TheChandrasekharMass- Homework 2

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1 Assignment 2 Astro 404 Kennedy Robinson

1.1 The Chandrasekhar mass limit for the white-dwarf models with $T = 0$ (ASTR 404)

The structure of a non-rotating spherically symmetric star is determined by the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -g\rho,$$

where P and ρ are the pressure and density, $g = \frac{d\varphi}{dr} = \frac{GM_r}{r^2}$ is the gravitational acceleration on the surface of a sphere of the radius r and mass

$$M_r = \int_0^r 4\pi r^2 \rho dr,$$

and φ is the gravitational potential on the sphere.

For a known distribution of gas particle's momentum $n(p)$, the pressure is given by the following integral:

$$P = \frac{1}{3}\int_0^\infty \frac{dn(p)}{dp} v(p) p dp,$$

where $v = \frac{dE(p)}{dp}$ is particle's velocity expressed via its energy $E(p) = \sqrt{m^2c^4 + p^2c^2}$.

White dwarfs are the stars in which the hydrostatic equilibrium is supported by the pressure of electron-degenerate gas. The zero-temperature assumption means that in their phase space electrons must occupy all available states with the minimum possible energy $E(p) \leq E(p_0)$. This simplifies the computation of their equation of state that is now defined by the Pauli exclusion and Heisenberg's uncertainty principles. According to them, only two electrons with opposite spins can occupy the elementary cell of the phase space h^3. Therefore, the number of electrons $dN(p)$ with momenta between p and $p + dp$ inside a volume V that can be combined in pairs with opposite spins and placed in the elementary cells is

$$dN(p) = 2\frac{4\pi p^2 dp V}{h^3},$$

which gives

$$\frac{dn(p)}{dp} = \frac{1}{V}\frac{dN(p)}{dp} = \frac{8\pi p^2}{h^3}.$$

The pressure of the electron-degenerate gas at $T = 0$ is therefore

$$P = \frac{8\pi}{3h^3}\int_0^{p_0}\frac{dE(p)}{dp}p^3 dp,$$

where p_0 is the Fermi momentum that is constrained by the electron number density

$$n = \frac{8\pi}{h^3}\int_0^{p_0} p^2 dp = \frac{8\pi}{3h^3}{p_0}^3.$$

The latter is related to the mass density as $\rho = \mu_{\mathrm{e}} M_{\mathrm{u}} n$, where $\mu_{\mathrm{e}} = (\Sigma X_i \frac{Z_i}{A_i})^{-1}$ is the electron mean molecular weight and M_{u} is the atomic mass unit, with X_i, Z_i, and A_i being the mass fraction, atomic number and mass number of the i-th nucleus. For a CO white dwarf, $\mu_{\mathrm{e}} = 2$.

1.1.1 This information is sufficient to estimate one of the most important quantity in the stellar astrophysics - the Chandrasekhar mass limit for white dwarfs, M_{Ch}.

[54]:
```
from __future__ import division
from sympy import *
import numpy as np
import matplotlib.pyplot as plt

# physical constants
import scipy.constants as sc

from scipy.integrate import odeint
from scipy import interpolate

# astronomical constants
from astropy import constants as ac

fsize=16
plt.rcParams['font.size'] = 14
plt.rcParams['font.family'] = 'serif'

init_printing()
```

[55]:
```
# x - dimensionless Fermi momentum
# p - momentum of an electron
# E - energy
# v = dE/dp - velocity
# P - pressure
# n - electron number density
# m - mass of electron
```

```
# c - the speed of light in vacuum
# h - Planck constant
# p0 - Fermi momentun of electrons
# mue - electron mean molecular weight
# NA - Avogadro's number
# AMU - atomic mass unit
x, p, E, v, P, n, m, c, h, p0, mue, G, NA, AMU = symbols('x p E v P n m c h p0␣
↪mue G NA AMU')
```

[56]:
```
E=sqrt(p**2*c**2+m**2*c**4)
print ("The electron energy:")
E
```

The electron energy:

[56]:

$$\sqrt{c^4m^2 + c^2p^2}$$

[57]:
```
v=diff(E,p)
print ("The electron velocity:")
v
```

The electron velocity:

[57]:

$$\frac{c^2p}{\sqrt{c^4m^2 + c^2p^2}}$$

[58]:
```
f=(8*pi/(3*h**3))*v*p**3
P_int=Integral(f,p)
print ("The electron pressure integral:")
P_int
```

The electron pressure integral:

[58]:

$$\int \frac{8\pi c^2p^4}{3h^3\sqrt{c^4m^2 + c^2p^2}}\,dp$$

This integral has to be taken from 0 to p_0. From its expression, it follows that $\frac{dP}{dx} = \frac{8\pi}{3}\frac{mc^2}{\lambda^3}\frac{x^4}{\sqrt{1+x^2}}$, where $x = \frac{p_0}{mc}$ is the dimensionless Fermi momentum and $\lambda = \frac{h}{mc}$ is the De Broglie wavelength of ultra-relativistic electrons.

[59]:
```
print ("The integrated pressure as a function of p0:")
P=integrate(f,(p,0,p0))
P
```

The integrated pressure as a function of p0:

[59]:

$$\frac{8\pi c^2\left(\frac{3c^3m^4\operatorname{asinh}\left(\frac{p_0}{cm}\right)}{8} - \frac{3c^2m^3p_0}{8\sqrt{1+\frac{p_0^2}{c^2m^2}}} - \frac{mp_0^3}{8\sqrt{1+\frac{p_0^2}{c^2m^2}}} + \frac{p_0^5}{4c^2m\sqrt{1+\frac{p_0^2}{c^2m^2}}}\right)}{3h^3}$$

[60]:
```
# x=p0/(m*c) is the relativization parameter
print ("The integrated pressure as a function of x:")
Px=simplify(P.subs(p0,c*m*x))
Px
```

The integrated pressure as a function of x:

[60]:
$$\frac{\pi c^5 m^4 \left(2x^5 - x^3 - 3x + 3\sqrt{x^2+1}\operatorname{asinh}(x)\right)}{3h^3\sqrt{x^2+1}}$$

[61]:
```
A=(pi/3)*(c**5*m**4)/h**3
A
```

[61]:
$$\frac{\pi c^5 m^4}{3h^3}$$

[62]:
```
A0=(A.subs({c:sc.c, m:sc.m_e, h:sc.h})).evalf()
print ("A0=",A0)
```

```
A0= 6.00233218566044e+21
```

1.1.2 In fact, we don't even need to know the pressure in order to find the Chandrasekhar mass limit. It is sufficient to know the pressure derivative with respect to the Fermi momentun that we have already found.

[63]:
```
f=(8*pi/(h**3))*p**2
n_int=Integral(f,p)
print ("The electron number density integral:")
n_int
```

The electron number density integral:

[63]:
$$\int \frac{8\pi p^2}{h^3}\,dp$$

[64]:
```
print ("The integrated electron number density:")
n_int=integrate(f,(p,0,p0))
nx=n_int.subs(p0,c*m*x)
nx
```

The integrated electron number density:

[64]:
$$\frac{8\pi c^3 m^3 x^3}{3h^3}$$

[65]:
```
print ("The integrated mass density:")
nrho=nx*mue*AMU
nrho
```

The integrated mass density:

[65]: $\displaystyle \frac{8\pi AMUc^3m^3muex^3}{3h^3}$

So, the mass density is $\rho = \frac{8\pi}{3}\frac{\mu_e M_u}{\lambda^3}x^3 = B_0\mu_e x^3$, where $M_e = N_A^{-1}$ is the atomic mass unit (AMU) in grams.

[66]:
```
print ("When calculating the AMU, the Avogadro number has to be multiplied by␣
↪1000 to transform grams to kilograms")
B=(8*pi/3)*(m*c/h)**3/(1e3*NA)
B
```

```
When calculating the AMU, the Avogadro number has to be multiplied by 1000 to
transform grams to kilograms
```

[66]: $\displaystyle \frac{0.00266666666666667\pi c^3m^3}{NAh^3}$

[67]:
```
B0=(B.subs({c:sc.c, m:sc.m_e, h:sc.h, NA:sc.N_A})).evalf()
print ("B0=",B0)
```

```
B0= 973932167.009161
```

