1 Theory

1. Let W_1 and W_2 be vector subspaces and $W = W_1 + W_2$. Show, by giving an example, that the decomposition of a vector $\mathbf{x} \in W$ is not unique, i.e.,

$$\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1^{'} + \mathbf{w}_2^{'},$$

where $\mathbf{w}_1 \neq \mathbf{w}_1^{'}$, $\mathbf{w}_2 \neq \mathbf{w_2}'$, $\mathbf{w}_1, \mathbf{w}_1^{'} \in W_1$, $\mathbf{w}_2, \mathbf{w}_2^{'} \in W_2$.

2. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}.$$

Determine bases for the column space, row space, null space, and left null space of A.

3. Let $V = \mathbb{R}^3$, let

$$\mathbf{u}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{u}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad , \mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix},$$

and define $W_1 = \text{span}\ (\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$. Find the orthogonal projection of $\mathbf{x}$ onto W_1. Also find the projection matrix $\mathbb{P}$ associated with this mapping.

4. Reconsider Problem 3. Find vectors such that $x = \mathbb{P}x$ and $x \neq \mathbb{P}x$ where the matrix $\mathbb{P}$ is the projection matrix from Problem 3.

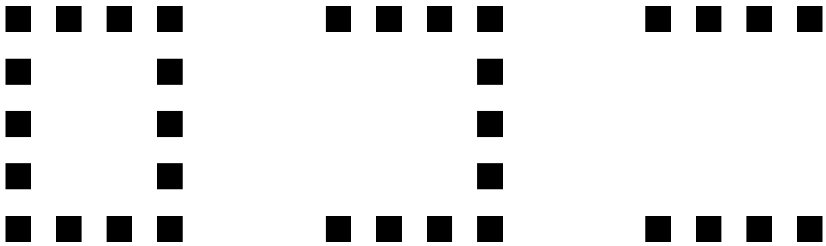
5. Determine the SVD of the data matrix

$$\begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix},$$

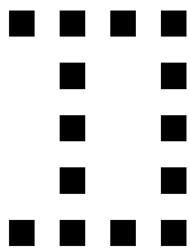
and compute the rank-one, -two, and -three approximations to A.

2 Computing

1. Consider the training set consisting of the following three patterns consisting of 5×4 arrays of black squares

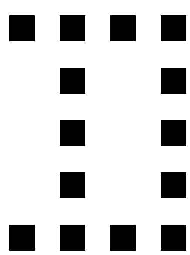
   

   Using Kohonen's novelty filter, find the novelty in the pattern

   Proceed by assuming that the black square entries have numerical value one and the blank entries have numerical value zero. Concatenate the columns of each pattern to make vectors in $\mathbb{R}^{20}$.

2. Compute the SVD of the matrix A whose entries come from the pattern

   and display the reconstructions A_1, A_2, A_3, A_4. Again, treat the squares as ones and the blanks as zeros. Your reconstructions should be matrices with numerical values.

3. This assignment requires the use of a MATLAB image. Choose your favorite image for this exercise (be sure to choose an image whose resolution is at least 300-by-300.). All the necessary MATLAB syntax is described as follows. To begin, load the MATLAB image into the matrix A using

```
>> A = imread('myImage.tif'); % adjust accordingly with the image extension
```

If your image is in color, A will have three dimensions where the first two give the resolution of the image and the last one contains a layer of red, a layer of green, and a layer of blue. To turn a color image into a monochrome one, use the MATLAB command

```
>> B = rgb2gray(A);
```

You don't necessarily have to work with a black and white picture, but it is definitely easier to start with one. The full and reduced SVD may then be executed simply by

```
>> A = double(A);
% data matrix has to be in double precision in order to perform mathematics on it
>> [U,S,V] = svd(A);
>> [U_thin,S_thin,V_thin] = svd(A,0); %% it is zero, not the letter "o"
%% ‘double’ converts any 8-bit single (uint8)
%% into 16-bit double precision
```

where U (resp. U_thin), S (resp. S_thin), and V (resp. V_thin) are the left-singular vectors, the singular values, and the right-singular vectors (resp. reduced). A rank-k approximation of the image may be found via

```
>> A_k = U(:,1:k)*S(1:k,1:k)*V(:,1:k)';
```

The resulting image may be displayed using

```
>> imagesc(A_k); % may use the option: axis off, axis square
```

or

```
>> image(A_k);
```

For better viewing, one can reset the colormap to gray scales with the command

```
>> imagesc(A_k), colormap(gray)
```

Now, do the following:

(a) Plot the singular value distribution of your image, where the x-axis is the counting index while the y-axis is the magnitude of the singular values. If we define the **cumulative energy** of $A \in \mathbb{R}^{m \times n}$ with rank r to be

$$E_k = \frac{\sum_{i=1}^{k} \sigma_i^2}{\sum_{i=1}^{r} \sigma_i^2}, \text{ where } k \leq r,$$

identify the number of singular values (σ_is) needed to retain at least 95% of the energy. This number is often called *numerical rank* of the matrix A.

(b) Compute the rank-10, rank-50, rank-100, and rank-200 approximations to your chosen image along with the *relative errors* of approximation (use the title, xlabel, and ylabel commands to specify appropriate information), display them on the same figure (using subplot), and interpret your results. (Recall that the *absolute* error of a rank-k approximation is measured by the $k+1^{\text{th}}$ singular value, so the *relative* error is given by σ_{k+1}/σ_1.)