Chapter 1

Vectors & Coordinates

About half of this chapter will be material (vector algebra and derivatives of vectors) that you've seen before. We'll review the basics, then dig more deeply into vectors in non-Cartesian coordinates and applications to physics.

- Hassani 1, 2
- Review:
  - Hassani 1.1, 2.1-2.2
  - SST 9.1-9.4, 10.1-10.5
  - OSTX III 1.1-1.3, 2.1-4, 3.1-2, 3.4

1.1 Definitions and arithmetic

- A vector in three(two) dimensions is an ordered triple(pair) of real numbers. The entries in the vector are called its **components**.
- Vectors add/subtract component-wise: if $\mathbf{a} = \left(a_x, a_y\right)$ and $\mathbf{b} = \left(b_x, b_y\right)$, then

  $$\mathbf{a} \pm \mathbf{b} = \left(a_x \pm b_x, a_y \pm b_y\right) .$$

  This is true regardless of which unit vectors are used. You can visualize it with the parallelogram rule.
- Multiplication of a vector by a scalar multiplies each component by that scalar:

  $$\alpha \mathbf{a} = \left(\alpha a_x, \alpha a_y\right) .$$

  It "scales" the vector by the factor α. If $\alpha < 0$, the direction is reversed and the magnitude is multiplied by $|\alpha|$.

- The **magnitude** of a vector is its Pythagorean length:

  $$\|\mathbf{a}\| = \sqrt{a_x^2 + a_y^2}$$

  in 2D and

  $$\|\mathbf{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

  in 3D. Some books with use single bars, $|\mathbf{a}|$, where we use double bars. We'll also sometimes write a for the magnitude of $\mathbf{a}$. When this notation is used it's especially important to use some distinguishing notation for vectors; otherwise, you'll quickly become confused between the scalar a and the vector $\mathbf{a}$.

- In 2D, we can write the components of $\mathbf{a}$ in polar form as $(a\cos\phi, a\sin\phi)$, where $a = \|\mathbf{a}\|$ and ϕ is the angle from the x axis, running CCW. There's also a polar form for 3D vectors, but we'll wait on that until we've laid some more foundation.

1.1.1 Unit vectors

In 2D, define $\hat{\boldsymbol{\imath}} = (1, 0)$ and $\hat{\boldsymbol{\jmath}} = (0, 1)$. Then any 2D vector $\mathbf{a} = \left(a_x, a_y\right)$ can be written as

$$\mathbf{a} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}}.$$

In 3D, we have $\hat{\boldsymbol{\imath}} = (1, 0, 0)$, $\hat{\boldsymbol{\jmath}} = (0, 1, 0)$, and $\hat{\boldsymbol{k}} = (0, 0, 1)$, and we can write any $\mathbf{a}$ as

$$\mathbf{a} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}.$$

The $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ system in 2D and $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ systems in 3D are orthogonal: $\hat{\boldsymbol{\imath}} \perp \hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{\imath}} \perp \hat{\boldsymbol{k}}$, and $\hat{\boldsymbol{\jmath}} \perp \hat{\boldsymbol{k}}$. Since these vectors also have unit magnitude (they're **normalized**) we call these systems **orthonormal**.

You might also see the symbols $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ in place of $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$. When working with generic unit vectors we'll often use the symbol $\hat{\boldsymbol{e}}$ (from German *einheit*, unit). For example, in an orthogonal system of unit vectors we have $\hat{\boldsymbol{e}}_i \perp \hat{\boldsymbol{e}}_j$ when $i \neq j$.

Given any vector $\mathbf{a}$, we can produce a unit vector in the direction of $\mathbf{a}$ by dividing by its magnitude,

$$\hat{\boldsymbol{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|}.$$

This procedure is called **normalization**.

1.2 Dot product

The dot product takes two vectors as arguments, and returns a scalar.

