A Crash Course in Linear Algebra for Differential Equations
Part II: Eigensystems

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1 The matrix eigenvalue problem

Here's the problem that motivates this whole document: solve the system of differential equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}.$$

In this equation, $\mathbf{u}(t)$ is a vector-valued function, $\mathbf{u}(t) = \begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_n(t) \end{bmatrix}$ and A is a square matrix. We are going to build a general solution to this equation as a linear combination of simpler solutions. The "simple" solutions we will look for are *exponential in time,*

$$\mathbf{u}(t) = \mathbf{v}e^{\lambda t}$$

where λ is a scalar constant and $\mathbf{v}$ a constant (*i.e.*, independent of time) vector, both λ and $\mathbf{v}$ to be determined.

Let's plug that trial solution $\mathbf{v}e^{\lambda t}$ into the equation. On the RHS, we have $A\mathbf{v}e^{\lambda t} = e^{\lambda t}A\mathbf{v}$. On the LHS, we have $\lambda\mathbf{v}e^{\lambda t}$. Set them equal, and find the equation

$$e^{\lambda t}\lambda\mathbf{v} = e^{\lambda t}A\mathbf{v}$$

or

$$A\mathbf{v} = \lambda\mathbf{v}.$$

This is a new kind of problem, to be solved for λ and $\mathbf{v}$. It is called an ***eigenvalue problem*** or ***eigenproblem*** or ***eigensystem***. The scalars λ are called eigenvalues, and the vectors $\mathbf{v}$ are called eigenvectors. It will help if we rewrite the equation as

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{1}$$

Clearly $\mathbf{v} = 0$ is always a solution, but isn't interesting or useful for anything; we'll ignore that trivial solution. From the Big Matrix Theorem, we'll have nontrivial solutions $\mathbf{v} \neq 0$ if and only if $A - \lambda I$ is singular, and in turn, if and only if $\det(A - \lambda I) = 0$. Our first job, then, is to solve the equation

$$\det(A - \lambda I) = 0 \tag{2}$$

for the eigenvalue λ. It might not be obvious to you what sort of equation this will be in general, so let's start with an example.

Example 1

Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. We need to solve the equation $\det(A - \lambda I) = 0$ for λ. Compute

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1$$
$$= \lambda^2 - 4\lambda + 3$$

and set to zero,

$$\lambda^2 - 4\lambda + 3 = 0.$$

The quadratic factors easily to $(\lambda - 1)(\lambda - 3)$ so the solutions are $\lambda = 1$ and $\lambda = 3$. These are the eigenvalues.

1.1 Eigenvalues and the characteristic polynomial

So for our two-by-two example the scary looking equation $\det(A - \lambda I) = 0$ turned out to be something quite familiar: a quadratic equation. This will be the case for *any* two-by-two matrix, since

$$\det \begin{pmatrix} A_{11}-\lambda & A_{12} \\ A_{21} & A_{22}-\lambda \end{pmatrix} = (A_{11}-\lambda)(A_{22}-\lambda) - A_{12}A_{21} \tag{3}$$

is always a quadratic polynomial in λ. What about a three-by-three matrix? Compute $\det(A - \lambda I)$ by expansion in minors,

$$\det \begin{pmatrix} A_{11}-\lambda & A_{12} & A_{13} \\ A_{21} & A_{22}-\lambda & A_{23} \\ A_{31} & A_{32} & A_{33}-\lambda \end{pmatrix} =$$
$$= (A_{11}-\lambda)\det \begin{pmatrix} A_{22}-\lambda & A_{23} \\ A_{32} & A_{33}-\lambda \end{pmatrix} - A_{12}\det \begin{pmatrix} A_{21} & A_{23} \\ A_{31} & A_{33}-\lambda \end{pmatrix} +$$
$$+ A_{13}\det \begin{pmatrix} A_{21} & A_{22}-\lambda \\ A_{31} & A_{32} \end{pmatrix}. \tag{4}$$

We've already shown that the first minor determinant is a quadratic, and it's multiplied by a first degree polynomial. The other minor determinants are first degree polynomials, and they're multiplied by constants. Therefore $\det(A - \lambda I)$ is cubic in λ for any three by three matrix A.

Repeat the same argument to show that $\det(A - \lambda I)$ is quartic for four by four A, quintic for five by five A, and so on. We've found the following important result:

Theorem 1 If A is an $N \times N$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree N.

This polynomial is called the ***characteristic polynomial*** of A. The eigenvalues are the roots of the characteristic polynomial. From theorem 1 and the Fundamental Theorem of Algebra we have the important corollary:

Corollary 1 An $N \times N$ matrix A has exactly N eigenvalues (counting multiplicity in the case of multiple roots). The eigenvalues may be complex numbers; if the elements of A are real then any complex eigenvalues must appear as conjugate pairs.

Example 2

Find the eigenvalues of $A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$. The characteristic polynomial is $p(\lambda) = (7-\lambda)\left(\lambda^2 - 4\lambda + 3\right)$ so the eigenvalues are $\lambda = 1$, $\lambda = 3$, and $\lambda = 7$.

Example 3

Find the eigenvalues of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The characteristic polynomial is $p(\lambda) = \lambda^2 + 1$ so the eigenvalues are $\lambda = \pm i$. The matrix is real, but the eigenvalues are complex.

The next example illustrates a nice shortcut for finding the eigenvalues of diagonal matrices.

Example 4

Find the eigenvalues of $A = \begin{pmatrix} 5 & 0 \\ 0 & 20 \end{pmatrix}$. The characteristic polynomial is $p(\lambda) = (5-\lambda)(20-\lambda)$ so the eigenvalues are the diagonal elements $\lambda = 5$ and $\lambda = 20$.

If you think about how the characteristic polynomial is computed for a general triangular matrix you'll quickly find the following useful theorem:

Theorem 2 The eigenvalues of a triangular matrix are the diagonal entries.

Since a diagonal matrix is a special case of a triangular matrix, the eigenvalues of any diagonal matrix are just the diagonal entries.

1.2 Finding eigenspaces and eigenvectors

We now know a procedure for finding the eigenvalues of a matrix: form the characteristic polynomial and then find its roots. We've thereby solved the first part of the eigenvalue problem, finding λ such that

$$(A - \lambda I)\, \mathbf{v} = 0 \tag{5}$$

has nontrivial solutions. The second part of the problem is to find the allowed vectors $\mathbf{v}$. We already know how to do this: all vectors that solve 5 are in the null space of $A - \lambda I$, and we know how to find the null space of a matrix. Be sure to understand that for each

λ we're not finding just one single $\mathbf{v}$, but rather an entire space of vectors, which we'll call the ***eigenspace*** associated with λ. An ***eigenvector*** will be any vector in the eigenspace. If the eigenspace is one dimensional, we can pick any one representative vector in the eigenspace and use it as a basis for that space. In cases where the eigenspace associated with eigenvalue λ has dimension $d > 1$, we'll choose any d linearly independent vectors from the eigenspace and use them as a basis. What we really mean by "find the eigenvectors associated with the eigenvalue λ" is to find a basis for the null space of $A - \lambda I$.

Keep in mind that eigenvectors are never uniquely determined. Even when an eigenspace is one dimensional, any vector in that space is an eigenvector: if $\mathbf{v}$is an eigenvector, then so would be $-\mathbf{v}$, $2\mathbf{v}$, $\sqrt{7}\mathbf{v}$, and so on. So saying we've found "the eigenvector" for an eigenvalue is a little misleading; we've really found an eigenspace and chosen a set of basis vectors for it. We're free to choose any member of the space as "the" eigenvector at our convenience. When the eigenspace has more than one dimension, we can take any basis vectors for that space and call them "the" eigenvectors.

With those preliminaries out of the way, let's work some examples.

Example 5

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

The characteristic polynomial is $p(\lambda) = \lambda^2 + 4\lambda + 3$ so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -3$. First consider $\lambda_1 = -1$: its eigenspace is the null space of

$$A + I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

A basis vector for that space is the $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Next consider $\lambda_2 = -3$: its eigenspace is the null space of

$$A + 3I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

A basis vector for that space is $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$.

