Numerical Analysis

Homework C

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Fit a third order polynomial, $P_3(x)$, to $f(x) = x^5$ on $[-1, 1]$ with least square error:

a) $P_3(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ derive and then solve the normal equations.

$$
\begin{aligned}
E &= \int_{-1}^{1} (x^5 - P_3(x))^2 \,\mathrm{d}x \\
&= \int_{-1}^{1} (x^5 - b_0 - b_1x - b_2x^2 - b_3x^3)^2 \,\mathrm{d}x \\
&= \int_{-1}^{1} (x^{10} - b_0x^5 - b_1x^6 - b_2x^7 - b_3x^8) + (-b_0x^5 + {b_0}^2 + b_0b_1x + b_0b_2x^2 + b_0b_3x^3) \\
&\qquad + (-b_1x^6 + b_0b_1x + {b_1}^2x^2 + b_1b_2x^3 + b_1b_3x^4) + (-b_2x^7 + b_0b_2x^2 + b_1b_2x^3 + {b_2}^2x^4 + b_2b_3x^5) \\
&\qquad + (-b_3x^8 + b_0b_3x^3 + b_1b_3x^4 + b_2b_3x^5 + {b_3}^2x^6) \,\mathrm{d}x \\
&= \left[\frac{x^{11}}{11} - \frac{b_0x^6}{6} - \frac{b_1x^7}{7} - \frac{b_2x^8}{8} - \frac{b_3x^9}{9} - \frac{b_0x^6}{6} + {b_0}^2x + \frac{b_0b_1x^2}{2} + \frac{b_0b_2x^3}{3} + \frac{b_0b_3x^4}{4} \right. \\
&\qquad - \frac{b_1x^7}{7} + \frac{b_0b_1x^2}{2} + \frac{{b_1}^2x^3}{3} + \frac{b_1b_2x^4}{4} + \frac{b_1b_3x^5}{5} - \frac{b_2x^8}{8} + \frac{b_0b_2x^3}{3} + \frac{b_1b_2x^4}{4} \\
&\qquad \left. + \frac{{b_2}^2x^5}{5} + \frac{b_2b_3x^6}{6} - \frac{b_3x^9}{9} + \frac{b_0b_3x^4}{4} + \frac{b_1b_3x^5}{5} + \frac{b_2b_3x^6}{6} + \frac{{b_3}^2x^7}{7} \right]_{-1}^{1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{11} - \frac{b_0}{6} - \frac{b_1}{7} - \frac{b_2}{8} - \frac{b_3}{9} - \frac{b_0}{6} + {b_0}^2 + \frac{b_0 b_1}{2} + \frac{b_0 b_2}{3} + \frac{b_0 b_3}{4} \\
&\qquad - \frac{b_1}{7} + \frac{b_0 b_1}{2} + \frac{{b_1}^2}{3} + \frac{b_1 b_2}{4} + \frac{b_1 b_3}{5} - \frac{b_2}{8} + \frac{b_0 b_2}{3} + \frac{b_1 b_2}{4} \\
&\qquad + \frac{{b_2}^2}{5} + \frac{b_2 b_3}{6} - \frac{b_3}{9} + \frac{b_0 b_3}{4} + \frac{b_1 b_3}{5} + \frac{b_2 b_3}{6} + \frac{{b_3}^2}{7} \\
&\qquad + \frac{1}{11} + \frac{b_0}{6} - \frac{b_1}{7} + \frac{b_2}{8} - \frac{b_3}{9} + \frac{b_0}{6} + {b_0}^2 - \frac{b_0 b_1}{2} + \frac{b_0 b_2}{3} - \frac{b_0 b_3}{4} \\
&\qquad - \frac{b_1}{7} - \frac{b_0 b_1}{2} + \frac{{b_1}^2}{3} - \frac{b_1 b_2}{4} + \frac{b_1 b_3}{5} + \frac{b_2}{8} + \frac{b_0 b_2}{3} - \frac{b_1 b_2}{4} \\
&\qquad + \frac{{b_2}^2}{5} - \frac{b_2 b_3}{6} - \frac{b_3}{9} - \frac{b_0 b_3}{4} + \frac{b_1 b_3}{5} - \frac{b_2 b_3}{6} + \frac{{b_3}^2}{7} \\
&= -\frac{4}{7} b_1 - \frac{4}{9} b_3 + 2{b_0}^2 + \frac{2}{3}{b_1}^2 + \frac{2}{5}{b_2}^2 + \frac{2}{7}{b_3}^2 + \frac{4}{3} b_0 b_2 + \frac{4}{5} b_1 b_3 + \frac{2}{11}
\end{aligned}
$$