- If $\mathbf{a}$ and $\mathbf{b}$ are represented in terms of their Cartesian components, define the dot product by:

  $$\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^{3} a_j b_j$$

- Properties: from the definition you can show that for all vectors $\mathbf{a}$ and $\mathbf{b}$ and scalars α,
  1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (*the dot product is commutative*)
  2. $(\alpha \mathbf{a}) \cdot \mathbf{b} = \alpha\, (\mathbf{a} \cdot \mathbf{b})$ (*the dot product commutes with scalar multiplication*)
  3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}$ (*the dot product distributes over addition*)
  4. $\mathbf{a} \cdot \mathbf{a} \geq 0$, and $\mathbf{a} \cdot \mathbf{a} = 0$ iff $\mathbf{a} = 0$ (*the dot product is positive definite)*

  Properties 2 and 3 can be summed up in a single equation: $\mathbf{a} \cdot (\beta \mathbf{b} + \gamma \mathbf{c}) = \beta\, (\mathbf{a} \cdot \mathbf{b}) + \gamma\, (\mathbf{a} \cdot \mathbf{c})$. *Dot products distribute over linear combinations.*

- Let θ_{ab} be the angle between $\mathbf{a}$ and $\mathbf{b}$. With the angle addition formula, the dot product can be shown to be:
  $$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos \theta_{ab}.$$
  Consequences:
  - $\mathbf{a} \cdot \mathbf{b} = 0$ iff $\mathbf{a} \perp \mathbf{b}$.
  - $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.
  - Cauchy-Schwarz inequality: $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \, \|\mathbf{b}\|$

- If $\mathbf{a}$ and $\mathbf{b}$ are represented in arbitrary unit vectors $\mathbf{e}_j$, then
  $$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j.$$
  If those unit vectors are orthonormal ($\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}$), then the preceding result simplifies to
  $$\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^{3} a_j b_j.$$

1.2.1 Applications of the dot product

The principal uses of the dot product will be:

- To find the component of $\mathbf{a}$ along a unit vector $\hat{\boldsymbol{p}}$, compute $\hat{\boldsymbol{p}} \cdot \boldsymbol{a}$.
- To find the magnitude of $\mathbf{a}$, compute $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.
- To check if $\mathbf{a} \perp \mathbf{b}$, check if $\mathbf{a} \cdot \mathbf{b} = 0$.
- With orthonormal unit vectors, any vector can be resolved into components as follows:
  $$\mathbf{a} = (\hat{\boldsymbol{e}}_1 \cdot \mathbf{a})\, \hat{\boldsymbol{e}}_1 + (\hat{\boldsymbol{e}}_2 \cdot \mathbf{a})\, \hat{\boldsymbol{e}}_2 + (\hat{\boldsymbol{e}}_3 \cdot \mathbf{a})\, \hat{\boldsymbol{e}}_3.$$

These have many applications in analyzing the geometry of physical problems. Spoiler: we'll develop an abstract generalization of the dot product that lets us apply these ideas to *functions* as well as "ordinary" 3D vectors. For now, we stick with the concrete.

1.2.1.1 A familiar application: block on an inclined plane

You've likely encountered the problem of a block sliding on an inclined plane. The key step is to resolve the downward gravitational force $\mathbf{F}_g$ into two components: $\mathbf{F}_\parallel$ down along the plane's surface and $\mathbf{F}_\perp$ normal to it. By contemplating triangles you can see that $\mathbf{F}_\parallel$ has magnitude $mg \sin\alpha$. The distance traveled down a frictionless plane starting from rest is then

$$s(t) = s_0 - \frac{1}{2} g t^2 \sin\alpha.$$

In more complicated problems it's harder to work out vector components by inspection. Following the general principle that "*algebra is a way to substitute computation for thinking*," we'll develop a systematic method for finding components. In the inclined plane example, we introduce an orthonormal set of unit vectors, a tangent vector $\hat{\boldsymbol{\tau}}$ down along the plane and a normal vector $\hat{\boldsymbol{n}}$ outward from the plane. By construction, we have $\hat{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{n}} = 0$. Write $\mathbf{F}$ in terms of these unit vectors,

$$\mathbf{F} = F_\tau \hat{\boldsymbol{\tau}} + F_n \hat{\boldsymbol{n}},$$

where the coefficients F_τ and F_n are to be determined. To find the components along each unit vector, use the dot product. Dot $\hat{\boldsymbol{\tau}}$ into $\mathbf{F}$,

