Complex Analysis: Exam 1B

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1. (20) Miscellaneous computations. Show all necessary steps.

   (a) Compute all values of $(1-i)^{\frac{4}{3}} = ((1-i)^4)^{\frac{1}{3}}$.

Begin by writing $z = 1-i$ in polar form. The modulus is $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Then, by using the relationships

$$\cos\theta = \frac{\text{Re}(z)}{|z|} \qquad \text{and} \qquad \sin\theta = \frac{\text{Im}(z)}{|z|},$$

we see that $\cos\theta = 1/\sqrt{2}$ and $\sin\theta = -1/\sqrt{2}$. From this set of equations we conclude that $\text{Arg}(z) = -\pi/4$. Therefore $z = \sqrt{2}e^{-i\pi/4}$. Raising z to the fourth power gives $z^4 = 4e^{-i\pi}$ = -4. Now all that's left to do is find the cube roots of z^4, which will require the use of

$$z^{1/m} = |z|^{1/m} e^{i(\theta + 2k\pi)/m} \qquad (k = 0, 1, 2, ..., m-1).$$

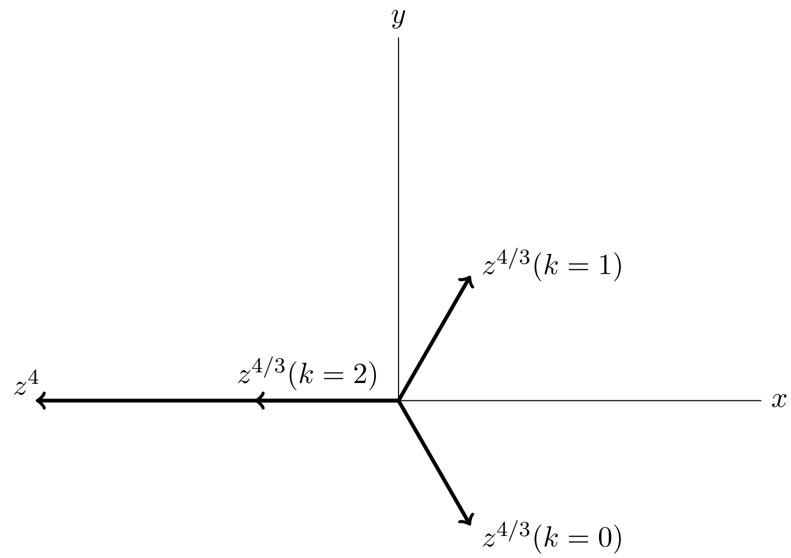
In our case $m = 3$ so that the above equation takes the form

$$(z^4)^{1/3} = \left|z^4\right|^{1/3} e^{i(-\pi + 2k\pi)/3} = 4^{1/3} e^{i\pi(2k-1)/3} \qquad (k = 0, 1, 2).$$

By plugging in the possible values of k, we get:

k	$z^{4/3}$
0	$4^{1/3}e^{-i\pi/3}$
2	$4^{1/3}e^{i\pi/3}$
2	$4^{1/3}e^{i\pi}$

A plot of these roots, along with the vector representing z^4, can be seen in the figure below.



(b) Compute $\int_0^{2\pi} \cos^5(x)\, dx$ using methods introduced in our class.

Start by simplifying the integrand using Euler's formula

$$\cos^5 x = \left[\frac{e^{ix}+e^{-ix}}{2}\right]^5 = \frac{1}{2^5}[e^{ix}+e^{-ix}]^5.$$

Expanding via the binomial formula gives

$$\begin{aligned}\cos^5 x &= \frac{1}{2^5}[e^{5ix} + 5e^{4ix}e^{-ix} + 10e^{3ix}e^{-2ix} + 10e^{2ix}e^{-3ix} + 5e^{ix}e^{-4ix} + e^{-5ix}] \\ &= \frac{1}{2^5}[(e^{5ix}+e^{-5ix}) + 5(e^{3ix}+e^{-3ix}) + 10(e^{ix}+e^{-ix})] \\ &= \frac{1}{2^5}[(2\cos 5x) + (2\cos 3x) + (2\cos x)] \\ &= \frac{1}{2^4}[\cos 5x + \cos 3x + \cos x].\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^{2\pi} \cos^5 x\, dx &= \frac{1}{2^4}\int_0^{2\pi} [\cos 5x + \cos 3x + \cos x]\, dx \\ &= \frac{1}{2^4}\left[\frac{1}{5}\sin 5x + \frac{5}{3}\sin 3x + 10\sin x\right]_0^{2\pi} \\ &= 0.\end{aligned}$$

(c) Find the partial fraction decomposition of $\frac{z^2+z+1}{(z-i)^2(z+2)}$. (You do not need to simplify the constants that you solve for.)

