.64 = 7.52 - 6.88$.

If the parent distribution were not normal, the cumulative distribution would not plot as a straight line on normal probability paper. Figures 4B.3*a* and 4B.3*b* show the curved lines that result from a light-tailed distribution and a heavy-tailed distribution, respectively. The third figure 4B.3*c* shows a plot for a sample in which all observations are random drawings from a normal distribution except for two outliers.

Application to Estimates from 2^k Designs

If all k factors in a two-level factorial or fractional factorial design are without influence, all the $2^k - 1$ estimates of effects and interactions would be expected to be approximately normal with mean zero, and variance $\{4\sigma^2/n\}$. Thus, if a normal plot of the estimated effects approximates a straight line, there would be no reason to believe that any of the true effects were nonzero. By contrast, points that fall well off the line to the right at the top and to the left at the bottom would suggest the existence of real effects. Examples of how this works in practice are seen in Figure 4.3.

Detection of Aberrant Observations

Consider the estimates from a 2^4 factorial design in Table 4B.2. A normal plot of these is shown in Figure 4B.4. The data suggest, though not strongly, that the main

TABLE 4B.2. Ordered estimates from a 2^4 design

Rank	Effect name	Estimate	Rank	Effect name	Estimate
1	2	−4.22	9	124	0.72
2	13	−2.49	10	12	0.91
3	234	−1.58	11	4	1.01
4	24	−1.18	12	123	1.20
5	1	−0.80	13	34	1.49
6	23	−0.80	14	1234	1.52
7	14	−0.58	15	3	3.71
8	134	0.40			

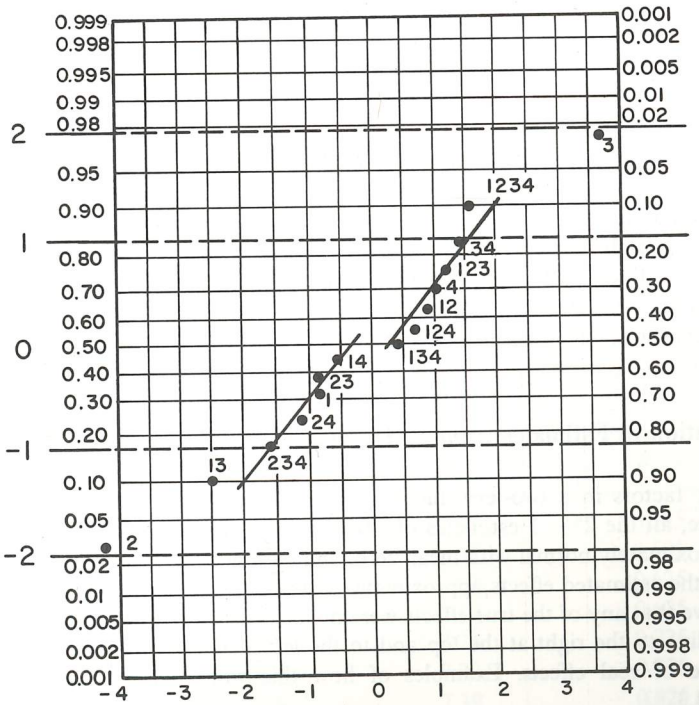

FIGURE 4B.4. Effect of an aberrant observation on a normal plot of factorial effect estimates from a 2^4 design.

TABLE 4B.3. Table of signs for the analysis of a 2^4 design. Matching the signs of biased effects with the run that probably causes the bias

