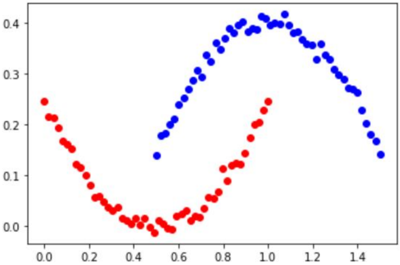
Deep Learning

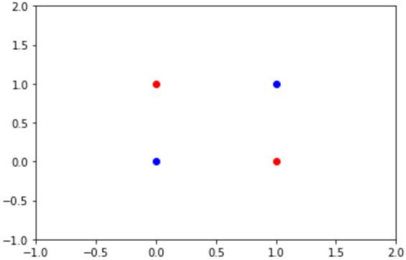
2 Network of Perceptrons

Dr. Konda Reddy Mopuri
Dept. of Artificial Intelligence
IIT Hyderabad
Jan-May 2023

Linear Classifiers: Shortcomings

- Lower capacity: data has to be linearly separable
- Some times no hyper-plane can separate the data (e.g. XOR data)





Pre-processing

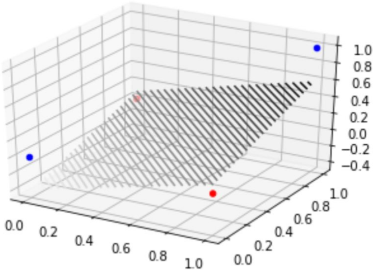
1. Sometimes, data specific pre-processing makes the data linearly separable

Pre-processing

1. Sometimes, data specific pre-processing makes the data linearly separable
2. Consider the xor case
   $\phi(\mathbf{x}) = \phi(x_u, x_v) = (x_u, x_v, x_u x_v)$

Pre-processing

1. Sometimes, data specific pre-processing makes the data linearly separable
2. Consider the xor case
   $\phi(\mathbf{x}) = \phi(x_u, x_v) = (x_u, x_v, x_u x_v)$
3. Consider the perceptron in the new space $f(\mathbf{x}) = \sigma(\mathbf{w}^T \phi(\mathbf{x}) + b)$



Pre-processing

1. (Recap: Polynomial regression): increasing the degree $\rightarrow$ increase the model capacity

Pre-processing

1. (Recap: Polynomial regression): increasing the degree $\rightarrow$ increase the model capacity
2. (Recap: Bias-Variance decomposition): to reduce the bias error, we increased the model capacity

Pre-processing

1. (Recap: Polynomial regression): increasing the degree $\rightarrow$ increase the model capacity
2. (Recap: Bias-Variance decomposition): to reduce the bias error, we increased the model capacity
3. Feature design (or pre-processing) may also be another way to reduce the capacity without affecting (or improving) the bias

Consider the XOR function

x_1	x_2	XOR
0	0	0
0	1	1
1	0	1
1	1	0

Consider the XOR function

- If we attempt to realize XOR function with a single perceptron

x_1	x_2	XOR	
0	0	0	$w_0 < 0$
0	1	1	$w_2 + w_0 \geq 0$
1	0	1	$w_1 + w_0 \geq 0$
1	1	0	$w_1 + w_2 + w_0 < 0$

Consider the XOR function

- If we attempt to realize XOR function with a single perceptron

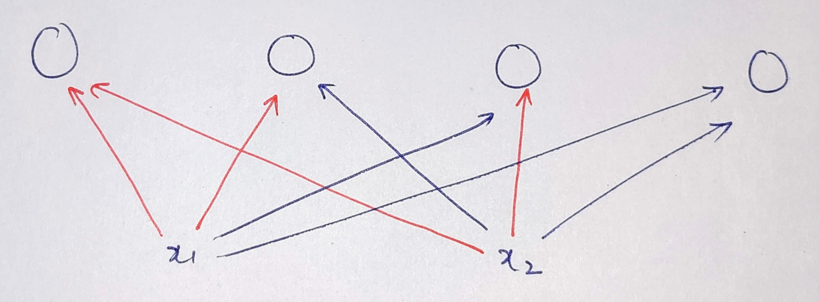
x_1	x_2	XOR	
0	0	0	$w_0 < 0$
0	1	1	$w_2 + w_0 \geq 0 \implies w_2 \geq -w_0$
1	0	1	$w_1 + w_0 \geq 0 \implies w_1 \geq -w_0$
1	1	0	$w_1 + w_2 + w_0 < 0 \implies w_1 + w_2 < -w_0$

Consider the XOR function

- Clearly, a single perception cannot represent the XOR function!