$$\begin{aligned} \hat{\boldsymbol{\tau}} \cdot \mathbf{F} &= F_\tau \left(\hat{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{\tau}}\right) + F_n \left(\hat{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{n}}\right) \\ &= 1 F_\tau + 0 F_n = F_\tau, \end{aligned}$$

so we find

$$F_\tau = \hat{\boldsymbol{\tau}} \cdot \mathbf{F}.$$

Similarly, dot $\hat{\boldsymbol{n}}$ into $\mathbf{F}$ to find the normal component,

$$F_n = \hat{\boldsymbol{n}} \cdot \mathbf{F}.$$

All of this can be represented compactly in the formula

$$\mathbf{F} = \left(\hat{\boldsymbol{\tau}} \cdot \mathbf{F}\right) \hat{\boldsymbol{\tau}} + \left(\hat{\boldsymbol{n}} \cdot \mathbf{F}\right) \hat{\boldsymbol{n}}.$$

With $\mathbf{F}$ appearing on both sides, here's what the formula means: given some $\mathbf{F}$ and an orthonormal unit vector set , resolve it into components along $\hat{\boldsymbol{\tau}}$ and $\hat{\boldsymbol{n}}$ by taking dot products $\hat{\boldsymbol{\tau}} \cdot \mathbf{F}$ and $\hat{\boldsymbol{n}} \cdot \mathbf{F}$.

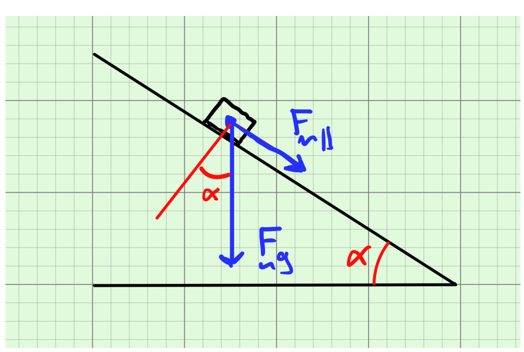

Figure 1.1: Gravitational force acting on a block on an inclined plane.

1.2.1.2 A less familiar application: polarized light

In an electromagnetics course you'll learn that light is a wave in an EM field. Plane EM waves are transverse: if a plane wave is traveling in the $x$direction (along $\hat{\boldsymbol{\imath}}$) then $\mathbf{E} \perp \hat{\boldsymbol{\imath}}$ and $\mathbf{B} \perp \hat{\boldsymbol{\imath}}$, which we can express using the dot product: $\mathbf{E} \cdot \hat{\boldsymbol{\imath}} =0$ and $\mathbf{B} \cdot \hat{\boldsymbol{\imath}} = 0$. It can also be shown that in a plane wave $\mathbf{E} \perp \mathbf{B}$. In the rest of this discussion we'll focus on the electric field and ignore the magnetic field. The polarization state of a wave is the direction of the electric field **E**. A typical light source (an LED bulb, a star, etc) emits light that is unpolarized: a superposition of different polarization states.

Although most (not all!) emitters produce unpolarized light, it is possible to obtain polarized light. Materials called polarizers transmit only light with a specific polarization state, absorbing all else. Unpolarized light entering a polarizer emerges polarized. Other processes can select for certain polarizations; these include scattering and reflection. We'll focus on polarizing materials, and consider a polarizer to be a "black box" that selects a particular polarization.

For a mathematical model of the action of a polarizer, we need two vectors: **E** and $\hat{\boldsymbol{p}}$.

1.2.2 Resolving a vector into orthogonal components

This is important enough to call out.

If $\{\hat{\boldsymbol{e}}_i\}_{i=1}^{n}$ is an orthonormal set of unit vectors in the space $\mathbb{R}^n$, then any vector $\mathbf{F} \in \mathbb{R}^n$ can be expanded uniquely as

$$\mathbf{F} = \sum_{j=1}^{n} F_j \hat{\boldsymbol{e}}_j,$$

where the j-th coefficient F_j is

$$F_j = \hat{\boldsymbol{e}}_j \cdot \mathbf{F}.$$

1.3 Cross product

Two vectors **a** and **b** define a parallelogram. A direction, or orientation, can be assigned to the parallelogram using the right-hand rule: start with your right fingers along **a**, curl your fingers into **b**, and your thumb indicates the direction of $\mathbf{a} \times \mathbf{b}$.