Example 6

Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

The characteristic polynomial is $p(\lambda) = \lambda^2 - 5\lambda + 4$ so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 4$. The eigenspace for λ_1 is the null space of

$$A - I = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

for which a basis vector is $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. The eigenspace for λ_2 is the null space of

$$A - 4I = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

for which a basis vector is $\mathbf{v}_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$.

So far, we've seen matrices whose eigenvalues are distinct. Since polynomials can have repeated roots, it's possible to see the same eigenvalue twice. It is in these cases where we really need to keep in mind the idea that eigenvectors are basis vectors for an eigenspace.

Example 7

Find eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is a triangular matrix so we can read the eigenvalues off the diagonal. We have $\lambda_1 = 2$ and $\lambda_2, \lambda_3 = 1$ with multiplicity two. What are the eigenspaces? For λ_1 the eigenspace is the null space of

$$A - 2I = \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

That eigenspace contains all vectors that are multiples of $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. The case $\lambda_2 = \lambda_3 = 1$ is more interesting: the eigenspace is the null space of

$$A - I = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is two dimensional. An obvious choice of basis vectors is $\mathbf{v}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ and $\mathbf{v}_3 = \begin{bmatrix} 3 & -1 & 0 \end{bmatrix}^T$.

In example 1.2eigenvalue of multiplicity two is associated with a two-dimensional eigenspace and so has two eigenvectors. Does an eigenvalue of multiplicity r always have an eigenspace of dimension r and therefore r eigenvectors? No, as we see in the next example.

Example 8

Find eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

As in example 1.2 the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 1$. You can check that an eigenvector for λ_1 is once again $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. The eigenspace for $\lambda_2 = \lambda_3 = 1$ is the null space of

$$A - I = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The null space is one dimensional with basis $\mathbf{v}_2 = \begin{bmatrix} 3 & -1 & 0 \end{bmatrix}^T$. The eigenvalue $\lambda = 1$ has multiplicity two but has a one-dimensional null space and only one eigenvector.

Evidently it's possible to have an eigenspace with dimension smaller than the multiplicity of its associated eigenvalue.

Definition 1 The *algebraic multiplicity* of an eigenvalue is its multiplicity as a root of the characteristic polynomial. The *geometric multiplicity* of an eigenvalue is the dimension of its eigenspace, and hence the number of linearly independent eigenvectors you can find for it.

Definition 2 An eigenvalue whose geometric multiplicity is less than its algebraic multiplicity is called *defective*. A matrix with at least one defective eigenvalue is called a *defective matrix*.

Defective matrices are somewhat tricker to work with than non-defective matrices, for reasons that will become clear in the next section.

1.3 Eigenvectors of a non-defective matrix are linearly independent

1.3.1 Simplest case: distinct eigenvalues

Suppose $A \in \mathbb{R}^{m\times m}$ has m distinct eigenvalues. Let's try to find a nontrivial LC of the eigenvectors that sums to zero: assume there are scalar constants $c_1, \cdots, c_m$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = 0.$$

If we can find such a LC with at least one nonzero coefficient, then the eigenvectors are LD; otherwise they are LI. We'll use a simple (though hardly obvious) trick: assume some nontrivial LC

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = 0$$

and operate from the left with $(A - \lambda_2 I)(A - \lambda_3 I)\cdots(A - \lambda_m I)$, finding

$$(A - \lambda_2 I)(A - \lambda_3 I)\cdots(A - \lambda_m I)(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m) = 0.$$

Notice that all eigenvalues are represented in the product of the $A - \lambda_k I$ matrices except λ_1. Multiplication by $(A - \lambda_k I)$ kills the term in the sum involving $\mathbf{v}_k$, so the surviving expression is

$$(A - \lambda_2 I)\cdots(A - \lambda_m I)\, c_1\mathbf{v}_1 = 0.$$

Now $\mathbf{v}_1$ is an eigenvector so it's not zero. It is also *not* in the null space of any $A - \lambda_k I$ except $k = 1$, so the sequence of matrix-vector multiplications doesn't kill it. The only possibility is $c_1 = 0$. Now repeat, using the operator $(A - \lambda_1 I)(A - \lambda_3 I) \cdots (A - \lambda_m I)$ that leaves out the second eigenvalue, and conclude $c_2 = 0$; continue through the eigenvalues to find *all* the coefficients are zero. Therefore the eigenvectors are linearly independent.

1.3.2 Repeated but non-defective eigenvalues

With repeated but non-defective eigenvalues, the proof of linear independence of the eigenvectors requires bookkeeping on the basis vectors for the multi-dimensional eigenspaces of the repeated eigenvalues. I refer you to a linear algebra textbook for the ugly details; suffice to say that the conclusion is that

- If $\lambda_i \neq \lambda_j$, then the vectors in the eigenspace of λ_i are LI of the vectors in the eigenspace of λ_j.
- If $A \in \mathbb{R}^{m \times m}$ is non-defective, then A has m LI eigenvectors.

1.4 Diagonalization

In this section we're working with a non-defective $m \times m$ matrix A; from the previous section we know that there are m LI eigenvectors. Take the m eigenvectors and glue them together as columns of a matrix V,

$$V = \left[\ \mathbf{v}_1 \,\middle|\, \mathbf{v}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{v}_m\ \right].$$

Now multiply V by A from the left,

$$\begin{aligned} AV &= \left[\ A\mathbf{v}_1 \,\middle|\, A\mathbf{v}_2 \,\middle|\, \cdots \,\middle|\, A\mathbf{v}_m\ \right] \\ &= \left[\ \lambda_1\mathbf{v}_1 \,\middle|\, \lambda_2\mathbf{v}_2 \,\middle|\, \cdots \,\middle|\, \lambda_m\mathbf{v}_m\ \right]. \end{aligned}$$

The result of operating on V with A is to multiply each column of V by the eigenvalue associated with the eigenvector in that column. Now, to scale each column of a matrix by specified factors you multiply that matrix from the *right* by the diagonal matrix with the scale factors on the diagonal (try it); in this case,

$$\left[\ \lambda_1\mathbf{v}_1 \,\middle|\, \lambda_2\mathbf{v}_2 \,\middle|\, \cdots \,\middle|\, \lambda_m\mathbf{v}_m\ \right] = \left[\ \mathbf{v}_1 \,\middle|\, \mathbf{v}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{v}_m\ \right] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}.$$

If we define Λ to be the diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}$$

then we can summarize the previous calculations by the formula

$$AV = V\Lambda.$$

It's very important to remember the order of operations: A multiplies V from the *left*, while Λ multiplies V from the *right*. It matters, because unlike ordinary multiplication between real or complex numbers, matrix multiplication is *not commutative*: in general, $V\Lambda \neq \Lambda V$.

