Solution of Laplace’s equation on a semi-infinite strip

Solve $\nabla^2 u = 0$ on the semi-infinite strip $0 \le x \le 1, y \ge 0$ with boundary conditions

$$u(x, 0) = U_0(x)$$
$$u(0, y) = u(1, y) = 0$$
$$u(x, y) \to 0 \text{ as } y \to \infty.$$

With these BCs, the solution is

$$u(x, y) = \sum_{n=1}^{\infty} c_n v_n(x) e^{-k_n y}$$

where $k_n = n\pi$ and $v_n(x) = \sin(k_n x)$. The coefficients c_n are obtained by orthogonal projection,

$$c_n = \frac{\langle v_n, U_0 \rangle}{\langle v_n, v_n \rangle}$$

with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx.$$

- ## Set up

  - ### Define the inner product to use

In[36]:=
```
IP[u_, v_] := Integrate[u v, {x, 0, 1}]
```

Be sure to use deferred assignment here (“:=” instead of “=”)

  - ### Define the eigenfunctions

In[37]:=
```
k_n_ = n Pi
```

Out[37]=

πn

In[38]:=
```
v_n_[x_] = Sin[k_n x]
```

Out[38]=

$\sin(\pi n x)$

Example 1

In this example, the boundary value at $y = 0$ is $U_0(x) = \sin(\pi x) + \frac{3}{4}\sin(2\pi x) + \frac{1}{5}\sin(3\pi x)$. This function is already written explicitly as a sum of eigenfunctions,

$$U_0(x) = v_1(x) + \frac{3}{4} v_2(x) + \frac{1}{5} v_3(x)$$

so the solution will be

$$u(x, y) = \sin(\pi x)\, e^{-\pi y} + \tfrac{3}{4} \sin(2 \pi x)\, e^{-2 \pi y} + \tfrac{1}{5} \sin(3 \pi x)\, e^{-3 \pi y}.$$

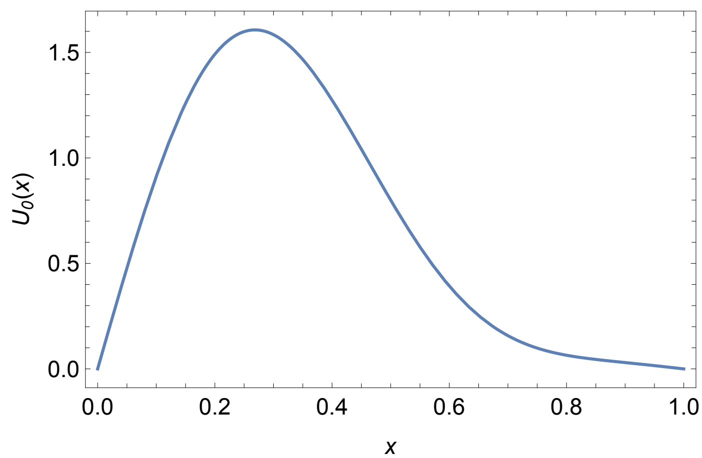
- Define the function value at the non-homogeneous boundary

In[39]:=
```
U0[x_] = v1[x] + 3 / 4 v2[x] + 1 / 5 v3[x]
```
Out[39]=

$$\sin(\pi x) + \frac{3}{4} \sin(2 \pi x) + \frac{1}{5} \sin(3 \pi x)$$

Show the function

In[40]:=
```
Plot[U0[x], {x, 0, 1}, FrameLabel → {x, "U0(x)"}]
```
Out[40]=



- Compute the coefficients

We’ll need to do some integrals. While Mathematica can do all the integrals easily, you’ll need to be careful: in some problems the result might come out in a form that’s difficult to use directly. This example is such a problem.

Do the integrals:

In[41]:=
```
cn_ = IP[vn[x], U0[x]] / IP[vn[x], vn[x]]
```
Out[41]=

$$-\frac{(n^4 - 10 n^2 + 249) \sin(\pi n)}{10 \pi \left(n^2 (n^2 - 7)^2 - 36\right) \left(\frac{1}{2} - \frac{\sin(2 \pi n)}{4 \pi n}\right)}$$

Notice the $\sin(n \pi)$ in the numerator: the numerator is zero whenever n is an integer.Let’s look at the denominator:

In[42]:=
```
Factor[Denominator[c_n]]
```
Out[42]=

$$\frac{5\,(n-3)\,(n-2)\,(n-1)\,(n+1)\,(n+2)\,(n+3)\,(2\,\pi\,n-\sin(2\,\pi\,n))}{2\,n}$$

The denominator is zero for $n = 1, 2, 3$, nonzero for all other natural numbers. At $n = 1, 2, 3$ the expression for the coefficients is indeterminate.