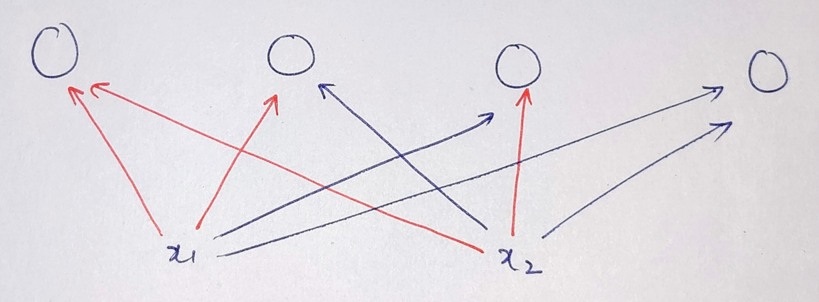
Let's see if multiple perceptions can do this

- Consider 4 perceptions



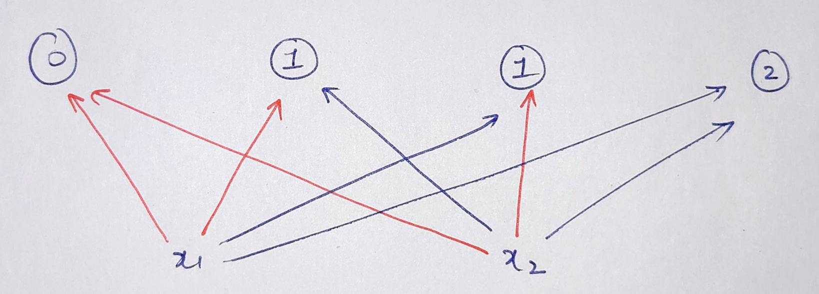
Let's see if multiple perceptions can do this

- Consider 4 perceptions
- $\color{red}{\rightarrow} = +1$ and $\color{blue}{\rightarrow} = -1$



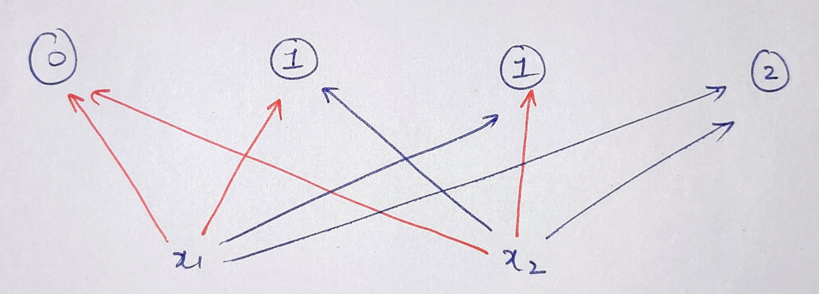
Let's see if multiple perceptions can do this