Since we've assumed A is not defective it has a complete set of m LI eigenvectors, so the columns of V are LI, so V is nonsingular, so V^{-1} exists. We can use V to transform from A to Λ or back as follows:

$$\Lambda = V^{-1}AV$$

or

$$A = V\Lambda V^{-1}.$$

A transformation of the form

$$A = SBS^{-1}$$

is called a ***similarity transformation*** between A and B, and any two matrices that can be transformed into one another by means of a similarity transformation are called ***similar matrices***. We write the similarity relation as $A \sim B$, read as "A is similar to B." So when V^{-1} exists, A is similar to the diagonal matrix Λ, and we say that A is ***diagonalizable***. The process of transforming a matrix to diagonal form via similarity transformation is called ***diagonalization.***

The whole procedure of diagonalization depends on the linear independence of the eigenvectors; otherwise, the matrix V isn't invertible. A defective $m \times m$ matrix won't have a full set of LI eigenvectors, so a defective matrix can't be diagonalized, and the eigenvalue factorization

$$A = V\Lambda V^{-1}$$

cannot exist for a defective matrix. For defective matrices, there is a more complicated, harder to compute, and less useful factorization known as the Jordan decomposition. *Every* matrix, defective or not, has a Jordan decomposition; for non-defective matrices the Jordan decomposition simply reduces to the eigenvalue decomposition. Unfortunately the Jordan decomposition is essentially impossible to compute in the presence of roundoff error, so while it is useful in theory it generally can't be used in real-world problems. For that reason, in the main part of these notes we restrict ourselves to non-defective matrices; the Jordan decomposition and its applications are discussed in Appendix C

1.5 Eigenvalues and eigenvectors of A^{-1}

If we know the eigenvalues and eigenvectors of A, it's simple to work out the eigenvalues and eigenvectors of A^{-1}. Suppose λ, $\mathbf{v}$ is an eigenpair of a nonsingular matrix A:

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Then

$$A^{-1}A\mathbf{v} = \lambda A^{-1}\mathbf{v}$$

$$A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$$

so λ^{-1}, $\mathbf{v}$ is an eigenpair of A^{-1}. The eigenvectors of A^{-1} are the same as the eigenvectors of A, while the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

Seen another way, suppose A is diagonalizable and nonsingular. Then

$$A^{-1} = \left(V\Lambda V^{-1}\right)^{-1}.$$

Use the identity $(AB)^{-1} = B^{-1}A^{-1}$ to find

$$\begin{aligned} A^{-1} &= \left(V^{-1}\right)^{-1}\Lambda^{-1}V^{-1} \\ &= V\Lambda^{-1}V^{-1}. \end{aligned}$$

In other words, if A is diagonalizable with V and Λ, then A^{-1} is diagonalizable with the same V but with Λ^{-1} instead of Λ.

1.6 Applications of diagonalization

Diagonalization is useful because it lets us transform a problem to a basis in which the problem is simple. For a warm-up problem I'll show you how to solve

$$A\mathbf{x} = \mathbf{b}$$

by diagonalizing A. Solving $A\mathbf{x} = \mathbf{b}$ by diagonalization is not a good method for solving equations in practice, but understanding how the idea works will help us understand how to solve $\mathbf{y}' = A\mathbf{y}$ by diagonalization. Diagonalization is a *very* useful method for that problem!

1.6.1 Solving $A\mathbf{x} = \mathbf{b}$ via diagonalization

Let A be a diagonalizable matrix with no zero eigenvalues. To solve the system $A\mathbf{x} = \mathbf{b}$, rewrite using the diagonalization of A,

$$V\Lambda V^{-1}\mathbf{x} = \mathbf{b}$$

multiply from the left by V^{-1},

$$\Lambda V^{-1}\mathbf{x} = V^{-1}\mathbf{b}$$

then multiply from the left by Λ^{-1},

$$V^{-1}\mathbf{x} = \Lambda^{-1}V^{-1}\mathbf{b}$$

and finally multiply from the left by V,

$$\mathbf{x} = V\Lambda^{-1}V^{-1}\mathbf{b}.$$

Example 9

Solve $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ using diagonalization. The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ has eigenvalue-eigenvector pairs $\lambda_1 = 3$, $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $\lambda_2 = 1$, $\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. So

$$V = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Compute $V^{-1}\mathbf{b}$ by solving $V\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \underset{\Longrightarrow}{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$y_2 = \frac{1}{2}$$

$$y_1 = 1 - y_2 = \frac{1}{2}$$

so

$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, compute $\Lambda^{-1}V^{-1}\mathbf{b} = \Lambda^{-1}\mathbf{y}$ or

$$\Lambda^{-1}V^{-1}\mathbf{b} = \frac{1}{2} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

The final step is to multiply by V:

$$\mathbf{x} = V\Lambda^{-1}V^{-1}\mathbf{b} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

You should compare to the solution obtained by Gaussian elimination; you'll find the same result, but with much less work.

At this point you're probably thinking that diagonalization is a completely stupid way to solve $A\mathbf{x} = \mathbf{b}$, and as a rule you'd be right. Instead of solving one system, we need to solve an eigenvalue problem and then compute $V^{-1}\mathbf{b}$ which if you recall we do by solving the system $V\mathbf{y} = \mathbf{b}$ for a temporary variable $\mathbf{y}$. But it's usually harder to solve $A\mathbf{v} = \lambda\mathbf{v}$ than to solve $A\mathbf{x} = \mathbf{b}$, and then solving $V\mathbf{y} = \mathbf{b}$ is usually just as expensive as solving $A\mathbf{x} = \mathbf{b}$. So using diagonalization is indeed a stupid method for solving linear systems of equations *unless* the eigenvalue problem is particularly simple *and* something about the eigenvectors makes solving $V\mathbf{y} = \mathbf{b}$ easier than solving the original system. This will almost never be the case with matrix-vector problems; however, certain boundary value problems and partial differential equation will reduce to an abstract problem having a similar structure, and this idea *will* be a practical solution method for those problems. Furthermore, we can use a very similar method to solve systems of ODEs, and again, in that case using diagonalization *is* often more practical than other methods.

Here's what's really going on with solving a system by diagonalization:

1. Computing $V^{-1}\mathbf{b}$ resolves $\mathbf{b}$ into a linear combination of eigenvectors
2. Computing $\Lambda^{-1}\left(V^{-1}\mathbf{b}\right)$ solves the problem in the eigenvector basis
3. Cmoputing $V\left(\Lambda^{-1}V^{-1}\mathbf{b}\right)$ converts the result back into the original basis

This can be summarized in the following diagram:

$$\mathbf{x} = \underbrace{V\left(\underbrace{\Lambda^{-1}\left(\underbrace{V^{-1}\mathbf{b}}_{\text{transform } \mathbf{b} \text{ to ev basis}}\right)}_{\text{solve uncoupled problems in ev basis}}\right)}_{\text{transform back to Cartesian basis}}. \tag{6}$$

Even though this procedure is a completely impractical way to solve $A\mathbf{x} = \mathbf{b}$, remember this general idea: ***transform to a basis where the problem becomes simple, solve the simple problem, then transform back to the original basis.*** We will use that idea throughout this course for problems much more difficult than $A\mathbf{x} = \mathbf{b}$.

1.6.2 Powers of A

Suppose that A is diagonalizable. Then it's easy to integer compute powers of A using the eigenvalue factorization. To compute A^p, write A as $V\Lambda V^{-1}$, and multiply p times,

$$\begin{aligned} A^p &= \underbrace{\left(V\Lambda V^{-1}\right)\left(V\Lambda V^{-1}\right)\cdots\left(V\Lambda V^{-1}\right)}_{p \text{ times}} \\ &= V\Lambda\left(V^{-1}V\right)\Lambda\left(V^{-1}V\right)\cdots\left(V^{-1}V\right)\Lambda V^{-1} \\ &= V\Lambda^p V^{-1}. \end{aligned}$$

Since Λ is diagonal, computing Λ^p is very simple: just take the p-th power of each of the diagonal entries. Computing A^p naively requires $p-1$ matrix-matrix multiplications; computing it with the eigenvalue decomposition takes just two.