The desired form is

$$R(z) \equiv \frac{z^2+z+1}{(z-i)^2(z+2)} = \frac{A_0^{(1)}}{z+2} + \frac{A_0^{(2)}}{(z-i)^2} + \frac{A_1^{(2)}}{z-i}.$$

With the function in this form, we are now able to make use of the general expression for the coefficients of the partial fraction decomposition of a rational function $R_{m,n}(z)$ (whose denominator degree $n = d_1 + d_2 + ... + d_r$ exceeds its numerator degree m):

$$A_s^{(j)} = \lim_{z \to \xi_j} \frac{1}{s!} \frac{d^s}{dz^2} \left[(z - \xi_j)^{d_j} R_{m,n}(z) \right],$$

where ξ_j are distinct roots and d_j is the degree of the root. We find that

$$A_0^{(1)} = \lim_{z \to -2} (z+2) R(z) = \lim_{z \to -2} \frac{z^2 + z + 1}{(z - i)^2} = \frac{3}{3 + 4i} = \frac{3}{3 + 4i} \left(\frac{3 - 4i}{3 - 4i} \right) = \frac{9}{25} - \frac{12}{25} i.$$

The next coefficient is

$$A_0^{(2)} = \lim_{z \to i} (z - i)^2 R(z) = \lim_{z \to i} \frac{z^2 + z + 1}{z + 2} = \frac{i}{i + 2} = \frac{i}{i + 2} \left(\frac{-i + 2}{-i + 2} \right) = \frac{1}{5} + \frac{2}{5} i.$$

Finally, we have

$$A_1^{(2)} = \lim_{z \to i} \frac{d}{dz} \left[(z - i)^2 R(z) \right] = \lim_{z \to i} \frac{d}{dz} \left[\frac{z^2 + z + 1}{z + 2} \right] = \lim_{z \to i} \left[\frac{(2z + 1)(z + 2) - (z^2 + z + 1)}{(z + 2)^2} \right]$$
$$= \lim_{z \to i} \left[\frac{z^2 + 4z + 1}{(z + 2)^2} \right] = \frac{4i}{3 + 4i} = \frac{4i}{3 + 4i} \left(\frac{3 - 4i}{3 - 4i} \right) = \frac{16}{25} + \frac{12}{25} i.$$

Putting everything together:

$$R(z) \equiv \frac{z^2 + z + 1}{(z - i)^2 (z + 2)} = \frac{\frac{9}{25} - \frac{12}{25} i}{z + 2} + \frac{\frac{1}{5} + \frac{2}{5} i}{(z - i)^2} + \frac{\frac{16}{25} + \frac{12}{25} i}{z - i}.$$

(d) Describe the set of points $|z| = 2|z - i|$.

The modulus of a complex number $z = x + iy$ is given by $|z| = (x^2 + y^2)^{1/2}$. To avoid dealing with the square roots, let us square both sides of the given equation and write out the moduli in terms of x and y:

$$\begin{aligned} x^2 + y^2 &= 4(x^2 + (y - 1)^2) \\ &= 4(x^2 + y^2 - 2y + 1). \end{aligned}$$

Grouping terms together yields

$$x^2 + \left(y^2 - \frac{8}{3} y \right) = -\frac{4}{3}.$$

We shall next complete the square by adding $(4/3)^2$ to each side

$$x^2 + \left(y^2 - \frac{8}{3} y + \frac{16}{9} \right) = -\frac{4}{3} + \frac{16}{9},$$

which simplifies to

$$x^2 + \left(y - \frac{4}{3} \right)^2 = \frac{4}{9}.$$

This is the equation for a circle centered about the point $(x, y) = (0, 4/3)$ and with radius $(4/9)^{1/2} = 2/3$. Therefore, the set of points that satisfy $|z| = 2|z - i|$ are all of the points on the circle centered about the point $(x, y) = (0, 4/3)$ and with radius $2/3$.

2. (5) Use the idea of local linear approximation to describe what happens to a small disc centered at $z_0 = 3 + 4i$ when it is substituted into $f(z) = 1/z$. Describe what happens in the language of translations, rotations, expansions, and/or contractions. Draw an appropriate diagram.

