Math 521 HW2

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1 Theory

1.1 Problem 1

Given vector space W_1 and W_2 where $W = W_1 + W_2$. It's not given that W_1 is the orthogonal complement of W_2 or vice-versa. Let us assume that W_2 is the orthogonal complement of W_1.
Hence $x = w_1 + w_2$ is a unique decomposition, where $x \in W$, $w_1 \in W_1$ $w_2 \in W_2$

Now lets assume that the decomposition is not unique i.e.
$x = w_1 + w_2 = w_1^{'} + w_2^{'}$ where $w_1^{'} \in W_1$ and $w_2^{'} \in W_2$

$=> z = w_1 - w_1^{'} = w_2^{'} - w_2$ So $z \in W_1$ and $z \in W_2$ which implies $z = \vec{O}$

That is a contradiction, hence the decomposition $x = w_1 + w_2$ is unique if W_1 is the orthogonal complement of W_2

If W_2 is not the complement of W_1 then the decomposition of x will not be unique.

Example: Let V be R^3. Lets define subspace W_1 be the X-Y plane $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ for all a,b $\in R$ and W_2 be the Y-Z plane $\begin{pmatrix} 0 \\ c \\ d \end{pmatrix}$ for all c,d $\in R$

Clearly $W_1 + W_2 = R^3$, that is all vectors in R^3 can be represented by vectors from W_1 and W_2 Let $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = w_1 + w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = w_1^{'} + w_2^{'}$

Hence this is an example of non unique decomposition because $W_1 \cap W_2 =$ y-axis

1.2 Problem 2

Determine column space, row space, null space and left null space basis of a given matrix

Given matrix A = $\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}$

The reduced row echelon form (rref) of A =

$\begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

Therefore the

Column space basis for A = $span(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix})$ since the pivot column is the first column.

Null space of A is determined by solving equation (1):

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1}$$

$=> x_1 - x_2 = 0$
$=> x_1 = x_2$
Therefore,

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ since x_2 is the free variable and x_1 is the pivot.

Therefore the null space basis for $A = span(\begin{pmatrix} 1 \\ 1 \end{pmatrix})$

Row space of A = column space of A^T

Left null space of A = null space of A^T

$A^T = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$

The reduced row echelon form (rref) of $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

Therefore, row space basis for $A = span(\begin{pmatrix} 1 \\ -1 \end{pmatrix})$

The left null space of A is determined by solving equation (2):

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{2}$$

$$=> x_1 = -2x_2 - 3x_3 \text{ or } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Therefore the left null space basis for $A = span(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix})$

1.3 Problem 3

We first create matrix A by column stacking $u^{(1)}$ and $u^{(2)}$.

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$$

The orthonormal bases of A by Gram-Schmidt process is

First orthnormal basis vector $u_1 = \frac{u^{(1)}}{||u^{(1)}||} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$$y_2 = u^{(2)} - (u^{(2)}.u_1)u_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - (\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} . \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}) \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$$

There the second basis vector $u_2 = \frac{y_2}{||y_2||} = \frac{1}{3\sqrt{5}} \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$

Stacking u_1 and u_2 into M

$$M = \begin{pmatrix} 0.44 & -0.59 \\ 0.89 & 0.29 \\ 0 & 0.75 \end{pmatrix}$$

Therefore the projection matrix P is

$$P = MM^T$$

$$=> P = \begin{pmatrix} 0.56 & 0.22 & -0.44 \\ 0.22 & 0.89 & 0.22 \\ -0.44 & 0.22 & 0.56 \end{pmatrix}$$

Given a vector $x = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

The projection of x on to A

$$Proj_A x = Px$$

$$=> Proj_A x = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

We notice here that $x = Px$ which means x is an eigen vector of P and x belongs to the plane spanned by the column space of P.

