1.2 NONDETERMINISM

Nondeterminism is a useful concept that has had great impact on the theory of computation. So far in our discussion, every step of a computation follows in a unique way from the preceding step. When the machine is in a given state and reads the next input symbol, we know what the next state will be—it is determined. We call this ***deterministic*** computation. In a ***nondeterministic*** machine, several choices may exist for the next state at any point.

Nondeterminism is a generalization of determinism, so every deterministic finite automaton is automatically a nondeterministic finite automaton. As the following figure shows, nondeterministic finite automata may have additional features.

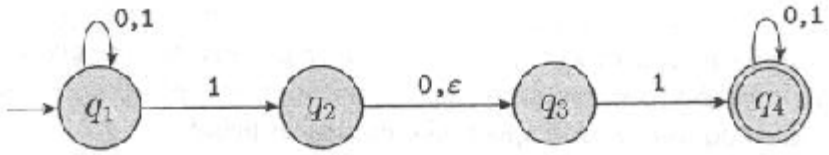


FIGURE 1.14
The nondeterministic finite automaton N_1

The difference between a deterministic finite automaton, abbreviated DFA, and a nondeterministic finite automaton, abbreviated NFA, is immediately apparent. First, every state of a DFA always has exactly one exiting transition arrow for each symbol in the alphabet. The nondeterministic automaton shown in Figure 1.14 violates that rule. State q_1 has one exiting arrow for 0, but it has two for 1; q_2 has one arrow for 0, but it has none for 1. In an NFA a state may have zero, one, or many exiting arrows for each alphabet symbol.

Second, in a DFA, labels on the transition arrows are symbols from the alphabet. This NFA has an arrow with the label ε. In general, an NFA may have arrows labeled with members of the alphabet or ε. Zero, one, or many arrows may exit from each state with the label ε.

How does an NFA compute? Suppose that we are running an NFA on an input string and come to a state with multiple ways to proceed. For example, say that we are in state q_1 in NFA N_1 and that the next input symbol is a 1. After reading that symbol, the machine splits into multiple copies of itself and follows *all* the possibilities in parallel. Each copy of the machine takes one of the possible ways to proceed and continues as before. If there are subsequent choices, the machine splits again. If the next input symbol doesn't appear on any of the arrows exiting the state occupied by a copy of the machine, that copy of the machine dies, along with the branch of the computation associated with it. Finally, if *any one* of these copies of the machine is in an accept state at the end of the input, the NFA accepts the input string.

If a state with an ε symbol on an exiting arrow is encountered, something similar happens. Without reading any input, the machine splits into multiple copies, one following each of the exiting ε-labeled arrows and one staying at the current state. Then the machine proceeds nondeterministically as before.

Nondeterminism may be viewed as a kind of parallel computation wherein several "processes" can be running concurrently. When the NFA splits to follow several choices, that corresponds to a process "forking" into several children, each proceeding separately. If at least one of these processes accepts then the entire computation accepts.

Another way to think of a nondeterministic computation is as a tree of possibilities. The root of the tree corresponds to the start of the computation. Every branching point in the tree corresponds to a point in the computation at which the machine has multiple choices. The machine accepts if at least one of the computation branches ends in an accept state, as shown in the following figure.

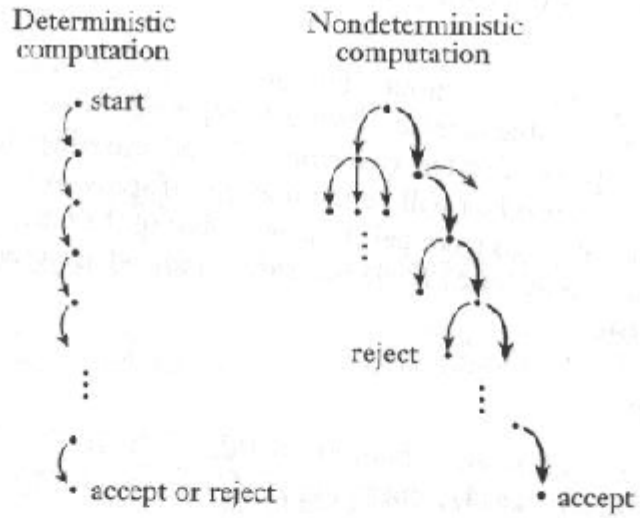


FIGURE 1.15
Deterministic and nondeterministic computations with an accepting branch

Let's consider some sample runs of the NFA N_1 shown in Figure 1.14. On input 010110 start in the start state q_1 and read the first symbol 0. From q_1 there is only one place to go on a 0, namely, back to q_1, so remain there.

Next read the second symbol 1. In q_1 on a 1 there are two choices: either stay in q_1 or move to q_2. Nondeterministically, the machine splits in two to follow each choice. Keep track of the possibilities by placing a finger on each state where a machine could be. So you now have fingers on states q_1 and q_2. An ε arrow exits state q_2 so the machine splits again; keep one finger on q_2, and move the other to q_3. You now have fingers on q_1, q_2, and q_3.

When the third symbol 0 is read, take each finger in turn. Keep the finger on q_1 in place, move the finger on q_2 to q_3, and remove the finger that has been on q_3. That last finger had no 0 arrow to follow and corresponds to a process that simply "dies." At this point you have fingers on states q_1 and q_3.

When the fourth symbol 1 is read, split the finger on q_1 into fingers on states q_1 and q_2, then further split the finger on q_2 to follow the ε arrow to q_3, and move the finger that was on q_3 to q_4. You now have a finger on each of the four states.

When the fifth symbol 1 is read, the fingers on q_1 and q_3 result in fingers on states q_1, q_2, q_3, and q_4, as you saw with the fourth symbol. The finger on state q_2 is removed. The finger that was on q_4 stays on q_4. Now you have two fingers on q_4, so remove one, because you only need to remember that q_4 is a possible state at this point, not that it is possible for multiple reasons.

When the sixth and final symbol 0 is read, keep the finger on q_1 in place, move the one on q_2 to q_3, remove the one that was on q_3, and leave the one on q_4 in

place. You are now at the end of the string, and you accept if some finger is on an accept state. You have fingers on states q_1, q_3, and q_4, and as q_4 is an accept state, N_1 accepts this string. The computation of N_1 on input 010110 is depicted in Figure 1.16.

What does N_1 do on input 010? Start with a finger on q_1. After reading the 0 you still have a finger only on q_1, but after the 1 there are fingers on q_1, q_2, and q_3 (don't forget the ε arrow). After the third symbol 0, remove the finger on q_3, move the finger on q_2 to q_3, and leave the finger on q_1 where it is. At this point you are at the end of the input, and as no finger is on an accept state, N_1 rejects this input.

