ECEN 314: Signals and Systems - Homework 1

- Date Assigned: Monday 08/31/2021
- Date Due: Wednesday, 09/07/2021

I Reading Exercise

Notes on mathematical preliminaries

II Problems

1. Express each of the following complex numbers in polar form and plot them in the complex plane, indicating the magnitude and angle (or, phase) of each complex number

   (a) $(\sqrt{3}+j^3)(1-j)$

   (b) $\frac{2-j(6/\sqrt{3})}{2+j(6/\sqrt{3})}$

   (c) $(\sqrt{3}+j)\ 2\sqrt{2}\ e^{-j\pi/4}$

   Solution :

   Problem 1 a)

   $$\begin{aligned}(\sqrt{3}+j^3)(1-j) &= (\sqrt{3}-j)(1-j) \quad [\text{since } j^3 = j^2 \times j = (-1)\times j] \\ &= (\sqrt{3}-\sqrt{3}j - j - 1) \\ &= (-1+\sqrt{3}) + j(-1-\sqrt{3})\end{aligned}$$

   $$\begin{aligned}\text{Magnitude} &= \sqrt{(-1+\sqrt{3})^2 + (-1-\sqrt{3})^2} \\ &= \sqrt{(1-2\sqrt{3}+3)+(1+2\sqrt{3}+3)} \quad [(a+b)^2 = (a^2+2ab+b^2)] \\ &= \sqrt{2+6} \ = \ \sqrt{8} \ = \ 2\sqrt{2}\end{aligned}$$

   $$\text{angle or phase} = \tan^{-1}\frac{(-1-\sqrt{3})}{(-1+\sqrt{3})} = -75^\circ$$

   Problem 1 b)

   $$\begin{aligned}\frac{2-j(6/\sqrt{3})}{2+j(6/\sqrt{3})} &= \frac{(2-j(6/\sqrt{3}))^2}{(2+j(6/\sqrt{3}))(2-j(6/\sqrt{3}))} \\ &= \frac{4-j(\frac{24}{\sqrt{3}})-\frac{36}{3}}{4+\frac{36}{3}} = \frac{-8-j8\sqrt{3}}{16} \\ &= -\frac{1}{2} - j\frac{\sqrt{3}}{2}\end{aligned}$$

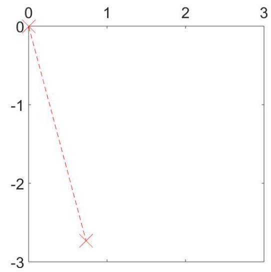

Figure 1: Problem 1 a)

$$\text{magnitude} = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$
$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$
$$\text{angle} = \tan^{-1}(\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}) = 240^\circ$$

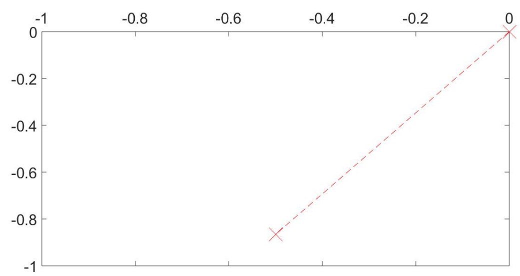

Figure 2: Problem 1 b)

Problem 1 c)

$(\sqrt{3} + j)2\sqrt{2}e^{-j\frac{\pi}{4}} = (2\sqrt{6} + j2\sqrt{2})e^{-j\frac{\pi}{4}}$

$|2\sqrt{6} + j2\sqrt{2}| = \sqrt{4 \cdot 6 + 4 \cdot 2} = \sqrt{24 + 8} = \sqrt{32} = 2\sqrt{8} = 4\sqrt{2}$

$\measuredangle(2\sqrt{6} + j2\sqrt{2}) = \tan^{-1}(\frac{2\sqrt{2}}{2\sqrt{6}}) = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$

$(2\sqrt{6} + j2\sqrt{2})e^{-j\frac{\pi}{4}} = 4\sqrt{2}e^{j\frac{\pi}{6}}e^{-j\frac{\pi}{4}} = 4\sqrt{2}e^{-j\frac{\pi}{12}}$

Hence, magnitude $= 4\sqrt{2}$

$\measuredangle = -\frac{\pi}{12} = -15^\circ$

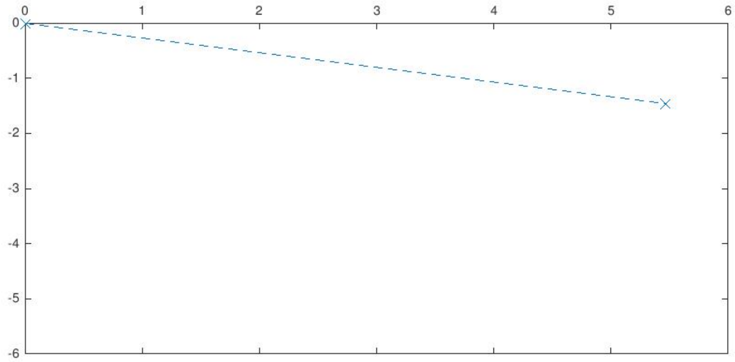

Figure 3: Problem 1 c)

