Complex Analysis: Exam 1B

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1. (20) Miscellaneous computations. Show all necessary steps.

   (a) Compute all values of $(1-i)^{\frac{4}{3}} = ((1-i)^4)^{\frac{1}{3}}$.

Begin by writing $z = 1 - i$ in polar form. The modulus is $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Then, by using the relationships

$$\cos\theta = \frac{\text{Re}(z)}{|z|} \qquad \text{and} \qquad \sin\theta = \frac{\text{Im}(z)}{|z|},$$

we see that $\cos\theta = 1/\sqrt{2}$ and $\sin\theta = -1/\sqrt{2}$. From this set of equations we conclude that $\text{Arg}(z) = -\pi/4$. Therefore $z = \sqrt{2}e^{-i\pi/4}$. Raising z to the fourth power gives $z^4 = 4e^{-i\pi}$ = -4. Now all that's left to do is find the cube roots of z^4, which will require the use of

$$z^{1/m} = |z|^{1/m}e^{i(\theta+2k\pi)/m} \qquad (k = 0, 1, 2, ..., m-1).$$

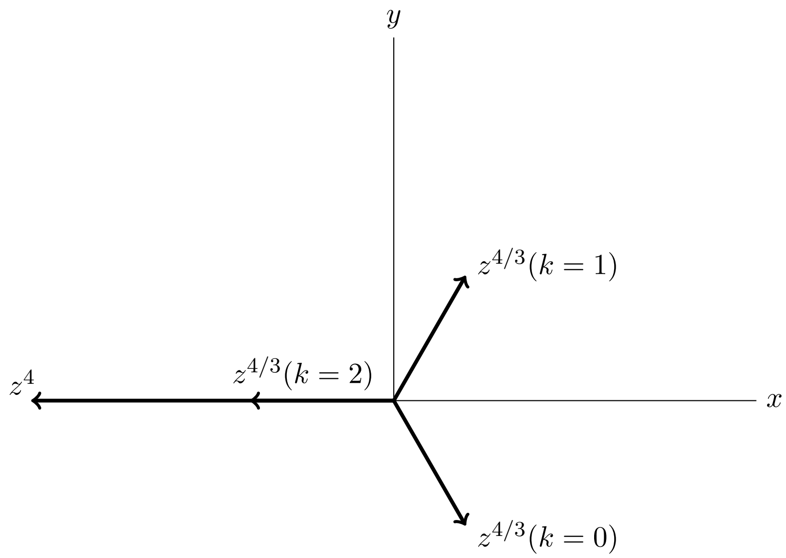
In our case $m = 3$ so that the above equation takes the form

$$(z^4)^{1/3} = \left|z^4\right|^{1/3} e^{i(-\pi+2k\pi)/3} = 4^{1/3}e^{i\pi(2k-1)/3} \qquad (k = 0, 1, 2).$$

By plugging in the possible values of k, we get:

k	$z^{4/3}$
0	$4^{1/3}e^{-i\pi/3}$
2	$4^{1/3}e^{i\pi/3}$
2	$4^{1/3}e^{i\pi}$

A plot of these roots, along with the vector representing z^4, can be seen in the figure below.



(b) Compute $\int_0^{2\pi} \cos^5(x)\,dx$ using methods introduced in our class.

Start by simplifying the integrand using Euler's formula

$$\cos^5 x = \left[\frac{e^{ix}+e^{-ix}}{2}\right]^5 = \frac{1}{2^5}[e^{ix}+e^{-ix}]^5.$$

Expanding via the binomial formula gives

$$\begin{aligned}\cos^5 x &= \frac{1}{2^5}[e^{5ix} + 5e^{4ix}e^{-ix} + 10e^{3ix}e^{-2ix} + 10e^{2ix}e^{-3ix} + 5e^{ix}e^{-4ix} + e^{-5ix}] \\ &= \frac{1}{2^5}[(e^{5ix}+e^{-5ix}) + 5(e^{3ix}+e^{-3ix}) + 10(e^{ix}+e^{-ix})] \\ &= \frac{1}{2^5}[(2\cos 5x) + (2\cos 3x) + (2\cos x)] \\ &= \frac{1}{2^4}[\cos 5x + \cos 3x + \cos x].\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^{2\pi} \cos^5 x\,dx &= \frac{1}{2^4}\int_0^{2\pi}[\cos 5x + \cos 3x + \cos x]\,dx \\ &= \frac{1}{2^4}\left[\frac{1}{5}\sin 5x + \frac{5}{3}\sin 3x + 10\sin x\right]_0^{2\pi} \\ &= 0.\end{aligned}$$

(c) Find the partial fraction decomposition of $\frac{z^2+z+1}{(z-i)^2(z+2)}$. (You do not need to simplify the constants that you solve for.)

