Some 2D orthogonal coordinate systems

PHYS/MATH 4325
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The most important 2D orthogonal coordinate systems are Cartesian and plane polar; you need to learn to work comfortably with both. Several others allow separation of variables of the wave and Schrodinger equations (if you don't know what "separation of variables" means, fear not, by the end of the semester you will). Don't worry: we won't do separation of variables in any of the oddball systems like bipolar coordinates; that's horribly messy and the applications are specialized. We will (later this semester) study separation of variables for partial differential equations in Cartesian, plane polar, and (in 3D) cylindrical and spherical coordinates; there are many important applications in these systems, from atoms to galaxies and many things in between.

Techniques based on functions of complex variables can construct very general orthogonal coordinate systems in 2D, in which the Laplace equation of electrostatics, magnetostatics, and ideal fluid flow is *easily* solvable. We will study complex variables in 4326, though we'll only briefly touch on that application of them (the technique is known as conformal mapping; it is, unfortunately, only applicable in 2D).

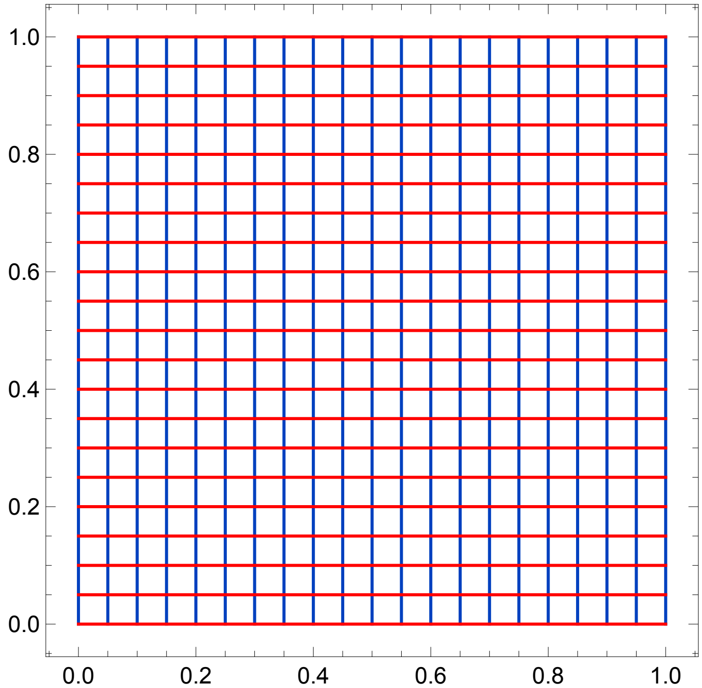
```
In[]:= plotCoords[R_, {u_, uMin_, uMax_, nu_ : 20}, {v_, vMin_, vMax_, nv_ : 20}, args___] :=
 Show[
  ParametricPlot[Table[R[u, v], {u, uMin, uMax, (uMax - uMin) / nu}],
   {v, vMin, vMax}, PlotStyle → RGBColor[0, 0.25, 0.75], args],
  ParametricPlot[Table[R[u, v], {v, vMin, vMax, (vMax - vMin) / nv}],
   {u, uMin, uMax}, PlotStyle → Red, args]
 ]
```

Cartesian coordinates (2D)

Too many applications to list

```
In[]:= r_cart2D[x_, y_] = {x, y}
Out[]= {x, y}
```

