Solution of the heat equation in 1D

In[1]:= □

Out[1]= Null□

```
In[2]:= IP[f_, g_] := Integrate[f g, {x, 0, 1}]
```

```
In[3]:= k_n_ = n Pi
```

Out[3]= πn

```
In[4]:= v_n_[x_] = Sin[k_n x]
```

Out[4]= $\sin(\pi n x)$

- Check orthogonality

```
In[5]:= J[m_, n_] = IP[v_m[x], v_n[x]]
```

Out[5]= $$\frac{n \sin(\pi m) \cos(\pi n) - m \cos(\pi m) \sin(\pi n)}{\pi m^2 - \pi n^2}$$

A simple example

```
In[6]:= U_0[x_] = v_1[x] + 1 / 3 v_4[x]
```

Out[6]= $$\sin(\pi x) + \frac{1}{3} \sin(4 \pi x)$$

- Find the coefficients

```
In[7]:= B_n_ = IP[v_n[x], U_0[x]] / IP[v_n[x], v_n[x]]
```

Out[7]= $$\frac{(n^2 + 44) \sin(\pi n)}{3 \pi (n^2 - 16) (n^2 - 1) \left(\frac{1}{2} - \frac{\sin(2 \pi n)}{4 \pi n}\right)}$$

This is messy!

```
In[8]:= B_n_ = Assuming[n ∈ Integers && n > 0, IP[v_n[x], U_0[x]] / IP[v_n[x], v_n[x]]]
```

Out[8]= 0

This misses the special cases!

```
In[9]:= B_n_ := IP[v_n[x], U_0[x]] / IP[v_n[x], v_n[x]]
```

In[10]:=
```
Table[{n, B_n}, {n, 1, 10}]
```
Out[10]=

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 0 \\ 4 & \frac{1}{3} \\ 5 & 0 \\ 6 & 0 \\ 7 & 0 \\ 8 & 0 \\ 9 & 0 \\ 10 & 0 \end{pmatrix}$$

- ## Form M-th partial sum of the solution

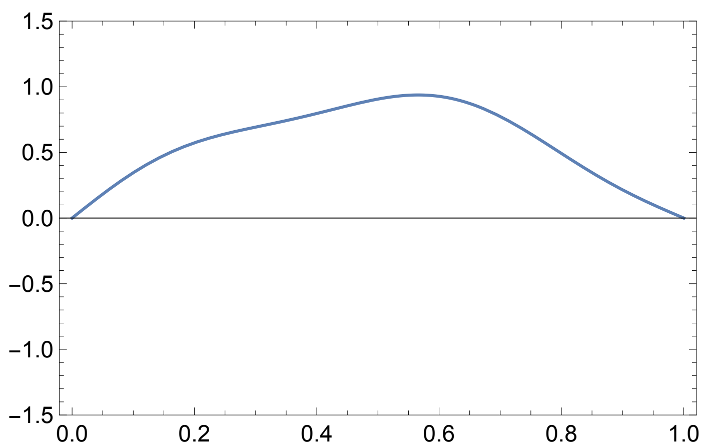
In[11]:=
```
uSum[M_, x_, t_] := Sum[B_n v_n[x] Exp[-k_n^2 t], {n, 1, M}]
```

In[12]:=
```
u4[x_, t_] = uSum[4, x, t]
```
Out[12]=

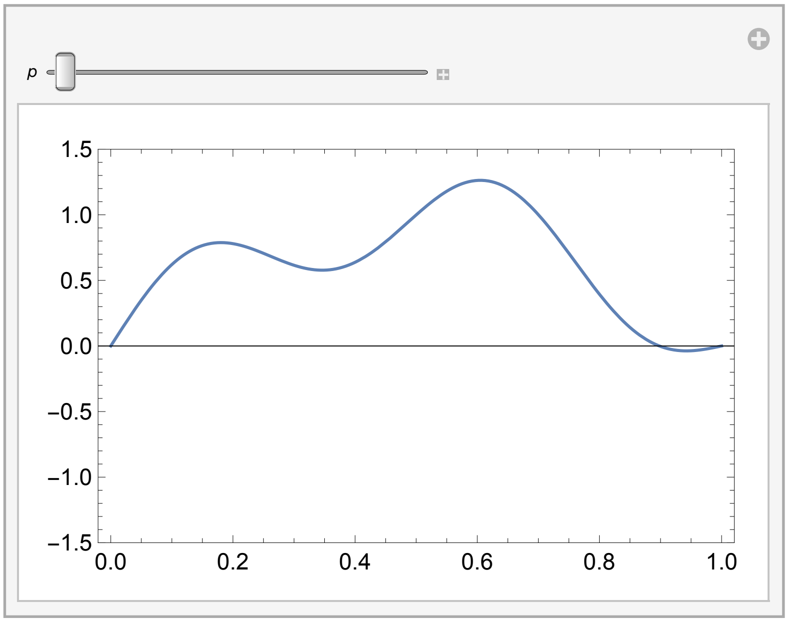
$$e^{-\pi^2 t} \sin(\pi x) + \frac{1}{3} e^{-16 \pi^2 t} \sin(4 \pi x)$$