The desired form is

$$R(z) \equiv \frac{z^2+z+1}{(z-i)^2(z+2)} = \frac{A_0^{(1)}}{z+2} + \frac{A_0^{(2)}}{(z-i)^2} + \frac{A_1^{(2)}}{z-i}.$$

With the function in this form, we are now able to make use of the general expression for the coefficients of the partial fraction decomposition of a rational function $R_{m,n}(z)$ (whose denominator degree $n = d_1 + d_2 + ... + d_r$ exceeds its numerator degree m):

$$A_s^{(j)} = \lim_{z\to\xi_j} \frac{1}{s!}\frac{d^s}{dz^2}\left[(z-\xi_j)^{d_j} R_{m,n}(z)\right],$$

where ξ_j are distinct roots and d_j is the degree of the root. We find that

$$A_0^{(1)} = \lim_{z\to-2}(z+2)R(z) = \lim_{z\to-2}\frac{z^2+z+1}{(z-i)^2} = \frac{3}{3+4i} = \frac{3}{3+4i}\left(\frac{3-4i}{3-4i}\right) = \frac{9}{25} - \frac{12}{25}i.$$

The next coefficient is

$$A_0^{(2)} = \lim_{z\to i}(z-i)^2R(z) = \lim_{z\to i}\frac{z^2+z+1}{z+2} = \frac{i}{i+2} = \frac{i}{i+2}\left(\frac{-i+2}{-i+2}\right) = \frac{1}{5} + \frac{2}{5}i.$$

Finally, we have

$$\begin{aligned} A_1^{(2)} &= \lim_{z\to i}\frac{d}{dz}\left[(z-i)^2R(z)\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{z^2+z+1}{z+2}\right] = \lim_{z\to i}\left[\frac{(2z+1)(z+2)-(z^2+z+1)}{(z+2)^2}\right] \\ &= \lim_{z\to i}\left[\frac{z^2+4z+1}{(z+2)^2}\right] = \frac{4i}{3+4i} = \frac{4i}{3+4i}\left(\frac{3-4i}{3-4i}\right) = \frac{16}{25} + \frac{12}{25}i. \end{aligned}$$

Putting everything together:

$$R(z) \equiv \frac{z^2+z+1}{(z-i)^2(z+2)} = \frac{\frac{9}{25}-\frac{12}{25}i}{z+2} + \frac{\frac{1}{5}+\frac{2}{5}i}{(z-i)^2} + \frac{\frac{16}{25}+\frac{12}{25}i}{z-i}.$$

(d) Describe the set of points $|z| = 2|z-i|$.

The modulus of a complex number $z = x + iy$ is given by $|z| = (x^2+y^2)^{1/2}$. To avoid dealing with the square roots, let us square both sides of the given equation and write out the moduli in terms of x and y

$$\begin{aligned} x^2 + y^2 &= 4(x^2 + (y-1)^2) \\ &= 4(x^2 + y^2 - 2y + 1). \end{aligned}$$

Grouping terms together yields

$$x^2 + \left(y^2 - \frac{8}{3}y\right) = -\frac{4}{3}.$$

We shall next complete the square by adding $(4/3)^2$ to each side

$$x^2 + \left(y^2 - \frac{8}{3}y + \frac{16}{9}\right) = -\frac{4}{3} + \frac{16}{9},$$

which simplifies to

$$x^2 + \left(y - \frac{4}{3}\right)^2 = \frac{4}{9}.$$

This is the equation for a circle centered about the point $(x, y) = (0, 4/3)$ and with radius $(4/9)^{1/2} = 2/3$. Therefore, the set of points that satisfy $|z| = 2|z-i|$ are all of the points on the circle centered about the point $(x, y) = (0, 4/3)$ and with radius $2/3$.

2. (5) Use the idea of local linear approximation to describe what happens to a small disc centered at $z_0 = 3 + 4i$ when it is substituted into $f(z) = 1/z$. Describe what happens in the language of translations, rotations, expansions, and/or contractions. Draw an appropriate diagram.

3. (10) Describe the projection on the Riemann Sphere of the set $\{z = x + iy : x \geq \sqrt{3}\}$. I encourage you to include a nice diagram.

4. (15) Choose one of the following explorations.

   (b) An exploration of *admissibility*. Recall that if $z = x + iy$, then $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$.

   i. (4) Substitute for x and y in $f(x, y) = x^2 - y^2 + i2xy$ and verify that f reduces to a function of z only, i.e. that there are no $\bar{z}$ terms remaining after simplification. Also use the Cauchy Riemann Equations to verify that f is analytic.

   ii. (4) Substitute for x and y in $g(x, y) = x^2 - y^2 + i3xy$ and verify that f does not reduce to a function of z only, i.e. that there are still $\bar{z}$ terms remaining after simplification. Also use the Cauchy Riemann Equations to verify that g is not analytic.

   iii. (7) *Admissibility* essentially boils down to the idea that $f(x, y) = u(x, y) + iv(x, y)$ does not depend on $\bar{z}$. A more precise way to say this is that $\frac{\partial f}{\partial \bar{z}} = 0$. Use the multivariable chain rule,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}},$$

   to show that $\frac{\partial f}{\partial \bar{z}} = 0$ if the Cauchy Riemann Equations are satisfied. This indicates that f does not depend on $\bar{z}$ if f is analytic.