- Let's have these thresholds



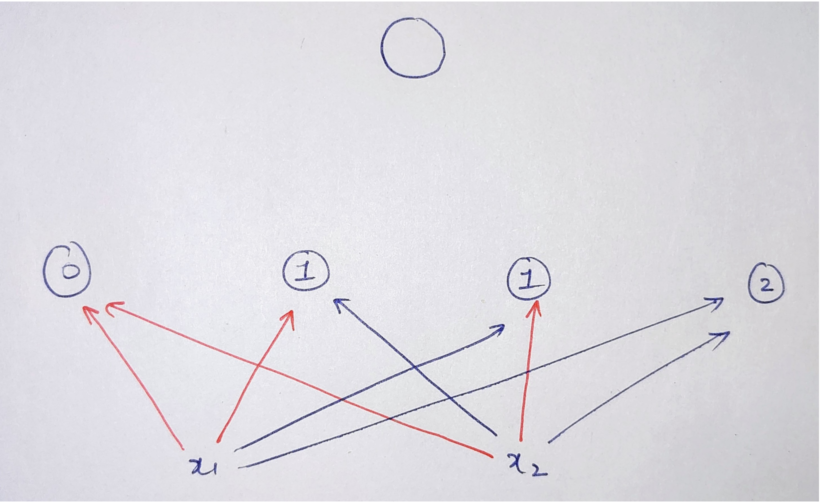
Let's see if multiple perceptions can do this

- Let's have these thresholds
- Notice, each of them fire for exactly one specific input pattern



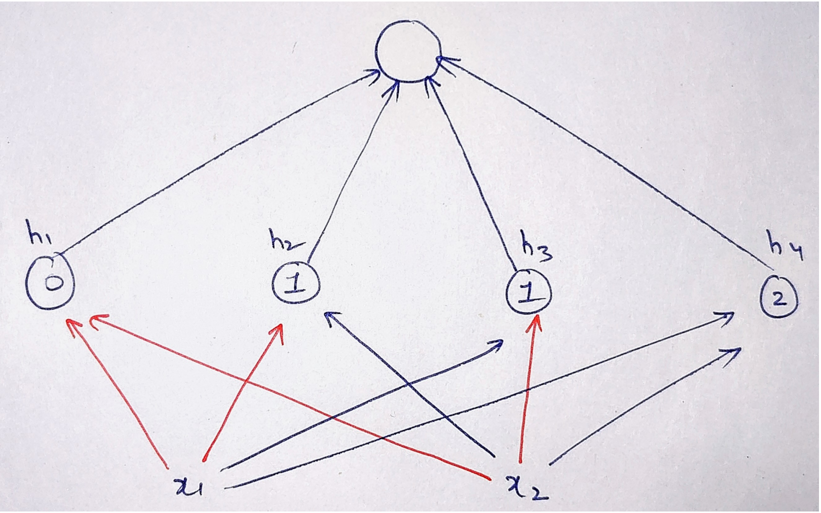
Let's see if multiple perceptions can do this

- Let's now add another perceptron



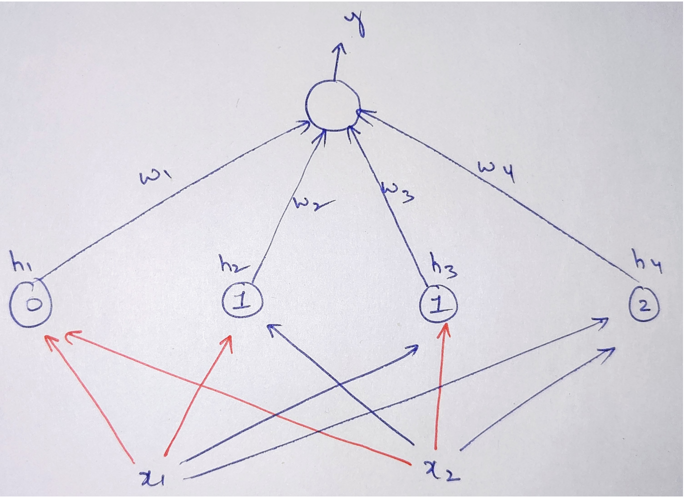
Let's see if multiple perceptions can do this

- Connect the previous (hidden) ones and call it the output perceptron



Let's see if multiple perceptions can do this

- See if we can find a set of weights (W_i) to represent the XOR function



Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0					
0	1					
1	0					
1	1					

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	
0	1					
1	0					
1	1					

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	
0	1	0	1	0	0	
1	0	0	0	1	0	
1	1	0	0	0	1	

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	w_1
0	1	0	1	0	0	w_2
1	0	0	0	1	0	w_3
1	1	0	0	0	1	w_4