- Definition: The cross product $\mathbf{a} \times \mathbf{b}$ is the vector in the direction given by the right-hand rule, having magnitude $\|\mathbf{a}\times\mathbf{b}\|$ equal to the area of the parallelogram.
- Consequences:
  1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
  2. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
  3. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

4. $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,|\sin\theta_{ab}|$

- You can compute the cross product using the determinant formula

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \left(a_y b_z - a_z b_y\right)\hat{\boldsymbol{\imath}} - \left(a_x b_z - a_z b_x\right)\hat{\boldsymbol{\jmath}} + \left(a_x b_y - a_y b_x\right)\hat{\boldsymbol{k}}.$$

1.4 Orthogonal curvilinear coordinates

If $\langle q_1, q_2, q_3\rangle$ are orthogonal curvilinear coordinates, then the scale factors h_j are

$$h_j = \left\|\frac{\partial\mathbf{r}}{\partial q_j}\right\|,$$

and the unit vectors tangent to the q_1 coordinate line, then

$$\hat{\boldsymbol{e}}_j = \frac{1}{h_j}\frac{\partial\mathbf{r}}{\partial q_j}.$$

1.4.1 The three most important coordinate systems

- Cartesian: $\mathbf{r}\left(z, y, z\right) = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$
  - $h_x = h_y = h_z = 1$
- Cylindrical polar coordinates: $\mathbf{r}\left(\rho, \phi, z\right) = \rho\left(\hat{\boldsymbol{\imath}}\cos\phi + \hat{\boldsymbol{\jmath}}\sin\phi\right) + z\hat{\boldsymbol{z}}$
  - $h_\rho = 1$; $\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\imath}}\cos\phi + \hat{\boldsymbol{\jmath}}\sin\phi$
  - $h_\phi = \rho$; $\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\imath}}\sin\phi + \hat{\boldsymbol{\jmath}}\cos\phi$
  - $h_z = 1$; $\hat{\boldsymbol{z}} = \hat{\boldsymbol{k}}$

  Notice that $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{z}}$. When $z = 0$, the system reduces to 2D polar ("plane polar") coordinates.
- Spherical polar coordinates: $\mathbf{r}\left(r, \theta, \phi\right) = r\left(\left(\hat{\boldsymbol{\imath}}\cos\phi + \hat{\boldsymbol{\jmath}}\sin\phi\right)\sin\theta + \hat{\boldsymbol{k}}\cos\theta\right)$
  - $h_r = 1$; $\hat{\boldsymbol{r}} = \left(\hat{\boldsymbol{\imath}}\cos\phi + \hat{\boldsymbol{\jmath}}\sin\phi\right)\sin\theta + \hat{\boldsymbol{k}}\cos\theta$
  - $h_\theta = r$; $\hat{\boldsymbol{\theta}} = \left(\hat{\boldsymbol{\imath}}\cos\phi + \hat{\boldsymbol{\jmath}}\sin\phi\right)\cos\theta - \hat{\boldsymbol{k}}\sin\theta$
  - $h_\phi = r\sin\theta$; $\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\imath}}\sin\phi + \hat{\boldsymbol{\jmath}}\cos\phi$

  Notice that $\mathbf{r} = r\hat{\boldsymbol{r}}$.

Except in Cartesian coordinates, the unit vectors depend on the coordinates. One consequence to be explored below is that if a particle's position is described in non-Cartesian coordinates, the unit vectors change as the particle moves.

Certainly in cylindrical and possibly in spherical coordinates you can reason through the geometry to figure out the unit vectors, and skip the calculation of the scale factors and partial derivatives. However, the geometrical reasoning gets much more complicated for coordinate systems such as ellipsoidal or parabolic coordinates, while the brute force calculation will always work. The calculation might be complicated, but it's mindless and can be done by a computer.

