

Foundations of Machine Learning

AI2000 and AI5000

FoML-34

Support Vector Machines (cntd.)

Optimization with inequality constraints

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So far in FoML

- Intro to ML and Probability refresher
- MLE, MAP, and fully Bayesian treatment
- Supervised learning
 - a. Linear Regression with basis functions
 - b. Bias-Variance Decomposition
 - c. Decision Theory - three broad classification strategies
 - d. Neural Networks
- Unsupervised learning
 - a. K-Means, Hierarchical, and GMM for clustering
- Kernelizing linear Models
 - a. Dual representation, Kernel trick



For today

- SVM (cntd.)
 - Optimization with inequality constraints

SVM for binary classification

$$\arg \min_{\mathbf{w}, b} \frac{1}{2} \|\mathbf{w}\|^2$$

$$t_n (\mathbf{w}^T \phi(\mathbf{x}_n) + b) \geq 1, \quad n = 1, \dots, N.$$

Constrained optimization (Quadratic programming) problem



Optimization with inequality constraints



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Earlier - equality constraints

- Maximize $f(x)$ with constraints $g(x)=0$
- We exploited: gradients are normal to the levelset $g(x)=0$
- → introduced a Lagrangian function $L(x, \lambda)$
- Stationary points of $L \rightarrow$ solution to the original problem



Optimization with inequality constraints

- Maximize $f(x)$ such that $g(x) \geq 0$
- Two possibilities
 - a. Stationary point lies in region $g(x) \geq 0$ (inactive constraints)
 - →
 - b. Stationary point lies on the boundary $g(x) = 0$ (active constraints)
 - →

$$\text{Primal Lagrangian} \quad L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x})$$



Optimization with inequality constraints

- Maximize $f(x)$ such that $g(x) \geq 0$
- Can be formulated as a max-min optimization problem

$$\max_{\mathbf{x}} \min_{\lambda} L(\mathbf{x}, \lambda) \text{ s.t. } \lambda \geq 0, g(\mathbf{x}) \geq 0, \lambda g(\mathbf{x}) = 0$$



Optimization with inequality constraints

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- Idea is to solve a dual Lagrangian (optimize w.r.t primal variable x for fixed values of λ)

$$\tilde{L}(\lambda) = \max_{\mathbf{x}} L(\mathbf{x}, \lambda) \text{ with } L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x})$$



Optimization with inequality constraints

$$\tilde{L}(\lambda) = \max_{\mathbf{x}} L(\mathbf{x}, \lambda) \text{ with } L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x})$$

- Work the dual Lagrangian analytically
 - Stationarity condition ($\nabla f(\mathbf{x}) = 0$) eliminates \mathbf{x}
 - \rightarrow function of λ
 - This forms an upper bound on the primal max-min problem (as a function of λ)
 - Minimize w.r.t. λ



Optimization with inequality constraints

- Duality gap

- For \mathbf{x}' that satisfies $g(\mathbf{x}') \geq 0$, we have $f(\mathbf{x}') \leq L(\mathbf{x}', \lambda) \leq \tilde{L}(\lambda)$



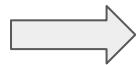
$$\mathbf{p}^* = \max_{\mathbf{x}, g(\mathbf{x}) \geq 0} f(\mathbf{x}) \leq \min_{\lambda} \tilde{L}(\lambda) = \mathbf{d}^*$$



Optimization with inequality constraints

- Duality gap

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$$\mathbf{p}^* = \max_{\mathbf{x}, g(\mathbf{x}) \geq 0} f(\mathbf{x}) \leq \min_{\lambda} \tilde{L}(\lambda) = \mathbf{d}^*$$

Most convex problems exhibit strong duality, i.e., $\mathbf{p}^* = \mathbf{d}^*$



Summary

- Primal problem maximize $f(x)$ subject to $g(x) \geq 0$
 $\max_{\mathbf{x}} \min_{\lambda} L(\mathbf{x}, \lambda) \text{ s.t. } \lambda \geq 0, g(\mathbf{x}) \geq 0, \lambda g(\mathbf{x}) = 0$
- Dual problem (find the lowest upper bound) $\min_{\lambda} \tilde{L}(\lambda)$ subject to $\lambda \geq 0$



Summary

- Primal problem maximize $f(x)$ subject to $g(x) \geq 0$



$$\max_{\mathbf{x}} \min_{\lambda} L(\mathbf{x}, \lambda) \text{ s.t. } \lambda \geq 0, g(\mathbf{x}) \geq 0, \lambda g(\mathbf{x}) = 0$$

- Dual problem (find the lowest upper bound) $\min_{\lambda} \tilde{L}(\lambda)$ **subject to** $\lambda \geq 0$

- Steps

- Define Lagrangian $L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x})$
- Compute the dual $\tilde{L}(\lambda)$
- Solve the dual problem $\lambda^* = \min_{\lambda} \tilde{L}(\lambda)$ **subject to** $\lambda \geq 0$
- Maximize the primal Lagrangian $\mathbf{x}^* = \arg \max_{\mathbf{x}} L(\mathbf{x}, \lambda^*)$



Next

- Kernel SVM