In[13]:=
```
showU[t_] := Plot[u4[x, t], {x, 0, 1}, PlotRange → {-1.5, 1.5}]
```

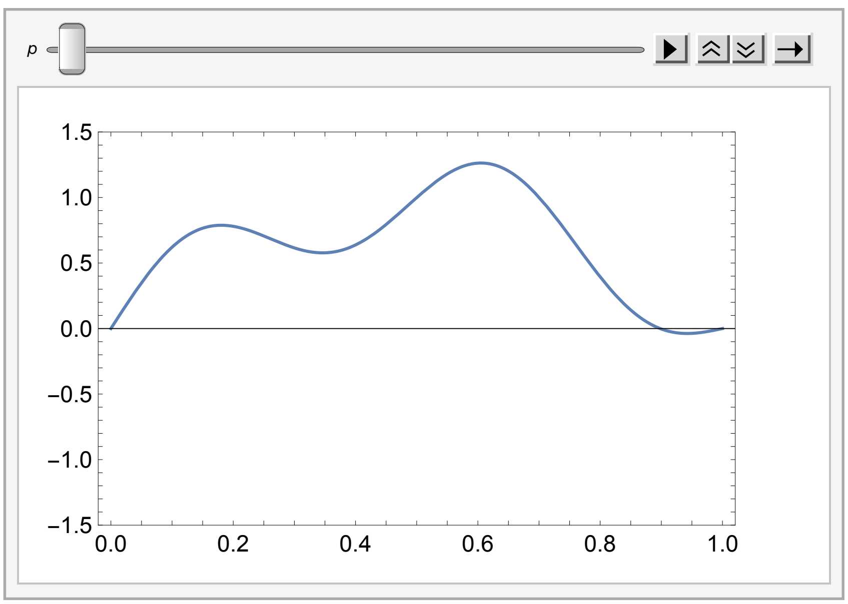
In[14]:=
```
showU[0.01]
```
Out[14]=



In[15]:=
```
Manipulate[showU[10^p], {p, -4, 0}]
```
Out[15]=



In[16]:=
```
Animate[showU[10^p], {p, -4, 0}, AnimationRunning → False]
```
Out[16]=

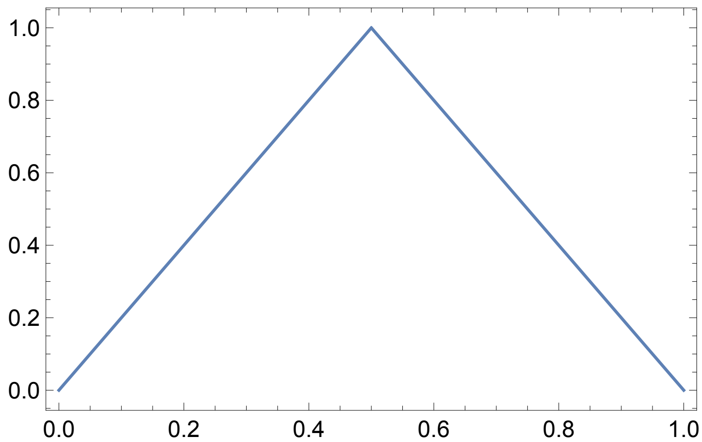


A not-so-simple example

In[17]:=
```
U₀[x_] = Piecewise[{{2 x, x ≤ 1 / 2}, {2 (1 - x), x > 1 / 2}}]
```
Out[17]=

$$\begin{cases} 2\,x & x \le \frac{1}{2} \\ 2\,(1-x) & x > \frac{1}{2} \end{cases}$$