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	$w_1 + w_0$
0	1	0	1	0	0	$w_2 + w_0$
1	0	0	0	1	0	$w_3 + w_0$
1	1	0	0	0	1	$w_4 + w_0$

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y	XOR
0	0	1	0	0	0	$w_1 + w_0$	0
0	1	0	1	0	0	$w_2 + w_0$	1
1	0	0	0	1	0	$w_3 + w_0$	1
1	1	0	0	0	1	$w_4 + w_0$	0

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y	XOR
0	0	1	0	0	0	$w_1 + w_0$	0
0	1	0	1	0	0	$w_2 + w_0$	1
1	0	0	0	1	0	$w_3 + w_0$	1
1	1	0	0	0	1	$w_4 + w_0$	0

$w_1 < -w_0$ $\quad w_2 \geq -w_0$ $\quad w_3 \geq -w_0$ $\quad w_4 < -w_0$

Let's see if multiple perceptions can do this

- Clearly possible to find such weights $\rightarrow$ represent the XOR function!

x_1	x_2	h_1	h_2	h_3	h_4	y	XOR
0	0	1	0	0	0	$w_1 + w_0$	0
0	1	0	1	0	0	$w_2 + w_0$	1
1	0	0	0	1	0	$w_3 + w_0$	1
1	1	0	0	0	1	$w_4 + w_0$	0

$w_1 < -w_0 \quad w_2 \geq -w_0 \quad w_3 \geq -w_0 \quad w_4 < -w_0$

What about other 2-input Boolean functions?

- Possible to represent!

What about other 2-input Boolean functions?

- Possible to represent!
- Leads to finding a different set of non-contradicting weights

What if there are more inputs?

- Can do the same with 2^n perceptions in the hidden layer and 1 in the output layer!

What did we just find?

- Any Boolean function of n inputs can be exactly represented with 2^n perceptions in the hidden layer and 1 in the output layer!

What did we just find?

- Any Boolean function of n inputs can be exactly represented with 2^n perceptions in the hidden layer and 1 in the output layer!
- Note that $2^n + 1$ is a sufficient but not necessary

What did we just find?

- Any Boolean function of n inputs can be exactly represented with 2^n perceptions in the hidden layer and 1 in the output layer!
- Note that $2^n + 1$ is a sufficient but not necessary
- Caveat: the size of the hidden layer grows exponentially!

Network of Perceptrons

- Generally referred to as MLP (Multi-Layered Network of Perceptrons)

Moving on from Boolean functions

- $y = f(x)$, where $x \in \mathcal{R}^n$ and $y \in \mathcal{R}$

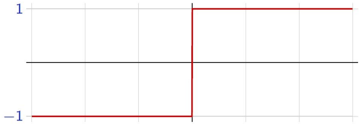
Moving on from Boolean functions

- $y = f(x)$, where $x \in \mathcal{R}^n$ and $y \in \mathcal{R}$
- Can MLPs represent such functions?

Threshold-ing is very harsh!

1. Perceptron's o/p is discontinuous!

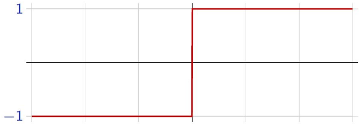
$$\sigma(x) = \begin{cases} 1 & \text{when } x \geq 0 \\ -1 & \text{else} \end{cases}$$



Threshold-ing is very harsh!

1. Perceptron's o/p is discontinuous!

$$\sigma(x) = \begin{cases} 1 & \text{when } x \geq 0 \\ -1 & \text{else} \end{cases}$$



2. Think of inputs -0.0001 and 0

Enough of Boolean functions!

1. Many real world problems have non-binary outputs

Enough of Boolean functions!

1. Many real world problems have non-binary outputs
2. Perceptron only gives two outputs!

Enough of Boolean functions!

1. Many real world problems have non-binary outputs
2. Perceptron only gives two outputs!
3. Sigmoid neuron

$$f(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{1} + \mathbf{e}^{-\mathbf{w}^{\mathbf{T}}\mathbf{x}}}$$