	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
Sign	−	+	+	+	+	−	−	−	−	+	+	+	(+)	−	+
Estimate	−0.80	−4.22	3.71	1.01	0.91	−2.49	−0.58	−0.80	−1.18	1.49	1.20	0.72	0.40	−1.58	1.52
1	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+
2	+	−	−	−	−	−	−	+	+	+	+	+	+	−	−
3	−	+	−	−	−	+	+	−	−	+	+	+	−	+	−
4	+	+	−	−	+	−	−	−	−	+	+	−	+	+	+
5	−	−	+	−	+	−	+	−	+	−	+	−	+	+	−
6	+	−	+	−	−	+	−	−	+	−	−	+	−	+	+
7	−	+	+	−	−	−	+	+	−	−	+	+	−	−	+
8	+	+	+	−	+	+	−	+	−	−	−	−	−	−	+
9	−	−	−	+	+	+	+	+	−	−	+	+	+	+	−
10	+	−	−	+	−	+	−	−	+	−	−	−	+	+	+
11	−	+	−	+	−	−	+	−	+	−	+	+	+	+	+
12	+	+	−	+	+	−	−	−	+	−	+	−	−	−	+
13	−	−	+	+	+	+	+	+	−	+	−	+	+	+	−
14	+	−	+	+	−	+	−	+	−	+	+	−	−	+	+
15	−	+	+	+	−	−	+	+	+	+	−	+	−	+	+
16	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

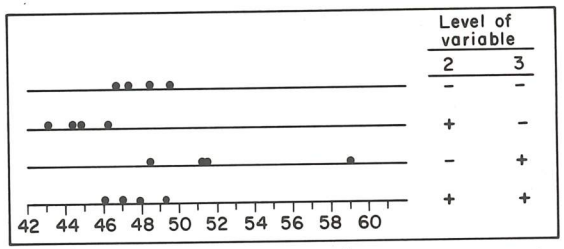

FIGURE 4B.5. Plot of replicate values when the 2^4 design is considered as a replicated 2^2 design in factors 2 and 3.

effect 2 (and possibly 3 also) is distinguishable from noise, but this plot also exhibits another striking characteristic. The inlying values, which should represent just noise, appear to be fitted not by one line, but by two parallel straight lines which "break apart" close to the zero value on the abscissa, and thus divide positive from negative effect estimates.

Now consider Table 4B.3, the body of which shows the $\pm$ signs which determine the effects, set out in the usual format. Suppose an observation were aberrant, for example, suppose observation 1 were too big by an amount δ. Then, from the table of signs, we see that effects 1, 2, 3, 4, 123, 124, 134, 234 would all be reduced by an amount $\delta/8$ and the remaining effects 12, 13, 14, 23, 24, 34, 1234 would all be increased by $\delta/8$. If we regard all the effects with the exception of the main effects 2 and 3 as possibly arising from noise, and consider the signs of the effects that remain, we see that, except for 134, these signs correspond to observation 13, suggesting that this observation is in error, and possibly too large. (Sometimes the complements of the signs make the match up, indicating a "too small" observation.)

Further, if we assume that factors x_2 and x_3 *are* the only ones of possible importance, the 2^4 design then becomes a replicated 2^2 in these factors. The replicate values are plotted in Figure 4B.5, from which it again seems likely that the thirteenth observation of 59.15 is in error.

APPENDIX 4C. CONFIDENCE REGIONS FOR CONTOUR PLANES (see Section 4.5)

We can construct confidence regions for the contour planes as follows. Consider, for example, the equation for the contour plane of the desired level of hue, that is,

$$20 = 16.95 - 5.64x_1 - 0.17x_4 + 5.42x_6.$$

Let

$$Z_H \equiv 20 - 16.95 + 5.64x_1 + 0.17x_4 - 5.42x_6.$$

For any particular point (x_1, x_4, x_6) which lies on the "true" contour plane for a response of 20 in hue we have

$$E(Z_H) = 0.$$

Also, we know that the estimate of variance of Z_H is

$$\hat{\sigma}^2_{Z_H} = (1 + x_1^2 + x_4^2 + x_6^2)s_2^2/64.$$

Thus a confidence region for all points (x_1, x_4, x_6) which satisfy hue = 20 is given by

$$-t_{0.025}(\nu)\hat{\sigma}_{Z_H} \le Z_H \le t_{0.025}(\nu)\hat{\sigma}_{Z_H}$$

where $t_{0.025}(\nu)$ is the upper $97\frac{1}{2}\%$ point of a t distribution with ν df and where $\nu = 60$ is the number of degrees of freedom on which $s_2^2 = 35.1219$ (from Table 4.7) is based.