1.7 Solving a system of differential equations by diagonalization

Now we come to the first real application of all this eigenstuff: solving a system of differential equations. Again we assume that A is diagonalizable, and write the system

$$\mathbf{y}' = A\mathbf{y}$$

as

$$\mathbf{y}' = V\Lambda V^{-1}\mathbf{y}.$$

Multiply from the left by V^{-1}

$$V^{-1}\mathbf{y}' = \Lambda V^{-1}\mathbf{y}$$

and introduce the temporary variable $\mathbf{z} = V^{-1}\mathbf{y}$. Now the system is

$$\mathbf{z}' = \Lambda\mathbf{z}$$

or

$$\begin{aligned} z_1' &= \lambda_1 z_1 \\ z_2' &= \lambda_2 z_2 \\ &\vdots \\ z_m' &= \lambda_m z_m \end{aligned} \tag{7}$$

which is an *uncoupled* system of equations! We know the solution:

$$\begin{aligned} z_1 &= e^{\lambda_1 t} c_1 \\ z_2 &= e^{\lambda_2 t} c_2 \\ &\vdots \\ z_m &= e^{\lambda_m t} c_m \end{aligned} \tag{8}$$

where the c_j's are constants to be determined from initial conditions. At this point we introduce some notation: for a *diagonal* matrix D, write the matrix

$$e^D = \begin{bmatrix} e^{D_{11}} & & & \\ & e^{D_{22}} & & \\ & & \ddots & \\ & & & e^{D_{nn}} \end{bmatrix}$$

in shorthand as e^D. We'll do a general definition of the matrix exponential later; for now, simply understand e^D as shorthand applicable to diagonal matrices only. With that shorthand, the solution (8) can be written compactly as

$$\mathbf{z} = e^{\Lambda t}\mathbf{c}.$$

Notice that $\mathbf{z}(0) = \mathbf{c}$, so that the coefficients $\mathbf{c}$ are determined easily from the initial conditions *as given in the eigenvector basis.*

Let's put it all together. The solution to the decoupled problem is

$$\mathbf{z}(t) = e^{\Lambda t}\mathbf{z}(0).$$

That's expressed in the eigenvector basis; to express our solution in the original Cartesian basis we multiply by V,

$$\mathbf{y}(t) = V\mathbf{z}(t) = Ve^{\Lambda t}\mathbf{z}(0).$$

Finally, $\mathbf{y}(0)$, the initial value in the original basis, is related to $\mathbf{z}(0)$ by

$$\mathbf{y}(0) = V\mathbf{z}(0)$$

or

$$\mathbf{z}(0) = V^{-1}\mathbf{y}(0)$$

so that the solution is

$$\boxed{\mathbf{y}(t) = Ve^{\Lambda t}V^{-1}\mathbf{y}(0)}.$$

We can summarize the meaning of this formula as follows:

$$\mathbf{y}(t) = V\underbrace{\left(\underbrace{e^{\Lambda t}\left(\underbrace{V^{-1}\mathbf{y}(0)}_{\text{transform initial values to ev basis}}\right)}_{\text{solve uncoupled problems in ev basis}}\right)}_{\text{transform back to Cartesian basis}}.$$

Notice the structural similarity to formula (6) for the solution to $A\mathbf{x} = \mathbf{b}$ through diagonalization. This is because we've used exactly the same approach to both problems: ***transform to a basis where the problem becomes simple, solve the simple problem, then transform back to the original basis***.

Example 10

Solve $\mathbf{y}' = A\mathbf{y}$ by diagonalization, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

From a previous example we know

$$V = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$V^{-1}\mathbf{y}(0) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then $e^{\Lambda t}V^{-1}\mathbf{y}(0)$ is

$$\frac{1}{2}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{3t} \\ e^{t} \end{bmatrix}$$

and the solution is finally

$$\begin{aligned}\mathbf{y}(t) &= Ve^{\Lambda t}V^{-1}\mathbf{y}(0) \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{3t} \\ e^{t} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{3t}+e^{t} \\ -e^{3t}+e^{t} \end{bmatrix}.\end{aligned}$$

We've found a simple formula expressing the solution to $\mathbf{y}' = A\mathbf{y}$, at least for the case where A is diagonalizable. Because we'll never compute V^{-1} explicitly it's best to think of the formula $\mathbf{y}(t) = Ve^{\Lambda t}V^{-1}\mathbf{y}(0)$ not so much as a formula you plug Λ, V, and $\mathbf{y}(0)$ into, but as a shorthand notation to remind us of the steps of a procedure for computing the solution $\mathbf{y}(t)$.

Notice that the rate constants in the solution to $\mathbf{y}' = A\mathbf{y}$ are the eigenvalues of A. So even without going through the full procedure of finding the solution, we get some important information about the solution just by finding the eigenvalues of A.

2 The Hermitian eigenvalue problem and orthogonal diagonalizability

Eigenvalue problems are particularly nice for an important class of matrices that arise frequently in physics and engineering: the Hermitian matrices.

2.1 The transpose of a matrix

Given an $m \times n$ matrix A, the ***transpose*** of A (written A^T) is the matrix obtained by exchanging rows and columns: for example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

The first row in A becomes the first column in A^T, and so on. In terms of components, the i, j entry of A becomes entry j, i in A^T, so that $A_{ij} = \left(A^T\right)_{ji}$.

By considering the entries in a matrix-matrix product,

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj},$$

we can see that transposing a product results in the reversed product of the transposes,

$$(AB)^T = B^T A^T. \tag{9}$$

An operation that will be important in what follows is the dot product with the result of a matrix-vector operation:

$$\mathbf{x}\cdot(A\mathbf{y}) = \mathbf{x}^T A\mathbf{y}.$$

The result is a real number. Take the transpose, and use equation 9 to rewrite,

$$\left(\mathbf{x}^T A\mathbf{y}\right)^T = \mathbf{y}^T A^T \mathbf{x}.$$

2.1.1 The conjugate transpose (adjoint) and the inner product between complex vectors

At this point we need to consider vectors and matrices with complex entries. Recall the the ***complex conjugate*** of a complex number $z = x + iy$ is denoted $\bar{z}$ and is computed by changing the sign of the imaginary part of z: $\bar{z} = x - iy$. The ***absolute value*** or ***magnitude*** of z is

$$|z| = \sqrt{\bar{z}z} = \sqrt{x^2 + y^2}$$

which is the length of z as regarded as a vector in the complex plane.

A useful operation on complex vectors and matrices will be the ***conjugate transpose*** or ***adjoint***, which we define to be the complex conjugate of the transpose (or, equivalently, the transpose of the conjugate; the order is irrelevant). If A is an $m \times n$ complex matrix, its adjoint is

$$A^* = \overline{(A^T)} = \left(\overline{A}\right)^T$$

and the i, j entry in A^* is

$$(A^*)_{ij} = \overline{A}_{ji}.$$

Vectors with complex components add, subtract, and multiply by scalars in the same way as do real vectors, except that complex addition, subtraction, and multiplication is used throughout. The dot product between complex vectors, however, presents a problem. Here's why: Let $\mathbf{a}$ be a complex vector. Then the usual definition of the magnitude of $\mathbf{a}$ as $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ breaks down, because $\mathbf{a} \cdot \mathbf{a}$ can be negative when $\mathbf{a}$ is complex. The definition

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^*\mathbf{a}} = \sqrt{\sum_{j=1}^{n} |a_i|^2}$$

does behave like a proper magnitude; it is defined for all $\mathbf{a}$, and is zero iff $\mathbf{a} = 0$. When working with complex vectors, I'll avoid the notation $\mathbf{a} \cdot \mathbf{b}$ and use either $\mathbf{a}^*\mathbf{b}$ (when conjugation is intended) or $\mathbf{a}^T\mathbf{b}$ (when conjugation is not intended; this case is rare).

We will say that complex vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal when $\mathbf{a}^*\mathbf{b} = 0$.

For complex vectors and matrices, we have the identities

$$(AB)^* = B^* A^*$$

and

$$\overline{(\mathbf{x}^* A\mathbf{y})} = (\mathbf{x}^* A\mathbf{y})^* = \mathbf{y}^* A^* \mathbf{x}.$$

2.1.2 Hermitian and real symmetric matrices

A matrix is called ***Hermitian*** if it is equal to its own adjoint: $A = A^*$. For example, the matrix

$$\begin{bmatrix} 2 & -1 & 4i \\ -1 & 3 & -1 \\ -4i & -1 & 4 \end{bmatrix}$$

is Hermitian. In terms of matrix entries, a matrix is Hermitian if $A_{ij} = \overline{A}_{ij}$ for all i, j.

A real Hermitian matrix is called ***real symmetric***, and is equal to its own transpose.

Two other complex matrix types that sometimes occur are:

- Complex symmetric: $A = A^T$ where A has a nonzero imaginary part. Such a matrix is ***not*** Hermitian.
- Anti-Hermitian: $A = -A^*$.