In[18]:=
```
Plot[U₀[x], {x, 0, 1}]
```
Out[18]=



• Find the coefficients

In[19]:=
```
Bn_ = IP[vn[x], U₀[x]] / IP[vn[x], vn[x]]
```
Out[19]=

$$\frac{2\left(2\sin\left(\frac{\pi n}{2}\right)-\sin(\pi n)\right)}{\pi^2 n^2\left(\frac{1}{2}-\frac{\sin(2\pi n)}{4\pi n}\right)}$$

This is messy!

In[20]:=
```
Bn_ = Assuming[n ∈ Integers && n > 0, IP[vn[x], U₀[x]] / IP[vn[x], vn[x]]]
```
Out[20]=

$$\frac{8\sin\left(\frac{\pi n}{2}\right)}{\pi^2 n^2}$$

This misses the special cases!

In[21]:=
```
Bn_ := IP[vn[x], U₀[x]] / IP[vn[x], vn[x]]
```

In[22]:=
```
Table[{n, B_n}, {n, 1, 10}]
```
Out[22]=

$$\begin{pmatrix} 1 & \frac{8}{\pi^2} \\ 2 & 0 \\ 3 & -\frac{8}{9\pi^2} \\ 4 & 0 \\ 5 & \frac{8}{25\pi^2} \\ 6 & 0 \\ 7 & -\frac{8}{49\pi^2} \\ 8 & 0 \\ 9 & \frac{8}{81\pi^2} \\ 10 & 0 \end{pmatrix}$$

- Form M-th partial sum of the solution

In[23]:=
```
uSum[M_, x_, t_] := Sum[B_n v_n[x] Exp[-k_n^2 t], {n, 1, M}]
```

In[24]:=
```
u3[x_, t_] = uSum[3, x, t]
```
Out[24]=

$$\frac{8 e^{-\pi^2 t} \sin(\pi x)}{\pi^2} - \frac{8 e^{-9\pi^2 t} \sin(3\pi x)}{9\pi^2}$$

In[25]:=
```
u5[x_, t_] = uSum[5, x, t]
```
Out[25]=

$$\frac{8 e^{-\pi^2 t} \sin(\pi x)}{\pi^2} - \frac{8 e^{-9\pi^2 t} \sin(3\pi x)}{9\pi^2} + \frac{8 e^{-25\pi^2 t} \sin(5\pi x)}{25\pi^2}$$

In[26]:=
```
u11[x_, t_] = uSum[11, x, t]
```
Out[26]=

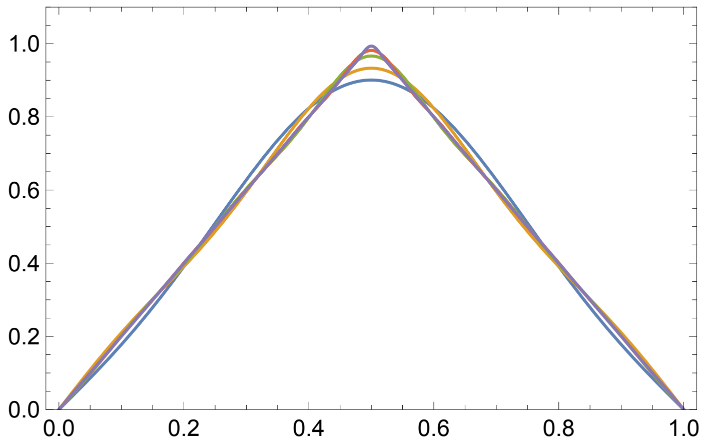
$$\frac{8 e^{-\pi^2 t} \sin(\pi x)}{\pi^2} - \frac{8 e^{-9\pi^2 t} \sin(3\pi x)}{9\pi^2} + \frac{8 e^{-25\pi^2 t} \sin(5\pi x)}{25\pi^2} - \frac{8 e^{-49\pi^2 t} \sin(7\pi x)}{49\pi^2} + \frac{8 e^{-81\pi^2 t} \sin(9\pi x)}{81\pi^2} - \frac{8 e^{-121\pi^2 t} \sin(11\pi x)}{121\pi^2}$$