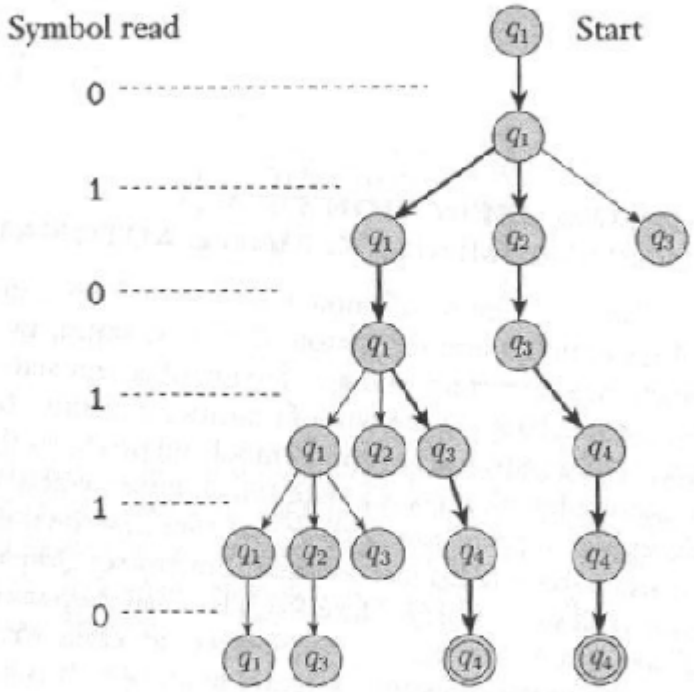


FIGURE 1.16
The computation of N_1 on input 010110

By continuing to experiment in this way, you will see that N_1 accepts all strings that contain either 101 or 11 as a substring.

Nondeterministic finite automata are useful in several respects. As we will show, every NFA can be converted into an equivalent DFA, and constructing NFAs is sometimes easier than directly constructing DFAs. An NFA may be much smaller than its deterministic counterpart, or its functioning may be easier to understand. Nondeterminism in finite automata is also a good introduction to nondeterminism in more powerful computational models because finite automata are especially easy to understand. Now we turn to several examples of NFAs.

EXAMPLE 1.14

Let A be the language consisting of all strings over $\{0,1\}$ containing a 1 in the third position from the end (e.g., 000100 is in A but 0011 is not). The following four-state NFA N_2 recognizes A.

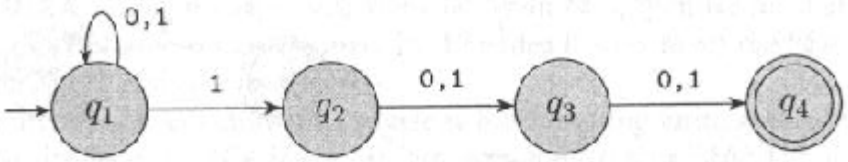


FIGURE 1.17
The NFA N_2 recognizing A

One good way to view the computation of this NFA is to say that it stays in the start state q_1 until it "guesses" that it is three places from the end. At that point, if the input symbol is a 1, it branches to state q_2 and uses q_3 and q_4 to "check" on whether its guess was correct.

As mentioned, every NFA can be converted into an equivalent DFA, but sometimes that DFA may have many more states. The smallest DFA for A contains eight states. Furthermore, understanding the functioning of the NFA is much easier, as you may see by examining the following figure for the DFA.

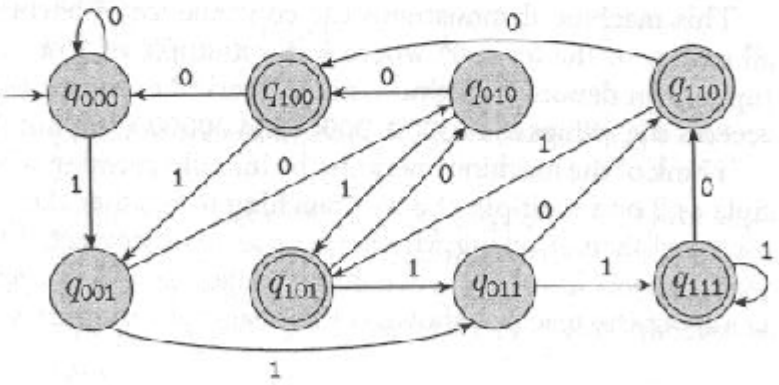


FIGURE 1.18
A DFA recognizing A

Suppose that we added ε to the labels on the arrows going from q_2 to q_3 and from q_3 to q_4 in machine N_2 in Figure 1.17. In other words, both arrows would then have the label 0, 1, ε instead of just 0, 1. What language would N_2 recognize with this modification? Try modifying the DFA in Figure 1.18 to recognize that language. ■

EXAMPLE 1.15

Consider the following NFA N_3 that has an input alphabet {0} consisting of a single symbol. An alphabet containing only one symbol is called a ***unary*** alphabet.

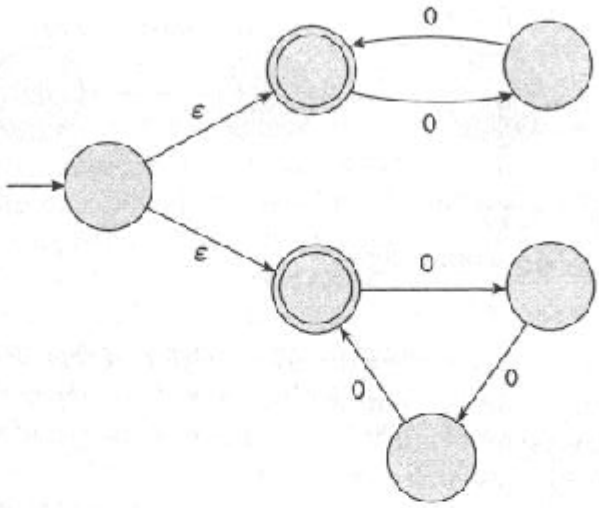


FIGURE 1.19
The NFA N_3

This machine demonstrates the convenience of having ε arrows. It accepts all strings of the form 0^k where k is a multiple of 2 or 3. (Remember that the superscript denotes repetition, not numerical exponentiation.) For example, N_3 accepts the strings ε, 00, 000, 0000, and 000000, but not 0 or 00000.

Think of the machine operating by initially guessing whether to test for a multiple of 2 or a multiple of 3 by branching into either the top loop or the bottom loop and then checking whether its guess was correct. Of course, we could replace this machine by one that doesn't have ε arrows or even any nondeterminism at all, but the machine shown is the easiest one to understand for this language. ■

EXAMPLE 1.16

We give another example of an NFA in the following figure. Practice with it to satisfy yourself that it accepts the strings ε, a, baba, and baa, but that it doesn't accept the strings b, bb, and babba. Later we use this machine to illustrate the procedure for converting NFAs to DFAs.

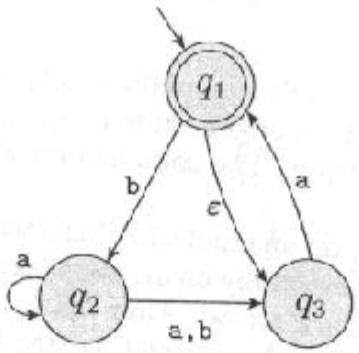


FIGURE 1.20
The NFA N_4

FORMAL DEFINITION OF A NONDETERMINISTIC FINITE AUTOMATON

The formal definition of a nondeterministic finite automaton is similar to that of a deterministic finite automaton. Both have states, an input alphabet, a transition function, a start state, and a collection of accept states. However, they differ in one essential way: in the type of transition function. In a DFA the transition function takes a state and an input symbol and produces the next state. In an NFA the transition function takes a state and an input symbol *or the empty string* and produces *the set of possible next states*. In order to write the formal definition, we need to set up some additional notation. For any set Q we write $\mathcal{P}(Q)$ to be the collection of all subsets of Q. Here $\mathcal{P}(Q)$ is called the ***power set*** of Q. For any alphabet Σ we write Σ_ε to be $\Sigma \cup \{\varepsilon\}$. Now we can easily write the formal description of the type of the transition function in an NFA. It is $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$, and we are ready to give the formal definition.