It follows that the (x_1, x_4, x_6) boundary for the confidence region for the hue = 20 contour plane is defined by

$$(20 - 16.95 + 5.64x_1 + 0.17x_4 - 5.42x_6)^2$$

$$= (1 + x_1^2 + x_4^2 + x_6^2)2^2(35.1219)/64$$

or

$$(3.05 + 5.64x_1 + 0.17x_4 - 5.42x_6)^2 = 2.195(1 + x_1^2 + x_4^2 + x_6^2).$$

This equation represents two hyperbolic surfaces which "sandwich" the estimated hue = 20 plane and go further away from it as we travel further away from the point $(\bar{x}_1, \bar{x}_4, \bar{x}_6)$. Points (x_1, x_4, x_6) that make left-hand side (LHS) < right-hand side (RHS) lie within the confidence region. Points for which LHS > RHS lie outside it.

EXERCISES

4.1. Explain briefly, in a few sentences, Yates' method of evaluating effects in a 2^k design.

4.2. A 2^4 design gave rise to the following contrasts in standard order (1, 2, 12, 3, 13, 23, 123, 4, 14, etc.): 18, 21, 3, 6, 0, 4, −1, 24, −8, 11, −9, 3, 1, 1, −2. Use a normal plot technique to analyze these data.

Note: A *contrast* is a linear combination of the y's whose coefficients add to zero. Thus the factorial design estimates of main effects and of interactions are all contrasts, but $\bar{y}$ is not.

4.3. A 2^4 factorial design yielded the following contrasts:

$A \leftarrow 0.4$	$AB \leftarrow 16.7$	$ABC \leftarrow -0.1$
$B \leftarrow -7.6$	$AC \leftarrow 3.1$	$ABD \leftarrow -4.7$
$C \leftarrow 14.1$	$AD \leftarrow 5.2$	$ACD \leftarrow 7.7$
$D \leftarrow 66.7$	$BC \leftarrow 8.3$	$BCD \leftarrow -2.3$
	$BD \leftarrow -3.6$	$ABCD \leftarrow 3.9$
	$CD \leftarrow 14.3$	

Interpret the results, employing a normal plot technique.

4.4. (a) If a full 2^k experiment is performed and each factor has levels -1 and 1, what are the relationships between the estimates of (i) the main effects of the k factors, and (ii) the coefficients β_i in a first-order model.
(b) How, in the same situation, are two-factor interactions represented in a regression model?

4.5. Imagine that individual trials comprising a 2^3 factorial design are run sequentially in standard order (in complete disregard for randomization !) while, simultaneously, a fourth variable disregarded by the experimenter takes the values shown in Table E4.5.
If the first-order model $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$ is fitted by least squares, determine the bias in the estimates of b_0, b_1, b_2, and b_3 of β_0, β_1, β_2, and β_3 induced by the failure to add a term $\beta_4 x_4$ to the model.

TABLE E4.5

x_1	x_2	x_3	x_4
−	−	−	−7
+	−	−	5
−	+	−	7
+	+	−	3
−	−	+	−3
+	−	+	−7
−	+	+	−5
+	+	+	7

4.6. Use Yates' algorithm to evaluate the effects from the observations in Table E4.6, obtained from a 2^3 factorial design.

TABLE E4.6

x_1	x_2	x_3	y
−	−	−	13
+	−	−	11
−	+	−	9
+	+	−	5
−	−	+	11
+	−	+	8
−	+	+	9
+	+	+	7

4.7. Imagine a 2^k factorial design with a center point replicated r times. What effect do these additional center observations have on the estimation of the factorial effects and on the overall mean?