In[43]:=
```
Factor[c_n]
```
Out[43]=

$$-\frac{2\,n\left(n^4-10\,n^2+249\right)\sin(\pi\,n)}{5\,(n-3)\,(n-2)\,(n-1)\,(n+1)\,(n+2)\,(n+3)\,(2\,\pi\,n-\sin(2\,\pi\,n))}$$

You can resolve the indeterminacy with L'Hopital's rule, for example

In[44]:=
```
Limit[c_n, n → 3]
```
Out[44]=

$$\frac{1}{5}$$

but it's usually simpler to just use deferred evaluation

In[45]:=
```
c_n_ := IP[v_n[x], U_0[x]] / IP[v_n[x], v_n[x]]
```

This way, the integral isn't evaluated until the value of n is known, at which point Mathematica can do an integral specialized to that value.

Let's see a few values:

In[46]:=
```
Table[{n, c_n}, {n, 1, 5}]
```
Out[46]=

$$\begin{pmatrix} 1 & 1 \\ 2 & \frac{3}{4} \\ 3 & \frac{1}{5} \\ 4 & 0 \\ 5 & 0 \end{pmatrix}$$

- ## Form the solution to Laplace's equation

Write a function to add up the first M terms in the partial sums.

In[47]:=
```
uSum[M_, x_, y_] := Sum[c_n v_n[x] Exp[-k_n y], {n, 1, M}]
```

In this case, we need only 3 terms.

In[48]:=
```
u3[x_, y_] = uSum[3, x, y]
```
Out[48]=

$$e^{-\pi\,y}\sin(\pi\,x)+\frac{3}{4}\,e^{-2\,\pi\,y}\sin(2\,\pi\,x)+\frac{1}{5}\,e^{-3\,\pi\,y}\sin(3\,\pi\,x)$$

That's the solution we worked out by hand.

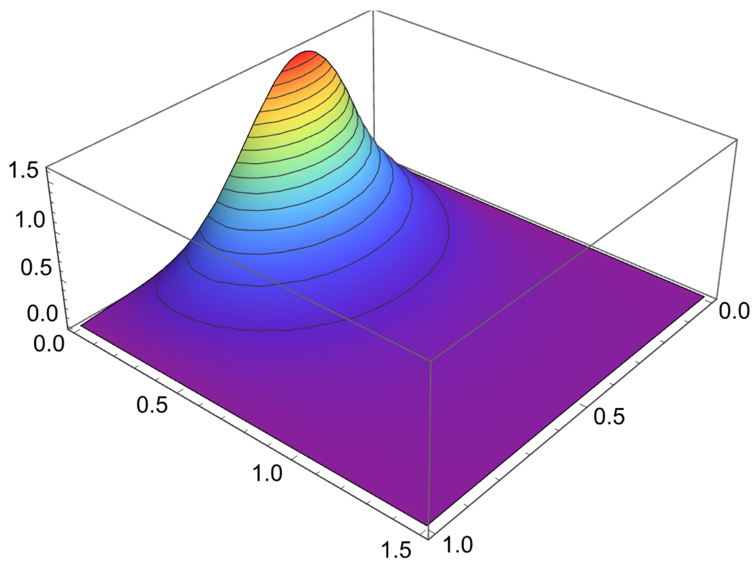
- ## Plot the solution

The problem is posed on a semi-infinite strip so we can't plot the whole domain. The slowest decaying term goes as $e^{-\pi y}$, which will decrease to $e^{-\pi} \approx 0.043$ at $y = 1$ and $e^{-3\pi/2} \approx 0.009$ at $y = \frac{3}{2}$. Since 1% accuracy is good enough for a rough plot, we'll plot $y \in \left[0, \frac{3}{2}\right]$ instead of $y \in [0, \infty]$.

Here's a 3D plot. The "ZMesh" option shows the contour levels on the surface. I chose the ViewPoint to show the shape nicely. You can use the mouse to rotate the image and see it from different angles.

In[78]:=
```
Plot3D[u3[x, y], {x, 0, 1}, {y, 0, 3 / 2},
 PlotRange → All, PlotTheme → "ZMesh", ViewPoint → {1.3, 2.4, 2}]
```

