

# Deep Learning

## 06 Backpropagation-2

Dr. Konda Reddy Mopuri  
Dept. of Artificial Intelligence  
IIT Hyderabad  
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- $$\frac{\partial \ell}{\partial b_i^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}} \frac{\partial s_i^{(l)}}{\partial b_i^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}} \quad (2)$$

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- Then wrt the parameters

$$\frac{\partial \ell}{\partial w_{i,j}^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}} x_j^{(l-1)} \text{ and } \frac{\partial \ell}{\partial b_i^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}}$$

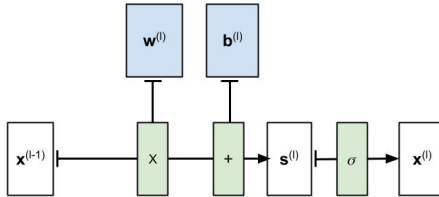
# Jacobian in Tensorial form

- $\psi : \mathcal{R}^N \rightarrow \mathcal{R}^M$  then  $\left[ \frac{\partial \psi}{\partial x} \right] = \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_M}{\partial x_1} & \cdots & \frac{\partial \psi_M}{\partial x_N} \end{bmatrix}$

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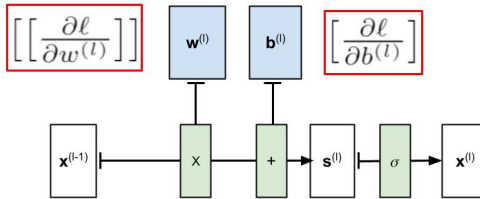
- $\psi : \mathcal{R}^N \rightarrow \mathcal{R}^M$  then  $\left[ \frac{\partial \psi}{\partial x} \right] = \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_M}{\partial x_1} & \cdots & \frac{\partial \psi_M}{\partial x_N} \end{bmatrix}$
- $\psi : \mathcal{R}^{N \times M} \rightarrow \mathcal{R}$  then  $\left[ \left[ \frac{\partial \psi}{\partial x} \right] \right] = \begin{bmatrix} \frac{\partial \psi}{\partial w_{1,1}} & \cdots & \frac{\partial \psi}{\partial w_{1,M}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi}{\partial w_{N,1}} & \cdots & \frac{\partial \psi}{\partial w_{N,M}} \end{bmatrix}$

# Forward Pass

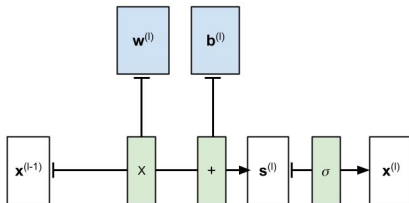




# Goal of Backward Pass

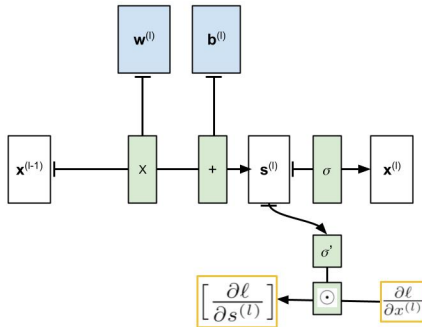


# Begin from succeeding layer

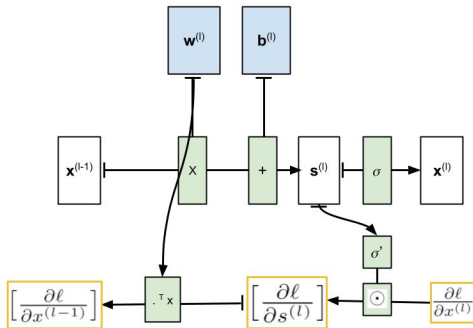


$$\frac{\partial \ell}{\partial \mathbf{x}^{(l)}}$$

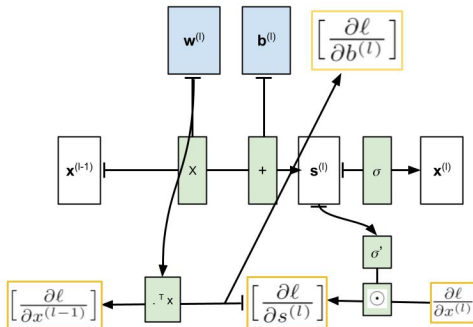
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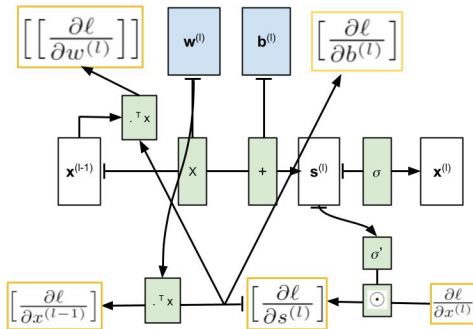
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# Update the parameters