We can make the local linear approximation $f(z) \approx f(z_0) + f'(z_0)(z - z_0)$ for a small disc, say of radius δ, centered at the point z_0 when it is substituted into some function $f(z)$. The term $f(z_0)$ generates a translation mapping, telling us where the new disk will be centered. The derivative of the function evaluated at z_0 can be written in polar form $f'(z_0) = re^{i\theta}$, where r generates a magnification mapping and $e^{i\theta}$ generates a rotation mapping. These latter two mappings are centered about the new disk due to the inclusion of the $(z - z_0)$ term.

For this problem, we have $z_0 = 3 + 4i$ and $f(z) = 1/z$, so

$$f(z_0) = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{3 - 4i}{3^2 + 4^2} = \frac{3}{25} - \frac{4}{25}i.$$

Clearly $f(z)$ is continuous at z_0. In fact, the inversion mapping $f(z) = 1/z$ is continuous everywhere in the complex plane except $z = 0$. Next, we find that the first derivative of $f(z)$ evaluated at z_0 is

$$f'(z_0) = \frac{d}{dz}\frac{1}{z}\bigg|_{z=z_0} = -\frac{1}{z_0^2} = -\frac{1}{(3+4i)^2} = \frac{1}{7 - 24i} = \frac{1}{7 - 24i}\frac{7 + 24i}{7 + 24i} = \frac{7}{25^2} + \frac{24}{25^2}i.$$

The modulus of this result is

$$r = |z| = \sqrt{\left(\frac{7}{25^2}\right)^2 + \left(\frac{24}{25^2}\right)^2} = \sqrt{\frac{25^2}{25^4}} = \frac{1}{25}.$$

Since an angle is fixed by its sine and cosine, θ is uniquely determined by the pair of equations

$$\cos\theta = \frac{x}{|z|} \qquad \text{and} \qquad \sin\theta = \frac{y}{|z|}.$$

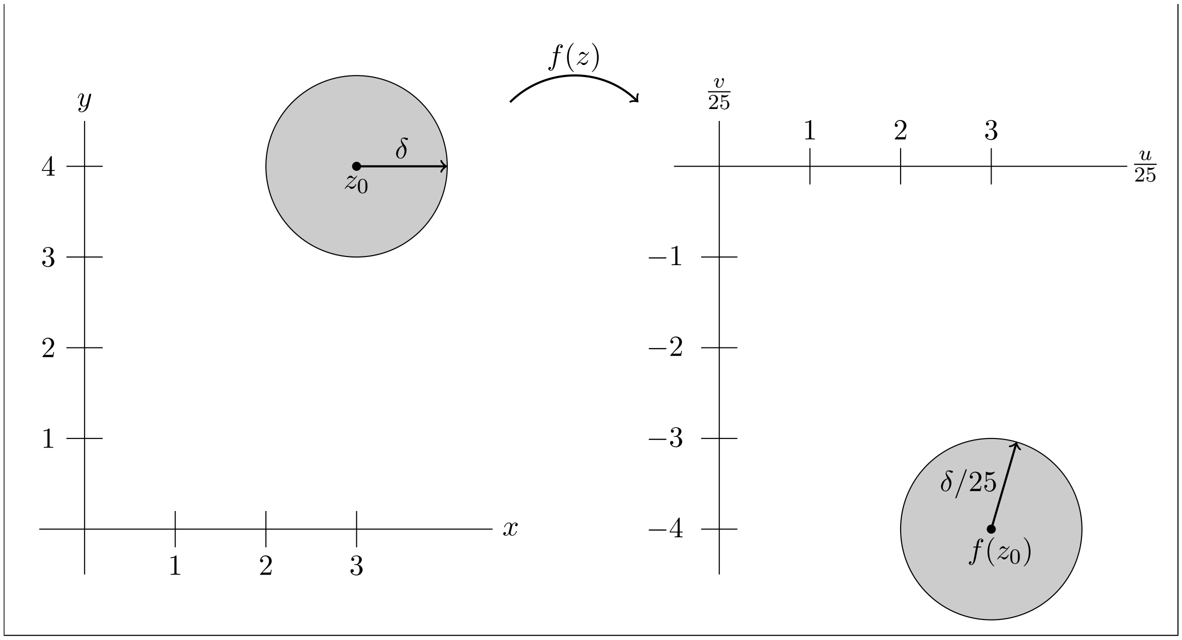
Plugging in the appropriate values gives

$$\cos\theta = \frac{7/25^2}{1/25} = \frac{7}{25} \qquad \text{and} \qquad \sin\theta = \frac{24/25^2}{1/25} = \frac{24}{25},$$

from which we have $\theta = 1.287$ rad $= 0.410\pi$ rad $= 73.74°$. Our local linear approximation for a small disk centered at the point $z_0 = 3 + 4i$ is thus

$$\frac{1}{z} \approx \left(\frac{3}{25} - \frac{4}{25}i\right) + \frac{1}{25}e^{i(0.410\pi)}(z - (3 + 4i)).$$