4.8. The primary purpose of quenching a steel is to produce martensite. The transition from austenite to martensite depends upon the steel's composition, which affects the temperature of the martensite formation. It has long been believed that carbon and other alloying elements lower the temperature range in which martensite forms. A preliminary 2^3 factorial study to determine the effects of carbon, manganese, and nickel on "martensite start temperature" produced the data in Table E4.8.
(a) Code the predictor variables (i.e., the test conditions).
(b) Draw a diagram of the design and mark the response observations (i.e., the start temperatures) on it.
(c) Estimate all the main effects, two-factor interactions, and the three-factor interaction.

TABLE E4.8

Test conditions			Start temperature
%C	%Mn	%Ni	(°F)
0.1	0.2	0.1	940
0.9	0.2	0.1	330
0.1	1.4	0.1	820
0.9	1.4	0.1	250
0.1	0.2	0.2	840
0.9	0.2	0.2	280
0.1	1.4	0.2	760
0.9	1.4	0.2	200

(d) Assume that the standard deviation of the response observations is $\sigma = 65$. Which estimates in (c) are large compared with their standard errors?

(e) What are the practical conclusions of your analysis?

TABLE E4.9

X_1	X_2	X_3	y
110	2500	2	37.7
110	1000	2	25.6
60	2500	2	22.2
110	1000	7.3	36.6
110	1000	7.3	34.4
110	2500	7.3	49.1
60	1000	2	12.5
60	2500	2	21.0
110	2500	7.3	46.0
60	2500	7.3	33.5
110	1000	2	28.2
110	2500	2	39.2
60	1000	2	12.2
60	1000	7.3	19.5
60	2500	7.3	35.0
60	1000	7.3	20.8

4.9. (Adapted from data provided by Mario Karfakis.) The data in Table E4.9 resulted from a study of the effect of three factors

X_1 = revolutions per minute (rpm),

X_2 = downward thrust (weight), (lb),

X_3 = drilling fluid flow rate,

on the rate of penetration, y, of a drill. The observations are listed in the random order in which they were taken. Code the X's, and identify the type of design used. Then carry out a suitable analysis and list the practical conclusions you reach.

4.10. Four variables were examined for their effects on the yield y of a chemical process:

x_1 = catalyst concentration,

x_2 = NaOH concentration,

x_3 = level of agitation,

x_4 = temperature.

TABLE E4.10

1	2	3	4	y	1	2	3	4	y
−	+	−	+	61	+	−	+	+	84
+	−	−	+	70	+	+	+	−	41
−	−	−	−	46	−	−	+	−	67
−	−	+	+	87	+	−	−	−	49
+	+	+	+	62	+	+	−	+	59
+	−	+	−	64	−	+	−	−	36
+	+	−	−	38	−	−	−	+	68
−	+	+	−	38	−	+	+	+	62

A 2^4 factorial design was employed. The trials were run in random order. Interpret the data, given in Table E4.10.

4.11. A manufacturer of paper bags wishes to check tear resistance, y, for which she has a long-used numerical scale. She examines three factors at each of two coded levels namely

x_1 = type of paper,

x_2 = humidity,

x_3 = direction of tear,

and obtains three replicate observations as shown in Table E4.11. (The runs have been rearranged for convenience from the random order in which they were performed.) The total sum of squares for the 24 observations is 496.87.

TABLE E4.11

x_1	x_2	x_3	Tear resistance			Row sums
−	−	−	3.8	3.1	2.2	9.1
+	−	−	6.6	8.0	6.8	21.4
−	+	−	3.4	1.7	3.8	8.9
+	+	−	6.8	8.2	6.0	21.0
−	−	+	2.3	3.1	0.7	6.1
+	−	+	4.7	3.5	4.4	12.6
−	+	+	2.1	1.1	3.6	6.8
+	+	+	4.2	4.7	2.9	11.8
						97.7

Perform a factorial analysis on the data, estimate σ^2 from the repeat runs, and formulate conclusions about the factors.

