4.8 Pushdown Automata

The pushdown automaton (PDA) has a one-way, read-only, potentially infinite input tape on which an input string is written (see Fig. 4.22); its head either advances to the right from the leftmost cell or remains stationary. It also has a stack, a storage medium analogous to the stack of trays in a cafeteria. The **stack** is a potentially infinite ordered collection of initially blank cells with the property that data can be pushed onto it or popped from it. Data is **pushed** onto the top of the stack by moving all existing entries down one cell and inserting the new element in the top location. Data is **popped** by removing the top element and moving all other entries up one cell. The control unit of a pushdown automaton is a finite-state machine. The full power of the PDA is realized only when its control unit is nondeterministic.

DEFINITION 4.8.1 *A* **pushdown automaton (PDA)** *is a six-tuple* $M = (\Sigma, \Gamma, Q, \Delta, s, F)$, *where* Σ *is the* **tape alphabet** *containing the blank symbol* β, Γ *is the* **stack alphabet** *containing the blank symbol* γ, Q *is the* **set of states**, $\Delta \subseteq (Q \times (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\}) \times Q \times (\Gamma \cup \{\epsilon\}))$ *is the set of* **transitions**, s *is the* **initial state**, *and* F *is the* **set of final states**. *We now describe transitions.*

If for state p, *tape symbol* x, *and stack symbol* y *the transition* $(p, x, y; q, z) \in \Delta$, *then if* M *is in state* p, $x \in \Sigma$ *is under its tape head, and* $y \in \Gamma$ *is at the top of its stack,* M *may pop* y *from its stack, enter state* $q \in Q$, *and push* $z \in \Gamma$ *onto its stack. However, if* $x = \epsilon$, $y = \epsilon$ *or* $z = \epsilon$, *then* M *does not read its tape, pop its stack or push onto its stack, respectively. The head on the tape either remains stationary if* $x = \epsilon$ *or advances one cell to the right if* $x \neq \epsilon$.

If at each point in time a unique transition $(p, x, y; q, z)$ *may be applied, the PDA is* **deterministic**. *Otherwise it is* **nondeterministic**.

The PDA M **accepts the input string** $w \in \Sigma^*$ *if when started in state* s *with an empty stack (its cells contain the* **blank stack symbol** γ*) and* w *placed left-adjusted on its otherwise blank tape (its blank cells contain the* **blank tape symbol** β*), the last state entered by* M *after reading the components of* w *and no other tape cells is a member of the set* F. M **accepts the language** $L(M)$ *consisting of all such strings.*

Some of the special cases for the action of the PDA M on empty tape or stack symbols are the following: if $(p, x, \epsilon; q, z)$, x is read, state q is entered, and z is pushed onto

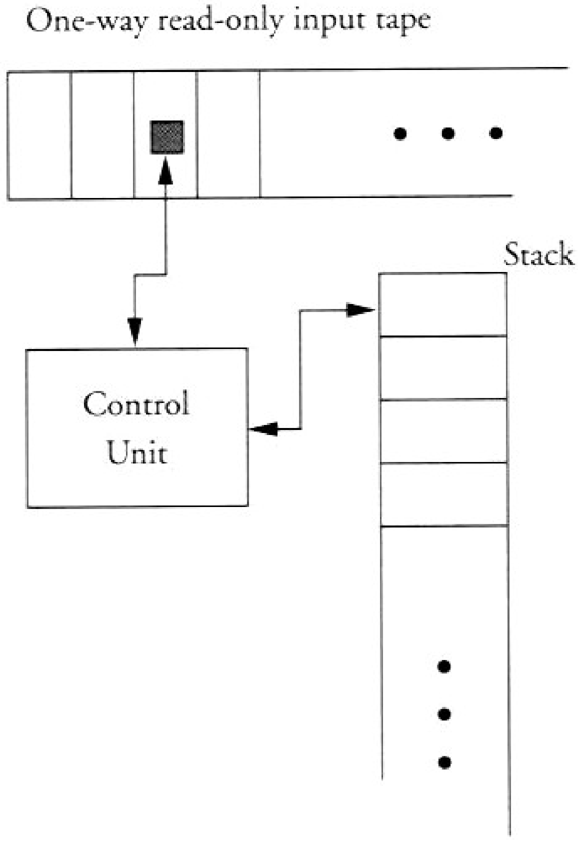

Figure 4.22 The control unit, one-way input tape, and stack of a pushdown automaton.

the stack; if $(p, x, y; q, \epsilon)$, x is read, state q is entered, and y is popped from the stack; if $(p, \epsilon, y; q, z)$, no input is read, y is popped, z is pushed and state q is entered. Also, if $(p, \epsilon, \epsilon; q, \epsilon)$, M moves from state p to q without reading input, or pushing or popping the stack.

Observe that if every transition is of the form $(p, x, \epsilon; q, \epsilon)$, the PDA ignores the stack and simulates an FSM. Thus, the languages accepted by PDAs include the regular languages.

We emphasize that a PDA is nondeterministic if for some state q, tape symbol x, and top stack item y there is more than one transition that M can make. For example, if Δ contains $(s, a, \epsilon; s, a)$ and $(s, a, a; r, \epsilon)$, M has the choice of ignoring or popping the top of the stack and of moving to state s or r. If after reading all symbols of $\boldsymbol{w}$ M enters a state in F, then M accepts $\boldsymbol{w}$.