1.4.2 Cartesian unit vectors represented in non-Cartesian coordinates

It's sometimes useful (see 1.5.3.3 below) to write the Cartesian unit vectors in terms of unit vectors in another system. We can do this with dot products: for example, to find the components of $\hat{\boldsymbol{\imath}}$ in the cylindrical unit vectors, write $\hat{\boldsymbol{\imath}}$ as

$$\hat{\boldsymbol{\imath}} = \left(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\rho}}\right) \hat{\boldsymbol{\rho}} + \left(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\phi}}\right) \hat{\boldsymbol{\phi}} + \left(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{z}}\right) \hat{\boldsymbol{z}}$$

and work through the algebra,

$$\begin{aligned} \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\rho}} &= \cos\phi \\ \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\phi}} &= -\sin\phi \\ \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{z}} &= 0, \end{aligned}$$

leading to the result

$$\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\rho}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi.$$

A similar calculation finds

$$\begin{aligned} \hat{\boldsymbol{\jmath}} &= \hat{\boldsymbol{\rho}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi \\ \hat{\boldsymbol{k}} &= \hat{\boldsymbol{z}}. \end{aligned}$$

The same procedure can be used to do any transformation between any two systems of orthonormal unit vectors.

1.5 Time derivatives of vectors and products

- Differentiate a vector component-by-component:

$$\frac{d\mathbf{f}}{dt} = \frac{d}{dt}\left[\sum_{j=1}^{3} f_j\left(t\right) \hat{\boldsymbol{e}}_j\left(t\right)\right]$$

$$= \sum_{j=1}^{3} \frac{df_j}{dt} \hat{\boldsymbol{e}}_j + \sum_{j=1}^{3} f_j \frac{d\hat{\boldsymbol{e}}_j}{dt}.$$

  When the unit vectors $\left\{\hat{\boldsymbol{e}}_j\right\}$ are constant in time, then the second term is zero.

- By grinding through the ordinary product rule, you can easily show the following:

$$\frac{d}{dt}(\alpha \mathbf{f}) = \alpha \frac{d\mathbf{f}}{dt} + \frac{d\alpha}{dt}\mathbf{f}$$

$$\frac{d}{dt}(\mathbf{f} \cdot \mathbf{g}) = \frac{d\mathbf{f}}{dt} \cdot \mathbf{g} + \mathbf{f} \cdot \frac{d\mathbf{g}}{dt}$$

$$\frac{d}{dt}(\mathbf{f} \cdot \mathbf{f}) = 2\mathbf{f} \cdot \frac{d\mathbf{f}}{dt}$$

$$\frac{d}{dt}(\mathbf{f} \times \mathbf{g}) = \frac{d\mathbf{f}}{dt} \times \mathbf{g} + \mathbf{f} \times \frac{d\mathbf{g}}{dt}$$

1.5.1 Applications

1.5.1.1 Angular momentum and torque

The angular momentum of a particle is $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. Differentiate $\mathbf{L}$, using the product rule for the cross product:

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= m\frac{d\mathbf{r}}{dt} \times \mathbf{v} + m\mathbf{r} \times \frac{d\mathbf{v}}{dt} \\ &= m\mathbf{v} \times \mathbf{v} + \mathbf{r} \times (m\mathbf{a}) \\ &= \mathbf{r} \times \mathbf{F};\end{aligned}$$

this is the familiar relation between angular momentum $\mathbf{L}$ and torque $\mathbf{r} \times \mathbf{F}$. When the force is entirely radial, the torque is zero, so $\frac{d\mathbf{L}}{dt} = 0$ and therefore $\mathbf{L} = \text{constant}$.

1.5.1.2 Motion in a magnetic field

The Lorentz force on a charged particle in a magnetic field is $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$; the nonrelativistic equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}.$$

We won't solve this equation now; however, we can deduce some properties of the motion even without a full solution.

- The acceleration is $\perp$ to the velocity.
- The magnitude of the velocity is constant. To prove this, compute

$$\frac{d}{dt}\left(\|\mathbf{v}\|^2\right) = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}$$

  and use the equation of motion and vector identities to find the result.