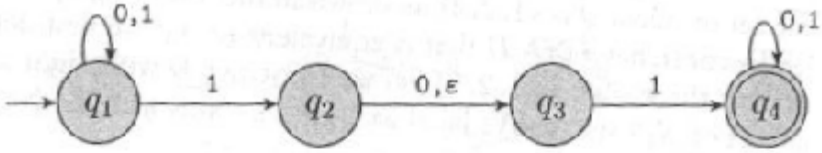
DEFINITION 1.17

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

EXAMPLE 1.18

Recall the NFA N_1:



The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0,1\}$,
3. δ is given as

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$

4. q_1 is the start state, and
5. $F = \{q_4\}$.

The formal definition of computation for an NFA also is similar to that for a DFA. Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and w a string over the alphabet Σ. Then we say that N ***accepts*** w if we can write w as $w = y_1 y_2 \cdots y_m$, where each y_i is a member of Σ_ε and a sequence of states $r_0, r_1, \ldots, r_m$ exists in Q with the following three conditions:

1. $r_0 = q_0$,
2. $r_{i+1} \in \delta(r_i, y_{i+1})$, for $i = 0, \ldots, m-1$, and
3. $r_m \in F$.

Condition 1 says that the machine starts out in the start state. Condition 2 says that state r_{i+1} is one of the allowable next states when N is in state r_i and reading y_{i+1}. Observe that $\delta(r_i, y_{i+1})$ is the *set* of allowable next states and so we say that r_{i+1} is a member of that set. Finally, Condition 3 says that the machine accepts its input if the last state is an accept state.

EQUIVALENCE OF NFAS AND DFAS

Deterministic and nondeterministic finite automata recognize the same class of languages. Such equivalence is both surprising and useful. It is surprising because NFAs appear to have more power than DFAs, so we might expect that NFAs recognize more languages. It is useful because describing an NFA for a given language sometimes is much easier than describing a DFA for that language.

Say that two machines are ***equivalent*** if they recognize the same language.

THEOREM 1.19

Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

PROOF IDEA If a language is recognized by an NFA, then we must show the existence of a DFA that also recognizes it. The idea is to convert the NFA into an equivalent DFA that simulates the NFA.

Recall the "reader as automaton" strategy for designing finite automata. How would you simulate the NFA if you were pretending to be a DFA? What do you need to keep track of as the input string is processed? In the examples of NFAs you kept track of the various branches of the computation by placing a finger on each state that could be active at given points in the input. You updated the fingers by moving, adding, and removing them according to the way the NFA operates. All you needed to keep track of was the set of states with fingers.

If k is the number of states of the NFA, it has 2^k subsets of states. Each subset corresponds to one of the possibilities that DFA must remember, so the DFA simulating the NFA will have 2^k states. Now we need to figure out which will be the start state and accept states of the DFA, and what will be its transition function. We can discuss this more easily after setting up some formal notation.

PROOF Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A. We construct a DFA M recognizing A. Before doing the full construction, let's first consider the easier case wherein N has no ε arrows. Later we take the ε arrows into account.

Construct $M = (Q', \Sigma, \delta', q_0', F')$.

1. $Q' = \mathcal{P}(Q)$.
   Every state of M is a set of states of N. Recall that $\mathcal{P}(Q)$ is the set of subsets of Q.
2. For $R \in Q'$ and $a \in \Sigma$ let $\delta'(R, a) = \{q \in Q | \, q \in \delta(r, a) \text{ for some } r \in R\}$.
   If R is a state of M, it is also a set of states of N. When M reads a symbol a in state R, it shows where a takes each state in R. Because each state may go to a set of states, we take the union of all these sets. Another way to write this expression is
   $$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a).$$ [4]
3. $q_0' = \{q_0\}$.
   M starts in the state corresponding to the collection containing just the start state of N.
4. $F' = \{R \in Q' | \, R \text{ contains an accept state of } N\}$.
   The machine M accepts if one of the possible states that N could be in at this point is an accept state.

[4] The notation $\bigcup_{r \in R} \delta(r, a)$ means: the union of the sets $\delta(r, a)$ for each possible r in R.

Now we need to consider the ε arrows. To do so we set up an extra bit of notation. For any state R of M we define $E(R)$ to be the collection of states that can be reached from R by going only along ε arrows, including the members of R themselves. Formally, for $R \subseteq Q$ let

$$E(R) = \{q \mid q \text{ can be reached from } R \text{ by traveling along 0 or more } \varepsilon \text{ arrows}\}.$$

Then we modify the transition function of M to place additional fingers on all states that can be reached by going along ε arrows after every step. Replacing $\delta(r, a)$ by $E(\delta(r, a))$ achieves this effect. Thus

$$\delta'(R, a) = \{q \in Q \mid q \in E(\delta(r, a)) \text{ for some } r \in R\}.$$

Additionally we need to modify the start state of M to move the fingers initially to all possible states that can be reached from the start state of N along the ε arrows. Changing q_0' to be $E(\{q_0\})$ achieves this effect. We have now completed the construction of the DFA M that simulates the NFA N.

The construction of M obviously works correctly. At every step in the computation of M on an input, it clearly enters a state that corresponds to the subset of states that N could be in at that point. Thus our proof is complete.

If the construction used in the preceding proof were more complex we would need to prove that it works as claimed. Usually such proofs proceed by induction on the number of steps of the computation. Most of the constructions that we use in this book are straightforward and so do not require such a correctness proof. To see an example of a more complex construction that we do prove correct turn to the proof of Theorem 1.28.

Theorem 1.19 states that every NFA can be converted into an equivalent DFA. Thus nondeterministic finite automata give an alternative way of characterizing the regular languages. We state this fact as a corollary of Theorem 1.19.

COROLLARY 1.20

A language is regular if and only if some nondeterministic finite automaton recognizes it.

One direction of the "if and only if" states that a language is regular if some NFA recognizes it. Theorem 1.19 shows that any NFA can be converted into an equivalent DFA, so if an NFA recognizes some language, so does some DFA, and hence the language is regular. The other direction states that a language is regular only if some NFA recognizes it. That is, if a language is regular, some NFA must be recognizing it. Obviously, this condition is true because a regular language has a DFA recognizing it and any DFA is also an NFA.

EXAMPLE 1.21

Let's illustrate the procedure of converting an NFA to a DFA using the machine N_4 that was given in Example 1.16. For clarity, we have relabeled the states of N_4 to be $\{1, 2, 3\}$. Thus in the formal description of $N_4 = (Q, \{\mathtt{a},\mathtt{b}\}, \delta, 1, \{1\})$, the set of states Q is $\{1, 2, 3\}$ as shown in the following figure.