1.2 The Chandrasekhar differential equation

The hydrostatic equilibrium equation can be written as

$$\frac{dP(x)}{dr} = \frac{dP}{dx}\frac{dx}{dr} = -\rho\frac{d\varphi}{dr}.$$

In the spherically symmetric case, the gravitational potential obeys the following (Poisson's) equation:

$$\Delta\varphi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = 4\pi G\rho.$$

After substituting the expressions for $\frac{dP}{dx}$ and $\rho(x)$ into Poisson's equation, it becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\sqrt{1+x^2}}{dr}\right) = -\frac{8\pi}{3}\frac{(\mu_e M_u)^2}{mc^2}\frac{4\pi G}{\lambda^3}x^3.$$

Introducing the Chandrasekhar function $y = \frac{\sqrt{1+x}}{y_0}$ and the dimensionless radius $\xi = \frac{r}{l_0}$, where

$$l_0 = \frac{1}{y_0}\left[\left(\frac{3}{8\pi}\right)\frac{mc^2\lambda^3}{4\pi G(\mu_e M_u)^2}\right]^{1/2},$$

we derive the white-dwarf structure equation

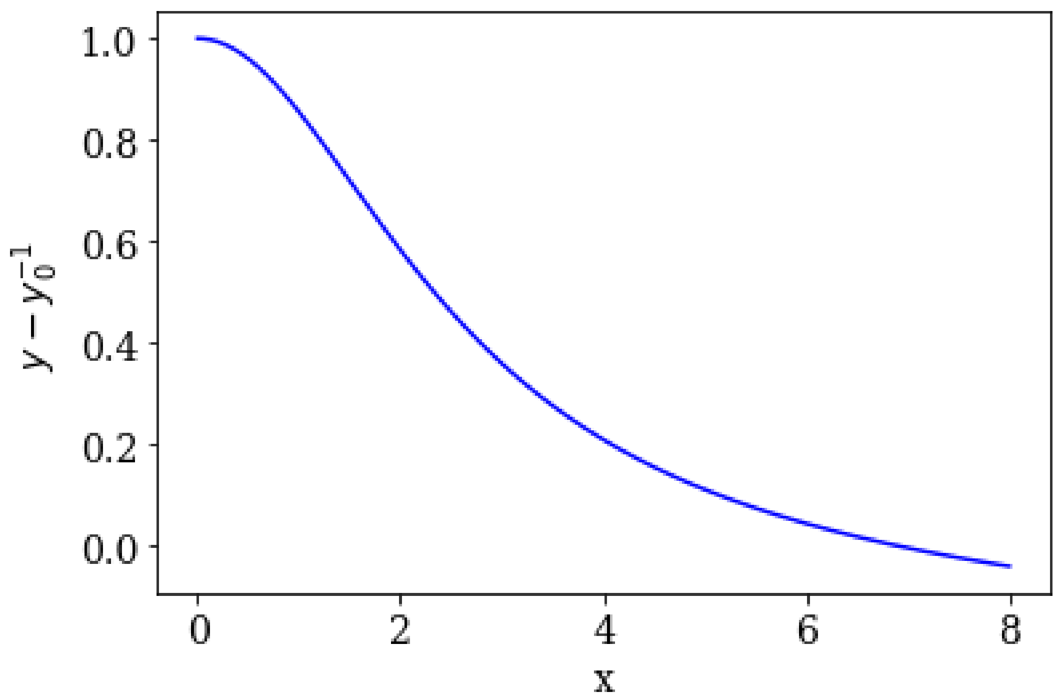
$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dy}{d\xi}\right) = -\left(y^2 - \frac{1}{y_0^2}\right)^{3/2}$$

that has to be solved for the initial conditions $y(0) = 1$ and $y'(0) = 0$. This is a one-parameteric equation with

$$y_0^2 = 1 + \left(\frac{\rho_c}{B_0 \mu_e}\right)^{2/3},$$

where ρ_c is the central density. In the Chandrasekhar (ultra-relativistic) limit $\rho_c \to \infty$.