1.5.2 Time derivatives of unit vectors

Let $\mathbf{e}$ be any unit vector. Then $\frac{d\hat{\boldsymbol{e}}}{dt} \perp \hat{\boldsymbol{e}}$. You can prove this using the product rule and the property $\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}} = 1$. Here's how to understand it: if $\frac{d\hat{\boldsymbol{e}}}{dt}$ had a component along $\hat{\boldsymbol{e}}$, then its magnitude would change; this is impossible, therefore $\frac{d\hat{\boldsymbol{e}}}{dt}$ must be $\perp \hat{\boldsymbol{e}}$.

In Cartesian coordinates the unit vectors are independent of time. In other coordinate systems, this will not be the case.

1.5.2.1 Cylindrical coordinates

Differentiate the expressions for $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$:

- $\frac{d\hat{\boldsymbol{\rho}}}{dt} = -\dot{\phi}\hat{\boldsymbol{\imath}} \sin\phi + \dot{\phi}\hat{\boldsymbol{\jmath}} \cos\phi$
- $\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\hat{\boldsymbol{\imath}} \cos\phi - \dot{\phi}\hat{\boldsymbol{\jmath}} \sin\phi$
- $\frac{d\hat{\boldsymbol{z}}}{dt} = \mathbf{0}$

You'll usually want these in terms of the cylindrical unit vectors. By inspection, you can spot that

$$\frac{d\hat{\boldsymbol{\rho}}}{dt} = \dot{\phi}\hat{\boldsymbol{\phi}}$$
$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\hat{\boldsymbol{\rho}}$$

In more complicated coordinate systems it won't be so easy to spot the expansion coefficients, but it's always possible to work them out using the dot product:

$$\frac{d\hat{\boldsymbol{\rho}}}{dt} = \left(\hat{\boldsymbol{\rho}} \cdot \frac{d\hat{\boldsymbol{\rho}}}{dt}\right)\hat{\boldsymbol{\rho}} + \left(\hat{\boldsymbol{\phi}} \cdot \frac{d\hat{\boldsymbol{\phi}}}{dt}\right)\hat{\boldsymbol{\phi}}$$

which you can show is equal to $\dot{\phi}\hat{\boldsymbol{\phi}}$.

In mechanics, we'll also want the second derivatives (used in accelerations). The hard part was computing the first derivatives; with those in hand, it follows that

$$\frac{d^2\hat{\boldsymbol{\rho}}}{dt^2} = \ddot{\phi}\hat{\boldsymbol{\phi}} - \dot{\phi}^2\hat{\boldsymbol{\rho}}$$
$$\frac{d^2\hat{\boldsymbol{\phi}}}{dt^2} = -\ddot{\phi}\hat{\boldsymbol{\rho}} - \dot{\phi}^2\hat{\boldsymbol{\phi}}$$

1.5.3 Newton's second law in cylindrical coordinates

For a particle with constant mass m, Newton's second law of motion is

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}.$$

In 1.4.1 we saw that $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$. The acceleration is

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= \ddot{\rho}\hat{\boldsymbol{\rho}} + 2\dot{\rho}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \rho\frac{d^2\hat{\boldsymbol{\rho}}}{dt^2} + \ddot{z}\hat{\mathbf{z}} \\ &= \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}.\end{aligned}$$

The equations of motion are

$$\begin{aligned} m\left(\ddot{\rho} - \rho\dot{\phi}^2\right) &= F_\rho = \hat{\boldsymbol{\rho}}\cdot\mathbf{F} \\ m\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right) &= F_\phi = \hat{\boldsymbol{\phi}}\cdot\mathbf{F} \\ m\ddot{z} &= F_z = \hat{\mathbf{z}}\cdot\mathbf{F}.\end{aligned}$$

This is a system of three coupled second order differential equations.

1.5.3.1 Angular momentum conservation again

The angular equation of motion is

$$m\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right) = F_\phi.$$

This is a second order differential equation involving both ρ and ϕ; it can't be solved without also solving the other two equations of motion. But notice that if we introduce $L_z = m\rho^2\dot{\phi}$ and differentiate

$$\frac{d}{dt}\left(m\rho^2\dot{\phi}\right) = m\rho^2\ddot{\phi} + 2m\rho\dot{\phi}\dot{\rho},$$

we find

$$\frac{dL_z}{dt} = \rho F_\phi.$$

When $F_\phi = 0$, the quantity L_z is constant in time.