- $W^{(l)} = W^{(l)} - \eta \left[ \left[ \frac{\partial \ell}{\partial w^{(l)}} \right] \right]$  and  $\mathbf{b}^{(l)} = \mathbf{b}^{(l)} - \eta \left[ \frac{\partial \ell}{\partial b^{(l)}} \right]$

# Observations

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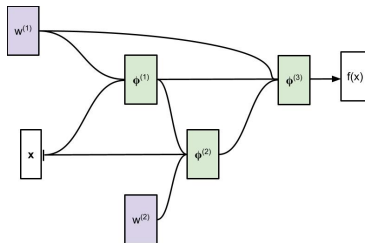
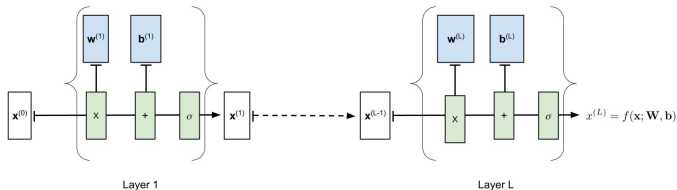
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- Heavy computations are with the linear operations
- Nonlinearities go into simple element wise operations
- In an untreated situation, BP Needs all the intermediate layer results to be in memory
- Takes twice the computations of forward pass

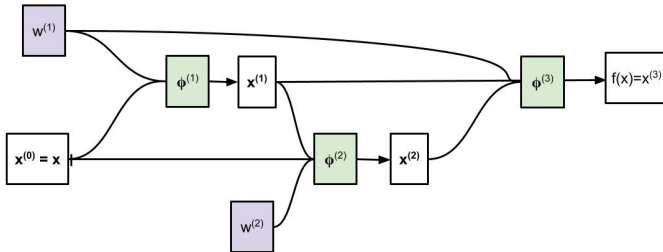
# Beyond MLP

- We can generalize MLP



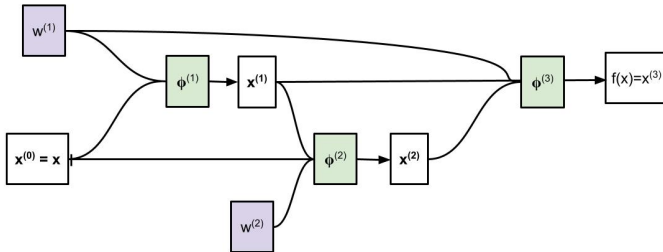
To an arbitrary Directed Acyclic Graph (DAG)

# Forward pass in the computational graph



●  $x^{(0)} = x$

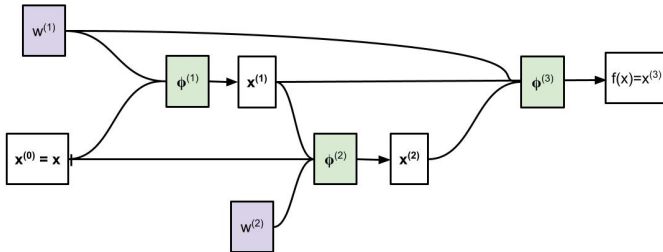
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- $x^{(0)} = x$
- $x^{(1)} = \phi^{(1)}(x^{(0)}; w^{(1)})$

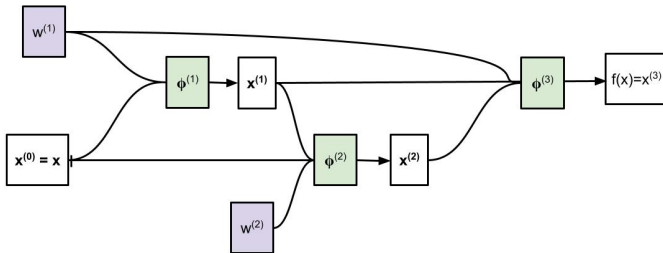


# Forward pass in the computational graph



- $x^{(0)} = x$
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- $x^{(2)} = \phi^{(2)}(x^{(0)}, x^{(1)}; w^{(2)})$