To construct a DFA D that is equivalent to N_4, we first determine D's states. N_4 has three states, $\{1, 2, 3\}$, so we construct D with eight states, one for each subset of N_4's states. We label each of D's states with the corresponding subset. Thus D's state set is

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

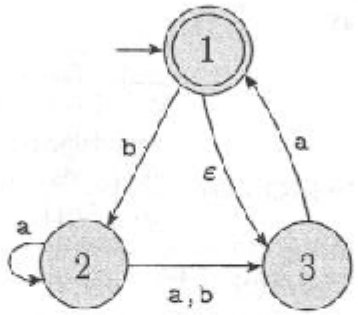


FIGURE 1.21
The NFA N_4

Next, we determine the start and accept states of D. The start state is $E(\{1\})$, the set of states that are reachable from 1 by traveling along ε arrows, plus 1 itself. An ε arrow goes from 1 to 3, so $E(\{1\}) = \{1, 3\}$. The new accept states are those containing N_4's accept state; thus $\{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$.

Finally, we determine D's transition function. Each of D's states goes to one place on input a, and one place on input b. We illustrate the process of determining the placement of D's transition arrows with a few examples.

In D, state $\{2\}$ goes to $\{2,3\}$ on input a, because in N_4, state 2 goes to both 2 and 3 on input a and we can't go farther from 2 or 3 along ε arrows. State $\{2\}$ goes to state $\{3\}$ on input b, because in N_4, state 2 goes only to state 3 on input b and we can't go farther from 3 along ε arrows.

State $\{1\}$ goes to $\emptyset$ on a, because no a arrows exit it. It goes to $\{2\}$ on b.

State $\{3\}$ goes to $\{1,3\}$ on a, because in N_4, state 3 goes to 1 on a and 1 in turn goes to 3 with an ε arrow. State $\{3\}$ on b goes to $\emptyset$.

State $\{1,2\}$ on a goes to $\{2,3\}$ because 1 points at no states with a arrows and 2 points at both 2 and 3 with a arrows and neither point anywhere with ε arrows. State $\{1,2\}$ on b goes to $\{2,3\}$. Continuing in this way we obtain the following diagram for D.

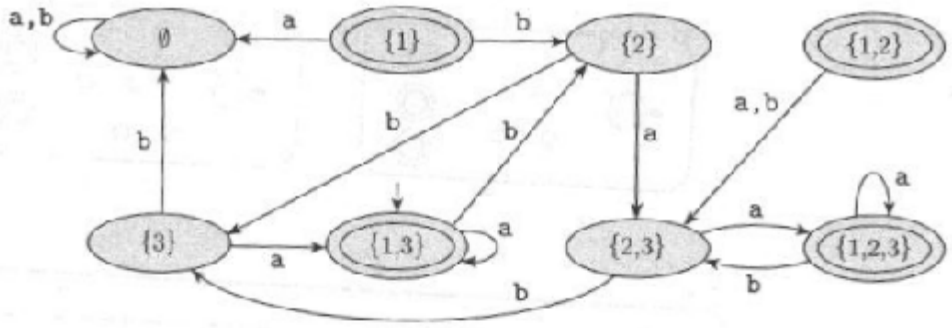


FIGURE **1.22**
A DFA D that is equivalent to the NFA N_4

We may simplify this machine by observing that no arrows point at states $\{1\}$ and $\{1, 2\}$, so they may be removed without affecting the performance of the machine. Doing so yields the following figure.

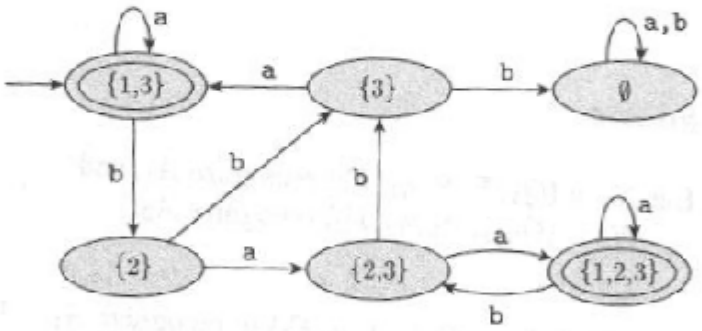


FIGURE **1.23**
DFA D after removing unnecessary states

CLOSURE UNDER THE REGULAR OPERATIONS

Now we return to the closure of the class of regular languages under the regular operations that we began in Section 1.1. Our aim is to prove that the union, concatenation, and star of regular languages are still regular. We abandoned the original attempt to do so when dealing with the concatenation operation was too complicated. The use of nondeterminism makes the proofs much easier.

First, let's consider again closure under union. Earlier we proved closure under union by simulating deterministically both machines simultaneously via a

Cartesian product construction. We now give a new proof to illustrate the technique of nondeterminism. Reviewing the first proof, on page 45, may be worthwhile to see how much easier and more intuitive the new proof is.

THEOREM 1.22

The class of regular languages is closed under the union operation.

PROOF IDEA We have regular languages A_1 and A_2 and want to prove that $A_1 \cup A_2$ is regular. The idea is to take two NFAs, N_1 and N_2 for A_1 and A_2, and combine them into one new NFA, N.

Machine N must accept its input if either N_1 or N_2 accepts this input. The new machine has a new start state that branches to the start states of the old machines with ε arrows. In this way the new machine nondeterministically guesses which of the two machines accepts the input. If one of them accepts the input, N will accept it, too.

We represent this construction in the following figure. On the left we indicate the start and accept states of machines N_1 and N_2 with large circles and some additional states with small circles. On the right we show how to combine N_1 and N_2 into N by adding additional transition arrows.

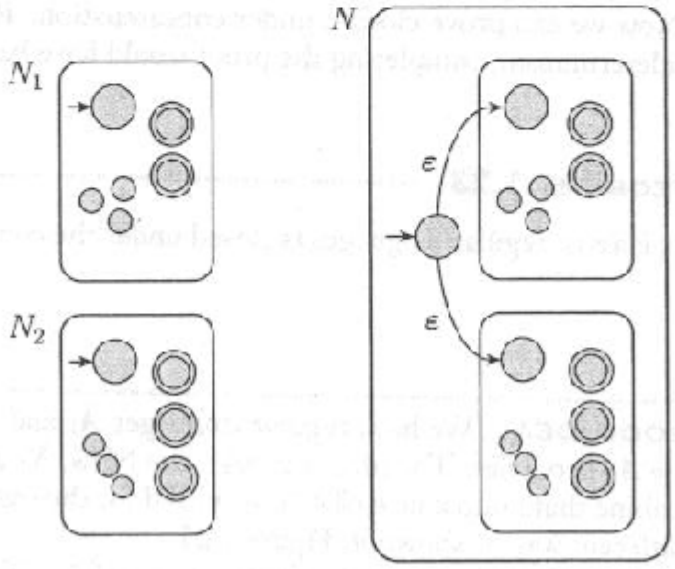


FIGURE 1.24
Construction of an NFA N to recognize $A_1 \cup A_2$

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
   The states of N are all the states of N_1 and N_2, with the addition of a new start state q_0.
2. The state q_0 is the start state of N.
3. The accept states $F = F_1 \cup F_2$.
   The accept states of N are all the accept states of N_1 and N_2. That way N accepts if either N_1 accepts or N_2 accepts.
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \\ \delta_2(q, a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon. \end{cases}$$

Now we can prove closure under concatenation. Recall that earlier, without nondeterminism, completing the proof would have been difficult.

