Eigenvalue problem examples

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Simple examples

First example

In[202]:= `A = {{-2, 1}, {1, -2}}`

Out[202]= $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

The Eigenvalue function returns the eigenvalues of a matrix

In[166]:= `Eigenvalues[A]`

Out[166]= $\{-3, -1\}$

The Eigenvectors function returns the eigenvectors in the *rows* of a matrix. NOTE: conventional notation puts them in the columns of a matrix.

In[167]:= `Eigenvectors[A]`

Out[167]= $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

The Eigensystem function returns the eigenvalues and eigenvectors

In[168]:= `Eigensystem[A]`

Out[168]= $\begin{pmatrix} -3 & -1 \\ \{-1, 1\} & \{1, 1\} \end{pmatrix}$

Second example

In[1]:= `A = {{2, 3}, {2, 1}}`

Out[1]= $\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$

In[2]:= `Eigensystem[A]`

Out[2]= $\begin{pmatrix} 4 & -1 \\ \{3, 2\} & \{-1, 1\} \end{pmatrix}$

Third example

In[203]:=
```
A = {{-2, 1, 0, 0}, {1, -2, 1, 0}, {0, 1, -2, 1}, {0, 0, 1, -2}}
```

Out[203]=
$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

In[205]:=
```
Eigenvalues[A]
```

Out[205]=
$$\left\{\frac{1}{2}\left(-5-\sqrt{5}\right), \frac{1}{2}\left(-3-\sqrt{5}\right), \frac{1}{2}\left(\sqrt{5}-5\right), \frac{1}{2}\left(\sqrt{5}-3\right)\right\}$$

In[206]:=
```
Eigenvectors[A]
```

Out[206]=
$$\begin{pmatrix} -1 & \frac{1}{2}\left(1+\sqrt{5}\right) & \frac{1}{2}\left(-1-\sqrt{5}\right) & 1 \\ 1 & \frac{1}{2}\left(1-\sqrt{5}\right) & \frac{1}{2}\left(1-\sqrt{5}\right) & 1 \\ -1 & \frac{1}{2}\left(1-\sqrt{5}\right) & \frac{1}{2}\left(\sqrt{5}-1\right) & 1 \\ 1 & \frac{1}{2}\left(1+\sqrt{5}\right) & \frac{1}{2}\left(1+\sqrt{5}\right) & 1 \end{pmatrix}$$

(Recall that the rows are the eigenvectors)

Fourth example: a real matrix can have complex eigenvalues

In[∘]:=
```
A = {{1, -1}, {1, 1}}
```

In[∘]:=
```
Eigensystem[A]
```

Fifth example: exact vs numerical calculations

In[229]:=
```
A = RandomInteger[{-10, 10}, {3, 3}]
```

Out[229]=
$$\begin{pmatrix} 0 & 8 & -9 \\ -8 & -4 & 10 \\ 3 & -3 & 4 \end{pmatrix}$$

In[230]:=
```
Eigenvalues[A, Cubics → True]
```

Out[230]=
$$\left\{\frac{35\left(1+i\sqrt{3}\right)}{2\sqrt[3]{86+\sqrt{50271}}}-\frac{1}{2}\left(1-i\sqrt{3}\right)\sqrt[3]{86+\sqrt{50271}}, \frac{35\left(1-i\sqrt{3}\right)}{2\sqrt[3]{86+\sqrt{50271}}}-\frac{1}{2}\left(1+i\sqrt{3}\right)\sqrt[3]{86+\sqrt{50271}}, \sqrt[3]{86+\sqrt{50271}}-\frac{35}{\sqrt[3]{86+\sqrt{50271}}}\right\}$$

That’s ugly. For most purposes it’s just as well to do the calculation numerically.

In[231]:=
```
Eigenvalues[N[A]]
```

Out[231]=
$$\{-0.799573999945097+10.3401139133858\,i, -0.799573999945097-10.3401139133858\,i, 1.599147998890194+0.\,i\}$$

Recall that a real cubic must have either one or three real roots. Numerically, these usually round to a real number plus a tiny imaginary part, of order machine epsilon. Use the Chop function to eliminate these artifacts.