2.2 Orthonormal matrices

An ***orthonormal*** matrix is a real matrix with the property that its transpose is its inverse: $Q^{-1} = Q^T$. Then it follows that $Q^TQ = QQ^T = I$. For this property to hold, it must be the case that the columns of Q are normalized and mutually orthogonal: if $\mathbf{q}_i$ and $\mathbf{q}_j$ are the i-th and j-th columns of Q, then

$$\mathbf{q}_i^T\mathbf{q}_j = \mathbf{q}_i \cdot \mathbf{q}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

2.3 The spectral theorem

Theorem 3 (Spectral theorem for Hermitian matrices). Let A be a complex matrix such that $A = A^*$. Then all eigenvalues of A are real, and eigenvectors associated with distinct eigenvalues are orthogonal.

Proof. Let $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$ be two eigenvalue-eigenvector pairs for a Hermitian matrix A. We can all agree with the statement that

$$\mathbf{v}_1^* A \mathbf{v}_2 - \mathbf{v}_1^* A \mathbf{v}_2 = 0. \tag{10}$$

Now let' s rearrange. The second term is $\mathbf{v}_1^* A \mathbf{v}_2 = (\mathbf{v}_2^* A^* \mathbf{v}_1)^*$ which, because A is Hermitian, also equal to $(\mathbf{v}_2^* A \mathbf{v}_1)^*$. Since the term is a scalar, its adjoint is simply its conjugate, so that $\mathbf{v}_1^* A \mathbf{v}_2 = \overline{(\mathbf{v}_2^* A \mathbf{v}_1)}$. Therefore equation 10 is

$$\mathbf{v}_1^* A \mathbf{v}_2 - \overline{(\mathbf{v}_2^* A \mathbf{v}_1)} = 0.$$

Next, use the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors to rewrite the above as

$$\mathbf{v}_1^* \lambda_2 \mathbf{v}_2 - \overline{(\mathbf{v}_2^* \lambda_1 \mathbf{v}_1)} = 0$$

$$\lambda_2 \left(\mathbf{v}_1^* \mathbf{v}_2 \right) - \overline{\lambda_1} \overline{\left(\mathbf{v}_2^* \mathbf{v}_1 \right)} = 0$$

so that, finally,

$$\left(\mathbf{v}_1^* \mathbf{v}_2 \right) \left(\lambda_2 - \overline{\lambda_1} \right) = 0. \tag{11}$$

We're done with calculating; it's time to think. Suppose the two eigenpairs are identical: $\mathbf{v}_1 = \mathbf{v}_2$ and $\lambda_1 = \lambda_2$, so that the equation becomes

$$\left(\mathbf{v}_1^* \mathbf{v}_1 \right) \left(\lambda_1 - \overline{\lambda_1} \right) = 0$$

$$\left\| \mathbf{v}_1 \right\|^2 \left(\lambda_1 - \overline{\lambda_1} \right) = 0.$$

Since $\|\mathbf{v}_1\|$ cannot be zero for an eigenvector (nonzero by definition), we must have $\lambda_1 = \overline{\lambda}_1$. Therefore λ_1 is real. Since we've chosen λ_1 arbitrarily, this conclusion must hold for every eigenvalue of A.

Next, suppose that $\lambda_1 \neq \lambda_2$. The factor $\lambda_2 - \overline{\lambda_1}$ is $\lambda_2 - \lambda_1$ in equation 11 because all eigenvalues are real, and is nonzero by the supposition that $\lambda_1 \neq \lambda_2$. Therefore, for equation 11 to hold we must have $\mathbf{v}_1^* \mathbf{v}_2 = 0$. □

2.4 Simultaneous diagonalizability and commutivity

2.4.1 An interlude in quantum mechanics

3 The exponential of a matrix

3.1 Series definition of e^A

The matrix exponential is defined by the infinite series

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \cdots = \sum_{j=0}^{\infty} \frac{A^j}{j!}.$$

3.1.1 The derivative of e^{At}

Now consider the exponential of a matrix A that is multiplied by a real parameter t. The function e^{At} is defined by the series

$$e^{At} = I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{6} + \cdots = \sum_{j=0}^{\infty} \frac{A^j t^j}{j!}.$$

Differentiate by t term-by-term to find

$$\begin{aligned} \frac{d}{dt}\left(e^{At} \right) &= \sum_{j=0}^{\infty} \frac{j A^j t^{j-1}}{j!} = A \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} \\ &= A e^{At}. \end{aligned}$$

Notice that it doesn't matter whether you write Ae^{At} or $e^{At}A$: because A commutes with every power of A, it also commutes with e^{At}.

3.1.2 $e^A e^B$ is not always e^{A+B}

Warning: not every property of the ordinary real (or complex) exponential carries over to the matrix exponential. In particular, the identity

$$e^a e^b = e^{a+b}$$

does ***NOT*** generalize to

$$e^A e^B = e^{A+B}.$$

To see why, compute the LHS by multiplying out the product of the two series,

$$\begin{aligned} e^A e^B &= \left(I + A + \frac{A^2}{2} + \cdots\right)\left(I + B + \frac{B^2}{2} + \cdots\right) \\ &= \left(I + (A+B) + AB + \frac{A^2}{2} + \frac{B^2}{2} + \cdots\right) \end{aligned}$$

and compute the RHS with the exponential series in $A + B$,

$$e^{A+B} = I + (A+B) + \frac{1}{2}\left(A^2 + AB + BA + B^2\right) + \cdots .$$

The series are equal through first order terms, but the second order terms are equal only in the case where

$$AB = \frac{1}{2}\left(AB + BA\right),$$

that is, only when $AB = BA$. Since will not be true in general, we have that in general

$$e^A e^B \neq e^{A+B}.$$

To prove the two sides were *not* equal I only needed to go to order 2 in the series. To determine when the series *are* equal is more difficult; surprisingly, it all comes down to whether the two matrices commute. I will state without proof the following theorem:

Theorem 4 The matrix exponential $\exp(A+B)$ is equal to $e^A e^B$ if and only if $AB = BA$.

3.1.3 The inverse of a matrix exponential

The inverse of a matrix exponential e^A is exactly what you'd expect it to be: e^{-A}. Since A and $-A$ commute, we have

$$\begin{aligned} e^A e^{-A} &= e^{A-A} \\ &= I. \end{aligned}$$

3.2 Calculation of e^{At} by factorization

When A is diagonalizable (*i.e.*, has an eigenvalue factorization) we can apply that factorization to compute all powers appearing in the series, and find

$$e^A = V e^{\Lambda} V^{-1}.$$

3.2.1 The exponential of a defective matrix

When A is defective, we must use the Jordan decomposition, in which case

$$e^A = Ve^J V^{-1}.$$

Warning: this formula is ***not*** suitable for practical calculations.