The disk in the uv-plane is centered at $(u, v) = (3/25, -4/25)$, has radius $\delta/25$, and a point in the neighborhood of z_0, upon being translated to $f(z_0)$, is rotated counterclockwise by 0.410π about the point $f(z_0)$. We can see what is happening geometrically in the figure below. Notice that the uv-plane is scaled by a factor of 25 since the mapping has contracted the neighborhood about z_0 by that amount.



3. (10) Describe the projection on the Riemann Sphere of the set $\{z = x + iy : x \geq \sqrt{3}\}$. I encourage you to include a nice diagram.

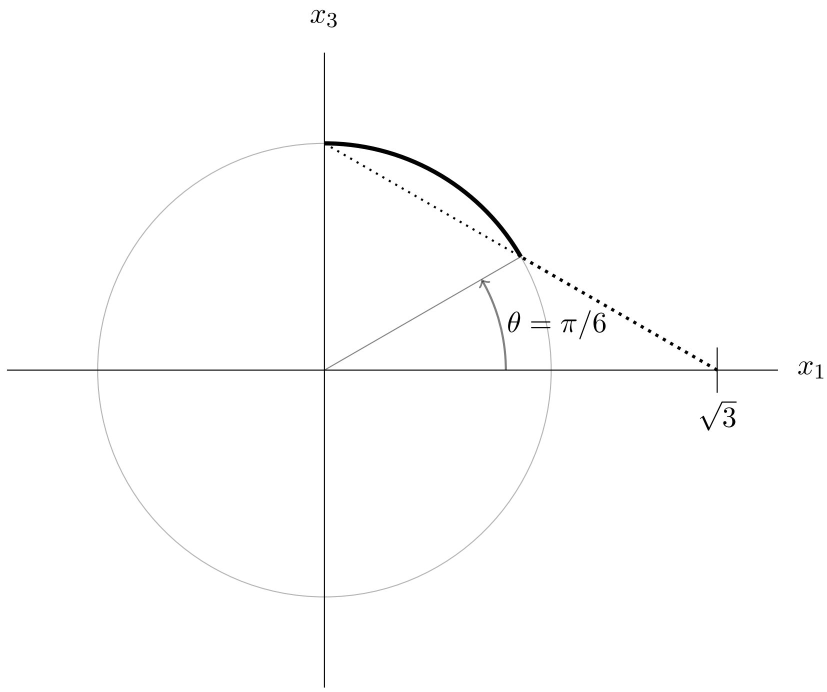
If $Z = (x_1, x_2, x_3)$ is the projection on the Riemann sphere of the point $z = x + iy$ in the complex plane, then

$$x_1 = \frac{2\,\mathrm{Re}\,z}{|z|^2+1}, \qquad x_2 = \frac{2\,\mathrm{Im}\,z}{|z|^2+1}, \qquad x_3 = \frac{|z|^2-1}{|z|^2+1}.$$