We now find the partial derivatives of E with respect to each b_i to get an inconsistent system of linear equations:

$$
\begin{aligned}
\frac{\partial E}{\partial b_0} &= 4b_0 + \frac{4}{3} b_2 \\
\frac{\partial E}{\partial b_1} &= -\frac{4}{7} + \frac{4}{3} b_1 + \frac{4}{5} b_3 \\
\frac{\partial E}{\partial b_2} &= \frac{4}{5} b_2 + \frac{4}{3} b_0 \\
\frac{\partial E}{\partial b_3} &= -\frac{4}{9} + \frac{4}{7} b_3 + \frac{4}{5} b_1
\end{aligned}
$$

Now, setting these equal to zero we have:

$$
\begin{aligned}
3b_0 + b_2 &= 0 \\
35b_1 + 21b_3 &= 15 \\
5b_0 + 3b_2 &= 0 \\
63b_1 + 45b_3 &= 35
\end{aligned}
$$

We are now able to represent this system of equations with matrices:

$$
\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 35 & 0 & 21 \\ 5 & 0 & 3 & 0 \\ 0 & 63 & 0 & 45 \end{bmatrix}
\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 15 \\ 0 \\ 35 \end{bmatrix}
$$

Now we multiply both sides on the left by the transpose of the 4×4 matrix in the equation:

$$\begin{bmatrix} 3 & 0 & 5 & 0 \\ 0 & 35 & 0 & 63 \\ 1 & 0 & 3 & 0 \\ 0 & 21 & 0 & 45 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 35 & 0 & 21 \\ 5 & 0 & 3 & 0 \\ 0 & 63 & 0 & 45 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 5 & 0 \\ 0 & 35 & 0 & 63 \\ 1 & 0 & 3 & 0 \\ 0 & 21 & 0 & 45 \end{bmatrix} \begin{bmatrix} 0 \\ 15 \\ 0 \\ 45 \end{bmatrix}$$

$$\begin{bmatrix} 34 & 0 & 18 & 0 \\ 0 & 5194 & 0 & 3570 \\ 18 & 0 & 10 & 0 \\ 0 & 3570 & 0 & 2466 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2730 \\ 0 \\ 1890 \end{bmatrix}$$

Using a calculator to solve this system yields:

$$\begin{aligned} b_0 &= 0 \\ b_1 &= -\frac{5}{21} \\ b_2 &= 0 \\ b_3 &= \frac{10}{9} \end{aligned}$$

Therefore our final solution is

$$P_3(x) = -\frac{5}{21}x + \frac{10}{9}x^3$$

b)Let $P_3(x) = a_0 l_0(x) + a_1 l_1(x) + a_2 l_2(x) + a_3 l_3(x)$ and derive and solve the *vastly nicer* normal equations for the Legendre Orthogonal Polynomials.

We must find $P_3(x)$ defined by:

$$P_3(x) = a_0 l_0 + a_1 l_1 + a_2 l_2 + a_3 l_3$$

To begin we find a_i for $i = 0, 1, 2, 3$. We do so using

$$a_i = \frac{\int_{-1}^{1} f(x) l_i(x)\,\mathrm{d}x}{\int_{-1}^{1} l_i(x)^2\,\mathrm{d}x}$$

In this manner we can work out the different a_i.

$$\begin{aligned}
a_0 &= \frac{\int_{-1}^{1} x^5\,\mathrm{d}x}{\int_{-1}^{1}\mathrm{d}x} = 0 \\
a_1 &= \frac{\int_{-1}^{1} x^5 x\,\mathrm{d}x}{\int_{-1}^{1} x^2\,\mathrm{d}x} = \frac{3}{7} \\
a_2 &= \frac{\int_{-1}^{1} x^5 (x^2 - \frac{1}{3})\,\mathrm{d}x}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2\,\mathrm{d}x} = 0 \\
a_3 &= \frac{\int_{-1}^{1} x^5 (x^3 - \frac{3}{5}x)\,\mathrm{d}x}{\int_{-1}^{1} (x^3 - \frac{3}{5}x)^2\,\mathrm{d}x} = \frac{10}{9}
\end{aligned}$$

Now we can plug in to find $P_3(x)$:

$$\begin{aligned}
P_3(x) &= \frac{3}{7}x + \frac{10}{9}(x^3 - \frac{3}{5}x) \\
&= -\frac{5}{21} + \frac{10}{9}x^3
\end{aligned}$$

c) Comparing our answers of parts **a** and **b**, we see they are exactly the same. This is great! Graphing our function over x^5 we have the following graph. This approximation looks pretty dang good.