4 The Laplace transform of e^{At} and applications

Let's just plug the matrix exponential into the Laplace transform formula and see what happens:

$$\mathcal{L}\left\{e^{At}\right\}(s) = \int_0^\infty e^{At} e^{-st}\, dt.$$

The first question is what to do with $e^{At}e^{st}$, a product of a matrix exponential with an ordinary scalar exponential. I'll work through the steps in detail to help you get comfortable with working with the matrix exponential. Suppose A has Jordan decomposition $A = VJV^{-1}$. Then $e^{At} = Ve^{Jt}V^{-1}$ and

$$\begin{aligned}
e^{At}e^{st} &= Ve^{Jt}V^{-1}e^{-st} \\
&= V\left(e^{Jt}e^{-st}\right)V^{-1} \quad \text{(since } e^{-st} \text{ is just a scalar)} \\
&= V\left(e^{Jt}e^{-Ist}\right)V^{-1} \quad \text{(since } Ie^{-st} = e^{-Ist}\text{)} \\
&= e^{(J-Is)t}.
\end{aligned}$$

With that done, the Laplace transform simplifies to

$$\mathcal{L}\left\{e^{At}\right\}(s) = \int_0^\infty e^{(A-Is)t}\, dt.$$

We have to do the integral, which is easy, working backwards from the derivative $\frac{d}{dt}\left[e^{At}\right] = Ae^{At}$,

$$\int_0^\infty e^{(A-Is)t}\, dt = (A - Is)^{-1} e^{(A-Is)t}\Big|_0^\infty.$$

Clearly we're in trouble if $(A - Is)$ isn't invertible; we'll look at that question in more detail later. We also have to worry about evaluation at the limits. Evaluation at the lower limit is easy: the exponential of zero is just the identity. The upper limit takes a little more thought. Intuitively, we'd say that $e^{(A-Is)t} \to 0$ as $t \to \infty$ if $A - Is$ is "negative", and blows up if $A - Is$ is "positive". But what does it even mean for a *matrix* to be "positive" or "negative"? So we need to be careful and work it through systematically. To keep it simple, let's assume that A isn't defective, and compute

$$e^{(A-Is)t} = Ve^{(\Lambda-Is)t}V^{-1}$$

$$= V \begin{pmatrix} e^{(\lambda_1 - s)t} & 0 & \\ 0 & e^{(\lambda_2 - s)t} & \\ & & \ddots \end{pmatrix} V^{-1}.$$

This will blow up as $t \to \infty$ if *any* of the exponents $\lambda_j - s$ is positive. It will go to the zero matrix if $s > \lambda_j$ for all j. Since with Laplace transforms we're often working in the complex plane, we have to be more precise and say that $\Re(s) > \Re(\lambda_i)$ for all j. I'll leave it to you to show that the same conclusion holds if A is defective, in which case we have to use the Jordan form $A = VJV^{-1}$ instead of the simpler diagonalization $A = V\Lambda V^{-1}$.

What if s is exactly equal to one of the eigenvalues, so that one of the diagonal entries is zero? Suppose that s is an eigenvalue. Then $\det(A - Is) = 0$, in which case $A - Is$ is singular. But remember the factor of $(A - Is)^{-1}$ that appeared when we integrated? That can't exist if $A - Is$ is singular, *i.e.,* if s is an eigenvalue of A.

Putting it all together, we can conclude that

$$\mathcal{L}\left\{e^{At}\right\}(s) = (Is - A)^{-1}$$

provided that

- $\Re(s)$ is greater than $\Re(\lambda)$ for all eigenvalues λ
- s is not an eigenvalue of A.

When s is an eigenvalue of A, the Laplace transform $\mathcal{L}\{e^{At}\}(s)$ has a pole (repeated eigenvalues have a pole of order equal to the multiplicity of the eigenvalue). If you've had a course in stability theory, you'll recall that the stability of a linear system can be described in terms of the locations of the poles of the Laplace transform. But the poles are just the eigenvalues of the system matrix A: linear stability theory is all about eigenvalues.

For comparison, recall the Laplace transform for the ordinary complex exponential,

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{(s - a)}$$

provided $\Re(s) > \Re(a)$.

The Laplace transforms for the matrix exponential and the ordinary exponential look very similar, but keep in mind the important distinction: the Laplace transform of e^{at} has poles where $s = a$, whereas the Laplace transform of e^{At} has poles where s is an *eigenvalue* of A, that is, where $A - Is$ is singular.

Laplace transform of a matrix exponential

$$\mathcal{L}\left\{e^{At}\right\} s = (Is - A)^{-1}$$

provided that

- $\Re(s)$ is greater than $\Re(\lambda)$ for all eigenvalues λ
- s is not an eigenvalue of A.

4.1 Solving $\mathbf{y}' = A\mathbf{y} + \mathbf{f}$ with Laplace transforms

The procedure for solving systems of equations with Laplace transforms is quite similar to that for solving single equations. Consider

$$\mathbf{y}' - A\mathbf{y} = \mathbf{f}(t)$$

$$\mathbf{y}(0) = \mathbf{y}_0$$

where A is an $n \times n$ matrix, $\mathbf{y}(t)$ an unknown vector-valued function that maps $\mathbb{R} \to \mathbb{R}^n$, $\mathbf{f}(t)$ a specifed vector valued function that maps $\mathbb{R} \to \mathbb{R}^n$, and $\mathbf{y}_0$ a speficied constant vector in $\mathbb{R}^n$.

The Laplace transform of $\mathbf{f}(t)$ is $\mathbf{F}(s)$, which is a vector-valued function mapping $\mathbb{R} \to \mathbb{R}^n$. Each component of $\mathbf{F}$ is simply the Laplace transform of the corresponding component of $\mathbf{f}$. For example, the vector-valued function $\begin{bmatrix} e^{-2t} & t & \sin(3t) \end{bmatrix}^T$ has the Laplace transform

$$\begin{pmatrix} e^{-2t} \\ t \\ \sin(3t) \end{pmatrix} \leftrightarrow \begin{pmatrix} \frac{1}{s+2} \\ \frac{1}{s^2} \\ \frac{3}{s^2+9} \end{pmatrix}.$$

Now that we know how to take the Laplace transform of a vector, let's solve the equation. Suppose $\mathbf{y}(t) \leftrightarrow \mathbf{Y}(s)$. Take Laplace transforms of both sides,

$$s\mathbf{Y}(s) - \mathbf{y}_0 - A\mathbf{Y}(s) = \mathbf{F}(s)$$

then solve for $\mathbf{Y}$,

$$\mathbf{Y} = (Is - A)^{-1}(\mathbf{y}_0 + \mathbf{F}(s)).$$

That's a pretty formula, but it's maybe not clear what you actually *do* with it. So let's do a specific example,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\mathbf{y}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathbf{f}(t) = \begin{pmatrix} et^{-2t} \\ t \end{pmatrix}.$$

Then the solution for $\mathbf{Y}(s)$ is

$$Y(s) = (Is - A)^{-1} \begin{pmatrix} 1 + \frac{1}{s+2} \\ 2 + \frac{1}{s^2} \end{pmatrix}$$

$$= \begin{pmatrix} s-2 & 1 \\ 1 & s-2 \end{pmatrix}^{-1} \begin{pmatrix} 1 + \frac{1}{s+2} \\ 2 + \frac{1}{s^2} \end{pmatrix}.$$

We know not to form the inverse; the use of inverse notation means to solve the linear system of equations using, for instance, Gaussian elimination. The calculations get messy so this is almost always better done by a symbolic math program.

Notice that when $\mathbf{f}(t) = 0$ we just have

$$\mathbf{Y}(s) = (Is - A)^{-1} \mathbf{y}_0.$$

But we know that $(Is - A)^{-1}$ is the Laplace transform of e^{At}, so the solution is

$$\mathbf{y}(t) = e^{At}\mathbf{y}_0.$$

The Laplace transform method carries over from single equations to systems of equations very cleanly. The calculations get messy, but that's what computers are for.

A Important notation

A.1 Abbreviations

- ew – eigenvalue (from German *Eigenwert*)
- ev – eigenvector (from German *Eigenvektor*)
- iff – "if and only if"
- LC – linear combination
- LD – linearly dependent
- LI – linearly independent

A.2 Mathematical notation

- $\in$ – Membership in a set. For example, $x \in \mathbb{R}$ means x is a member of the set $\mathbb{R}$
- $\forall$ – For all
- $\exists$ – There exists
- A^{-1} – The inverse of the matrix A
- A^T – The transpose of the matrix A
- A^{-T} – The inverse transpose of the matrix A
- A^* – The adjoint (conjugate transpose) of the matrix A
- A^{-*} – The inverse adjoint of the matrix A
- $\mathbb{C}$ – The set of complex numbers
- $\mathbb{C}^n$ – The set of complex vectors of length n
- $\mathbb{C}^{m \times n}$ – The set of complex $m \times n$ matrices

- $\mathbb{N}_0$ – The set of natural ("counting") numbers, starting with zero
- $\mathbb{N}_1$ – The set of natural ("counting") numbers, starting with one
- $\mathbb{R}$ – The set of real numbers
- $\mathbb{R}^n$ – The set of real vectors of length n
- $\mathbb{R}^{m\times n}$ – The set of real $m \times n$ matrices
- $\mathbb{Z}$ – The set of integers (from the German *Zahlen*)
- $\bar{z}$ – complex conjugate of z. For a and b real, the conjugate of $z = a + ib$ is $a - ib$.