In[232]:=
```
Chop[Eigenvalues[N[A]]]
```

Out[232]= $\{-0.79957\,39994\,5097 + 10.34011\,39133\,858\,i,\ -0.79957\,39994\,5097 - 10.34011\,39133\,858\,i,\ 1.59914\,79989\,0194\}$

- Matrix structures with special properties

  - A ***real symmetric matrix*** ($A^T = A$) will always have ***real eigenvalues*** and ***orthogonal eigenvectors***

    This will be important in applications (including coupled RC circuits, heat transfer, waveguides, and eddy currents), and we'll prove it in class. For now, here are some examples.

    1. Example

In[233]:=
```
A = {{4, 1}, {1, 3}}
```

Out[233]= $\begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$

In[234]:=
```
Eigensystem[A]
```

Out[234]= $\begin{pmatrix} \frac{1}{2}(7+\sqrt{5}) & \frac{1}{2}(7-\sqrt{5}) \\ \{\frac{1}{2}(1+\sqrt{5}),\ 1\} & \{\frac{1}{2}(1-\sqrt{5}),\ 1\} \end{pmatrix}$

2. Example

In[190]:=
```
A = {{-2, 1, 0}, {1, -2, 1}, {0, 1, -2}}
```

Out[190]= $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$

In[191]:=
```
Eigensystem[A]
```

Out[191]= $\begin{pmatrix} -2-\sqrt{2} & -2 & \sqrt{2}-2 \\ \{1,\ -\sqrt{2},\ 1\} & \{-1,\ 0,\ 1\} & \{1,\ \sqrt{2},\ 1\} \end{pmatrix}$

  - A ***real antisymmetric*** matrix ($A^T = -A$) will always have ***imaginary eigenvalues*** and ***orthogonal eigenvectors***

    This is also important in applications (including rotational dynamics, quantum computing, and wave propagation).

    1. Example

In[188]:=
```
A = {{0, 1}, {-1, 0}}
```

Out[188]= $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

In[189]:= `Eigensystem[A]`

Out[189]= $\begin{pmatrix} i & -i \\ \{-i, 1\} & \{i, 1\} \end{pmatrix}$

2. Example

In[245]:= `A = {{0, -2, 0}, {2, 0, -3}, {0, 3, 0}}`

Out[245]= $\begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & -3 \\ 0 & 3 & 0 \end{pmatrix}$

In[246]:= `Eigensystem[A]`

Out[246]= $\begin{pmatrix} i\sqrt{13} & -i\sqrt{13} & 0 \\ \left\{-\frac{2}{3}, \frac{i\sqrt{13}}{3}, 1\right\} & \left\{-\frac{2}{3}, -\frac{i\sqrt{13}}{3}, 1\right\} & \{3, 0, 2\} \end{pmatrix}$

Note: zero is both a real number and an imaginary number (it's the intersection of the real and imaginary axes in the complex plane).

• Finding the characteristic polynomial

Occasionally you'll want to work directly with the characteristic polynomial

In[247]:= `A = RandomInteger[{-10, 10}, {8, 8}]`

Out[247]= $\begin{pmatrix} -7 & -5 & 4 & 0 & 9 & 6 & -8 & 3 \\ 4 & 6 & 3 & 5 & -10 & 4 & 5 & -8 \\ -6 & -6 & 7 & -8 & 4 & -9 & -4 & 0 \\ -4 & 3 & -7 & 5 & 1 & 8 & 1 & 4 \\ -2 & -10 & 6 & 10 & 7 & 10 & 3 & 0 \\ 6 & 9 & 10 & -9 & 7 & 4 & -2 & -2 \\ 8 & -3 & -10 & -4 & -7 & 1 & 7 & -6 \\ 7 & 6 & 6 & 4 & 0 & 8 & -10 & 7 \end{pmatrix}$

In[248]:= `p[λ_] = CharacteristicPolynomial[A, λ]`

Out[248]= $\lambda^8 - 36\lambda^7 + 415\lambda^6 + 1926\lambda^5 - 86\,369\lambda^4 + 103\,720\lambda^3 + 5\,946\,826\lambda^2 - 33\,583\,760\lambda + 292\,662\,472$

Occasionally you'll be able to find the roots exactly; with a random matrix, that's almost certainly not the case.

