Linear algebra: Core definitions

Notation

- $\mathbb{R}^m$ and $\mathbb{C}^m$ are the m-dimensional spaces of real and complex numbers, respectively.
  - The term "scalar" will refer to a member of $\mathbb{R}$ or $\mathbb{C}$ depending whether we're working in $\mathbb{R}^m$ or $\mathbb{C}^m$.
- In what follows, $S = \{\mathbf{a}_j\}_{j=1}^n$ is a set of vectors, with each $\mathbf{a}_j \in \mathbb{C}^m$ or $\mathbb{R}^m$.

Definition: Linear combination and span

Definition (Linear combination and span)

A ***linear combination*** (LC) of the members of S is a scalar-weighted sum such as

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots + \mathbf{a}_n x_n,$$

where the n coefficients $\{x_j\}_{j=1}^{m}$ are scalars. The ***span*** of S, denoted $\operatorname{span} S$, is the set of all linear combinations that can be formed by the members of S.

Definition: Vector space and subspace

Definition (Vector space and subspace)

A set V of vectors is called a ***vector space*** if it obeys the axioms listed in Appendix B of the notes. We needn't go into all the axioms for this course; as long as one defines addition and scalar multiplication "reasonably," then it is sufficient to consider the ***closure conditions***

- $\mathbf{x} + \mathbf{y} \in V$ for all $\mathbf{x}, \mathbf{y} \in V$
- $\alpha\mathbf{x} \in V$ for all $\mathbf{x} \in V$ and for all scalars α

and the ***zero condition***,

- V contains the zero vector $\mathbf{0}$.

A subset of V that is itself a vector space is called a ***subspace***.

Definition: Matrix-vector product

Definition

Form the matrix $A \in \mathbb{C}^{m \times n}$ by packing the members of S into the columns of A. Given a vector $\mathbf{x} \in \mathbb{C}^n$, the ***matrix-vector product*** $A\mathbf{x}$ is the linear combination of the $\mathbf{a}$'s weighted by the entries in $\mathbf{x}$,

$$A\mathbf{x} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots + \mathbf{a}_n x_n.$$

Example: Matrix-vector product

- Let $A = \begin{bmatrix} -2 & 1 \\ 3 & 5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 4 & 7 \end{bmatrix}^T$. Then

$$A\mathbf{x} = 4 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + 7 \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 47 \end{bmatrix}.$$

Compare to the "dot product" approach,

$$A\mathbf{x} = \begin{bmatrix} -2 \cdot 4 + 1 \cdot 7 \\ 3 \cdot 4 + 5 \cdot 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 47 \end{bmatrix}.$$

Definition: Linear dependence and independence

Definition

The set S is called ***linearly dependent*** (LD) if there is a *nonzero* vector $\mathbf{x} \in \mathbb{C}^n$ such that

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots + \mathbf{a}_n x_n = \mathbf{0}.$$

If no such vector exists, then we call S ***linearly independent*** (LI).

Example: Linear dependence and independence

- Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. The set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is LD because $\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 = \mathbf{0}$ for nonzero vector $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} s$, where s is any constant.
- Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. The set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is LI because the linear combination

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 2x_2 \end{bmatrix}$$

cannot be zero unless $x_1 = x_2 = 0$.

Definition: Null space of a matrix

Definition (Null space)

The ***null space*** of a matrix $A \in \mathbb{C}^{m \times n}$, denoted $\operatorname{null}(A)$, is the set of vectors $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = \mathbf{0}$.

- Notice that $\operatorname{null}(A)$ *always* contains the zero vector. If $\operatorname{null}(A)$ contains *only* the zero vector, that null space is called ***trivial***; otherwise, it is called ***nontrivial***.
- It's not hard to show that $\operatorname{null}(A)$ obeys the closure conditions and is therefore a vector space.
- Notice that $\operatorname{null}(A)$ is trivial iff the columns of A are LI.

Example: Null spaces

- Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $\text{null}\,(A)$ consists of all scalar multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$,

$$\text{null}\,(A) = \left\{ c \begin{bmatrix} -2 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}.$$

- Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$. Then $\text{null}\,(A)$ is trivial.

Definition: Range space of a matrix

Definition (Range space)

The ***range space*** of a matrix $A \in \mathbb{C}^{m \times n}$, denoted $\operatorname{range}(A)$, is the set of vectors $\mathbf{y} \in \mathbb{C}^m$ that can be formed by linear combination of the columns of A. In formal notation,

$$\operatorname{range}(A) = \{\mathbf{y} \in \mathbb{C}^m : \exists \mathbf{x} \in \mathbb{C}^n \text{ for which } A\mathbf{x} = \mathbf{y}\}.$$

- It's not hard to show that $\operatorname{range}(A)$ obeys the closure conditions and is therefore a vector space.

Example: Range spaces

- Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $\operatorname{range}(A)$ consists of all scalar multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

$$\operatorname{range}(A) = \left\{ c \begin{bmatrix} 1 \\ 2 \end{bmatrix} : c \in \mathbb{R} \right\}.$$

- Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$. Then $\operatorname{range}(A)$ consists of all linear combinations $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, which is $\mathbb{R}^2$.