1 Integration

1.1 Line, area, and volume elements

1.2 Line integrals

1.3 Area, surface, and volume integrals

1.4 Some integration tricks

1.4.1 Poisson's Gaussian integral trick

The integral $G = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ appears frequently in applications, especially in statistical physics and quantum physics. The indefinite integral can't be evaluated by any standard methods; Liouville proved that the function e^{-x^2} has no closed-form antiderivative. No clever substitution will get you there. However, the *definite* integral of e^{-x^2} over the real line can be computed exactly by means of a diabolically clever trick by Poisson.[1]

We'll follow Poisson by squaring G:

$$G^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right).$$

In a definite integral the variable is a dummy; in the second copy of the integral we replace x by y,

$$G^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right).$$

But this is equal to the double integral

$$G^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy = \int_{\mathbb{R}^2} e^{-\left(x^2+y^2\right)}\,dx\,dy.$$

[1] The value of the Gaussian integral was first obtained by Laplace, using an equally clever but less simple trick. I'll teach you Poisson's trick because it's simpler. For more about the history of this integral and a few other tricks for its evaluation, refer to https://www.york.ac.uk/depts/maths/histstat/normal_history.pdf. My knowledge of French is next to nothing, but even I can understand Sturm's comment about the method: "*...déterminée par M. Poisson à l'aide d'un procédé très-remarquable*."

Change to polar coordinates,

$$G^2 = \int_0^{2\pi} \int_0^\infty h_\rho h_\phi e^{-\rho^2} \, d\rho \, d\phi = \int_0^{2\pi} \int_0^\infty \rho e^{-\rho^2} \, d\rho \, d\phi = 2\pi \int_0^\infty \rho e^{-\rho^2} \, d\rho.$$

That last integral can be done with the substitution $u = \rho^2$, $du = 2\rho \, d\rho$, after which we have

$$G^2 = \pi \int_0^1 e^{-u} \, du = \pi.$$

Take the square root to find $G = \sqrt{\pi}$. We've done it:

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$

Beautiful!

Problem 1 Show that $\int_{-\infty}^{\infty} e^{-\alpha x^2} \, dx = \sqrt{\frac{\pi}{\alpha}}$.

Problem 2 Explain why $\int_{-\infty}^{\infty} x^n e^{-x^2} \, dx$ is zero for all odd n.

Problem 3 Use completion of the square to evaluate $\int_{-\infty}^{\infty} e^{\alpha x} e^{-\beta x^2} \, dx$.

Problem 4 In a system in thermodynamic equilibrium at temperature T, the probability of finding a particle with energy in the interval $(E, E + dE)$ is $p(E) \, dE = Ce^{-\beta E} g(E) \, dE$, where $\beta = 1/k_B T$, $g(E) \, dE$ is proportional the number of possible states having energy in the range $(E, E + dE)$, and C is a normalization constant. The normalization constant will be fixed by the condition

$$\int p(E) \, dE = 1.$$

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1.4.1.1 Multidimensional Gaussian integrals

The Gaussian function of $\|\mathbf{r}\|$,

$$f(\mathbf{r}) = e^{-\alpha \|\mathbf{r}\|^2} = e^{-\alpha \mathbf{r} \cdot \mathbf{r}},$$

can be evaluated in any number of dimensions. Let's work in $\mathbb{R}^n$, in which case

$$f(\mathbf{r}) = e^{-\alpha\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)} = e^{-\alpha x_1^2} e^{-\alpha x_2^2} \cdots e^{-\alpha x_n^2}.$$

Integrate over all space:

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\alpha \mathbf{r} \cdot \mathbf{r}} d^n \mathbf{r} &= \left(\int_{-\infty}^{\infty} e^{-\alpha x_1^2} \, dx_1\right) \left(\int_{-\infty}^{\infty} e^{-\alpha x_2^2} \, dx_2\right) \cdots \left(\int_{-\infty}^{\infty} e^{-\alpha x_n^2} \, dx_n\right) \\ &= \left(\sqrt{\frac{\pi}{\alpha}}\right)^n. \end{aligned}$$

Let's generalize slightly, by letting the surfaces of constant f be hyperellipsoids rather than hyperspheres:

$$f(\mathbf{r}) = e^{-\alpha_1 x_1^2} e^{-\alpha_2 x_2^2} \cdots e^{-\alpha_n x_2^2}.$$

Use the reasoning above to show that

$$\int_{\mathbb{R}^n} f(\mathbf{r}) \, d^n \mathbf{r} = \frac{\pi^{n/2}}{\sqrt{\alpha_1 \alpha_2 \cdots \alpha_n}}.$$

1.4.2 Feynman's trick

Richard Feynman didn't invent this trick; it was known long before Feynman was born, and he learned it from a textbook. However, Feynman used it frequently and made it widely known among physicists, so it's often called Feynman's trick.