1.4 Problem 4

From problem 3 we already found that if:

$x = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ and

$P = \begin{pmatrix} 0.56 & 0.22 & -0.44 \\ 0.22 & 0.89 & 0.22 \\ -0.44 & 0.22 & 0.56 \end{pmatrix}$

then $x = Px$

However if we take

$x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ then

$Px = \begin{pmatrix} 0.55 \\ 2.22 \\ 0.55 \end{pmatrix} \neq x$

1.5 Problem 5

Determine the SVD of A

$$A = \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

$A^T A = \begin{pmatrix} 6 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 6 \end{pmatrix}$

To get the eigenvalue we set: $det(A^T A - \lambda I) = 0$

$$=> det(\begin{pmatrix} 6 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}) = 0$$

$$=> (6 - \lambda)((4 - \lambda)(6 - \lambda) - 0) - 0 + (-3)(3(4 - \lambda)) = 0$$

$$=> ((6 - \lambda)^2 - 9)(4 - \lambda) = 0$$

$$=> \lambda = 9, 4 \text{ or } 3$$

For $\lambda = 9$

$A^T A - \lambda I = \begin{pmatrix} -3 & 0 & -3 \\ 0 & -5 & 0 \\ -3 & 0 & -3 \end{pmatrix}$

The rref of the above matrix = $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

To find the null space of the above matrix we do the following:

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

=> $x_1 = -x_3$
=> $x_2 = 0$

Therefore the null space of $A^T A - \lambda I = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

So $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ since we have to normalize the null space, as the v_i vectors are orthonormal.

Similarly for $\lambda_2 = 4$ we have:

$$A^T A - \lambda_2 I = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix}$$

The rref of the above matrix = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

To find the null space of the above matrix we do the following:

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

=> $x_1 = 0$
=> $x_3 = 0$

Therefore the null space of $A^T A - \lambda_2 I = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

So $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

And finally for $\lambda_3 = 3$

$$A^T A - \lambda_3 I = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & -3 \end{pmatrix}$$

The rref of the above matrix $= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

To find the null space of the above matrix we do the following:

$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$
$=> x_1 = x_3$
$=> x_2 = 0$

Therefore the null space of $A^T A - \lambda_3 I = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

So $v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

The S matrix is constructed by takeing the square root of the eigenvalues λs and putting them in a diagonal matrix as follows:

$$S = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$$

To get he U matrix we do the following:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3} \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2} \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$u_3 = \frac{1}{\sigma_3} A v_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}$$

$$u_4 = \frac{NS(A^T)}{det(NS(A^T)}$$

$$rref(A^T) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$Therefore, \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

$=> x_1 = x_4$
$x_2 = -3x_4$
$x_3 = -x_4$

Therefore the null space of $A^T = \begin{pmatrix} 1 \\ -3 \\ -1 \\ 1 \end{pmatrix}$

$$u_4 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 \\ -3 \\ -1 \\ 1 \end{pmatrix}$$

Finally SVD of $A = U\Sigma V^T = \begin{pmatrix} -0.7071 & -0.5 & 0.4082 & 0.2887 \\ 0 & -0.5 & 0 & -0.8660 \\ -0.7071 & 0.5 & -0.4082 & -0.2887 \\ 0 & -0.5 & -0.8165 & 0.2887 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$
$\begin{pmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & -0.7071 \end{pmatrix}$

$$A_1 = u_1\sigma_1 v_1^T = \begin{pmatrix} -0.7071 \\ 0 \\ -0.7071 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix} \begin{pmatrix} 0.7071 & 0 & -0.7071 \end{pmatrix} = \begin{pmatrix} -1.5 & 0 & 1.5 \\ 0 & 0 & 0 \\ -1.5 & 0 & 1.5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} = \begin{pmatrix} -0.7071 & -0.5 \\ 0 & -0.5 \\ -0.7071 & 0.5 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -1.5 & -1 & 1.5 \\ 0 & -1 & 0 \\ -1.5 & 1 & 1.5 \\ 0 & -1 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix} = \begin{pmatrix} -0.7071 & -0.5 & 0.4082 \\ 0 & -0.5 & 0 \\ -0.7071 & 0.5 & -0.4082 \\ 0 & -0.5 & -0.8165 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1.732 \end{pmatrix}$$

$$\begin{pmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & -0.7071 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

We notice that by Rank-3 we fully recover the original matrix.