THEOREM 1.23

The class of regular languages is closed under the concatenation operation.

PROOF IDEA We have regular languages A_1 and A_2 and want to prove that $A_1 \circ A_2$ is regular. The idea is to take two NFAs, N_1 and N_2 for A_1 and A_2, and combine them into a new NFA N as we did for the case of union, but this time in a different way, as shown in Figure 1.25.

Assign N's start state to be the start state of N_1. The accept states of N_1 have additional ε arrows that nondeterministically allow branching to N_2 whenever N_1 is in an accept state, signifying that it has found an initial piece of the input that constitutes a string in A_1. The accept states of N are the accept states of N_2 only. Therefore it accepts when the input can be split into two parts, the first accepted by N_1 and the second by N_2. We can think of N as nondeterministically guessing where to make the split.

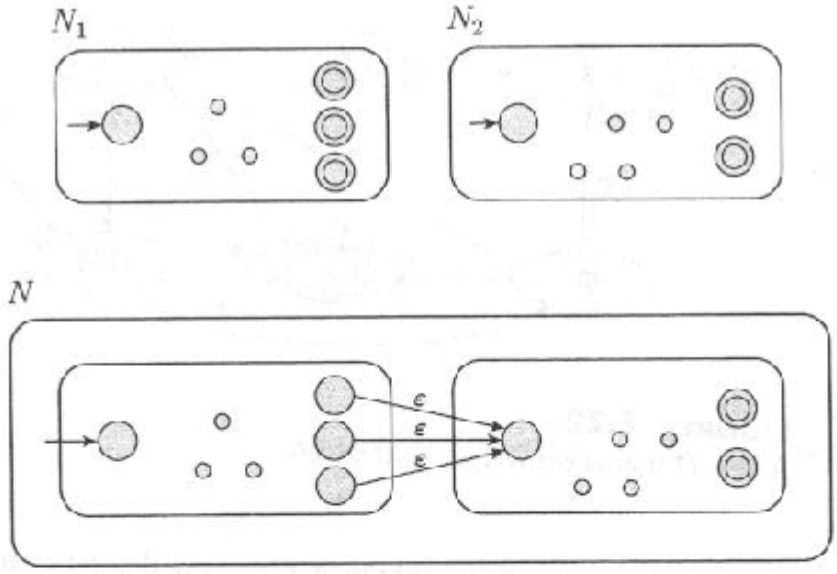


FIGURE 1.25
Construction of N to recognize $A_1 \circ A_2$

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$.

1. $Q = Q_1 \cup Q_2$.
   The states of N are all the states of N_1 and N_2.
2. The state q_1 is the same as the start state of N_1.
3. The accept states F_2 are the same as the accept states of N_2.
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q, a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q, a) \cup \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q, a) & q \in Q_2. \end{cases}$$

1.3 REGULAR EXPRESSIONS

In arithmetic, we can use the operations $+$ and $\times$ to build up expressions such as

$$(5 + 3) \times 4$$

Similarly, we can use the regular operations to build up expressions describing languages, which are called ***regular expressions***. An example is:

$$(0 \cup 1)0^*.$$

The value of the arithmetic expression is the number 32. The value of a regular expression is a language. In this case the value is the language consisting of all strings starting with a 0 or a 1 followed by any number of 0s. We get this result by dissecting the expression into its parts. First, the symbols 0 and 1 are shorthand for the sets $\{0\}$ and $\{1\}$. So $(0 \cup 1)$ means $(\{0\} \cup \{1\})$. The value of this part is the language $\{0,1\}$. The part 0^* means $\{0\}^*$, and its value is the language consisting of all strings containing any number of 0s. Second, like the $\times$ symbol in algebra, the concatenation symbol $\circ$ often is implicit in regular expressions. Thus $(0 \cup 1)0^*$ actually is shorthand for $(0 \cup 1) \circ 0^*$. The concatenation attaches the strings from the two parts to obtain the value of the entire expression.

Regular expressions have an important role in computer science applications. In applications involving text, users may want to search for strings that satisfy certain patterns. Regular expressions provide a powerful method for describing such patterns. Utilities such as AWK and GREP in UNIX, modern programming languages such as PERL, and text editors all provide mechanisms for the description of patterns using regular expressions.

EXAMPLE 1.25

Another example of a regular expression is

$$(0 \cup 1)^*$$

It starts with the language $(0 \cup 1)$ and applies the $*$ operation. The value of this expression is the language consisting of all possible strings of 0s and 1s. If $\Sigma = \{0,1\}$, we can write Σ as shorthand for the regular expression $(0 \cup 1)$. More generally, if Σ is any alphabet, the regular expression Σ describes the language consisting of all strings of length 1 over this alphabet, and Σ^* describes the language consisting of all strings over that alphabet. Similarly Σ^*1 is the language that contains all strings that end in a 1. The language $(0\Sigma^*) \cup (\Sigma^*1)$ consists of all strings that either start with a 0 or end with a 1. ∎

In arithmetic, we say that $\times$ has precedence over $+$ to mean that, when there is a choice, we do the $\times$ operation first. Thus in $2+3\times4$ the 3×4 is done before the addition. To have the addition done first we must add parentheses to obtain $(2+3)\times4$. In regular expressions, the star operation is done first, followed by concatenation, and finally union, unless parentheses are used to change the usual order.

FORMAL DEFINITION OF A REGULAR EXPRESSION

DEFINITION 1.26

Say that R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

In items 1 and 2, the regular expressions a and ε represent the languages $\{a\}$ and $\{\varepsilon\}$, respectively. In item 3, the regular expression $\emptyset$ represents the empty language. In items 4, 5, and 6, the expressions represent the languages obtained by taking the union or concatenation of the languages R_1 and R_2, or the star of the language R_1, respectively.

Don't confuse the regular expressions ε and $\emptyset$. The expression ε represents the language containing a single string, namely, the empty string, whereas $\emptyset$ represents the language that doesn't contain any strings.

Seemingly, we are in danger of defining the notion of regular expression in terms of itself. If true, we would have a ***circular definition***, which would be in-

valid. However, R_1 and R_2 always are smaller than R. Thus we actually are defining regular expressions in terms of smaller regular expressions and thereby avoiding circularity. A definition of this type is called an ***inductive definition.***

Parentheses in an expression may be omitted. If they are, evaluation is done in the precedence order: star, then concatenation, then union.

When we want to make clear a distinction between a regular expression R and the language that it describes, we write $L(R)$ to be the language of R.

EXAMPLE 1.27

In the following examples we assume that the alphabet Σ is $\{0,1\}$.