We now give two examples of PDAs and the languages they accept. The first accepts palindromes of the form $\{\boldsymbol{w}c\boldsymbol{w}^R\}$, where $\boldsymbol{w}^R$ is the reverse of $\boldsymbol{w}$ and $\boldsymbol{w} \in \{a, b\}^*$. The state diagram of its control unit is shown in Fig. 4.23. The second PDA accepts those strings over $\{a, b\}$ of the form $a^n b^m$ for which $n \geq m$.

EXAMPLE 4.8.1 *The PDA* $M = (\Sigma, \Gamma, Q, \Delta, s, F)$, *where* $\Sigma = \{a, b, c, \beta\}$, $\Gamma = \{a, b, \gamma\}$, $Q = \{s, p, r, f\}$, $F = \{f\}$ *and* Δ *contains the transitions shown in Fig. 4.24, accepts the language* $L = \{\boldsymbol{w}c\boldsymbol{w}^R\}$.

The PDA M of Figs. 4.23 and 4.24 remains in the **stacking state** s while encountering a's and b's on the input tape, pushing these letters (the order of these letters on the stack is the reverse of their order on the input tape) onto the stack (Rules (a) and (b)). If it encounters an

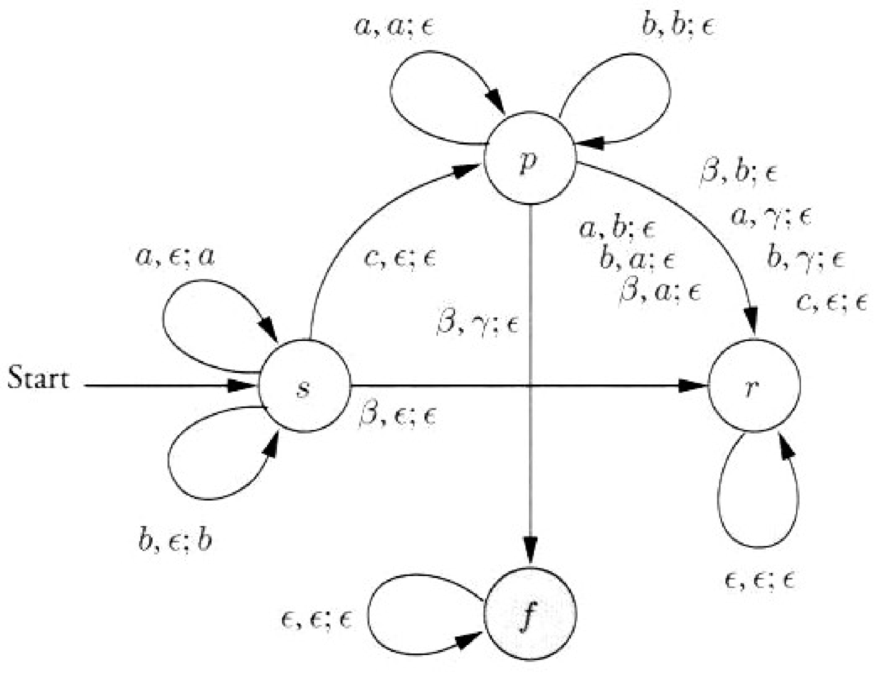

Figure 4.23 State diagram for the pushdown automaton of Fig. 4.24 which accepts $\{wcw^R\}$. An edge label $a, b; c$ between states p and q corresponds to the transition $(p, a, b; q, c)$.

instance of letter c while in state s, it enters the **possible accept state** p (Rule (c)) but enters the **reject state** r if it encounters a blank on the input tape (Rule (d)). While in state p it pops an a or b that matches the same letter on the input tape (Rules (e) and (f)). If the PDA discovers blank tape and stack symbols, it has identified a palindrome and enters the **accept state** f (Rule (g)). On the other hand, if while in state p the tape symbol and the symbol on the top of the stack are different or the letter c is encountered, the PDA enters the reject state r (Rules (h)–(n)). Finally, the PDA does not exit from either the reject or accept states (Rules (o) and (p)).

	Rule	*Comment*		*Rule*	*Comment*
(a)	$(s, a, \epsilon; s, a)$	push a	(i)	$(p, b, a; r, \epsilon)$	reject
(b)	$(s, b, \epsilon; s, b)$	push b	(j)	$(p, \beta, a; r, \epsilon)$	reject
(c)	$(s, c, \epsilon; p, \epsilon)$	accept?	(k)	$(p, \beta, b; r, \epsilon)$	reject
(d)	$(s, \beta, \epsilon; r, \epsilon)$	reject	(l)	$(p, a, \gamma; r, \epsilon)$	reject
(e)	$(p, a, a; p, \epsilon)$	accept?	(m)	$(p, b, \gamma; r, \epsilon)$	reject
(f)	$(p, b, b; p, \epsilon)$	accept?	(n)	$(p, c, \epsilon; r, \epsilon)$	reject
(g)	$(p, \beta, \gamma; f, \epsilon)$	accept	(o)	$(r, \epsilon, \epsilon; r, \epsilon)$	stay in reject state
(h)	$(p, a, b; r, \epsilon)$	reject	(p)	$(f, \epsilon, \epsilon; f, \epsilon)$	stay in accept state

Figure 4.24 Transitions for the PDA described by the state diagram of Fig. 4.23.