2. Using Euler's relation, derive the following relationships.
   (a) $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
   (b) $\sin\theta\ \sin\phi = \frac{1}{2}\cos(\theta-\phi) - \frac{1}{2}\cos(\theta+\phi)$

   Solution :
   Problem 2 a)
   prove: $\cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta)$

$$
\begin{aligned}
\cos^2\theta &= (\frac{1}{2}(e^{j\theta}+e^{-j\theta}))^2 = \frac{1}{4}(e^{2j\theta}+2e^{j\theta}e^{-j\theta}+e^{-2j\theta}) \\
&= \frac{1}{4}(2+e^{2j\theta}+e^{-2j\theta}) = \frac{1}{4}(2+2\cos 2\theta) \\
&= \frac{1}{2}(1+\cos\theta)
\end{aligned}
$$

   Problem 2 b)
   prove : $\sin\theta\sin\phi = \dfrac{1}{2}\cos(\theta-\phi) - \dfrac{1}{2}\cos(\theta+\phi)$

   Proof :

$$
\begin{aligned}
\sin\theta\sin\phi &= \frac{1}{2j}(e^{j\theta}-e^{-j\theta})\frac{1}{2j}(e^{j\phi}-e^{-j\phi}) \\
&= \frac{-1}{4}(e^{j\theta}-e^{-j\theta})(e^{j\phi}-e^{-j\phi}) \\
&= \frac{-1}{4}(e^{j(\theta+\phi)}-e^{j(\theta-\phi)}-e^{-j(\theta-\phi)}+e^{-j(\theta+\phi)}) \\
&= \frac{-1}{4}(\{e^{j(\theta+\phi)}+e^{-j(\theta+\phi)}\}-\{e^{j(\theta-\phi)}+e^{-j(\theta-\phi)}\}) \\
&= \frac{-1}{4}(2\cos(\theta+\phi)-2\cos(\theta-\phi)) \\
&= \frac{1}{2}(\cos(\theta-\phi)-\cos(\theta+\phi))
\end{aligned}
$$

3. Derive the following relations where z, z_1 and z_2 are arbitrary complex numbers.

   (a) $(e^z)^* = e^{z^*}$

   (b) $z_1 z_2^* + z_1^* z_2 = 2\Re\{(z_1 z_2^*)\} = 2\Re\{z_1^* z_2\}$

   (c) $|z_1 z_2^* + z_1^* z_2| \leq 2|z_1 z_2|$

Solution :

Problem 3 a)

$(e^z)^* = (e^{z^*})$

Proof:

$$\begin{aligned} e^z &= e^{a+jb} = e^a(\cos(b) + j\sin(b)) \\ (e^z)^* &= e^a(\cos(b) + j\sin(b))^* = e^a(\cos(b) - j\sin(b)) = e^a(\cos(-b) + j\sin(-b)) \\ &= e^a e^{-jb} = e^{a-jb} = e^{(a+jb)^*} = e^{z^*} \end{aligned}$$

Problem 3 b)

$z_1 {z_2}^* + {z_1}^* z_2 = 2R(z_1 {z_2}^*)$

Proof:

Let $r_1 = magnitude(z_1), \quad \theta_1 = \measuredangle z_1, \quad r_2 = magnitude(z_2), \quad \theta_2 = \measuredangle z_2$

$$\begin{aligned} z_1 {z_2}^* + {z_1}^* z_2 &= r_1 r_2 e^{j(\theta_1 - \theta_2)} + r_1 r_2 e^{-j(\theta_1 - \theta_2)} \\ &= 2 r_1 r_2 \cos(\theta_1 - \theta_2) \end{aligned}$$

$$\begin{aligned} 2Re\{z_1 {z_2}^*\} &= 2Re\{r_1 r_2 e^{j(\theta_1 - \theta_2)}\} \\ &= 2Re\{r_1 r_2(\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2))\} \\ &= 2 r_1 r_2(\cos(\theta_1 - \theta_2)) \end{aligned}$$

Hence proved

Problem 3 c)

$|z_1 {z_2}^* + {z_1}^* z_2| \leq 2|z_1 z_2|$

Proof :

Note that $2|z_1 z_2| = 2r_1 r_2$

$|z_1 {z_2}^* + {z_1}^* z_2| = |2r_1 r_2 \cos(\theta_1 - \theta_2)| = 2r_1 r_2 |\cos(\theta_1 - \theta_2)|$

But we know from trigonometry that $0 \leq |\cos(\theta_1 - \theta_2)| \leq 1$

Hence $0 \leq 2r_1 r_2 |\cos(\theta_1 - \theta_2)| \leq 2r_1 r_2 = 2|z_1 z_2|$

4. Using the geometric sum formulas, evaluate each of the following sums and express your answer in Cartesian form.
   (a) $\sum_{n=0}^{9} \left(\frac{1}{2}\right)^n e^{-j2\pi n/10}$ (you can use a calculator to evaluate the expression in the end)
   (b) $\sum_{k=0}^{N-1} e^{\frac{j2\pi k}{N}}$
   (c) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{2}\right)$

Solution :

