Foundations of Machine Learning
AI2000 and AI5000

FoML-06
Linear Regression

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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So far in FoML

- What is ML and the learning paradigms
- Probability refresher
- MLE, MAP, and fully Bayesian treatment

Linear Regression

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Linear Regression

- Dataset D $= \{ (\underline{x}_1, t_1) (\underline{x}_2, t_2) \dots (\underline{x}_N, t_N) \}$

$\underline{w} \in \mathbb{R}^D$

- Input variable $x_i \in \mathbb{R}^D$
- Output variable $t_i \in \mathbb{R}$
- Simplest linear model

$$\hat{t}_i = y(\underline{x}_i, \underline{w}) = w_0 + w_1 \underline{x_i}_1 + w_2 \underline{x_i}_2 + \dots + w_D \underline{x_i}_D$$

$$= w_0 + \underline{w}^T x_i$$

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Linear Regression

$w_0 = 0$

$x \in \mathbb{R}$

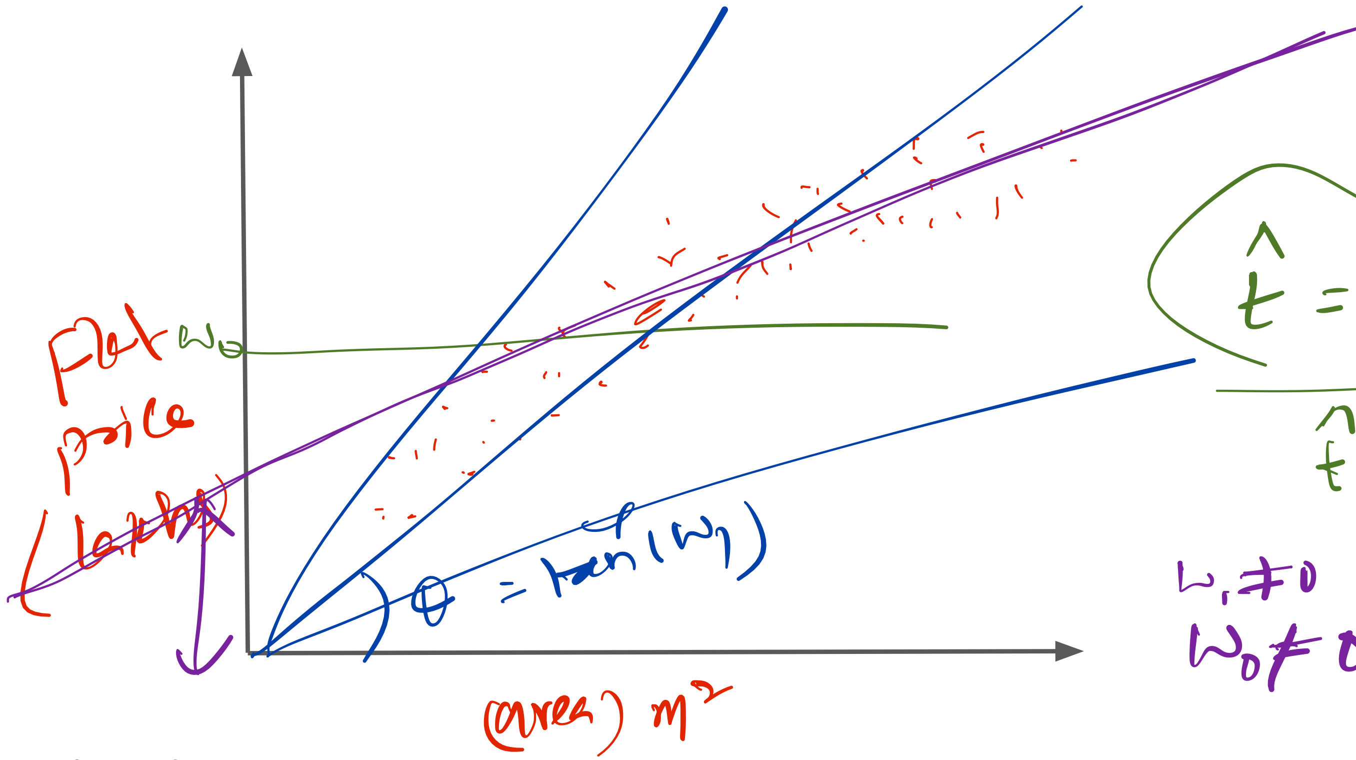
$t \in \mathbb{R}$

$\underline{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$

$w_1 = 0$

$$\boxed{\hat{t} = w_0 + w_1 x}$$

$\hat{t} = w_0$



$w_1 \neq 0$

$w_0 \neq 0$

Linear Basis function Models

- Fix the number of parameters M s.t. $w_0 \in \mathbb{R}, \underline{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_{M-1} \end{bmatrix} \in \mathbb{R}^{M-1}$
- Choose M-1 basis functions **x**: $\phi_i(\underline{x}) \in \mathbb{R} \quad i = 1, 2, \ldots M-1$

  $\phi_i(\underline{x}) : \mathbb{R}^D \longrightarrow \mathbb{R}$
- Mapping/Approximation:

$$y(\mathbf{x}, \mathbf{w}) = w_0 + w_1 \phi_1(\underline{x}) + w_2 \phi_2(\underline{x}) + \cdots + w_{M-1} \phi_{M-1}(\underline{x})$$

$$y(\underline{x}, \underline{w}) = \hat{t} = w_0 + \underline{\phi}^T \cdot \underline{w} \qquad \underline{\phi} = \begin{bmatrix} \phi_1(\underline{x}) \\ \vdots \\ \phi_{M-1}(\underline{x}) \end{bmatrix}$$

$$= \underline{\tilde{\phi}}^T \cdot \underline{\tilde{w}}$$

$$\underline{\tilde{w}} = \begin{bmatrix} w_0 \\ \underline{w}_1 \\ \vdots \\ \underline{w}_{M-1} \end{bmatrix} \qquad \underline{\tilde{\phi}} = \begin{bmatrix} \phi_0(\underline{x}) = 1 \\ \phi_1(\underline{x}) \\ \vdots \\ \phi_{M-1}(\underline{x}) \end{bmatrix}$$

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Example Basis functions

- Components of input

$\underline{x} = (x_1\ x_2 \cdots x_D)^T$ $M = D$

$\hat{t}(\underline{x}, \underline{w}) = w_0 + \sum_{i=1}^{M-1} w_i \phi_i(\underline{x})$ where $\phi_i(\underline{x}) = \underline{x}_i$

- Powers of input

$x \in \mathbb{R}$

$\underline{w} \in \mathbb{R}^M$

$\phi_i(x) = x^i$

$$\hat{t}(\underline{w}, x) = \sum_{i=0}^{M-1} w_i \cdot \phi_i(x)$$

$$= w_0 + w_1 x + w_2 x^2 + \cdots + w_{M-1} x^{M-1}$$

$$= \underline{w}^T \cdot \underline{\phi}(x)$$

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Example Basis Function

- Gaussian basis functions

$$\phi_i(\underline{x}) = e^{-\frac{1}{2}(\underline{x}-\mu_i)^T \Sigma_i^{-1}(\underline{x}-\mu_i)}$$

$\underline{x} \in \mathbb{R}^D$

$$\hat{t}(\underline{x}, \underline{w}) = w_0 + \sum_{i=1}^{M-1} w_i \phi_i(\underline{x})$$

