Accuracy of series approximations to the error function

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Here we compare two series approximations to the error function

- The Maclaurin series $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$ which is *convergent* for all x but *useful* for $|x| \lesssim 2$.
- The asymptotic series $\operatorname{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n x^{2n}}\right]$ which is *divergent* for all x but is *useful* for $|x| \gtrsim 3 - 4$.

Note: For high-accuracy calculations, neither of these methods should be used as-is. An acceleration method such as the Shanks transformation (https://en.wikipedia.org/wiki/Shanks_transformation) should be used. Alternatively, a error-minimizing method such as a Chebyshev polynomial expansion can be used.

Error in Maclaurin expansion for erf(x)

This series converges for all x, so in principle can produce accurate results for any x. However, for large x it can take very many terms to reach a small error, making it unsatisfactory as a practical method for $|x|$ larger than about 2.

The Maclaurin series for erf(x) is a convergent alternating series, so the error after N terms is bounded by the $N+1$-th term.

In[109]:=
```
e_n_[x_] = 2 / Sqrt[Pi] x^(2 n + 1) / n! / (2 n + 1)
```

Out[109]=
$$\frac{2\,x^{2n+1}}{\sqrt{\pi}\,(2n+1)\,n!}$$

In[110]:=
```
eHalf = Table[{n, e_n[1/2]}, {n, 0, 100}];
```

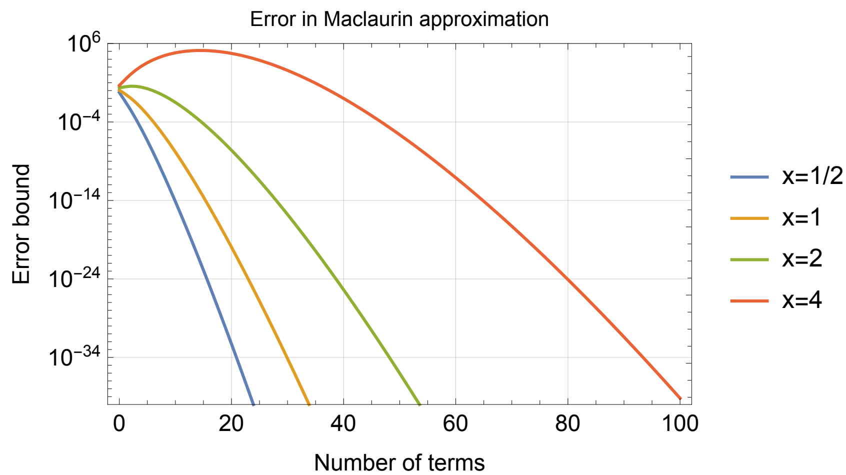
In[111]:=
```
e1 = Table[{n, e_n[1]}, {n, 0, 100}];
```

In[112]:=
```
e2 = Table[{n, e_n[2]}, {n, 0, 100}];
```

In[113]:=
```
e4 = Table[{n, e_n[4]}, {n, 0, 100}];
```

In[114]:=
```
ListLogPlot[{eHalf, e1, e2, e4}, GridLines → Automatic,
 Joined → True, PlotLegends → {"x=1/2", "x=1", "x=2", "x=4"},
 FrameLabel → {"Number of terms", "Error bound"},
 PlotLabel → "Error in Maclaurin approximation", PlotRange → {10^-40, 10^6}]
```

Out[114]=



Error in asymptotic expansion for erf(x)

This series diverges for all x, but a partial sum can still provide an accurate approximation for $|x|$ larger than about 4.

In[115]:=
```
ae_n_[x_] = 1/Sqrt[Pi]/x^(2 n + 1) (2 n - 1)!!/2^n Exp[-x^2]
```

Out[115]=
$$\frac{2^{-n}\, e^{-x^2}\, (2\, n-1)!!\, x^{-2\, n-1}}{\sqrt{\pi}}$$

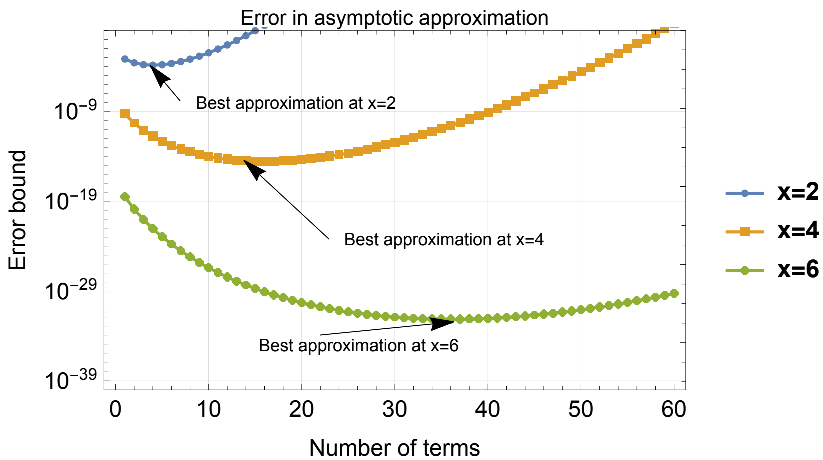
In[116]:=
```
ae2 = Table[{n, ae_n[2]}, {n, 1, 60}];
```

In[117]:=
```
ae4 = Table[{n, ae_n[4]}, {n, 1, 60}];
```

In[118]:=
```
ae6 = Table[{n, ae_n[6]}, {n, 1, 60}];
```

In[119]:=
```
ae8 = Table[{n, ae_n[8]}, {n, 1, 60}];
```

In[121]:=
```
ListLogPlot[{ae2, ae4, ae6}, PlotRange → {10^-40, 1}, Joined → True,
 PlotMarkers → {Automatic, Tiny}, FrameLabel → {"Number of terms", "Error bound"},
 PlotLabel → "Error in asymptotic approximation",
 PlotLegends → {"x=2", "x=4", "x=6"}, GridLines → Automatic]
```



The asymptotic series for erf(x) does ***not*** converge as $n \to \infty$. However, a truncated series can still provide a good approximation to erf(x). For a chosen value of x, add terms to the approximation as long as the terms decrease in magnitude.

Note: The phrase “best approximation” in the labels above means that this the smallest error that can be obtained using the asymptotic series without modification. A method such as an iterated Shanks transformation can be used to obtain smaller errors.