[68]:
```
# solution of the above second order ODE for y:

y02=1e-10 # 1/y0**2 a small y02 corresponds to ultra-relativistic limit
y0=np.sqrt(y02)

def g(y, x):
    y0 = y[0]
    y1 = y[1]
    y2 = -2*y1/x - (abs(y0**2-y02))**1.5
    return y1, y2

# Initial conditions on y, y' at x=0
init = 1.0, 0.0

# First integrate from 0 to 2
x0=1.0
y00=1.0
while y00 > y0:
    x0=2.0*x0
    x = np.linspace(1e-20,x0,1000)
    sol=odeint(g, init, x)
    y00=sol[-1,0]

plt.plot(x, sol[:,0]-y0, color='b')
plt.xlabel("x")
plt.ylabel("$y - y_0^{-1}$")
plt.show()
```



[69]:
```
# We interpolate the solution to find its and its derivative's values at the␣
↪surface
f = interpolate.interp1d(sol[:,0]-y0,x)
g = interpolate.interp1d(x,sol[:,1])
print (f(0))
print (sol[-1,0]-y0)
print (-f(0)**2*g(f(0)))
```

```
6.896615745806944
-0.0403664781397189
2.0182375566600235
```

The white-dwarf mass is $M = \int_0^R 4\pi r^2 \rho dr$. Using a solution of the differential equation, the mass can be presented as

$$M = \frac{4\pi B_0 L_0}{\mu_e^2}\left[-\xi^2 y'(\xi)\right],$$

where the coefficient B_0 has already been calculated, and L_0 is calculated now.

[70]:
```
# Don't forget to multiply NA by 1e3 to transform grams to kilograms
L = sqrt(3*h**3/(2*c*G))*(1e3*NA)/(4*pi*m)
L
```

[70]:

$$\frac{125.0\sqrt{6}NA\sqrt{\frac{h^3}{Gc}}}{\pi m}$$

[71]:
```
L0=(L.subs({NA:sc.N_A, h:sc.h, c:sc.c, G:sc.G, m:sc.m_e})).evalf()
L0
```

[71]: 7769060.49428712

[72]:
```
#R=y0*L0*f(0)/ac.R_sun.value
#print ("The radius (in solar units) for the white-dwarf model with T=0 K is")
#print ("R = ",R)
```