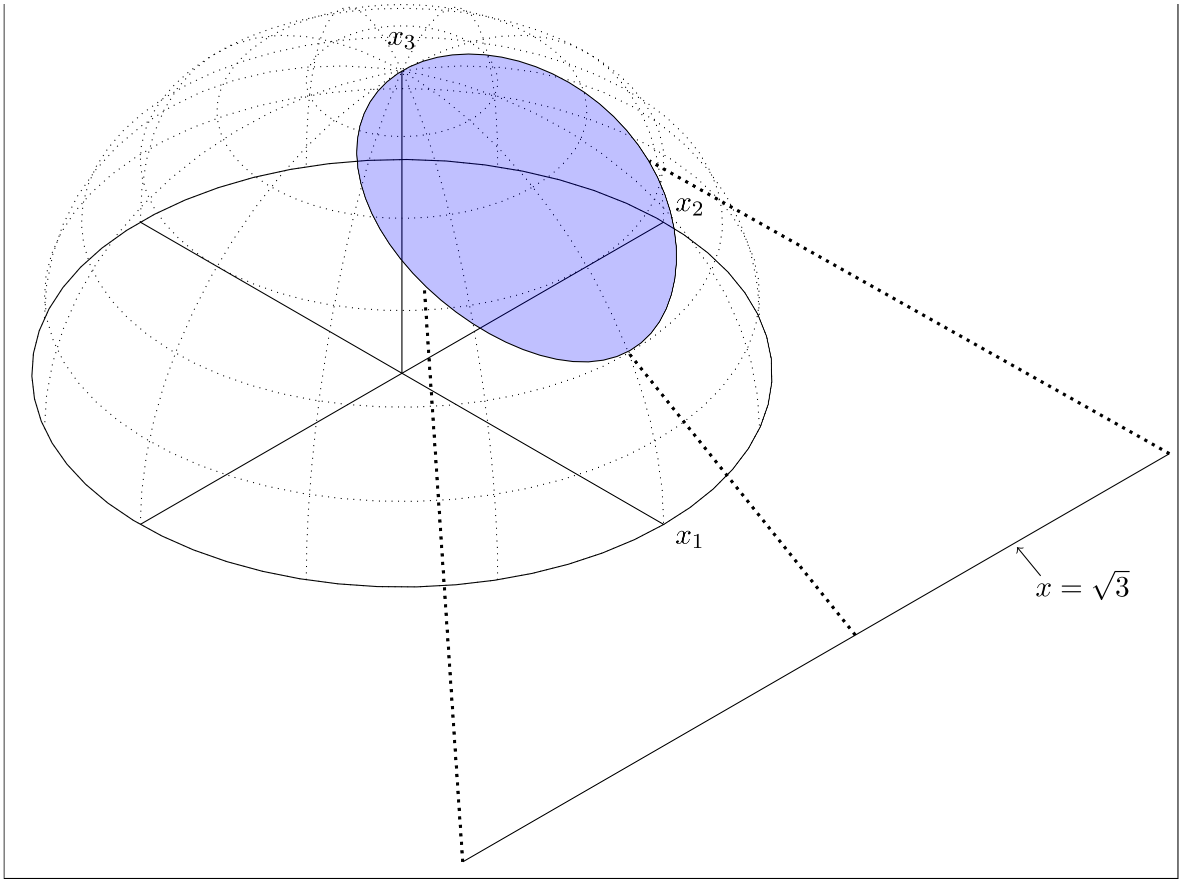
Let us first start by determining what the line $(x, y) = (\sqrt{3}, y)$ projects to on the Riemann sphere. Since all lines and cirlces in the z-plane correspond under stereographic projection to cirlces on the Riemann sphere, we are really just trying to figure out *which* circle the line projects to. The point Z has coordinates

$$x_1 = \frac{2\sqrt{3}}{4+y^2}, \qquad x_2 = \frac{2y}{4+y^2}, \qquad x_3 = \frac{2+y^2}{4+y^2}.$$

When $y = 0$, we find that $Z = (\sqrt{3}/2, 0, 1/2)$. In the limit that $y \to \pm\infty$, the projected points approach the north pole $Z \to (0, 0, 1)$ from the $\pm$ y-direction. The norm of the separation vector $(\sqrt{3}/2, 0, 1/2) - (0, 0, 1) = (\sqrt{3}/2, 0, -1/2)$ is 1, meaning that the circle projected on the surface has radius 0.5 (note that here we aren't considering the distance *on* the sphere, but in the 3-space the sphere is embedded in). We could also set the y-derivative of the x_2 equation to zero to find the minimum and maximum values of x_2 (which turn out to be $x_2 = \pm 0.5$). So, for $x = \sqrt{3}$, we get a circle of radius 0.5 centered at $Z = (1/2, 0, \sqrt{3}/2)$. A slice of the sphere corresponding to $x_2 = 0$ is shown below. The dark arc on the circle from $\theta = \pi/6$ to $\theta = \pi/2$, as measured from the positive x_1-axis, displays the portion of the sphere the circle is on.



If we were to continue doing this for every value of x greater than $\sqrt{3}$, we would find that every subsequent line maps to a circle inside the circle mapped out by the line $x = \sqrt{3}$. Thus, the set $\{z = x + iy : x \geq \sqrt{3}\}$ projects to the circle of radius 0.5 centered at $Z = (1/2, 0, \sqrt{3}/2)$ and all points on the sphere within that circle. A visualization of this projection is shown below, wherein the blue circle and the black line bordering it are the projections of the given set. The line $x = \sqrt{3}$ is shown in the bottom right, along with three points and their projections (connected by the dotted lines).