4.12. Look again at the data of Exercise 4.9. Fit the first-order model $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$ by least squares, and perform the usual analyses including an overall test for lack of fit. Your conclusions should be consistent with those from the factorial analysis.

4.13. To the data Exercise 4.10 fit the model

$$y = \beta_0 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_{23} x_2 x_3 + \beta_{24} x_2 x_4 + \beta_{34} x_3 x_4 + \varepsilon.$$

Provide the associated analysis of variance table, estimate σ^2, and check the fitted coefficients for significance.

4.14. Look at the data of Exercise 4.11. We could, formally, fit a regression equation to these data but they are much more suitably analyzed by factorial methods. Why?

TABLE E4.15

Run order	x_1	x_2	x_3	y
3	−1	−1	−1	49
12	1	−1	−1	48
11	−1	1	−1	53
8	1	1	−1	54
4	−1	−1	1	57
5	1	−1	1	58
9	−1	1	1	61
7	1	1	1	55
6	0	0	0	55
2	0	0	0	54
1	0	0	0	56
10	0	0	0	54

4.15. (a) Fit the model $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$ to the data given in Table E4.15. Test for lack of fit. What final model would you adopt, if any?

(b) This design has 12 observations; exhibit the vectors $\mathbf{c}_i$ for 12 meaningful orthogonal contrasts which exist and which provide a complete transformation of the original data set $y_1, y_2, \ldots, y_{12}$.

4.16. (Source: Maximum data through a statistical design, by C. D. Chang, O. K. Kononenko, and R. E. Franklin, Jr., *Industrial and Engineering Chemistry*, **52**, November 1960, 939–942. Material reproduced and adapted by permission of the copyright holder, the American Chemical Society. The data were obtained under a grant from the Sugar Research Foundation, Inc., now the World Sugar Research Organization, Ltd.) Table E4.16 shows the results obtained from a 2^4 factorial design plus six center points. Four response variables were observed. The trials were performed in random order but are listed in standard order. Estimate all factorial main effects and all interactions for every response, and evaluate the appropriate standard errors. Which variables appear to be effective? So far, the six center points have been used only to estimate the four σ_i^2, the variances of the four responses. Calculate the values of the contrasts

$$CC_i = (\text{average response at factorial points}) - (\text{average response at center points})$$

and determine their standard errors. Which responses show evidence of pure quadratic curvature? (*Note*: In general if there are k factors, such a contrast

TABLE E4.16. A 2^4 factorial design with six center points; four responses are shown

Trial number	Factor levels x_1, NH_3	x_2, T	x_3, H_2O	x_4, P	Responses y_1, PDA	y_2, DMP	y_3, PD	y_4, R
1	−1	−1	−1	−1	1.8	58.2	24.7	84.7
2	1	−1	−1	−1	4.3	23.4	45.5	73.2
3	−1	1	−1	−1	0.4	21.9	8.6	30.9
4	1	1	−1	−1	0.7	21.8	9.1	31.6
5	−1	−1	1	−1	0.3	14.3	75.5	90.1
6	1	−1	1	−1	4.5	6.3	86.5	96.3
7	−1	1	1	−1	0.0	4.5	10.0	14.5
8	1	1	1	−1	1.6	21.8	50.1	73.5
9	−1	−1	−1	1	1.3	46.7	43.3	91.3
10	1	−1	−1	1	4.2	53.2	39.7	97.1
11	−1	1	−1	1	1.9	23.7	5.4	31.0
12	1	1	−1	1	0.7	40.3	9.7	50.7
13	−1	−1	1	1	0.0	7.5	78.8	86.3
14	1	−1	1	1	2.3	13.3	77.8	93.4
15	−1	1	1	1	0.8	49.3	21.1	71.2
16	1	1	1	1	7.3	20.1	37.8	65.2
17[a]	0	0	0	0	5.0	32.8	45.1	82.8
	Pure error estimates of σ_i^2, s_i^2 =				1.08	9.24	15.37	8.18