A.3 Summary of important facts

A.3.1 Eigensystems

- The eigenvalues λ of the $m \times m$ matrix A are the roots of the degree m polynomial $\det(A - \lambda I)$.
  - An $m \times m$ matrix has m eigenvalues (counting multiplicity)
  - The eigenvalues of a real matrix will be real or complex conjugate pairs
- The eigenvector(s) corresponding to λ are in the null space of $(A - \lambda I)$
- Eigenvectors corresponding to distinct eigenvalues are linearly independent
- If $A \in \mathbb{C}^{m\times m}$ has m distinct eigenvalues, then its m eigenvectors are linearly independent. Consequently, the matrix V whose columns are the eigenvectors is nonsingular.
- An eigenvalue of algebraic multiplicity m_a will have an eigenspace of dimension $m_g \leq m_a$. If the "geometric multiplicity" m_g is less than the algebraic multiplicity m_a, then the eigenvalue is called defective.
  - A matrix with one or more defective eigenvalues cannot be diagonalized.
  - A non-defective matrix can be diagonalized.
- The eigenvectors of an Hermitian matrix are orthogonal, and the eigenvalues are real.
  - An Hermitian matrix A is orthogonally diagonalizable: $A = Q\Lambda Q^*$.

B Practical algorithms for solving eigensystems

It's very important to understand that the procedure shown in 1.1 and 1.2 is **not** a practical algorithm for solving real-world eigenvalue problems. Unless the matrix is small or has a simple structure (such as a diagonal matrix) it is surprisingly expensive even to form the characteristic polynomial, and then finding its roots accurately can be quite difficult[1].

So how are eigensystems really solved? It's complicated; to quote from the excellent textbook *Numerical Linear Algebra* by L. N. Trefethen and D. Bau: "*...the best algorithms for finding eigenvalues are powerful, yet particularly far from obvious.*" The details are intricate and the methods are ingenious, but speaking generally the most commonly used algorithms are of one of two types.

- For "small" eigenvalue problems ($N \lesssim 10000$ or so) one constructs a sequence of similarity transformations that converge to a triangular matrix. Recall that similarity transformations leave the eigenvalues unchanged, and the eigenvalues can then be read off as the diagonal elements of the result. The eigenvectors can be constructed from the composition of all the similarity transformations used.
- For very large matrices, we can project the eigenvalue problem onto a lower-dimensional space and solve the projected problem (which is small enough to use simpler methods). The trick is to construct a lower dimensional space that nearly contains the most important eigenvectors; this is done iteratively.

You can trust that the Mathematica functions `Eigenvalues`, `Eigenvectors`, and `Eigensystem` will solve eigenvalue problems using reliable algorithms.

C Defective matrices and the Jordan decomposition

Whether or not a matrix is defective will determine whether or not it's possible to decouple fully a system of differential equations. If A is not defective, then we'll be able to decouple $\mathbf{y}' = A\mathbf{y}$. If it is defective, the system can't be fully decoupled; we'll see that we can *almost* decouple it, and we'll find out how to construct a solution with a little more work than needed for a non-defective system.

We will start our study of non-diagonalizable matrices with a simple but extremely important example. A **Jordan block** is a matrix with the structure

$$J_\lambda = \begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \lambda & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{pmatrix}.$$

[1]Recall that there is no exact formula for the roots of general polynomials above degree four, and even the cubic and quartic formulas can be difficult to use in practice.

All entries on the main diagonal are λ, all entries on the first diagonal above the main are 1, and all other entries are zero.

All eigenvalues of J_λ are λ, with algebraic multiplicity equal to the size of the Jordan block. There is only one eigenvector: $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T$. With only one eigenvector, the eigenspace is one dimensional and the geometric multiplicity of the eigenvector is one. Therefore, J_λ is defective whenever it's larger than 1×1, and it is not diagonalizable.

We will see later that even when a matrix is not diagonalizable, we can always find a similary transformation to a matrix built out of Jordan blocks. The reason this is useful is that differential equations $\mathbf{y}' = J_\lambda \mathbf{y}$ can be solved by a procedure similar to backsubstitution for solving algebraic equations. This isn't quite as nice as working with a diagonal system, but it will work.

C.1 "Almost-diagonalization" with Jordan blocks: the Jordan decomposition

We can compute matrix exponentials (and hence solve differential equation) with Jordan blocks. The next step is to see how to express a defective matrix in terms of Jordan blocks.

Suppose a matrix has non-defective eigenvalues λ_1, $\lambda_2, \cdots, \lambda_{p-1}$, and then defective eigenvalues λ_p, $\lambda_{p+1}, \cdots$ with multiplicities m_p, m_{p+1}, $\cdots$. Form the diagonal matrix Λ with $\lambda_1, \cdots, \lambda_{p-1}$ on the diagonal. For each of the defective eigenvalues λ_i, form an $m_i \times m_i$ Jordan block J_{λ_i}. Then pack them into a *block-diagonal* matrix

$$J = \begin{pmatrix} \Lambda & 0 & 0 & & \\ 0 & J_{\lambda_p} & 0 & & \\ 0 & 0 & J_{\lambda_{p+1}} & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix}.$$

This is called the Jordan matrix. Alternatively, considering each non-defective eigenvalue λ_i to have a one-by-one Jordan block $[\lambda_i]$, we can write the Jordan matrix as

$$J = \begin{pmatrix} J_{\lambda_1} & & & & & \\ & J_{\lambda_2} & & & & \\ & & \ddots & & & \\ & & & J_{\lambda_p} & & \\ & & & & J_{\lambda_{p+1}} & \\ & & & & & \ddots \end{pmatrix}.$$

Example 11

Suppose A has non-defective eigenvalues $1, 2, 2, 3$ and defective eigenvalues

4, 4, 5, 5, 5. Then the Jordan matrix is

$$J = \begin{pmatrix} 1 & & & & & & & & \\ & 2 & & & & & & & \\ & & 2 & & & & & & \\ & & & 3 & & & & & \\ & & & & 4 & 1 & & & \\ & & & & & 4 & & & \\ & & & & & & 5 & 1 & \\ & & & & & & & 5 & 1 \\ & & & & & & & & 5 \end{pmatrix}.$$

When a matrix A is defective, we can't find an invertible V such that $A = V\Lambda V^{-1}$. But we can find the next best thing: an invertible V such that $A = VJV^{-1}$. This is called the Jordan decomposition, Jordan form, or Jordan canonical form. A proof of the existence of the Jordan decomposition is beyond the scope of this course.

C.1.1 Computing the Jordan decomposition

The matrix V in a Jordan form will satisfy the equation

$$AV = VJ.$$

We can always form J once we've found the eigenvalues and the geometric and algebraic multiplicities.

C.1.2 Numerical calculation of the Jordan form

The Jordan form is theoretically very useful. Unfortunately, it's difficult to compute except with *exact* arithmetic, so in any calculation done with floating point numbers on a computer it's effectively impossible to use. Here's why: the system

$$(A - \lambda I)\,\mathbf{v}_i = \mathbf{v}_j$$

involves a singular matrix (since $\det(A - \lambda I) = 0$ when λ is an eigenvalue). The *slightest* error during the solution of this system – such as the inevitable rounding-off in a computer calculation – can ruin the result.

C.2 Differential equations involving Jordan blocks

Consider the system of differential equations

$$\mathbf{y}' = J_\lambda \mathbf{y}$$

where J_λ is an $n \times n$ Jordan block. Because J_λ is upper triangular, we can solve the system first for $y_n(t)$, then $y_{n-1}(t)$, and so on, analogous to using backsubstitution for solving an upper triangular algebraic system.