Problem 4 a)

$$\begin{aligned}
\sum_{n=0}^{9} \left(\frac{1}{2}\right)^n e^{-j2\pi n/10} &= \sum_{n=0}^{9} \left(\frac{1}{2} e^{-j2\pi/10}\right)^n \\
&= \frac{1 - (\frac{1}{2} e^{-j2\pi/10})^{10}}{1 - \frac{1}{2} e^{-j2\pi/10}} \\
&= \frac{1 - (\frac{1}{2})^{10}}{1 - \frac{1}{2} e^{-j2\pi/10}} \\
&= 1.3491 - 0.6658i
\end{aligned}$$

Problem 4 b)

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}} = \frac{(1 - e^{j2\pi})}{1 - e^{j\frac{2\pi}{N}}} = \frac{1 - 1}{1 - e^{j\frac{2\pi}{N}}} = 0$$

Problem 4 c)

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}\cos\left(\frac{\pi n}{2}\right) &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{e^{j\frac{\pi}{2}}}{2}\right)^{n}+\sum_{n=0}^{\infty}\left(\frac{e^{-j\frac{\pi}{2}}}{2}\right)^{n}\right] \\
&= \frac{1}{2}\left[\frac{1}{1-\frac{e^{j\frac{\pi}{2}}}{2}}+\frac{1}{1-\frac{e^{-j\frac{\pi}{2}}}{2}}\right] \\
&= \frac{1}{2}\left[\frac{1}{1-\frac{j}{2}}+\frac{1}{1+\frac{j}{2}}\right] \\
&= \frac{1}{2}\left[\frac{1+\frac{j}{2}+1-\frac{j}{2}}{1+\frac{1}{4}}\right] \\
&= \frac{1}{2}\left(\frac{2}{\frac{5}{4}}\right) = \frac{1}{2}\left(\frac{8}{5}\right) = \frac{8}{10} = \frac{4}{5}
\end{aligned}$$

5. Evaluate each of the following integrals and express your answer in Cartesian form.
   (a) $\int_0^6 e^{j\pi t/2}\ dt$
   (b) $\int_0^\infty e^{-2t}\ \sin(3t)\ dt$
   (c) $\int_0^1 te^{-j\omega t}\ dt$. Your answer should be in terms of ω.

Solution :

Problem 5 a)

$$\begin{aligned}
\int_0^6 e^{j\frac{\pi t}{2}}dt &= \frac{2}{j\pi}[e^{j\frac{\pi t}{2}}]_0^6 = \frac{2}{j\pi}[e^{j3\pi}-1] \\
&= \frac{2}{j\pi}[-1-1] = \frac{-4}{j\pi} = \frac{-4j}{j^2\pi} = \frac{4j}{\pi}
\end{aligned}$$

Problem 5 b)

$$\begin{aligned}
\int_0^\infty e^{-2t}\sin(3t)dt &= \frac{1}{2j}\int_0^\infty e^{-2t}(e^{j3t}-e^{-j3t})dt \\
&= \frac{1}{2j}\left[\frac{1}{-2+3j}[e^{(-2+3j)t}]_0^\infty - \frac{1}{-2-3j}[e^{(-2-3j)t}]_0^\infty\right] \\
&= \frac{1}{2j}\left(\frac{1}{-2+3j}(0-1)-\frac{1}{-2-3j}(0-1)\right) \\
&= \frac{1}{2j}\left(\frac{-1}{-2+3j}+\frac{1}{-2-3j}\right) \\
&= \frac{1}{2j}\left(\frac{2+3j-2+3j}{4+9}\right) = \frac{1}{2j}\left(\frac{6j}{13}\right) = \frac{3}{13}
\end{aligned}$$

Problem 5 c)

$$
\begin{aligned}
\int_0^1 te^{-jwt}dt &= \left[(\frac{t}{-jw} - \frac{1}{(jw)^2})e^{-jwt} \right]_0^1 \quad \text{[Integration by parts]} \\
&= \big((\frac{1}{-jw} + \frac{1}{(w)^2})e^{-jw}\big) - \big(\frac{1}{w^2}\big) \\
&= \big((\frac{j}{w} + \frac{1}{(w)^2})e^{-jw}\big) - \big(\frac{1}{w^2}\big) \\
&= \frac{1+jw}{w^2}e^{-jw} - \frac{1}{w^2} = \frac{1+jw}{w^2}(\cos(w) - j\sin(w)) - \frac{1}{w^2} \\
&= \frac{\cos(w) - j\sin(w) + jw\cos(w) + w\sin(w)}{w^2} - \frac{1}{w^2} \\
&= \frac{\cos(w) + w\sin(w) - 1 + j(w\cos(w) - \sin(w))}{w^2}
\end{aligned}
$$

6. Find all the fifth roots of $z = 2e^{j\pi/4}$.

Solution :

Problem 6

$z = 2e^{j\frac{\pi}{4}}$

The fifth roots are of the form $re^{j\theta}$ such that

$r = 2^{1/5}, \quad \theta = \dfrac{\frac{\pi}{4} + 2k\pi}{5}, \; k = 0, 1, 2, 3, 4$