Out[78]=



Example 2

- ## Define the function value at the non-homogeneous boundary

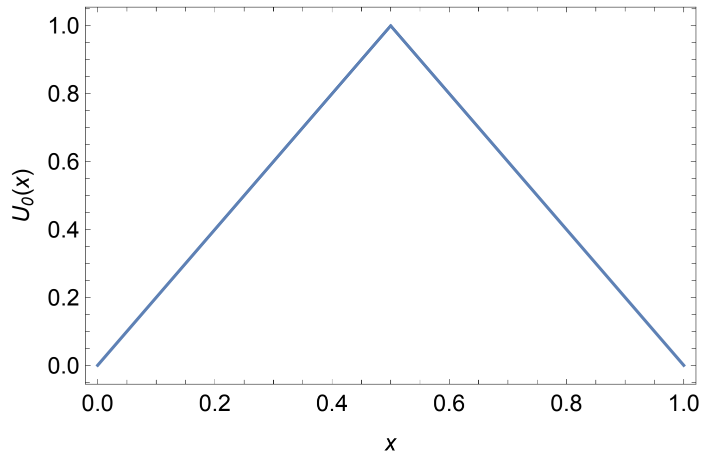
In[79]:=
```
U0[x_] = Piecewise[{{2 x, x < 1 / 2}, {2 (1 - x), x ≥ 1 / 2}}]
```

Out[79]=

$$\begin{cases} 2x & x < \frac{1}{2} \\ 2(1-x) & x \geq \frac{1}{2} \end{cases}$$

Show the function

In[80]:=
```
Plot[U0[x], {x, 0, 1}, FrameLabel → {x, "U0(x)"}]
```
Out[80]=



• Compute the coefficients

In this case, we'll be ok without deferred evaluation

In[81]:=
```
cn_ = IP[vn[x], U0[x]] / IP[vn[x], vn[x]]
```
Out[81]=

$$\frac{2\left(2\sin\left(\frac{\pi n}{2}\right)-\sin(\pi n)\right)}{\pi^2 n^2\left(\frac{1}{2}-\frac{\sin(2\pi n)}{4\pi n}\right)}$$

It can be cleaned up a bit by specifying that *n* is an integer

In[82]:=
```
cn_ = Assuming[n ∈ Integers, IP[vn[x], U0[x]] / IP[vn[x], vn[x]]]
```
Out[82]=

$$\frac{8\sin\left(\frac{\pi n}{2}\right)}{\pi^2 n^2}$$

Show a few:

In[83]:=
```
Table[{n, cn}, {n, 1, 5}]
```
Out[83]=

$$\begin{pmatrix} 1 & \frac{8}{\pi^2} \\ 2 & 0 \\ 3 & -\frac{8}{9\pi^2} \\ 4 & 0 \\ 5 & \frac{8}{25\pi^2} \end{pmatrix}$$

• Form the solution to Laplace's equation

Write a function to add up the first *M* terms in the partial sums.

In[ ]:=
```
uSum[M_, x_, y_] := Sum[cn vn[x] Exp[-kn y], {n, 1, M}]
```

Show a sum with a few terms

In[84]:=
```
u7[x_, y_] = uSum[7, x, y]
```

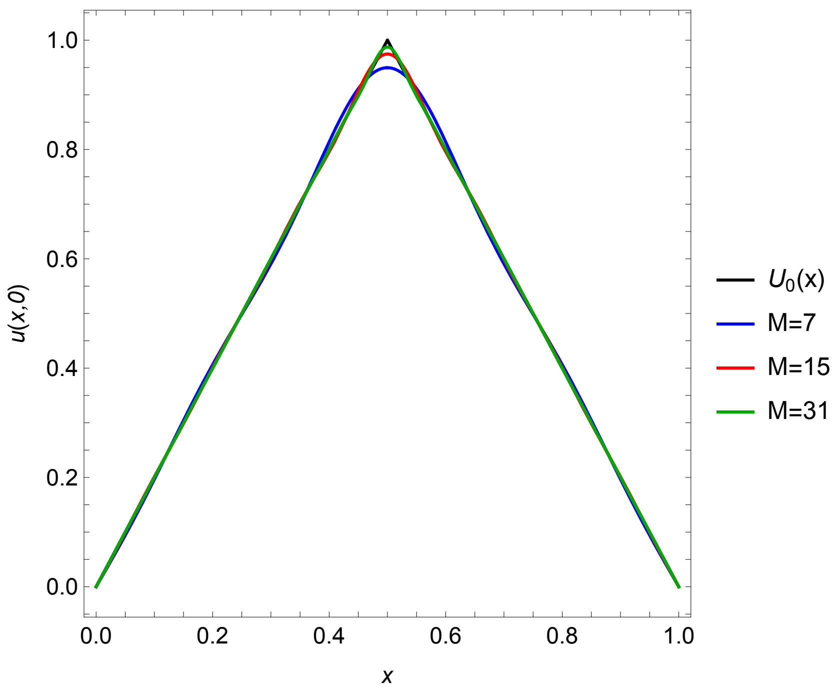
Out[84]=

$$\frac{8\,e^{-\pi y}\sin(\pi x)}{\pi^2} - \frac{8\,e^{-3\pi y}\sin(3\pi x)}{9\pi^2} + \frac{8\,e^{-5\pi y}\sin(5\pi x)}{25\pi^2} - \frac{8\,e^{-7\pi y}\sin(7\pi x)}{49\pi^2}$$

To see how many terms are needed, let’s plot the value at $y = 0$.

