Complex Analysis: Exam 2B

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1. (10) Harmonic functions.

   (a) Find a harmonic function $\phi(x, y)$ on the annulus $1 < |z - (1 + i)| < 3$ which satisfies $\phi \equiv 1$ on $|z - (1 + i)| = 1$ and $\phi \equiv -1$ on $|z - (1 + i)| = 3$.

   For any value of the constants A and B, the function

   $$\phi(x, y) = A \operatorname{Log} |z - (1 + i)| + B$$

   is harmonic, and we only have to adjust A and B so as to achieve the prescribed boundary conditions. We require that

   $$\begin{aligned} A \operatorname{Log} |2 - (1 + i)| + B &= 1, \\ A \operatorname{Log} |4 - (1 + i)| + B &= 3. \end{aligned}$$

   Since $|2 - (1 + i)| = |1 - i| = \sqrt{2}$ and $|4 - (1 + i)| = |3 + i| = \sqrt{10}$, we have

   $$\begin{aligned} A \operatorname{Log} \sqrt{2} + B &= 1, \\ A \operatorname{Log} \sqrt{10} + B &= 3. \end{aligned}$$

   Subtracting the first equation from the second yields

   $$A(\operatorname{Log} \sqrt{10} - \operatorname{Log} \sqrt{2}) = A \operatorname{Log} (5^{1/2}) = \frac{1}{2} A \operatorname{Log} 5 = 2$$

   from which we have $A = 4/\operatorname{Log} 5$. Substitution of A back into either boundary condition equation gives

   $$\frac{4}{\operatorname{Log} 5} \operatorname{Log} \sqrt{2} + B = \frac{\operatorname{Log} 4}{\operatorname{Log} 5} + B = 1.$$

   Clearly then $B = 1 - \operatorname{Log} 4/\operatorname{Log} 5$, and our desired function is

   $$\begin{aligned} \phi(x, y) &= \frac{4}{\operatorname{Log} 5} \operatorname{Log} |z - (1 + i)| + 1 - \frac{\operatorname{Log} 4}{\operatorname{Log} 5} \\ &= \frac{4 \operatorname{Log} |z - (1 + i)| + \operatorname{Log} (5/4)}{\operatorname{Log} 5}. \end{aligned}$$

   (b) Find a harmonic function $\phi(x, y)$ on the half plane $y > 0$ with $\phi \equiv 1$ on $\{(x, y) :$

$y = 0, x < -2\}$, $\phi \equiv 0$ on $\{(x, y) : y = 0, -2 < x < 3\}$,and $\phi \equiv -1$ on $\{(x, y) : y = 0, x > 3\}$.

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2. (5) Derive the formula

$$\tanh^{-1}(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right).$$

Be sure to make note during the derivation of any step where you are choosing a branch.

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3. (5) Show that

$$\left| \int_\gamma Log(z) dz \right| \leq \frac{\pi^2}{2},$$

where γ is the arc of the unit circle contained in the right half plane $\{(x, y) : x \geq 0\}$.

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4. (10) Compute

$$\int_\Gamma \overline{z}^2 \, dz$$

where Γ is the perimeter of the square with vertices $-1 - i, 1 - i, 1 + i$, and $-1 + i$ in that order.

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5. (10) Suppose that $p(z)$ is an nth degree polynomial for some $n \geq 1$. Show that if C_R represents a circle of radius R centered at some point in the complex plane, then

$$\int_{C_R} \frac{p'(z)}{p(z)} dz,$$

counts the number of roots of $p(z)$ contained in the interior of the circle C_R. For simplicity you may assume that $p(z)$ has n distinct roots. (Hint: It helps to create and compute some simple specific examples.)

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