The starting point is Leibniz's formula for differentiating an integral with respect to a parameter: If we have a definite integral E that depends on a parameter α,

$$E(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(\alpha, x)\, dx,$$

then its derivative with respect to α is

$$\frac{dE}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}\, dx + f(\alpha, b(\alpha))\, b'(\alpha) - f(\alpha, a(\alpha))\, a'(\alpha).$$

This is a function of α; we'll call it $H(\alpha)$. In the most common applications, the limits $a(\alpha)$ and $b(\alpha)$ will be constants; hereafter we'll assume that's true; then $a'(\alpha)$ and $b'(\alpha)$ are zero. In that case, Leibniz's formula simplifies to

$$H(\alpha) = E'(\alpha) = \int_a^b \frac{\partial f}{\partial \alpha}\, dx.$$

We now have two integrals:

$$H(\alpha) = \int_a^b \frac{\partial f}{\partial \alpha}(\alpha, x)\, dx$$

and

$$E(\alpha) = \int_a^b f(\alpha, x)\, dx,$$

related by $H(\alpha) = E'(\alpha)$. The idea behind Feynman's trick is this: suppose H is hard to do, but E is easy. Then compute $E(\alpha)$, and obtain $H(\alpha) = E'(\alpha)$ by differentiation.

1.4.2.1 Simple examples

Let's see it in action; in this first example we'll start with an easy integral and find a harder integral. The easy integral will be

$$E(\alpha) = \int_0^\infty e^{-\alpha x}\, dx = \frac{1}{\alpha}.$$

Differentiate the integral,

$$E'(\alpha) = \int_0^\infty \frac{\partial}{\partial \alpha}\left(e^{-\alpha x}\right) dx = -\int_0^\infty x e^{-\alpha x}\, dx$$

and the result,

$$E'(\alpha) = -\frac{1}{\alpha^2}.$$

Setting both forms of $E'(\alpha)$ equal to one another, we've computed the harder integral

$$\int_0^\infty xe^{-\alpha x}\,dx = \frac{1}{\alpha^2}.$$

In practice, though, we'll usually be given a hard integral $H(\alpha)$ and have to spot how to work backwards to an easier integral $E(\alpha)$. Suppose we need to compute

$$H(\alpha) = \int_{-\infty}^\infty x^2 e^{-\alpha x^2}\,dx.$$

This can be computed by two rounds of integration by parts; it's easier to use Feynman's trick. Recognize the integrand as the derivative of a simpler function,

$$x^2 e^{-\alpha x^2} = \frac{\partial}{\partial\alpha}\left(-e^{-\alpha x^2}\right).$$

This leads us to the easier integral

$$E(\alpha) = -\int_{-\infty}^\infty e^{-\alpha x^2}\,dx = -\sqrt{\frac{\pi}{\alpha}}$$

(computed in problem 1). Our hard integral is obtained by differentiation:

$$H(\alpha) = E'(\alpha) = \frac{\sqrt{\pi}}{2\alpha^{3/2}}.$$

1.4.2.2 Repeated differentiation

Sometimes you'll need to differentiate $E(\alpha)$ more than once to recover the hard integral. Consider the hard integral

$$H(\alpha) = \int_0^\infty x^4 e^{-\alpha x}\,dx,$$

which can be recognized as the *fourth* derivative of

$$E(\alpha) = \int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}.$$

Therefore,

$$H(\alpha) = E^{(4)}(\alpha) = \frac{4!}{\alpha^5}.$$

Problem 5 Use mathematical induction to prove the general formula $\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$ for $n \in \mathbb{N}_0$.

Problem 6 Use repeated differentiation to evaluate $\int_{-\infty}^\infty x^4 e^{-\alpha x^2}\,dx$ and $\int_{-\infty}^\infty x^6 e^{-\alpha x^2}\,dx$.

Problem 7 Use repeated differentiation to evaluate the integral $H(n) = \int_{-\pi}^{\pi} x^2 \cos(nx)\,dx$.

1.4.2.3 Inserting parameters

Sometimes there's no obvious parameter in an integral. Consider the hard integral

$$H = \int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)^2},$$

and compare to the easy integral

$$E = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

To produce the x^2 in the numerator, insert a parameter next to the x^2 in the denominator:

$$H(\alpha) = \int_{-\infty}^{\infty} \frac{x^2 dx}{(1+\alpha x^2)^2}$$

and

$$E(\alpha) = \int_{-\infty}^{\infty} \frac{dx}{1+\alpha x^2} = \frac{\pi}{\sqrt{\alpha}}.$$

Notice that $H(\alpha) = -E'(\alpha)$, so that the hard integral is

$$H(\alpha) = \frac{\pi}{2\alpha^{3/2}}.$$

If you want the original integral H, just evaluate $H(\alpha)$ at $\alpha = 1$:

$$H = \frac{\pi}{2}.$$

Problem 8 Suppose you're given an "easy" integral $E = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^4} = \frac{5\pi}{16}$. (Here the word "easy" is used in a relative sense). Use Feynman's trick to derive

$$H = \int_{-\infty}^{\infty} \frac{x^4\,dx}{(1+x^2)^6} = \frac{3\pi}{256}.$$

1.5 The Dirac delta function