1. $0^*10^* = \{w \mid w \text{ has exactly a single } 1\}$.
2. $\Sigma^*1\Sigma^* = \{w \mid w \text{ has at least one } 1\}$.
3. $\Sigma^*001\Sigma^* = \{w \mid w \text{ contains the string } 001 \text{ as a substring}\}$.
4. $(\Sigma\Sigma)^* = \{w \mid w \text{ is a string of even length}\}$.[5]
5. $(\Sigma\Sigma\Sigma)^* = \{w \mid \text{the length of } w \text{ is a multiple of three}\}$.
6. $01 \cup 10 = \{01, 10\}$.
7. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1 = \{w \mid w \text{ starts and ends with the same symbol}\}$.
8. $(0 \cup \varepsilon)1^* = 01^* \cup 1^*$.
   The expression $0 \cup \varepsilon$ describes the language $\{0, \varepsilon\}$, so the concatenation operation adds either 0 or ε before every string in 1^*.
9. $(0 \cup \varepsilon)(1 \cup \varepsilon) = \{\varepsilon, 0, 1, 01\}$.
10. $1^*\emptyset = \emptyset$.
    Concatenating the empty set to any set yields the empty set.
11. $\emptyset^* = \{\varepsilon\}$.
    The star operation puts together any number of strings from the language to get a string in the result. If the language is empty, the star operation can put together 0 strings, giving only the empty string.

If we let R be any regular expression, we have the following identities. They are good tests of whether you understand the definition.

$R \cup \emptyset = R$.
Adding the empty language to any other language will not change it.

$R \circ \varepsilon = R$.
Adding the empty string to any string will not change it.

[5] The *length* of a string is the number of symbols that it contains.

However, exchanging $\emptyset$ and ε in the preceding identities may cause the equalities to fail.

$R \cup \varepsilon$ may not equal R.
For example, if $R = 0$, then $L(R) = \{0\}$ but $L(R \cup \varepsilon) = \{0, \varepsilon\}$.

$R \circ \emptyset$ may not equal R.
For example, if $R = 0$, then $L(R) = \{0\}$ but $L(R \circ \emptyset) = \emptyset$.

Regular expressions are useful tools in the design of compilers for programming languages. Elemental objects in a programming language, called ***tokens***, such as the variable names and constants, may be described with regular expressions. For example, a numerical constant that may include a fractional part and/or a sign may be described as a member of the language

$$\{+, -, \varepsilon\}\left(D\,D^* \cup D\,D^*.D^* \cup D^*.D\,D^*\right),$$

where $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is the alphabet of decimal digits. Examples of generated strings are: `72`, `3.14159`, `+7.`, and `-.01`.

Once the syntax of the tokens of the programming language have been described with regular expressions, automatic systems can generate the ***lexical analyzer***, the part of a compiler that initially processes the input program.

EQUIVALENCE WITH FINITE AUTOMATA

Regular expressions and finite automata are equivalent in their descriptive power. This fact is rather remarkable, because finite automata and regular expressions superficially appear to be rather different. However, any regular expression can be converted into a finite automaton that recognizes the language it describes, and vice versa. Recall that a regular language is one that is recognized by some finite automaton.

THEOREM 1.28

A language is regular if and only if some regular expression describes it.

This theorem has two directions. We state and prove each direction as a separate lemma.

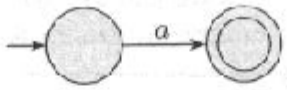
LEMMA 1.29

If a language is described by a regular expression, then it is regular.

PROOF IDEA Say that we have a regular expression R describing some language A. We show how to convert R into an NFA recognizing A. By Corollary 1.20, if an NFA recognizes A then A is regular.

PROOF Let's convert R into an NFA N. We consider the six cases in the formal definition of regular expressions.

1. $R = a$ for some a in Σ. Then $L(R) = \{a\}$, and the following NFA recognizes $L(R)$.



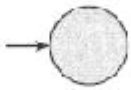
   Note that this machine fits the definition of an NFA but not that of a DFA because it has some states with no exiting arrow for each possible input symbol. Of course, we could have presented an equivalent DFA here but an NFA is all we need for now, and it is easier to describe.

   Formally, $N = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\})$, where we describe δ by saying that $\delta(q_1, a) = \{q_2\}$, $\delta(r, b) = \emptyset$ for $r \neq q_1$ or $b \neq a$.

2. $R = \varepsilon$. Then $L(R) = \{\varepsilon\}$, and the following NFA recognizes $L(R)$.

   Formally, $N = (\{q_1\}, \Sigma, \delta, q_1, \{q_1\})$, where $\delta(r, b) = \emptyset$ for any r and b.

3. $R = \emptyset$. Then $L(R) = \emptyset$, and the following NFA recognizes $L(R)$.



   Formally, $N = (\{q\}, \Sigma, \delta, q, \emptyset)$, where $\delta(r, b) = \emptyset$ for any r and b.

4. $R = R_1 \cup R_2$.
5. $R = R_1 \circ R_2$.
6. $R = R_1^*$.

For the last three cases we use the constructions given in the proofs that the class of regular languages is closed under the regular operations. In other words, we construct the NFA for R from the NFAs for R_1 and R_2 (or just R_1 in case 6) and the appropriate closure construction.

That ends the first part of the proof of Theorem 1.28, giving the easier direction of the if and only if. Before going on to the other direction let's consider some examples whereby we use this procedure to convert a regular expression to an NFA.

EXAMPLE 1.30

We convert the regular expression $(ab \cup a)^*$ to an NFA in a sequence of stages. We build up from the smallest subexpressions to larger subexpressions until we have an NFA for the original expression, as shown in the following diagram. Note that this procedure generally doesn't give the NFA with the fewest states. In this example, the procedure gives an NFA with eight states, but the smallest equivalent NFA has only two states. Can you find it?

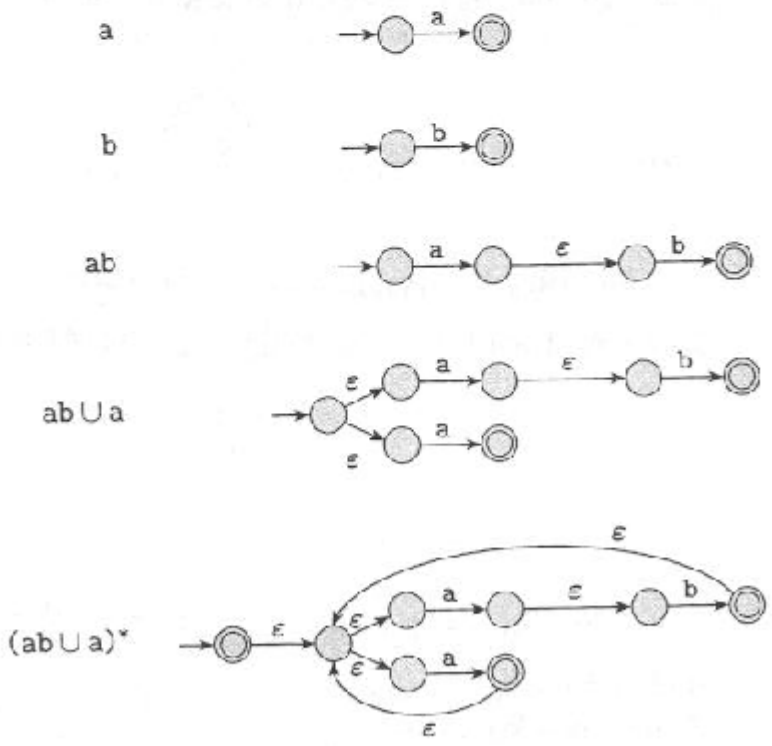


FIGURE 1.27
Building an NFA from the regular expression $(ab \cup a)^*$

EXAMPLE 1.31

In this second example we convert the regular expression $(a \cup b)^*aba$ to an NFA. A few of the minor steps are not shown.