In[27]:=
```
u21[x_, t_] = uSum[21, x, t];
```

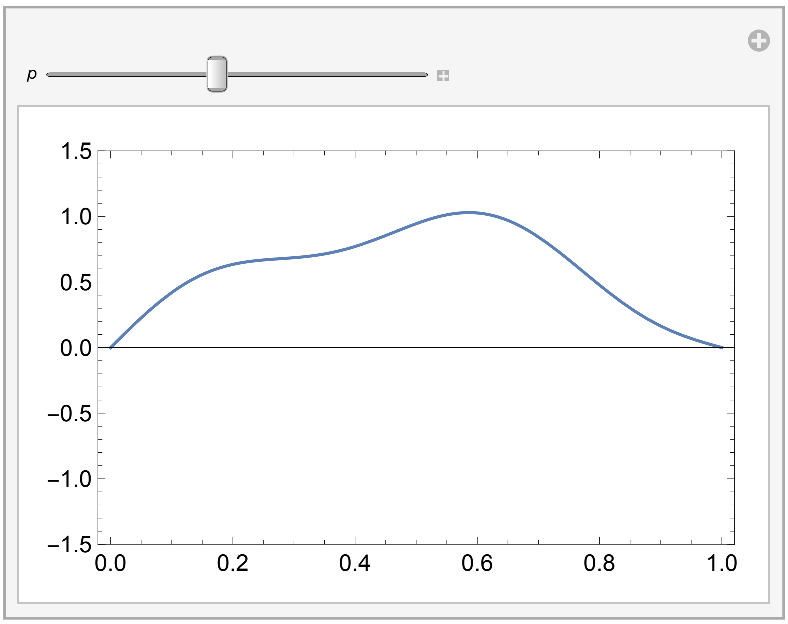
In[28]:=
```
u61[x_, t_] = uSum[61, x, t];
```

In[ ]:=
```
showU[t_] := Plot[{u3[x, t], u5[x, t], u11[x, t], u21[x, t], u61[x, t]},
  {x, 0, 1}, PlotRange → {0, 1.1}]
```

In[ ]:=
```
showU[0.0]
```
Out[ ]=



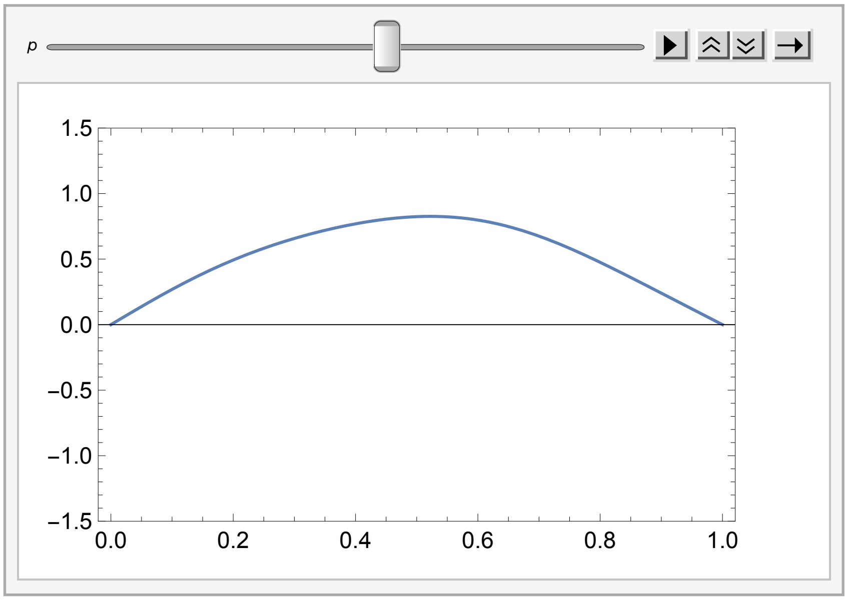
In[ ]:=
```
Manipulate[showU[10^p], {p, -4, 0}]
```
Out[ ]=



In[ ]:=

```
Animate[showU[10^p], {p, -4, 0}, AnimationRunning → False]
```

Out[ ]=



In[ ]:=

```
Animate[showU[t], {t, 0, 1}, AnimationRunning → False]
```

Out[ ]=