Consider the 3×3 case

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

We work from bottom up, each step requiring the solution of a single ODE. The equation for y_3 is

$$y_3' = \lambda y_3$$

and has solution

$$y_3(t) = c_3 e^{\lambda t}$$

where c_3 is to be determined from the initial conditions.

On to the next row: The equation for y_2 is

$$\begin{aligned} y_2' &= \lambda y_2 + y_3 \\ &= \lambda y_2 + c_3 e^{\lambda t}. \end{aligned}$$

This is now an inhomogeneous equation, but notice that the rate constant in the exponential in the inhomogeneous term is the same as the coefficient of the unknown. Physically, this corresponds to a resonance condition. Recall from basic differential equations that in such cases you need to use a particular solution of the form

$$y_p = Ate^{\lambda t}$$

so that y_2 will not be linearly dependent on y_3. Plug this particular solution into the equation to determine A (method of undetermined coefficients),

$$\underbrace{Ae^{\lambda t} + At\lambda e^{\lambda t}}_{y_p'} = \underbrace{At\lambda e^{\lambda t} + c_3 e^{\lambda t}}_{\lambda y_p + c_3 e^{\lambda t}}$$

with the result that $A = c_3$ and $y_p = c_3 te^{\lambda t}$. Now we can always add to y_p any solution to the homogeneous equation, so the general solution for y_2 is

$$y_2(t) = c_2 e^{\lambda t} + c_3 te^{\lambda t}.$$

The constant c_2 is to be determined from initial conditions. The constant c_3 is the same constant that appears in y_3 (since it came from y_3's appearance as an inhomogeneous term in the equation for y_2).

Keep going. The equation for y_1 is

$$y_1' = \lambda y_1 + c_3 te^{\lambda t} + c_2 e^{\lambda t}.$$

The particular solution has the form

$$y_p = At^2 e^{\lambda t} + Bte^{\lambda t}.$$

Use the method of undetermined coefficients to find A:

$$\underbrace{2Ate^{\lambda t} + A\lambda t^2 e^{\lambda t} + Be^{\lambda t} + B\lambda te^{\lambda t}}_{y_p'} = \underbrace{A\lambda t^2 e^{\lambda t} + B\lambda te^{\lambda t} + c_3 te^{\lambda t} + c_2 e^{\lambda t}}_{\lambda y_p + y_2}$$

$$\begin{cases} 2A + \lambda B & = B\lambda + c_3 \\ B & = c_2 \end{cases}$$

so

$$y_p = \left(\frac{1}{2}c_3 t^2 + c_2 t\right) e^{\lambda t}$$

and after adding a solution to the homogeneous equation,

$$y_1(t) = \left(c_1 + c_2 t + \frac{1}{2}c_3 t^3\right) e^{\lambda t}.$$

Collect together the solutions for y_1, y_2, y_3 in vector form:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 t + \frac{1}{2}c_3 t^2 \\ c_2 + c_3 t \\ c_3 \end{pmatrix} e^{\lambda t}.$$

It's worth noticing that if we form the matrix

$$\begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

then the solution can be written even more cleanly as

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

C.3 The general solution to $\mathbf{y}' = J_\lambda \mathbf{y}$

When J_λ is a Jordan block of size $m \times m$, you can find the solution by carrying out more steps of the backsubstitution procedure used in the 3×3 case: Solve the homogeneous equation

$$y_m' = \lambda y_m$$

to find

$$y_m(t) = c_m e^{\lambda t}.$$

Thereafter, for $i = m-1, m-2, \cdots, 1$, solve the inhomogeneous system

$$y_i' = \lambda y_i + y_{i-1},$$

taking care to use the correct form for the particular solution at each stage. What you will find is that the general solution is

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{m-1} \\ y_m \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{m-2}}{(m-2)!} & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{m-3}}{(m-3)!} & \frac{t^{m-3}}{(m-3)!} \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & 1 & t \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{m-1} \\ c_m \end{pmatrix}.$$

The i,j entry in the matrix above is $\frac{t^{j-i}}{(j-i)!}$ for $j \geq i$ (diagonal or above), and zero for any $j < i$ (below diagonal). If you understand mathematical induction, it's not hard to set up an inductive proof of this general formula.

C.4 The matrix exponential of a Jordan block

Suppose we evaluate $e^{J_\lambda t}$ using the series definition of the matrix exponential

$$e^{J_\lambda t} = I + J_\lambda t + \frac{1}{2} J_\lambda^2 t^2 + \frac{1}{6} J_\lambda^3 t^3 + \cdots.$$

I'll work out a few terms explicitly for the 3×3 case:

$$I + J_\lambda t = \begin{pmatrix} 1+\lambda t & t & 0 \\ 0 & 1+\lambda t & t \\ 0 & 0 & 1+\lambda t \end{pmatrix}$$

$$I + J_\lambda t + \frac{1}{2} J_\lambda^2 t^2 = \begin{pmatrix} 1+\lambda t + \frac{1}{2}\lambda t^2 & t\,(1+\lambda t) & \frac{t^2}{2} \\ 0 & 1+\lambda t + \frac{1}{2}\lambda^2 t^2 & t\,(1+\lambda t) \\ 0 & 0 & 1+\lambda t + \frac{1}{2}\lambda^2 t^2 \end{pmatrix}$$

$$I + J_\lambda t + \frac{1}{2} J_\lambda^2 t^2 + \frac{1}{6} J_\lambda^3 t^3 = \begin{pmatrix} 1+\lambda t + \frac{1}{2}\lambda t^2 + \frac{1}{6}\lambda^3 t^3 & t\left(1+\lambda t + \frac{1}{2}\lambda^2 t^2\right) & \frac{t^2}{2}\,(1+\lambda t) \\ 0 & 1+\lambda t + \frac{1}{2}\lambda^2 t^2 + \frac{1}{6}\lambda^3 t^3 & t\left(1+\lambda t + \frac{1}{2}\lambda^2 t^2\right) \\ 0 & 0 & 1+\lambda t + \frac{1}{2}\lambda^2 t^2 + \frac{1}{6}\lambda^3 t^3 \end{pmatrix}$$

See the pattern? The diagonal entries are going towards the MacLaurin series for $e^{\lambda t}$. The off-diagonals are going to $e^{\lambda t}$ times t^{j-i} divided by $(j-i)!$, precisely $e^{\lambda t}$ times the funny matrix we found while solving the differential equation. Again, to prove this rigorously you would use induction. We can recognize that general solution to $\mathbf{y}' = J_\lambda \mathbf{y}$ is expressed in terms of the matrix exponential,

$$\mathbf{y}\,(t) = e^{J_\lambda t}\mathbf{c}$$

where $\mathbf{c}$ is to be determined from the initial conditions.

The matrix exponential isn't quite as simple for a Jordan block as it is for a diagonal matrix; there are the odd off-diagonal terms involving powers of t. But it's still a simple, easily remembered pattern: the i,j entry of $e^{J_\lambda t}$ is

$$\left(e^{J_\lambda t}\right)_{i,j} = \begin{cases} \frac{t^{j-i} e^{\lambda t}}{(j-i)!} & j \geq i \quad \text{(diagonal and above)} \\ 0 & j < i \quad \text{(below diagonal)} \end{cases}.$$

We now know how to solve systems of ODEs where the coefficient matrix is a Jordan block as well as when the coefficient is diagonalizable. The next step is to see how to transform systems involving non-diagonalizable matrices into systems involving Jordan blocks.

D Further reading

- There are many introductory linear algebra textbooks. One of the better ones is [**?**]. For a concise and inexpensive introduction to linear algebra, it's hard to go wrong with the Schaum's Outline, [**?**].
- A linear algebra textbook aimed specifically at applications to differential equations is [**?**].
- For more information on computing with matrices, see [**?**], or the more advanced [**?**].
- Introductory books for the more mathematically mature[2] student are [**?**] and [**?**].

References

[2]"Mathematically mature" means "totally comfortable working with proofs"