[a]Average yield of six trials.

has expectation

$$E(CC) = \beta_{11} + \beta_{22} + \cdots + \beta_{kk}.$$

Thus if the CC contrast is compared to its standard error, a test for H_0: $\Sigma\beta_{ii} = 0$ versus H_1: $\Sigma\beta_{ii} \neq 0$ may be made.) Suppose you fitted a first-order model to each response. Would the center point average response be counted as one data point or as six data points. (The answer is six; why?) A reviewer of the original article suggested that the responses might have been transformed via

$$\omega_i = \sin^{-1}\left(\frac{y_i}{100}\right)^{1/2}$$

which is a transformation often appropriate when the data are percentages, as here, or proportions (when the division by 100 would not be needed). Is such a transformation useful in the sense that it renders a simpler interpretation of the factors' effects? Transform all the response values and repeat the factorial analysis to find out. (The y_2 data will be used again in Exercise 11.16 and will be combined there with additional data, for a second-order surface fitting exercise.)

4.17. The following estimates of effects resulted from a 2^5 experiment:

$E \leftarrow -224$	$BCD \leftarrow -18$	$BE \leftarrow 29$
$C \leftarrow -153$	$ABDE \leftarrow -14$	$DE \leftarrow 30$
$ABCDE \leftarrow -77$	$D \leftarrow -9$	$ABCE \leftarrow 31$
$ACE \leftarrow -58$	$B \leftarrow -6$	$BCE \leftarrow 39$
$AD \leftarrow -54$	$CD \leftarrow -4$	$ACDE \leftarrow 47$
$BC \leftarrow -53$	$ABC \leftarrow 0$	$AC \leftarrow 53$
$ABD \leftarrow -34$	$AE \leftarrow 2$	$ABCD \leftarrow 58$
$ACD \leftarrow -33$	$BD \leftarrow 7$	$AB \leftarrow 64$
$BDE \leftarrow -28$	$CDE \leftarrow 12$	$CE \leftarrow 83$
$ABE \leftarrow -22$	$BCDE \leftarrow 16$	$A \leftarrow 190$
	$ADE \leftarrow 21$	

(a) Plot these contrasts on normal probability paper.
(b) Which effects do you think are distinguishable from noise?
(c) Replot the remaining contrasts.
(d) Estimate the standard deviation of the contrasts from plot (c). See p. 131.

[Partial answers: (b) E, C, A. (d) Roughly 40.]

CHAPTER 5

Blocking and Fractionating 2^k Factorial Designs

5.1. BLOCKING THE 2^6 DESIGN

In the 2^6 design used above for illustration, the runs were made in a random order; the design was *fully randomized*. However, it might have been much better to run the arrangement in a series of *randomized blocks*. Suppose, for example, we expected trouble because of appreciable inhomogeneity of the raw material. Suppose further that a blender was available, large enough to produce homogeneous blends of raw material sufficient for making eight batches, that is, for conducting 8 of the 64 runs. Then the 64-run experiment might have been arranged using 8 blends (blocks) of homogeneous raw material within each of which 8 runs were made. The difficulty would be that differences between blends could bias the estimates of effects. R. A. Fisher found a way of circumventing this biasing problem by making the (often reasonable) assumption that the effects of blocks would be approximately additive, that is, that the differences between blends would cause the response results simply to be raised or lowered by a fixed (but unknown) amount.

Two Blocks

We introduce the method in a simpler context. Suppose first that we would like to make the 64 runs in two blocks (I and II) each with 32 runs. (Such an arrangement would be appropriate if we had a blender large enough to provide homogeneous raw material for 32 runs.) Consider the partial table of signs, Table 5.1, which shows the 2^6 design laid out in standard order. Suppose now that we allocate runs for which the product column **123456** has a minus sign to block I, and runs for which **123456** has a plus sign to block II. Then, following Fisher, we say that the block effect (denoted by B)