Hyper parameters

μ_i, Σ_i, M

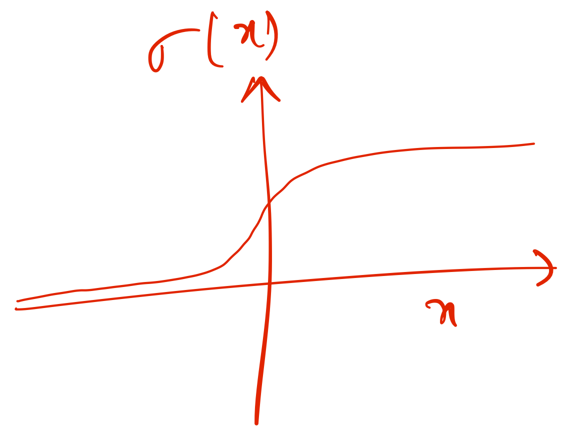
Example Basis Function

- Logistic sigmoid basis functions

$$\hat{t}_i = w_0 + \sum_{i=1}^{m} w_i \cdot \phi_i(x)$$

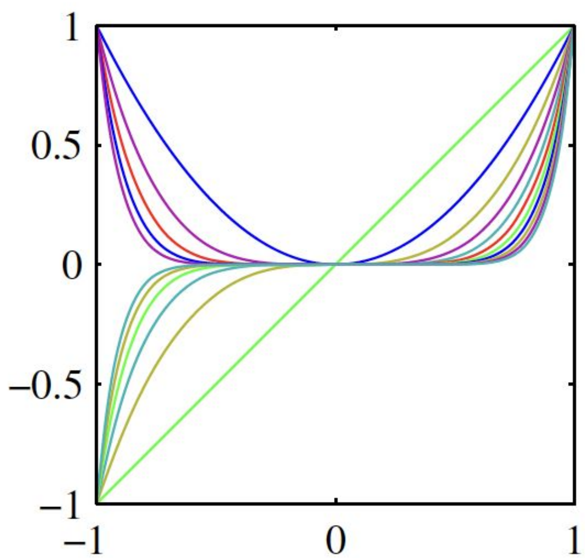
$$\phi_i(x) = \sigma\left(\frac{x-\mu_i}{s_i}\right)$$

$$x, \mu_i, s_i \in \mathbb{R}$$

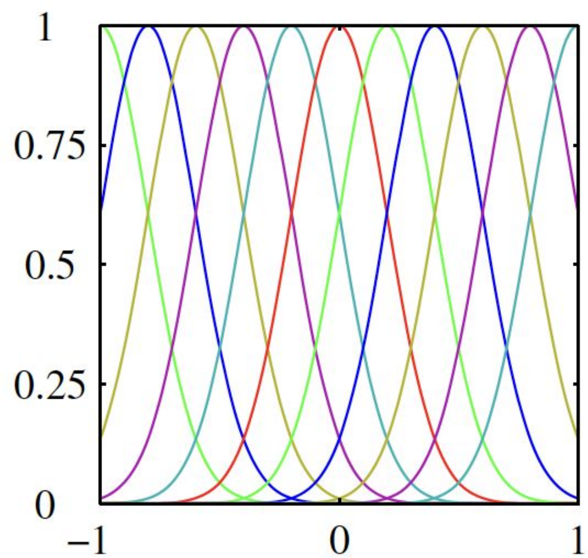


Example Basis Function

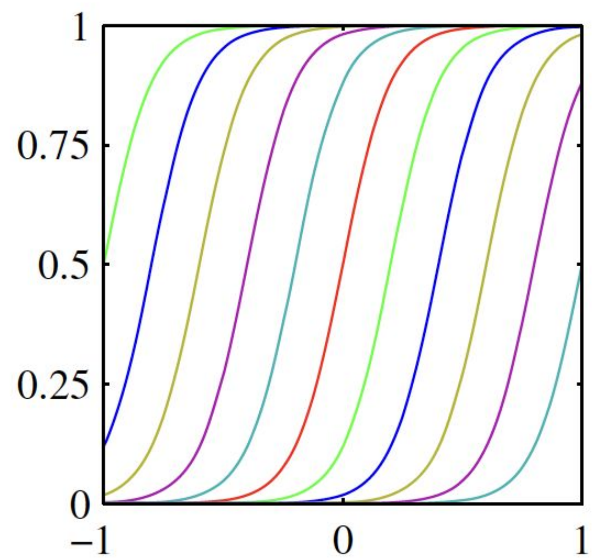
$$\phi_i(x) = \sigma\left(\frac{x-\mu_i}{s_i}\right)$$