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- $x^{(2)} = \phi^{(2)}(x^{(0)}, x^{(1)}; w^{(2)})$
- $f(x) = x^{(3)} = \phi^{(3)}(x^{(1)}, x^{(2)}; w^{(1)})$

# Notation: Jacobian of a general transformation

if  $(a_1 \dots a_Q) = \phi(b_1 \dots b_R)$  then we use the notation (3)

$$\left[ \frac{\partial a}{\partial b} \right] = J_\phi^T = \begin{bmatrix} \frac{\partial a_1}{\partial b_1} & \cdots & \frac{\partial a_Q}{\partial b_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_1}{\partial b_R} & \cdots & \frac{\partial a_Q}{\partial b_R} \end{bmatrix} \quad (4)$$

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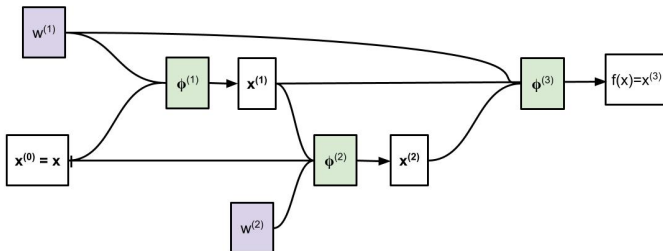
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if  $(a_1 \dots a_Q) = \phi(b_1 \dots b_R; c_1 \dots c_S)$  then we use the notation (5)

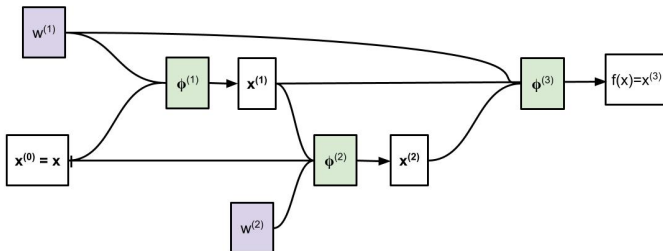
$$\left[ \frac{\partial a}{\partial c} \right] = J_{\phi|c}^T = \begin{bmatrix} \frac{\partial a_1}{\partial c_1} & \dots & \frac{\partial a_Q}{\partial c_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_1}{\partial c_S} & \dots & \frac{\partial a_Q}{\partial c_S} \end{bmatrix} \quad (6)$$

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- From the loss equation, we can compute  $\left[ \frac{\partial \ell}{\partial x^{(3)}} \right]$

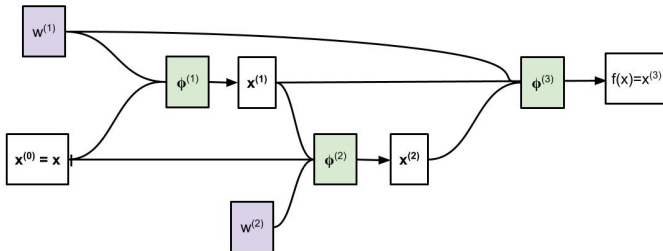
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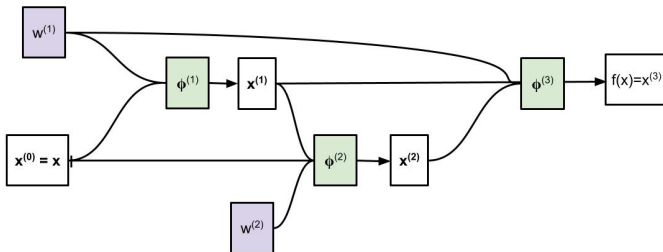


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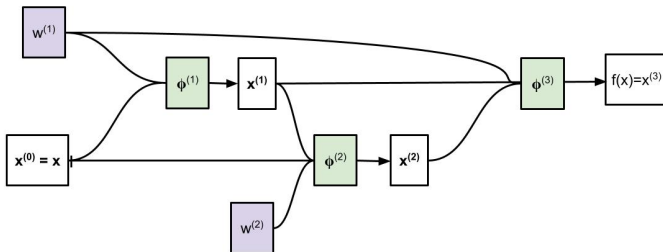
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