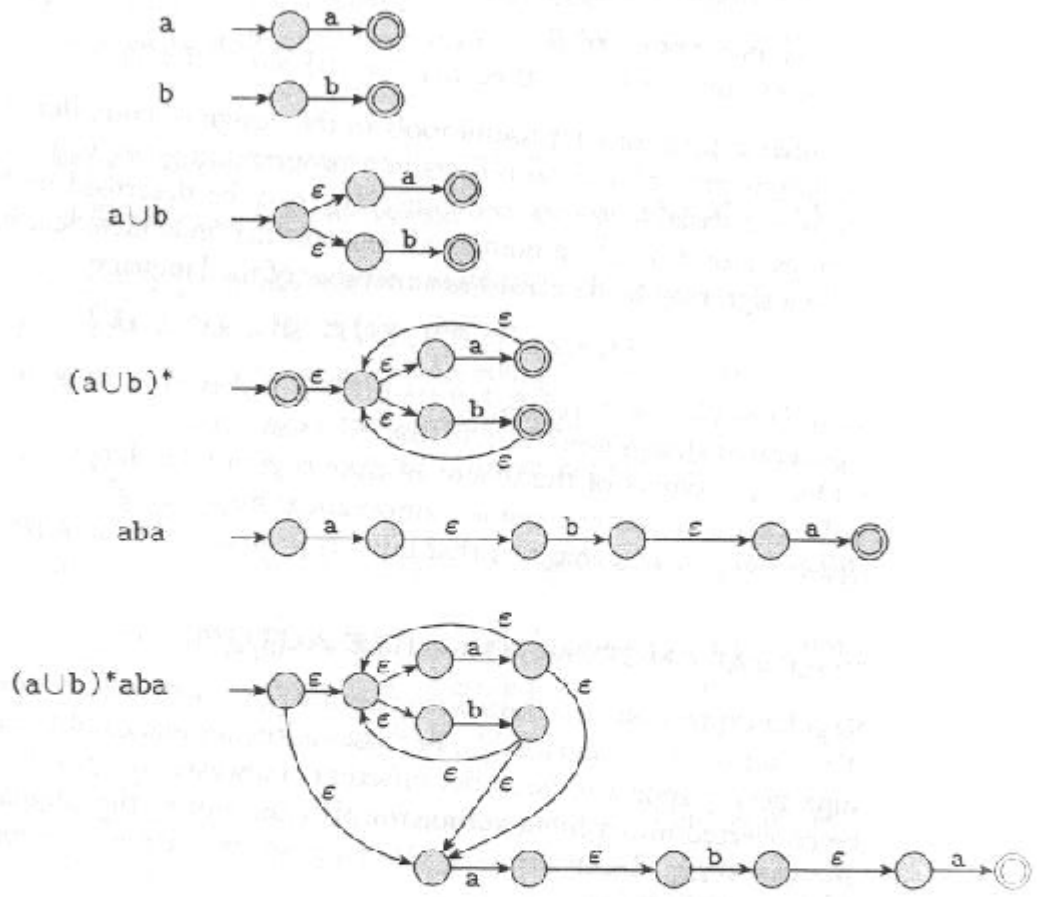


FIGURE 1.28
Building an NFA from the regular expression $(a \cup b)^*aba$

Now let's turn to the other direction of the proof of Theorem 1.28.

LEMMA 1.32

If a language is regular, then it is described by a regular expression.

PROOF IDEA We need to show that, if a language A is regular, a regular expression describes it. Because A is regular, it is accepted by a DFA. We describe a procedure for converting DFAs into equivalent regular expressions.

We break this procedure into two parts, using a new type of finite automaton called a ***generalized nondeterministic finite automaton***, GNFA. First we show how to convert DFAs into GNFAs and then GNFAs into regular expressions.

Generalized nondeterministic finite automata are simply nondeterministic finite automata wherein the transition arrows may have any regular expressions as labels, instead of only members of the alphabet or ε. The GNFA reads blocks of symbols from the input, not necessarily just one symbol at a time as in an ordinary NFA. The GNFA moves along a transition arrow connecting two states by reading a block of symbols from the input, which themselves constitute a string described by the regular expression on that arrow. A GNFA is nondeterministic and so may have several different ways to process the same input string. It accepts its input if its processing can cause the GNFA to be in an accept state at the end of the input. The following figure presents an example of a GNFA.

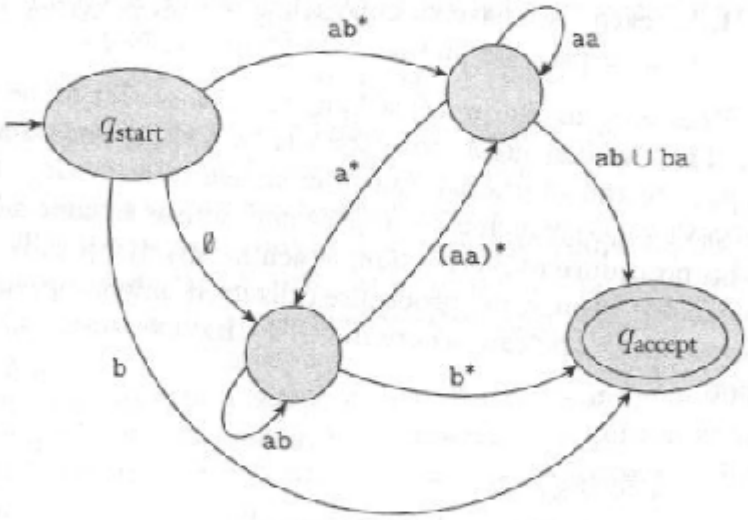


FIGURE 1.29
A generalized nondeterministic finite automaton

For convenience we require that GNFAs always have a special form that meets the following conditions.

- The start state has transition arrows going to every other state but no arrows coming in from any other state.
- There is only a single accept state, and it has arrows coming in from every other state but no arrows going to any other state. Furthermore, the accept state is not the same as the start state.
- Except for the start and accept states, one arrow goes from every state to every other state and also from each state to itself.

We can easily convert a DFA into a GNFA in the special form. We simply add a new start state with an ε arrow to the old start state and a new accept state with ε arrows from the old accept states. If any arrows have multiple labels (or if there are multiple arrows going between the same two states in the same direction), we replace each with a single arrow whose label is the union of the previous labels. Finally, we add arrows labeled $\emptyset$ between states that had no arrows. This last step won't change the language recognized because a transition labeled with $\emptyset$ can never be used. From here on we assume that all GNFAs are in the special form.

