Complex Analysis: Exam 1B

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1. (20) Miscellaneous computations. Show all necessary steps.

   (a) Compute all values of $(1-i)^{\frac{4}{3}} = ((1-i)^4)^{\frac{1}{3}}$.

Begin by writing $z = 1-i$ in polar form. The modulus is $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Then, by using the relationships

$$\cos\theta = \frac{\mathrm{Re}(z)}{|z|} \qquad \text{and} \qquad \sin\theta = \frac{\mathrm{Im}(z)}{|z|},$$

we see that $\cos\theta = 1/\sqrt{2}$ and $\sin\theta = -1/\sqrt{2}$. From this set of equations we conclude that $\mathrm{Arg}(z) = -\pi/4$. Therefore $z = \sqrt{2}e^{-i\pi/4}$. Raising z to the fourth power gives $z^4 = 4e^{-i\pi}$. Now all that's left to do is find the cube roots of z^4, which will require the use of

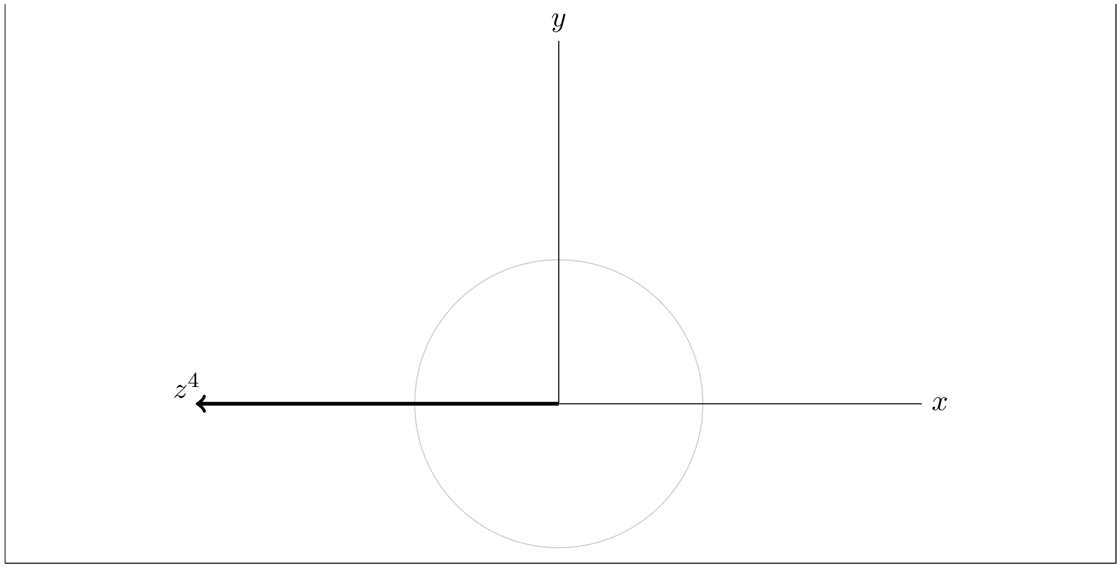
$$z^{1/m} = |z|^{1/m} e^{i(\theta + 2k\pi)/m} \qquad (k = 0, 1, 2, ..., m-1).$$

In our case $m = 3$ so that the above equation takes the form

$$(z^4)^{1/3} = \left|z^4\right|^{1/3} e^{i(-\pi + 2k\pi)/3} = 4^{1/3} e^{i\pi(2k-1)/3} \qquad (k = 0, 1, 2).$$

By plugging in the possible values of k, we get

k	$z^{4/3}$
0	$4^{1/3}e^{-i\pi/3}$
1	$4^{1/3}e^{i\pi/3}$
2	$4^{1/3}e^{i\pi}$



(b) Compute $\int_0^{2\pi} \cos^5(x)dx$ using methods introduced in our class.

(c) Find the partial fraction decomposition of $\dfrac{z^2+z+1}{(z-i)^2(z+2)}$. (You do not need to simplify the constants that you solve for.)

(d) Describe the set of points $|z| = 2|z-i|$.

2. (5) Use the idea of local linear approximation to describe what happens to a small disc centered at $z_0 = 3+4i$ when it is substituted into $f(z) = 1/z$. Describe what happens in the language of translations, rotations, expansions, and/or contractions. Draw an appropriate diagram.

3. (10) Describe the projection on the Riemann Sphere of the set $\{z = x+iy : x \geq \sqrt{3}\}$. I encourage you to include a nice diagram.

4. (15) Choose one of the following explorations.

(b) An exploration of *admissibility*. Recall that if $z = x+iy$, then $x = \frac{1}{2}(z+\bar{z})$ and $y = \frac{1}{2i}(z-\bar{z})$.

i. (4) Substitute for x and y in $f(x,y) = x^2y^2 + i2xy$ and verify that f reduces to a function of z only, i.e. that there are no z terms remaining after simplification. Also use the Cauchy Riemann Equations to verify that f is analytic.

ii. (4) Substitute for x and y in $g(x,y) = x^2y^2 + i3xy$ and verify that f does not reduce to a function of z only, i.e. that there are still z terms remaining after simplification. Also use the Cauchy Riemann Equations to verify that g is not analytic.

iii. (7) *Admissibility* essentially boils down to the idea that $f(x,y) = u(x,y) + iv(x,y)$ does not depend on z. A more precise way to say this is that $\frac{\partial f}{\partial \bar{z}} = 0$. Use the multivariable chain rule,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}},$$

to show that $\frac{\partial f}{\partial \bar{z}} = 0$ if the Cauchy Riemann Equations are satisfied. This indicates that f does not depend on z if f is analytic.