$$\phi_i(x) = x^i$$



$$\phi_i(x) = e^{-\frac{1}{2\sigma_i^2}(x-\mu_i)^2}$$



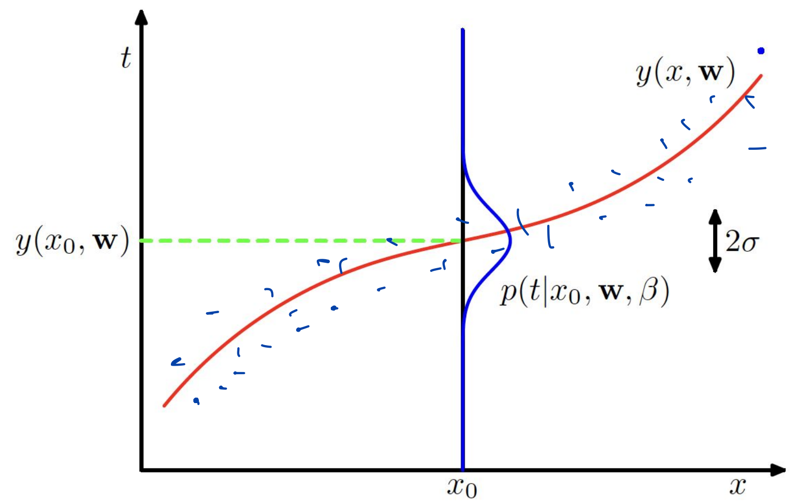
Linear Regression via MLE

Linear Regression

- Given data D $D = \{(x_1, t_1), (x_2, t_2), \ldots (x_N, t_N)\} = \{\mathbf{x}, \mathbf{t}\}$

Input variables x

Target variables t



Linear Model with basis functions

$$y(\mathbf{x}, \mathbf{w}) = w_0 + \sum_{i=1}^{M-1} \phi_i(x) w_i$$

Maximum Likelihood

- Assume Gaussian noise around the target

$$t = y(x, \mathbf{w}) + \sigma \cdot \epsilon, \quad \epsilon \in \mathcal{N}(0,1)$$

$\beta^{-1} = \sigma^2$

Maximum Likelihood

- Assume Gaussian noise around the target

$$t = y(x, \mathbf{w}) + \sigma \cdot \epsilon, \quad \epsilon \in \mathcal{N}(0, 1)$$

$$p(t|x, \mathbf{w}, \beta) =$$

Data matrix

Targets vector

ML: sum of squares error

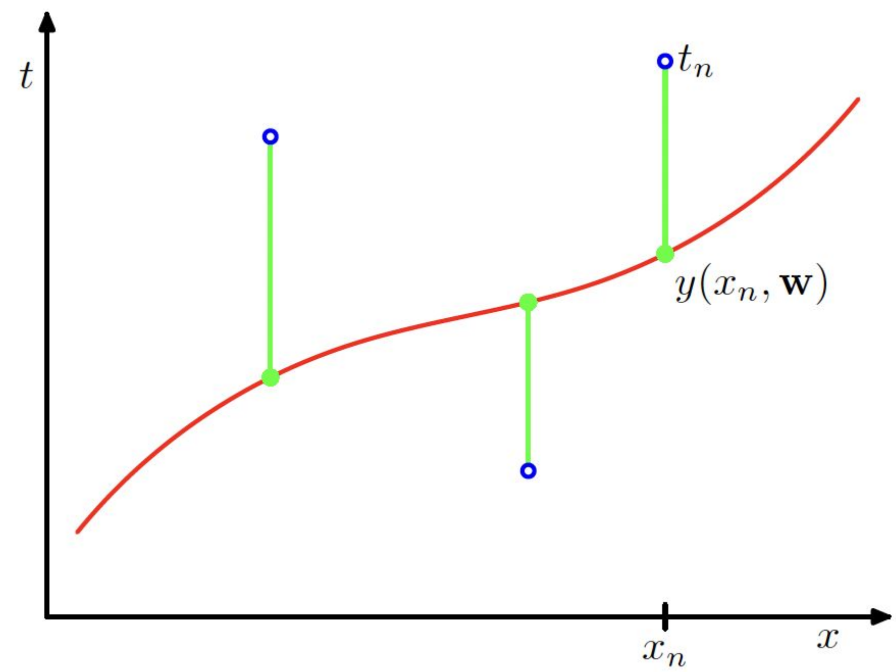
- Likelihood

$$p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta) = \prod_{i=1}^{N} \mathcal{N}(t_i|\mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}_i), \beta^{-1})$$

NLL =

Sum-of-squared error E_D (w) =

ML: sum of squares error



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ML Estimates

- Minimize the NLL (or, the sum of squared errors)

ML Estimates

- Optimal w* satisfies

$$\mathbb{E}[t'|\mathbf{x}', \mathbf{w_{ML}}] =$$

Next
SGD