4. (15) Choose one of the following explorations.

(b) An exploration of *admissibility*. Recall that if $z = x + iy$, then $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$.

i. (4) Substitute for x and y in $f(x, y) = x^2 - y^2 + i2xy$ and verify that f reduces to a function of z only, i.e. that there are no $\bar{z}$ terms remaining after simplification. Also use the Cauchy Riemann Equations to verify that f is analytic.

Using the relationships $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$, we obtain

$$
\begin{aligned}
f(x, y) &= \left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 + i2\left(\frac{z + \bar{z}}{2}\right)\left(\frac{z + \bar{z}}{2i}\right) \\
&= \frac{1}{4}(z^2 + \bar{z}^2 + 2z\bar{z}) + \frac{1}{4}(z^2 + \bar{z}^2 - 2z\bar{z}) + \frac{1}{2}(z^2 - \bar{z}^2) \\
&= \frac{1}{4}(z^2 + \bar{z}^2 + 2z\bar{z} + z^2 + \bar{z}^2 - 2z\bar{z} + 2z^2 - 2\bar{z}^2) \\
&= \frac{1}{4}(4z^2) \\
&= z^2.
\end{aligned}
$$

Now, we turn our attention to the Cauchy-Riemann equations

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},
$$

where u and v denote the real and imaginary parts, respectively, of $f(x, y)$. For our function:

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial y} = 2x,$$
$$\frac{\partial u}{\partial y} = -2y, \qquad \frac{\partial v}{\partial x} = 2y.$$

Since the first partial derivatives are continuous and satisfy the Cauchy-Riemann equations at all points of $\mathbb{C}$, f is entire.

ii. (4) Substitute for x and y in $g(x,y) = x^2 - y^2 + i3xy$ and verify that f does not reduce to a function of $\bar{z}$ only, i.e. that there are still $\bar{z}$ terms remaining after simplification. Also use the Cauchy Riemann Equations to verify that g is not analytic.

Using the relationships $x = \frac{1}{2}(z+\bar{z})$ and $y = \frac{1}{2i}(z-\bar{z})$, we obtain

$$\begin{aligned}
g(x,y) &= \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 + i3\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z+\bar{z}}{2i}\right) \\
&= \frac{1}{4}(z^2+\bar{z}^2+2z\bar{z}) + \frac{1}{4}(z^2+\bar{z}^2-2z\bar{z}) + \frac{3}{4}(z^2-\bar{z}^2) \\
&= \frac{1}{4}(z^2+\bar{z}^2+2z\bar{z}+z^2+\bar{z}^2-2z\bar{z}+3z^2-3\bar{z}^2) \\
&= \frac{1}{4}(5z^2-\bar{z}^2) \\
&= \frac{5}{4}z^2 - \frac{1}{4}\bar{z}^2.
\end{aligned}$$

Evaluating the Cauchy-Riemann equations for this function yields

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial y} = 3x,$$
$$\frac{\partial u}{\partial y} = -2y, \qquad \frac{\partial v}{\partial x} = 3y.$$

Since the Cauchy-Riemann equations are satsified nowhere, the function $g(x,y)$ is nowhere analytic.

iii. (7) *Admissibility* essentially boils down to the idea that $f(x,y) = u(x,y) + iv(x,y)$ does not depend on z. A more precise way to say this is that $\frac{\partial f}{\partial \bar{z}} = 0$. Use the multivariable chain rule,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}},$$

to show that $\frac{\partial f}{\partial \bar{z}} = 0$ if the Cauchy Riemann Equations are satisfied. This indicates that f does not depend on $\bar{z}$ if f is analytic.

First, note that $\partial x/\partial \bar{z} = 1/2$ and $\partial y/\partial \bar{z} = i/2$. Next, substitute $f = u + iv$

into the given equation:

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} \\ &= \frac{1}{2}\frac{\partial(u+iv)}{\partial x} + \frac{i}{2}\frac{\partial(u+iv)}{\partial y} \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\ &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right].\end{aligned}$$

Regarding the last line, the expressions in parentheses are zero, and thus $\frac{\partial f}{\partial \bar{z}} = 0$, when the Cauchy-Riemann equations are satisfied. We conclude that if a function is analytic, then it has no $\bar{z}$-dependence.