Now we show how to convert a GNFA into a regular expression. Say that the GNFA has k states. Then, because a GNFA must have a start and an accept state and they must be different from each other, we know that $k \geq 2$. If $k > 2$, we construct an equivalent GNFA form with $k - 1$ states. This step can be repeated on the new GNFA until it is reduced to two states. If $k = 2$, the GNFA has a single arrow that goes from the start state to the accept state. The label of this arrow is the equivalent regular expression. For example, the stages in converting a DFA with three states to an equivalent regular expression are shown in the following figure.

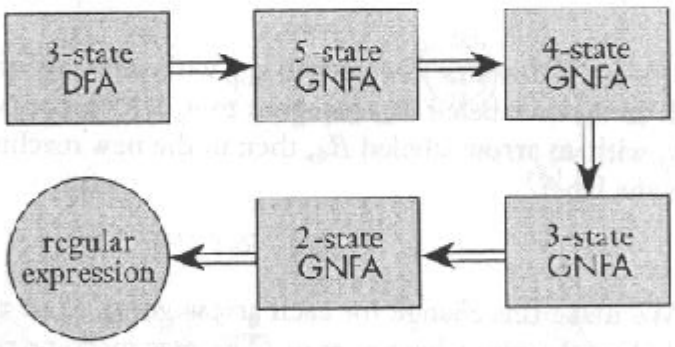


FIGURE 1.30
Typical stages in converting a DFA to a regular expression

The crucial step is in constructing an equivalent GNFA with one fewer state when $k > 2$. We do so by selecting a state, ripping it out of the machine, and repairing the remainder so that the same language is still recognized. Any state will do, provided that it is not the start or accept state. We are guaranteed that such a state will exist because $k > 2$. Let's call the removed state q_{rip}.

After removing q_{rip} we repair the machine by altering the regular expressions that label each of the remaining arrows. The new labels compensate for the absence of q_{rip} by adding back the lost computations. The new label going from a

state q_i to a state q_j is a regular expression that describes all strings that would take the machine from q_i to q_j either directly or via q_{rip}. We illustrate this approach in the following figure.

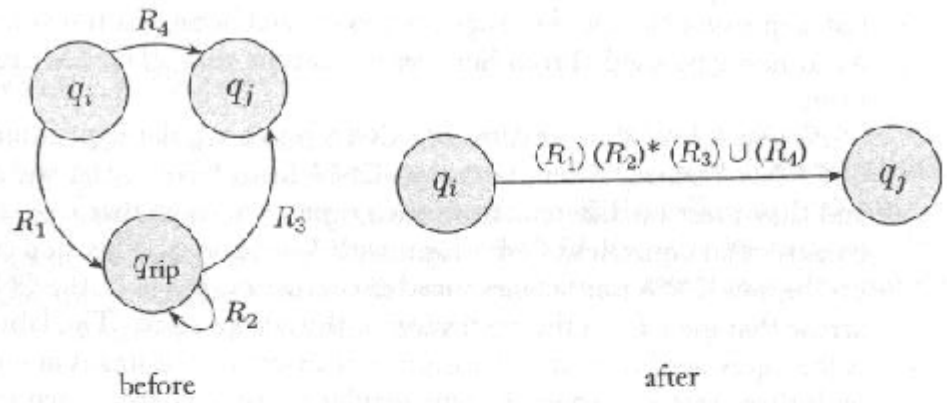


FIGURE 1.31
Constructing an equivalent GNFA with one fewer state

In the old machine if q_i goes to q_{rip} with an arrow labeled R_1, q_{rip} goes to itself with an arrow labeled R_2, q_{rip} goes to q_j with an arrow labeled R_3, and q_i goes to q_j with an arrow labeled R_4, then in the new machine the arrow from q_i to q_j gets the label

$$(R_1)(R_2)^*(R_3) \cup (R_4).$$

We make this change for each arrow going from any state q_i to any state q_j, including the case where $q_i = q_j$. The new machine recognizes the original language.

PROOF Let's now carry out this idea formally. First, to facilitate the proof, we formally define the new type of automaton introduced. A GNFA is similar to a nondeterministic finite automaton except for the transition function, which has the form

$$\delta: \big(Q - \{q_{\text{accept}}\}\big) \times \big(Q - \{q_{\text{start}}\}\big) \longrightarrow \mathcal{R}.$$

The symbol $\mathcal{R}$ is the collection of all regular expressions over the alphabet Σ, and q_{start} and q_{accept} are the start and accept states. If $\delta(q_i, q_j) = R$, the arrow from state q_i to state q_j has the regular expression R as its label. The domain of the transition function is $\big(Q - \{q_{\text{accept}}\}\big) \times \big(Q - \{q_{\text{start}}\}\big)$ because an arrow connects every state to every other state, except that no arrows are coming from q_{accept} or going to q_{start}.

DEFINITION 1.33

A ***generalized nondeterministic finite automaton***, $(Q, \Sigma, \delta, q_{\text{start}}, q_{\text{accept}})$, is a 5-tuple where

1. Q is the finite set of states,
2. Σ is the input alphabet,
3. $\delta\colon (Q - \{q_{\text{accept}}\}) \times (Q - \{q_{\text{start}}\}) \longrightarrow \mathcal{R}$ is the transition function,
4. q_{start} is the start state, and
5. q_{accept} is the accept state.

A GNFA accepts a string w in Σ^* if $w = w_1 w_2 \cdots w_k$, where each w_i is in Σ^* and a sequence of states $q_0, q_1, \ldots, q_k$ exists such that

1. $q_0 = q_{\text{start}}$ is the start state,
2. $q_k = q_{\text{accept}}$ is the accept state, and
3. for each i, we have $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$; in other words, R_i is the expression on the arrow from q_{i-1} to q_i.

Returning to the proof of Lemma 1.32, we let M be the DFA for language A. Then we convert M to a GNFA G by adding a new start state and a new accept state and additional transition arrows as necessary. We use the procedure CONVERT(G), which takes a GNFA and returns an equivalent regular expression. This procedure uses ***recursion***, which means that it calls itself. An infinite loop is avoided because the procedure calls itself only to process a GNFA that has one fewer state. The case where the GNFA has two states is handled without recursion.

CONVERT(G):

1. Let k be the number of states of G.
2. If $k = 2$, then G must consist of a start state, an accept state, and a single arrow connecting them and labeled with a regular expression R.
   Return the expression R.
3. If $k > 2$, we select any state $q_{\text{rip}} \in Q$ different from q_{start} and q_{accept} and let G' be the GNFA $(Q', \Sigma, \delta', q_{\text{start}}, q_{\text{accept}})$, where

   $$Q' = Q - \{q_{\text{rip}}\},$$

   and for any $q_i \in Q' - \{q_{\text{accept}}\}$ and any $q_j \in Q' - \{q_{\text{start}}\}$ let

   $$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4),$$

   for $R_1 = \delta(q_i, q_{\text{rip}})$, $R_2 = \delta(q_{\text{rip}}, q_{\text{rip}})$, $R_3 = \delta(q_{\text{rip}}, q_j)$, and $R_4 = \delta(q_i, q_j)$.
4. Compute CONVERT(G') and return this value.