1.5.3.2 Uniform circular motion

Consider uniform circular motion in the plane $z = 0$, centered on the origin. Cylindrical coordinates are the obvious choice for this problem; the radial coordinate ρ remains constant while the angular coordinate varies linearly with time,

$$\rho\left(t\right) = a; \quad \phi\left(t\right) = \omega t + \phi_0; \quad z\left(t\right) = 0$$

Then $\dot{\rho} = 0$, $\ddot{\rho} = 0$, $\dot{\phi} = \omega$, and $\ddot{\phi} = 0$. Plug into the equations of motion, finding

$$\begin{aligned} -m\rho\omega^2 &= F_\rho \\ F_\phi &= 0 \\ F_z &= 0.\end{aligned}$$

The first of these equations is the well-known condition for the centripetal force required to hold a particle in circular motion.

1.5.3.3 Equation of motion of an ideal pendulum

It will be convenient work in the $x - y$ plane but to orient the x axis downwards and the y axis pointing to the right. Then ϕ is the angle (CCW) from the downward vertical, and the gravitational force is $\mathbf{F} = mg\hat{\boldsymbol{\imath}}$. Cylindrical coordinates are the natural choice: the pendulum moves, the radial coordinate $\rho(t)$ remains constant and the angular coordinate varies.

The forces on the pendulum are gravitational, $\mathbf{F}_{\text{grav}} = mg\hat{\boldsymbol{\imath}}$, and an unknown, variable, tension in the pendulum rod, $\mathbf{F}_{\text{tens}} = F_{\text{tens}}\hat{\boldsymbol{\rho}}$. Since we're working in cylindrical coordinates, use the method of 1.4.2 to write the gravitational force as

$$\mathbf{F}_{\text{grav}} = mg\hat{\boldsymbol{\rho}}\cos\phi - mg\hat{\boldsymbol{\phi}}\sin\phi.$$

In an ideal pendulum, the tension is a *constraint* that will be chosen to keep the length of the pendulum rod constant. The total force is then

$$\mathbf{F} = (mg\cos\phi + F_{\text{tens}})\,\hat{\boldsymbol{\rho}} - mg\hat{\boldsymbol{\phi}}\sin\phi$$

Write out the equations of motion (ignoring the unused coordinate z),

$$m\left(\ddot{\rho} - \rho\dot{\phi}^2\right) = mg\cos\phi + F_{\text{tens}}$$

$$m\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right) = -mg\sin\phi.$$

Since $\rho(t)$ is the constant ℓ, these simplify to

$$-m\ell\dot{\phi}^2 = mg\cos\phi + F_{\text{tens}}$$

$$m\ell\ddot{\phi} = -mg\sin\phi.$$

These are two differential equations with, apparently, only one unknown: $\phi(t)$. However, the tension F_{tens} is a free variable that we choose to maintain a perfact balance with the radial component of gravity and the fictitious radial force $m\ell\dot{\phi}^2$. The first equation is then satisfied by assumption, leaving the equation of motion for $\phi(t)$,

$$\ddot{\phi} = -\frac{g}{\ell}\sin\phi.$$

This equation is nonlinear and can't be solved exactly in closed form; a useful approach when $|\phi| \ll 1$ is to use the small angle approximation $\sin\phi \approx \phi$ to derive a linear equation

$$\ddot{\phi} = -\frac{g}{\ell}\phi.$$

This is the harmonic oscillator equation $\ddot{\phi} + \omega_0^2\phi = 0$ with natural frequency $\omega_0 = \sqrt{g/\ell}$; we'll study this in great depth in (**??**). For now, recall from elementary physics that the solution to the simple harmonic oscillator equation is

$$\phi(t) = A\cos(\omega_0 t + \delta_0),$$

where the amplitude A and phase δ_0 are found from the initial conditions $\phi(0)$ and $\phi'(0)$. The period of the motion (in the small angle approximation only) is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{\ell}{g}}.$$

You will see later how to derive corrections to this formula when the amplitude is large enough that the small angle approximation is no longer valid.