[75]:
```
x_=np.linspace(0,0.9,100)
R=y0*L0*f(x1)/ac.R_sun.value
```

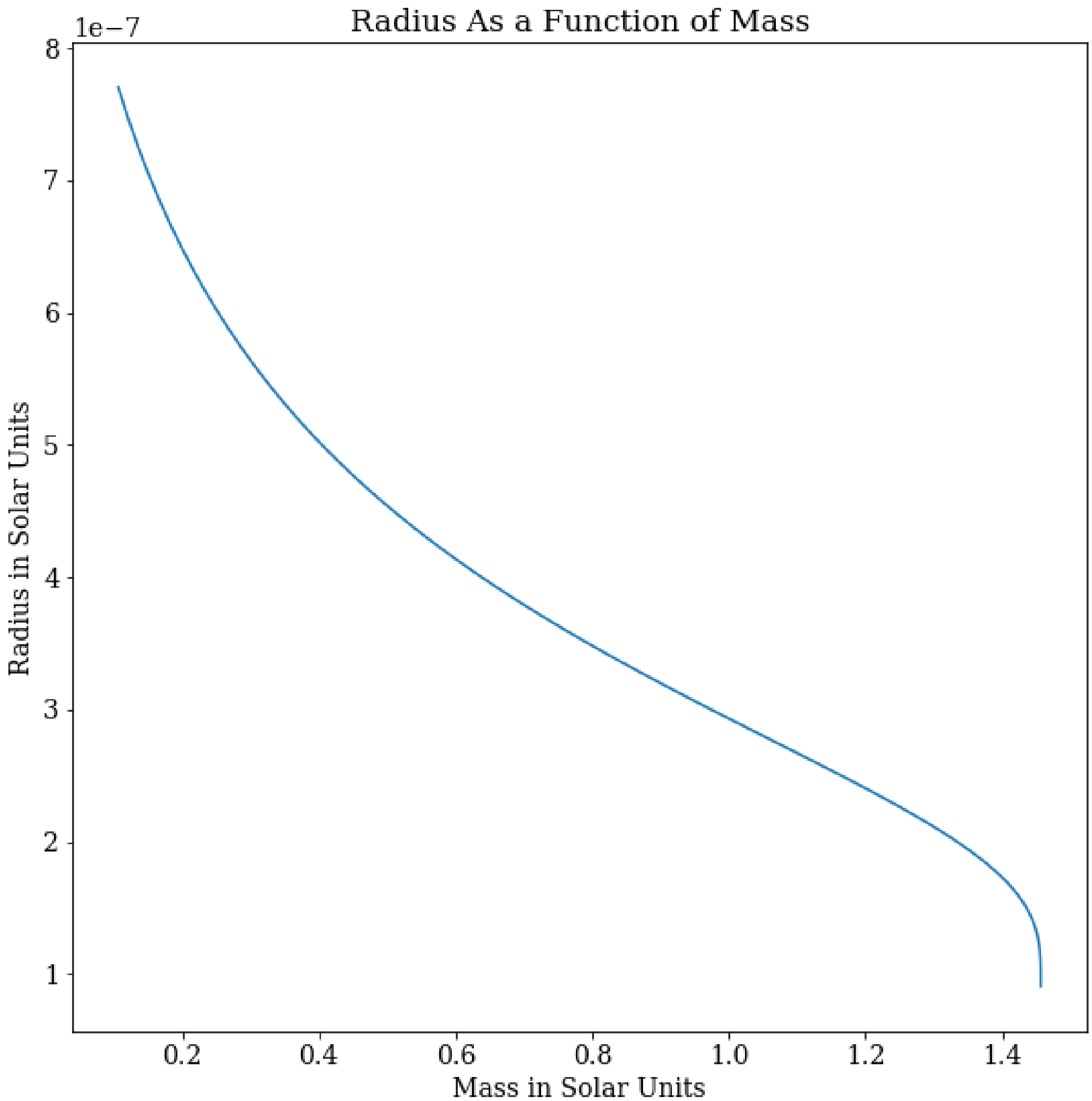
[76]:
```
mu1=-f(0)**2*g(f(0))
mue=2.0 # for a CO WD
M=4*pi*B0*L0**3/mue**2*mu1
print ("The Chandrasekhar mass limit (in solar masses) for the white-dwarf␣
↪model with T=0 K is")
MCh = M.evalf()/ac.M_sun.value
print ("MCh = ",M.evalf()/ac.M_sun.value)
print(MCh)
```

```
The Chandrasekhar mass limit (in solar masses) for the white-dwarf model with
T=0 K is
MCh =  1.45629871125147
1.45629871125147
```

[77]:
```
mu1=-f(x_)**2*g(f(x_))
mue=2.0 # for a CO WD
M=4*pi*B0*L0**3/mue**2*mu1
M1=[M[i].evalf()/ac.M_sun.value for i in range(len(M))]
#print ("The Chandrasekhar mass limit (in solar masses) for the white-dwarf␣
↪model with T=0 K is")
#MCh = M.evalf()/ac.M_sun.value
#print ("MCh = ",M.evalf()/ac.M_sun.value)
#print(MCh)
```

[80]:
```
plt.close(1); plt.figure(1,figsize=(10,10))
plt.plot(M1,R[::-1])
plt.xlabel("Mass in Solar Units")
plt.ylabel("Radius in Solar Units")
plt.title("Radius As a Function of Mass")
```

[80]: Text(0.5, 1.0, 'Radius As a Function of Mass')



The relationship between the mass and the radius of a white dwarfs is derived from the sum of its energy. The energy is approximated from the sum of the gravitation potential which has a -1/R relation and its kinetic energy, primarily coming from the motion of electrons. Compression of a white dwarf will increase the number of electrons in a given volume, increasing the kinetic energy of the electrons and hence increasing the pressure. This electron degeneracy pressure is what supports the white dwarf from gravitational collapse. This pressure is dependent only on the density and not on the temperature. Hence, when the mass is increased, the volume decreases.

A result of being supported by electron degeneracy pressure is having a limiting mass that cannot be exceeded without collapsing the white dwarf.

1.2.1 Use solutions of the above differential equation to plot the mass-radius relation for the zero-T white dwarfs.

[ ]:

```
```