In[250]:= `Solve[p[λ] == 0, λ]`

Out[250]= $\{\{\lambda \to \boxed{11.9\ldots}\}, \{\lambda \to \boxed{18.3\ldots}\}, \{\lambda \to \boxed{-10.1\ldots - 3.74\ldots i}\},$
$\{\lambda \to \boxed{-10.1\ldots + 3.74\ldots i}\}, \{\lambda \to \boxed{1.44\ldots - 5.80\ldots i}\},$
$\{\lambda \to \boxed{1.44\ldots + 5.80\ldots i}\}, \{\lambda \to \boxed{11.5\ldots - 13.9\ldots i}\}, \{\lambda \to \boxed{11.5\ldots + 13.9\ldots i}\}\}$

No luck. Find the eigenvalues numerically.

In[252]:=
```
Chop[Eigenvalues[N[A]]]
```

Out[252]= $\{18.31779\,90615\,311,\ 11.49979\,00579\,026 + 13.91028\,78290\,266\,i,\ 11.49979\,00579\,026 - 13.91028\,78290\,266\,i,\ 11.91854\,11677\,6,\ -10.05442\,55246\,581 + 3.74313\,41747\,6642\,i,\ -10.05442\,55246\,581 - 3.74313\,41747\,6642\,i,\ 1.43646\,53521\,0998 + 5.80427\,50443\,9488\,i,\ 1.43646\,53521\,0998 - 5.80427\,50443\,9488\,i\}$

- ## Complex matrices

  - ### An example from applications

The Pauli matrix σ_z appears in the theory of spin, and is used in the dynamics of qubits in quantum computing.

In[256]:=
```
A = {{0, -I}, {I, 0}}
```

Out[256]= $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Notice that $\sigma_x = \sigma_x{}^*$: this is the complex equivalent of symmetry.

In[257]:=
```
Eigensystem[A]
```

Out[257]= $\begin{pmatrix} -1 & 1 \\ \{i,\ 1\} & \{-i,\ 1\} \end{pmatrix}$

The eigenvalues are real.

  - ### A random example

Make a random 4 by 4 complex matrix

In[262]:=
```
A = RandomComplex[{-3 - 3 I, 3 + 3 I}, {4, 4}]
```

Out[262]= $\begin{pmatrix} -0.84149\,07133\,30602 - 0.15477\,68935\,86106\,i & -2.25139\,67836\,9716 + 0.92249\,28317\,75253\,i & -0.18273\,308 \\ -1.39034\,49008\,064 - 2.28952\,17716\,8824\,i & -1.46625\,00332\,1617 - 1.44088\,18496\,5601\,i & -1.44292\,44 \\ -1.45824\,47193\,7467 - 1.65366\,08440\,947\,i & -2.89274\,98495\,0246 + 2.48320\,50041\,9668\,i & 2.02909\,438 \\ -0.37589\,67805\,80415 + 0.98711\,24121\,57851\,i & -1.00348\,24390\,4687 - 0.65306\,98505\,23703\,i & 2.13110\,836 \end{pmatrix}$

Show it to lower precision

In[266]:=
```
NumberForm[A, 6]
```

Out[266]//NumberForm=

$$\begin{pmatrix} -0.841491 - 0.154777\,i & -2.2514 + 0.922493\,i & -0.182733 + 0.416639\,i & -0.313447 - 0.804938\,i \\ -1.39034 - 2.28952\,i & -1.46625 - 1.44088\,i & -1.44292 - 1.81986\,i & -2.68272 + 1.01113\,i \\ -1.45824 - 1.65366\,i & -2.89275 + 2.48321\,i & 2.02909 + 2.00864\,i & -1.84838 - 0.64608\,i \\ -0.375897 + 0.987112\,i & -1.00348 - 0.65307\,i & 2.13111 - 0.427105\,i & 0.72545 + 0.0321654\,i \end{pmatrix}$$

In[263]:=
```
Eigenvalues[A]
```

Out[263]= $\{-4.73306\,09341\,0498 - 1.93293\,99051\,2679\,i,\ 4.09014\,51828\,4266 - 0.57646\,33925\,32776\,i,\ 1.83134\,02417\,5689 + 2.77241\,85286\,9245\,i,\ -0.74162\,04830\,32747 + 0.18212\,90637\,75185\,i\}$

In[267]:=
```
NumberForm[Eigenvalues[A], 6]
```

Out[267]//NumberForm=

$\{-4.73306 - 1.93294\,i,\ 4.09015 - 0.576463\,i,\ 1.83134 + 2.77242\,i,\ -0.74162 + 0.182129\,i\}$