In[•]:=
```
plotCoords[r_cart2D, {x, 0, 1}, {y, 0, 1}]
```

Out[•]=


- ## Plane polar coordinates

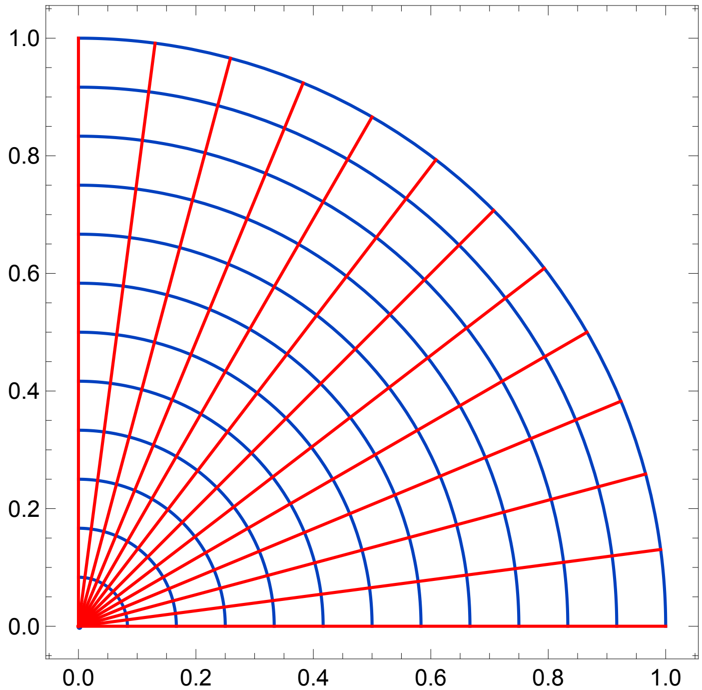
Too many applications to list

In[•]:=
```
r_pol[ρ_, ϕ_] = {ρ Cos[ϕ], ρ Sin[ϕ]}
```

Out[•]= $\{\rho \cos(\phi), \rho \sin(\phi)\}$

In[•]:=
```
plotCoords[r_pol, {ρ, 0, 1, 12}, {ϕ, 0, Pi/2, 12}]
```

Out[•]=



- ## Elliptical coordinates

See Hassani, prob 2.32

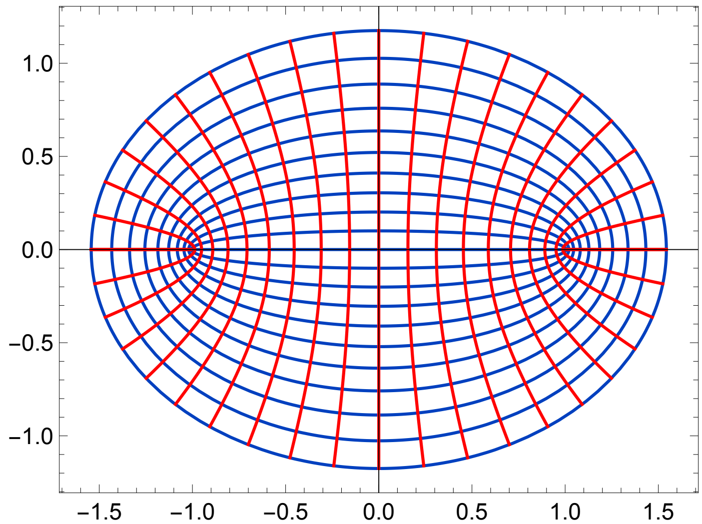
Some applications:

- Waveguide with elliptical cross-section
- Ideal flow out of a slit
- Electrostatic potential from an infinitely long strip with finite width

In[•]:=
```
r_ell[u_, v_] = {Cosh[u] Cos[v], Sinh[u] Sin[v]}
```

Out[•]= $\{\cosh(u)\cos(v), \sinh(u)\sin(v)\}$

In[•]:=
```
plotCoords[r_ell, {u, -1, 1}, {v, 0, Pi}]
```

Out[•]=


- ## Plane parabolic coordinates

See Hassani, prob 2.33

Some applications:

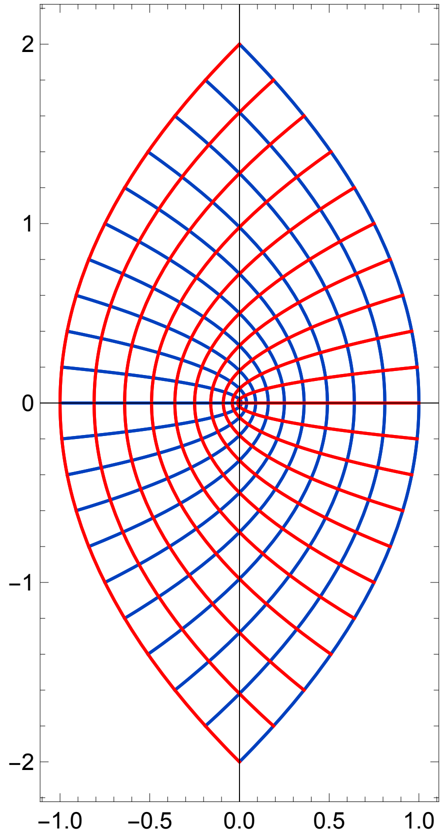
- Waveguide with parabolic cross-section
- Parabolic reflector

In[•]:=
```
r_par[u_, v_] = {u^2 - v^2, 2 u v}
```

Out[•]= $\{u^2 - v^2, 2\,u\,v\}$

In[∘]:=
```
plotCoords[r_par, {u, -1, 1}, {v, -1, 1}]
```

Out[∘]=



- Bipolar coordinates (2D)

See Hassani, problem 2.34

Some applications

- Electrostatic potential due to two cylinders
- Magnetostatic potential due to two wires
- Ideal flow around two cylinders

In[∘]:=
```
r_bp[u_, v_] = {Sinh[u], Sin[v]} / (Cosh[u] + Cos[v])
```

Out[∘]= $\left\{\frac{\sinh(u)}{\cosh(u)+\cos(v)}, \frac{\sin(v)}{\cosh(u)+\cos(v)}\right\}$

In[∘]:=
```
plotCoords[r_bp, {u, -ArcSinh[2], ArcSinh[2]},
  {v, 0, 2 Pi}, PlotRange → {{-2, 2}, {-2, 2}}]
```

Out[∘]=