In[104]:=
```
Plot[{U₀[x], uSum[7, x, 0], uSum[15, x, 0], uSum[31, x, 0]},
 {x, 0, 1}, FrameLabel → {x, "u(x,0)"}, AspectRatio → 1,
 PlotStyle → {Black, Blue, Red, Darker[Green]},
 PlotLegends → Placed[{"U₀(x)", "M=7", "M=15", "M=31"}, Right]]
```

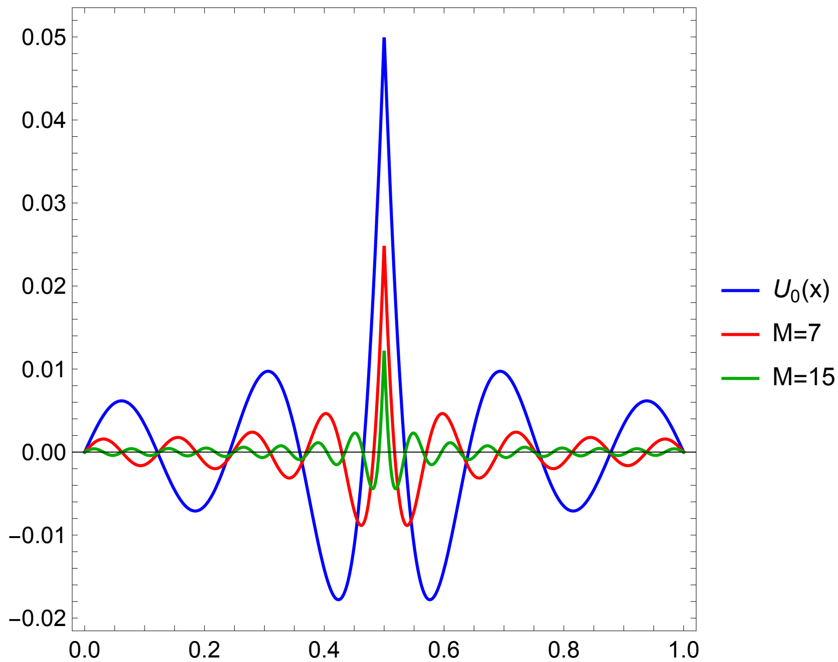
Out[104]=



In[106]:=

```
Plot[{U₀[x] - uSum[7, x, 0], U₀[x] - uSum[15, x, 0], U₀[x] - uSum[31, x, 0]}, {x, 0, 1},
 PlotRange → All, AspectRatio → 1, PlotStyle → {Blue, Red, Darker[Green]},
 PlotLegends → Placed[{"U₀(x)", "M=7", "M=15", "M=31"}, Right]]
```

Out[106]=



With 31 terms we get to just over 1% accuracy, so let's use that.

In[114]:=

```
u31[x_, y_] = uSum[31, x, y]
```

Out[114]=

$$\frac{8 e^{-\pi y} \sin(\pi x)}{\pi^2} - \frac{8 e^{-3\pi y} \sin(3\pi x)}{9\pi^2} + \frac{8 e^{-5\pi y} \sin(5\pi x)}{25\pi^2} - \frac{8 e^{-7\pi y} \sin(7\pi x)}{49\pi^2} + \frac{8 e^{-9\pi y} \sin(9\pi x)}{81\pi^2} - \frac{8 e^{-11\pi y} \sin(11\pi x)}{121\pi^2} + \frac{8 e^{-13\pi y} \sin(13\pi x)}{169\pi^2} - \frac{8 e^{-15\pi y} \sin(15\pi x)}{225\pi^2} + \frac{8 e^{-17\pi y} \sin(17\pi x)}{289\pi^2} - \frac{8 e^{-19\pi y} \sin(19\pi x)}{361\pi^2} + \frac{8 e^{-21\pi y} \sin(21\pi x)}{441\pi^2} - \frac{8 e^{-23\pi y} \sin(23\pi x)}{529\pi^2} + \frac{8 e^{-25\pi y} \sin(25\pi x)}{625\pi^2} - \frac{8 e^{-27\pi y} \sin(27\pi x)}{729\pi^2} + \frac{8 e^{-29\pi y} \sin(29\pi x)}{841\pi^2} - \frac{8 e^{-31\pi y} \sin(31\pi x)}{961\pi^2}$$

- Plot the solution

In[119]:=

```
Plot3D[u31[x, y], {x, 0, 1}, {y, 0, 3 / 2},
 PlotRange → All, PlotTheme → "ZMesh", ViewPoint → {1.3, 2.4, 2}]
```

Out[119]=