Math 521 HW3

Raj Mohanty
raj.mohanty@student.csulb.edu

1 Theory

1.1 Problem 1

To prove $\nabla_v(v,v) = 2v$

Let $v = [v_1, v_2, v_3 ... v_n]$
L.H.S $= \nabla_v(v,v) = \dfrac{d}{dv}(v^T v)$
Let $\alpha = v^T v = \sum_{i=1}^{n}\sum_{j=1}^{n} v_i v_j$
Differentiating w.r.t to the kth element of v we have:
$\dfrac{d\alpha}{dv_k} = \sum_{i=1}^{n} v_j + \sum_{i=1}^{n} v_i$
for all k = 1,2,3,...n, we have
$\dfrac{d\alpha}{dv_k} = v^T + v^T = 2v^T = 2v =$ R.H.S
To prove $\nabla_v(v,Cv) = 2Cv$

L.H.S $= \nabla_v(v,Cv) = \nabla_v v^T Cv$
Let $\alpha = v^T Cv = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} v_i v_j$
Differentiating w.r.t to the kth element of v we have:
$\dfrac{d\alpha}{dv_k} = \sum_{i=1}^{n} c_{kj} v_j + \sum_{i=1}^{n} c_{ik} v_i$
for all k = 1,2,3,...n, we have
$\dfrac{d\alpha}{dv_k} = v^T C^T + v^T C = v^T 2C = 2Cv =$ R.H.S (Since $C = C^T$)

1.2 Problem 2

To prove $(\phi^{(1)}, C\phi^{(2)}) = (C\phi^{(1)}, \phi^{(2)})$
R.H.S $= (C\phi^{(1)}, \phi^{(2)}) = C\phi^{(1)}.\phi^{(2)} = \phi^{(1)T} C^T \phi^{(2)}$
$= \phi^{(1)T} C \phi^{(2)}$ (Since $C = C^T$ as C is symmetric)
$= (\phi^{(1)}, C\phi^{(2)}) =$ L.H.S

1.3 Problem 4

Determine the eigenvalues and eigenvectors of X^TX and XX^T given

$$X = \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$
$$X^TX = \begin{pmatrix} 6 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 6 \end{pmatrix}$$

To get the eigenvalue we set: $det(X^TX - \lambda I) = 0$

$$=> det(\begin{pmatrix} 6 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}) = 0$$

$$=> (6-\lambda)((4-\lambda)(6-\lambda) - 0) - 0 + (-3)(3(4-\lambda)) = 0$$

$$=> ((6-\lambda)^2 - 9)(4-\lambda) = 0$$

$$=> \lambda = 9, 4 \text{ or } 3$$

For $\lambda = 9$

$$A^TA - \lambda I = \begin{pmatrix} -3 & 0 & -3 \\ 0 & -5 & 0 \\ -3 & 0 & -3 \end{pmatrix}$$

The rref of the above matrix $= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

To find the null space of the above matrix we do the following:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$=> x_1 = -x_3$

$=> x_2 = 0$

Therefore the null space of $X^TX - \lambda I = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

So $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ since we have to normalize the null space, as the v_i vectors are orthonormal.

Similarly for $\lambda_2 = 4$ we have:

$$X^T X - \lambda_2 I = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix}$$

The rref of the above matrix $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

To find the null space of the above matrix we do the following:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$=> x_1 = 0$
$=> x_3 = 0$

Therefore the null space of $X^T X - \lambda_2 I = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

So $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

And finally for $\lambda_3 = 3$

$$X^T X - \lambda_3 I = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & -3 \end{pmatrix}$$

The rref of the above matrix $= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

To find the null space of the above matrix we do the following:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$=> x_1 = x_3$
$=> x_2 = 0$

Therefore the null space of $X^TX - \lambda_3 I = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

So $v_3 = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

The S matrix is constructed by takeing the square root of the eigenvalues λs and putting them in a diagonal matrix as follows:

$$S = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$$

The eigenvalues of XX^T are same as X^TX
For $\lambda = 9$

$XX^T - \lambda I = \begin{pmatrix} -3 & 1 & 3 & 0 \\ 1 & -8 & -1 & 1 \\ 3 & -1 & -3 & 0 \\ 0 & 1 & 0 & -6 \end{pmatrix}$

The rref of the above matrix $= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Therefore the null space of $XX^T - \lambda_1 I = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

So $u_1 = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 4$

$XX^T - \lambda_2 I = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 1 & -3 & -1 & 1 \\ 3 & -1 & 2 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

The rref of the above matrix $= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Therefore the null space of $XX^T - \lambda_2 I = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

So $u_2 = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

For $\lambda_3 = 3$

$XX^T - \lambda_3 I = \begin{pmatrix} 3 & 1 & 3 & 0 \\ 1 & -2 & -1 & 1 \\ 3 & -1 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

The rref of the above matrix $= \begin{pmatrix} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Therefore the null space of $XX^T - \lambda_3 I = \begin{pmatrix} -0.5 \\ 0 \\ 0.5 \\ 1 \end{pmatrix}$

So $u_3 = \sqrt{\frac{2}{3}}\begin{pmatrix} 0.5 \\ 0 \\ -0.5 \\ 1 \end{pmatrix}$

To get he U matrix we do the following:

$$u_1 = \frac{1}{\sigma_1} X v_1 = \frac{1}{3}\begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} X v_2 = \frac{1}{2} \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$u_3 = \frac{1}{\sigma_3} X v_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}$$

$$u_4 = \frac{NS(X^T)}{det(NS(X^T)}$$

$$rref(X^T) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$Therefore, \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

$=> x_1 = x_4$
$x_2 = -3x_4$
$x_3 = -x_4$

Therefore the null space of $A^T = \begin{pmatrix} 1 \\ -3 \\ -1 \\ 1 \end{pmatrix}$

$$u_4 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 \\ -3 \\ -1 \\ 1 \end{pmatrix}$$

Finally SVD of $X = U\Sigma V^T = \begin{pmatrix} -0.7071 & -0.5 & 0.4082 & 0.2887 \\ 0 & -0.5 & 0 & -0.8660 \\ -0.7071 & 0.5 & -0.4082 & -0.2887 \\ 0 & -0.5 & -0.8165 & 0.2887 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & -0.7071 \end{pmatrix}$

1.4 Problem 5

If we take thin SVD
$A = \Sigma V^T$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & -0.7071 \end{pmatrix}$$

This requires 9 multiplications and 0 additions
$A = U^T \Sigma$

$$\begin{pmatrix} -0.7071 & 0 & -0.7071 & 0 \\ -0.5 & -0.5 & 0.5 & -0.5 \\ 0.4082 & 0 & -0.4082 & -0.8165 \end{pmatrix} \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

This requires 36 multiplications and 27 additions.

$A = \Sigma V^T$ is better. This is because we are dealing with a diagonal matrix Σ which reduces computations.